Serre dimension and Euler class groups of overrings of polynomial rings

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Abstract

Let R be a commutative Noetherian ring of dimension d and $B = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ a Laurent polynomial ring over R. If $A = B[Y, f^{-1}]$ for some $f \in R[Y]$, then we prove the following results:

(i) If f is a monic polynomial, then Serre dimension of A is $\leq d$. In case $n = 0$, this result is due to Bhatwadekar, without the condition that f is a monic polynomial.

(ii) The p-th Euler class group $E^p(A)$ of A, defined by Bhatwadekar and Raja Sridharan, is trivial for $p \geq max\{d+1, \dim A - p + 3\}$. In case $m = n = 0$, this result is due to Mandal-Parker.

1 Introduction

In this paper, we will assume that all rings are commutative Noetherian of finite Krull dimension, all modules are finitely generated and all projective modules are of constant rank. **Throughout this paper, R will denote a ring of dimension d and B will denote the Laurent polynomial ring $R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ over R.**

Let P be a projective R-module. An element $p \in P$ is said to be *unimodular* if there exist $\phi \in Hom(P, R)$ such that $\phi(p) = 1$. We write $\mathrm{Um}(P)$ for the set of all unimodular elements of P. We say that *Serre dimension* of R is $\leq t$ if every projective R-module of rank $\geq t + 1$ has a unimodular element.

Quillen [21] and Suslin [24] proved Serre's conjecture that projective modules over polynomial rings $k[X_1, \ldots, X_m]$ over a field k are free for all $m \geq 1$ or equivalently Serre dimension of $k[X_1, \ldots, X_m]$ is 0. It is a classical result due to Serre [23] that Serre dimension of R is $\leq d$. Plumstead ([20], Theorem 2) generalized Serre's result by proving that Serre dimension of $R[Y]$ is $\leq d$. Rao ([22], Theorem 1.1) generalized Plumstead's result and proved that if C is a birational overring of $R[Y]$, i.e. $R[Y] \subset C \subset S^{-1}R[Y]$, where S is the set of all non-zerodivisors of $R[Y]$, then Serre dimension of C is $\leq d$. As a consequence of Rao's result, we get that Serre dimension of $R[Y, f^{-1}] \leq d$ for any non-zerodivisor $f \in R[Y]$.

Bhatwadekar-Roy ([6], Theorem 3.1) generalized Plumstead's result and proved that Serre dimension of $R[X_1, \ldots, X_m]$ is $\leq d$ for any $m \geq 1$. Bhatwadekar-Lindel-Rao ([2], Theorem 4.1) generalized Bhatwadekar-Roy's result and proved that Serre dimension of $B := R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ is $\leq d$.

Bhatwadekar ([1], Theorem 3.5) further generalized above result of Bhatwadekar-Roy to polynomial extensions over a birational overring of $R[Y]$. More precisely, he proved that if C is a birational overring of $R[Y]$, then Serre dimension of $C[X_1,\ldots,X_m]$ is $\leq d$. As a consequence of this result, we get that Serre dimension of $R[X_1,\ldots,X_m,Y,f^{-1}]$ is $\leq d$ for any non-zerodivisor $f\in R[Y]$.

It is natural to ask if analogue of Bhatwadekar's result [1] is true for Laurent polynomial rings. More precisely, we can ask the following.

Question 1.1 *Let C be a birational overring of $R[Y]$. Is Serre dimension of $C[X_1,\ldots,X_m,Y_1^{\pm 1},\ldots,Y_n^{\pm 1}]$ $\leq d$?*

We answer this question when $C=R[Y,f^{-1}]$ with $f\in R[Y]$ a monic polynomial. Note that Lindel [13] gave another proof of Bhatwadekar-Lindel-Rao's result ([2], Theorem 4.1) mentioned above. Our proof closely follows Lindel's idea. We state our result.

Theorem 1.2 *Let $A=B[Y,f^{-1}]$, where $f\in R[Y]$ is a monic polynomial. Then Serre dimension of A is $\leq d$.*

Assume $\dim R=d\geq 3$ and p is a positive integer such that $p\geq d-p+3$. Then Bhatwadekar and Raja Sridharan have defined the p-th Euler class group $E^p(R)$ of R which is an additive abelian group. We will not give the explicit definition of $E^p(R)$ (see [5], section 4 for definition). Rather we will describe the elements of $E^p(R)$, since this suffices for our purpose. Let I be an ideal of R of height p such that the R/I-module I/I^2 is generated by p elements. Let $\phi:(R/I)^p\twoheadrightarrow I/I^2$ be a surjection, giving a set of p generators of R/I-module I/I^2. The surjection ϕ induces an element of the p-th Euler class group $E^p(R)$, denote it by the pair (I,ϕ). Further, it follows using *moving lemma and addition principle*, that every element of $E^p(R)$ is a pair (I,ϕ) for some height p ideal I of R and some surjection $\phi:(R/I)^p\twoheadrightarrow I/I^2$. Bhatwadekar and Raja Sridharan ([5], Theorem 4.2) proved that there exist a surjection $\Phi:R^p\twoheadrightarrow I$ which is a lift of ϕ, i.e. $\Phi\otimes A/I=\phi$, if and only if the associated element (I,ϕ) of the group $E^p(R)$ is the trivial element (identity element 0 of $E^p(R)$).

It is well known that a projective R-module of rank d need not, in general, have a unimodular element. The significance of Euler class group theory is demonstrated by the following result of Bhatwadekar-Raja Sridharan [3], where they proved that for a rank d projective R-module P with trivial determinant, the precise obstruction for P to have a unimodular element lies in $E^d(R)$. More precisely, given a pair (P,χ), where $\chi:\wedge^d P\stackrel{\sim}{\to} R$ is an isomorphism, they associate an element $e(P,\chi)$ of the Euler class group $E^d(R)$ and prove that P has a unimodular element if and only if $e(P,\chi)$ is the trivial element of $E^d(R)$. Such an obstruction theory is not known for projective R-modules of rank $d-1$ except for some special class of rings. When $R=S[Y]$ is a polynomial ring in one variable over some subring S of R, then Das [7] proved that for a rank $d-1$ projective R-module Q with trivial determinant, the precise obstruction for Q to have a unimodular element lies in $E^{d-1}(R)$.

Let I be an ideal of $R[Y]$ containing a monic polynomial in the variable Y. Assume $R[Y]/I$-module I/I^2 is generated by p elements, where $p\geq \dim(R[Y]/I)+2$. Then Mandal ([14], Theorem 2.1) proved that any surjection $\phi:(R[Y]/I)^p\twoheadrightarrow I/I^2$ can be lifted to a surjection $\Phi:R[Y]^p\twoheadrightarrow I$. Let $P=Q\oplus R$ be a projective R-module of rank p and $\psi:P[Y]/IP[Y]\twoheadrightarrow I/I^2$ be a surjection, then Bhatwadekar-Raja Sridharan ([5], Proposition 3.3) proved that ψ lifts to a surjection $\Psi:P[Y]\twoheadrightarrow I$, thus generalizing Mandal's result. If we further assume that height of I is p and $2p \geq \dim R[Y]+3$, then to the surjection ϕ, we can associate an element $(I,\phi) \in E^p(R[Y])$. Since Φ is a surjective lift of ϕ, by ([5], Theorem 4.2), we get that (I,ϕ) is a trivial element of $E^p(R[Y])$.

Let $A = R[X_1,\ldots,X_m]$ be a polynomial ring over R and I an ideal of A of height $\geq d+1$. Let $p \geq max\{\dim(A/I)+2, d+1\}$ be an integer and $\phi : (A/I)^p \twoheadrightarrow I/I^2$ be a surjection. Since height of $I > d$, by Suslin (2.5), there exist an automorphism Θ of A such that $\Theta(I)$ contains a monic polynomial in X_m with coefficients from $R[X_1,\ldots,X_{m-1}]$. Therefore replacing I by $\Theta(I)$, we may assume that I contains a monic polynomial in X_m. By Mandal ([14], Theorem 2.1) mentioned above, ϕ can be lifted to a surjection $\Phi : A^p \twoheadrightarrow I$. Therefore if we further assume that $p \geq max\{\dim A - p+3, d+1\}$, then by ([5], Theorem 4.2), the associated element (I,ϕ) of $E^p(A)$ is trivial. Since any element of $E^p(A)$ is a pair (I,ϕ) for some height p ideal I of A, we get that the p-th Euler class group $E^p(A)=0$. In particular, $E^{d+1}(R[Y])=0$ for $d \geq 2$. This result is generalized by Mandal-Parker ([17], Theorem 3.1) where they prove that $E^{d+1}(R[Y,f^{-1}])=0$ for $d \geq 2$ and $f \in R[Y]$. We generalize Mandal-Parker's result as follows.

Theorem 1.3 *Let $A = B[Y,f^{-1}]$ for some $f \in R[Y]$ and p an integer such that $p \geq max\{\dim A - p+2, d+1\}$. Let $P = Q \oplus R$ be a projective R-module of rank p and I a proper ideal of A of height $\geq d+1$. Assume there is a surjection $\phi : P \otimes A/I(P \otimes A) \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\Phi : P \otimes A \twoheadrightarrow I$. As a consequence, taking P to be free, we get that any p generators of I/I^2 can be lifted to p generators of I.*

The following result is a direct consequence of (1.3).

Corollary 1.4 *Let $A = B[Y,f^{-1}]$ for some $f \in R[Y]$ and p an integer such that $p \geq max\{\dim A - p+3, d+1\}$. Then the p-th Euler class group $E^p(A)$ of A is zero.*

Let I be an ideal of $R[Y]$ containing a monic polynomial and P a projective R-module of rank p with $p \geq \dim(R[Y]/I)+2$. Let $\phi : P[Y]/IP[Y] \twoheadrightarrow I/I^2$ and $\delta : P \twoheadrightarrow I(0) := \{f(0) | f \in I\}$ be two surjections such that $\phi(0) = \delta \otimes R/I(0)$. Then Mandal ([15], Theorem 2.1) proved that there exists a surjection $\Phi : P[Y] \twoheadrightarrow I$ such that $\Phi \otimes R[Y]/I = \phi$ and $\Phi(0) = \delta$, thus answering a question of Nori [15], in case the ideal I contains a monic polynomial.

Above result of Mandal on homotopy section was generalised by Kumar-Mandal ([10], Theorem 1.2) to Laurent polynomial case as follows: Let I be an ideal of $R[Y,Y^{-1}]$ containing a monic polynomial f in $R[Y]$ with $f(0)=1$. Let P be a projective R-module of rank p with $p \geq \dim(R[Y,Y^{-1}]/I)+2$. Let $\phi : P[Y,Y^{-1}]/IP[Y,Y^{-1}] \twoheadrightarrow I/I^2$ and $\delta : P \twoheadrightarrow I(1) := \{g(Y=1) | g \in I\}$ be two surjections such that $\phi(1) = \delta \otimes R/I(1)$. Then there exists a surjection $\Phi : P[Y,Y^{-1}] \twoheadrightarrow I$ such that $\Phi \otimes R[Y,Y^{-1}]/I = \phi$ and $\Phi(1) = \delta$.

We prove the following result which is an analogue of Kumar-Mandal's result.

Theorem 1.5 *Let $A = B[Y,f^{-1}]$, where $f \in R[Y]$ is a monic polynomial with $f(1)$ a unit in R. Let I be an ideal of A of height $\geq d+1$ and P a projective B-module of rank $\geq max\{d+1, \dim(A/I)+2)\}$. Let*

$\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow I/I^2$ and $\delta : P \twoheadrightarrow I(1)$ be two surjections such that $\delta \otimes I(1)/I(1)^2 = \phi \otimes A/(Y-1)$, where $I(1) = \{g(Y=1) | g \in I\}$ is an ideal of B. Then there exists a surjection $\Psi : P[Y, f^{-1}] \twoheadrightarrow I$ such that $\Psi \otimes A/I = \phi$ and $\Psi(1) = \delta$.

2 Preliminaries

In this section, we note down some results for later use. For a ring A, $\operatorname{ht} I$ will denote the height of an ideal I of A. We begin by stating a result of Lindel ([13], Lemma 1.1).

Proposition 2.1 *Let A be a ring, Q an A-module and $s \in A$ such that Q_s is free A_s-module of rank r. Then there exist $p_1, \ldots, p_r \in Q$, $\phi_1, \ldots, \phi_r \in Q^*$ and $t \geq 1$ such that*

(i) $0 :_A s'A = 0 :_A {s'}^2 A$, where $s' = s^t$.

(ii) $s'Q \subset F$ and $s'Q^* \subset G$, where $F = \sum_{i=1}^r Ap_i \subset Q$ and $G = \sum_{i=1}^r A\phi_i \subset Q^*$.

(iii) $(\phi_i(p_j))_{1 \leq i,j \leq r} = diagonal\,(s', \ldots, s')$. We say F and G are s'-dual submodules of Q and Q^* respectively.

Definition 2.2 (i) Let A be a ring, M an A-module and $\delta : A \to A$ an endomorphism. We say maps $\xi : M \to M$ and $\xi^* : M^* \to M^*$ are δ-semi-linear if ξ and ξ^* are group homomorphisms with respect to addition operation and $\xi(\alpha m) = \delta(\alpha)\xi(m)$, $\xi^*(\alpha\phi) = \delta(\alpha)\xi^*(\phi)$ for any $m \in M$, $\phi \in M^*$ and $\alpha \in A$.

(ii) Let I be an ideal of A and $s \in A$. An endomorphism $h : A \to A$ is called $s^t I$-analytic ($t \in \mathbb{N}$), if $h(s) = s$ and $h(a) - a \in s^t I$ for all $a \in A$ with $0 :_A s^{t-1} = 0 :_A s^t$.

The following result is due to Lindel ([13], Lemma 1.4)

Lemma 2.3 *Let A be a ring, I an ideal in A and M an A-module such that M_s is free of rank r for some $s \in A$. Then by (2.1), there exist $p_1, \ldots, p_r \in M$ and $\phi_1, \ldots, \phi_r \in M^*$ satisfying properties $(i - iii)$ of (2.1).*

Let $F = \sum_{i=1}^r Ap_i \subset M$ and $G = \sum_{i=1}^r A\phi_i \subset M^*$ be submodules of M and M^* respectively. Assume $h : A \to A$ is an $s^{2t}I$-analytic endomorphism of A. Then there exist h-semi-linear maps $\xi : M \to M$ and $\xi^* : M^* \to M^*$ with the following properties:

(i) $\xi(p) - p \in s^t IF$, $\xi^*(\phi) - \phi \in s^t IG$ and $\xi^*(\phi)\xi(p) = h(\phi(p))$ for all $p \in M$ and $\phi \in M^*$.

(ii) If N and N' are submodules of F and G respectively such that $F \subset N \subset M$ and $G \subset N' \subset M^*$, then $\xi(N) = N$ and $\xi^*(N') = N'$.

The following result on fiber product is well known. For a reference (see [16], Proposition 2.2.1).

Proposition 2.4 *Let A be a ring and $f, g \in A$ be such that $fA + gA = A$. Let M and N be two A-modules. Suppose $\phi : M_f \to N_f$ is an A_f-homomorphism and $\psi : M_g \to N_g$ is an A_g-homomorphism such that $\phi_g = \psi_f$. Then*

(i) there exist an A-homomorphism $\xi : M \to N$ such that $\xi_f = \phi$ and $\xi_g = \psi$.

(ii) if ϕ and ψ are surjective, then ξ is surjective.

In case f is a polynomial variable over R, the following result is implicit in Suslin ([25], Lemma 6.2) and is known as Suslin's monic polynomial theorem. The proof of Suslin's monic polynomial thorem works in our case also.

Theorem 2.5 *Let $R[X_1, \ldots, X_m, Y]$ be a polynomial ring over R. Let $f \in R[Y] - R$ and $A = R[X_1, \ldots, X_m, f^{-1}]$. Let I be an ideal of A of height $> d$. Then there exist a positive integer N such that for any set of integers $s_i > N$, if ϕ is the $R[f^{-1}]$-automorphism of $R[X_1, \ldots, X_m, f^{-1}]$ defined by $\phi(X_i) = X_i + f^{-s_i}$ for all i, then $\phi(I)$ contains a monic polynomial in f^{-1} with coefficients from $R[X_1, \ldots, X_m]$.*

The following result is implicit in Mandal's result ([14], Lemma 2.3).

Lemma 2.6 *Let I be an ideal of B of height $> d$ and $n > 0$. Then there exist a $R[Y_n^{\pm 1}]$-automorphism Θ of B such that $\Theta(I)$ contains a monic polynomial in Y_n of the form $1 + Y_n h$ for some $h \in R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}, Y_n]$.*

The following result is due to Bhatwadekar-Lindel-Rao ([2], Theorem 4.1).

Theorem 2.7 *Let P be a projective B-module of rank $> d$. Then P has a unimodular element.*

The following result is due to Bhatwadekar-Raja Sridharan ([4], Proposition 3.3).

Proposition 2.8 *Let I be an ideal of $R[X]$ containing a monic polynomial and $P = Q \oplus A$ a projective R-module of rank r, where $r \geq \dim(R[X]/I) + 2$. Let $\phi : P[X] \twoheadrightarrow I/I^2$ be a surjection. Then ϕ can be lifted to a surjection $\Phi : P[X] \twoheadrightarrow I$.*

The following result is due to Dhorajia-Keshari ([8], Theorem 3.12). We will only state the part needed here.

Theorem 2.9 *Let $A = R[X_1, \ldots, X_m, Y_1, \ldots, Y_n, (f_1 \ldots f_n)^{-1}]$ with $f_i \in R[Y_i]$ and P a projective A-module of rank $r \geq d + 1$. Then P is cancellative, i.e. $P \oplus A^t \overset{\sim}{\to} Q \oplus A^t$ for some integer $t > 0$ implies $P \overset{\sim}{\to} Q$.*

Definition 2.10 For an integer $n > 0$, a sequence of elements $a_1, \ldots, a_n$ in R is said to be a *regular sequence* of length n if a_i is a non-zerodivisor in $R/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, n$.

Let I be an ideal of R. We say I is *set theoretically* generated by n elements $f_1, \ldots, f_n \in R$ if $\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}$.

Assume height of I is n. Then I is said to be a *complete intersection* ideal if I is generated by a regular sequence of length n. Further, I is said to be a *locally complete intersection* ideal if $I_{\mathfrak{p}}$ is a complete intersection ideal of height n for all prime ideals $\mathfrak{p}$ of R containing I. ■

The following result is due to Mandal-Roy ([18], Theorem 2.1). See also ([14] Theorem 6.2.2).

Theorem 2.11 *Let $J \subset I$ be two ideals of $R[X]$ such that I contains a monic polynomial. Assume $I = (f_1, \ldots, f_n) + I^2$ and $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Then J is generated by n elements. As a consequence, since $\sqrt{I} = \sqrt{J}$, I is set-theoretically generated by n elements.*

The following result is due to Ferrand and Szpiro. For a proof see [27] or [19].

Theorem 2.12 *Let I be a locally complete intersection ideal of R of height $n \geq 2$ with $\dim(R/I) \leq 1$. Then there is a locally complete intersection ideal $J \subset R$ of height n such that*
(i) $\sqrt{I} = \sqrt{J}$ and
(ii) J/J^2 is a free R/J-module of rank n.

The following result is easy to prove, hence we omit the proof.

Lemma 2.13 *Let $f \in R[T] - R$. Then*
(i) If I is a proper ideal of $R[T, f^{-1}]$, then $\operatorname{ht} I = \operatorname{ht}(I \cap R[T])$.
(ii) If I is a proper ideal of $R[f, f^{-1}]$, then $\operatorname{ht} I = \operatorname{ht}(I \cap R[f^{-1}])$.

Lemma 2.14 *Let I be an ideal of $A = R[T, f^{-1}]$, where $f \in R[T] - R$. If $J = I \cap R[f^{-1}]$, then $\operatorname{ht} J = \operatorname{ht} I$.*

Proof Assume that I is a prime ideal. If we write $\mathfrak{a} = I \cap R$, then $\operatorname{ht} I = \operatorname{ht} IA_{\mathfrak{a}}$ and $\operatorname{ht} J = \operatorname{ht} JR_{\mathfrak{a}}[f^{-1}]$. Hence we assume that $(R, \mathfrak{a})$ is a local ring. Further if $I = \mathfrak{a}A$ is an extended ideal, then $\operatorname{ht} I = \operatorname{ht} \mathfrak{a} = \operatorname{ht} J$. Hence assume that $I \neq \mathfrak{a}A$. In this case $\operatorname{ht} I = \operatorname{ht} \mathfrak{a} + 1$. Since $R/\mathfrak{a}$ is a field, we get that $R/\mathfrak{a}[f, f^{-1}] \to R/\mathfrak{a}[T, f^{-1}]$ is an integral extension. Hence $\operatorname{ht} I/\mathfrak{a} = \operatorname{ht} \widetilde{J}/\mathfrak{a}$, where $\widetilde{J} = I \cap R[f, f^{-1}]$. Therefore $\operatorname{ht} I = \operatorname{ht} \mathfrak{a} + 1 = \operatorname{ht} \widetilde{J} = \operatorname{ht} J$, by (2.13). The general case follows by noting that $\operatorname{ht} I = \operatorname{ht} \sqrt{I}$, $\sqrt{I} = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_r$, $\sqrt{J} = \mathcal{P}'_1 \cap \ldots \cap \mathcal{P}'_r$, where $\mathcal{P}'_i = \mathcal{P}_i \cap R[f^{-1}]$ and $\operatorname{ht} \mathcal{P}_i = \operatorname{ht} \mathcal{P}'_i$. ■

Lemma 2.15 *Let R be a ring of dimension d, $B = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ and $A = B[Y, f^{-1}]$, where $f \in R[Y] - R$. Let I be an ideal of A of height $> d$. Then there exist an integer $N > 0$ such that for any set of integers s_i, l_i all bigger than N, if ϕ is an $R[Y, f^{-1}]$-automorphism of A, defined by $\phi(X_i) = X_i + f^{-s_i}$ and $\phi(Y_i) = Y_i f^{l_i}$, then $\phi(I)$ contains $1 + fh$ for some $h \in B[Y]$.*

Proof We induct on n. Assume that $n = 0$. If $I_1 = I \cap B[f^{-1}]$, then by (2.14), $\operatorname{ht} I_1 = \operatorname{ht} I > d$. Applying (2.5) to the ring $B[f^{-1}] = R[X_1, \ldots, X_m, f^{-1}]$, we can find a positive integer N_1 such that for any integers $s_i > N_1$, if ϕ_1 is the $R[f^{-1}]$-automorphism of $B[f^{-1}]$ defined by $\phi_1(X_i) = X_i + f^{-s_i}$ for $1 \leq i \leq m$, then $\phi_1(I_1)$ contains a monic polynomial, say F of degree u, in the variable f^{-1} with coefficients from B. Since ϕ_1 naturally extends to an $R[Y, f^{-1}]$-automorphism of A, we get $\phi_1(I)$ contains F and hence it contains $f^u F$ which is of the form $1 + fg$ for some $g \in B[Y]$.

Assume that $n > 0$. Define $L_{Y_n}(I)$ and $L_{Y_n^{-1}}(I)$ as the set of highest degree coefficients and lowest degree coefficients respectively of elements in I as a Laurent polynomial in the variable Y_n. It is easy to see that $L_{Y_n}(I)$ and $L_{Y_n^{-1}}(I)$ are ideals of $C[Y, f^{-1}]$, where $C = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}]$. By ([14], Lemma 3.1), we get that height of the ideals $L_{Y_n}(I)$ and $L_{Y_n^{-1}}(I)$ are $\geq \operatorname{ht} I$.

If we write $L = L_{Y_n}(I) \cap L_{Y_n^{-1}}(I)$, then L is an ideal of $C[Y, f^{-1}]$ of height $\geq \operatorname{ht} I > d$. Hence by induction on n, there exist an integer N_2 such that for any set of integers s_i, l_i all bigger than N_2, if θ_1 is an $R[Y, f^{-1}]$-automorphism of $C[Y, f^{-1}]$ defined by $\theta_1(X_i) = X_i + f^{-s_i}$ and $\theta_1(Y_j) = Y_j f^{l_j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, then $\theta_1(L)$ contains a polynomial $\widetilde{h} = 1 + fh'$ for some $h' \in C[Y]$.

We extend θ_1 to an $R[Y_n^{\pm 1}, Y, f^{-1}]$-automorphism of A. We can find a polynomial G in $\theta_1(I)$ of the form $G = \widetilde{h} + h_1 Y_n + \cdots + h_t Y_n^t$ for some $t \in \mathbb{N}$, $h_i \in C[Y, f^{-1}]$ and $\widetilde{h}$ as above. We can choose an integer $N_3 = \max \{\text{power of } f^{-1} \text{ occurring in } G\}$ such that for any integer $l_n > N_3$, if θ_2 is an $C[Y, f^{-1}]$-automorphism of A defined by $\theta_2(Y_n) = Y_n f^{l_n}$, then $\theta_2(G) = 1 + fh$ for some $h \in B[Y]$.

We note that $\theta_2\theta_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by $X_i \mapsto X_i + f^{-s_i}$ and $Y_j \mapsto Y_j f^{l_j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Taking $N = max\{N_2, N_3\}$ completes the proof. ∎

Proposition 2.16 *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial and I an ideal of A of height $> d$. Then there exist an integer $N > 0$ such that for any set of integers t_i, s_i, l_i all bigger than N, the $R[Y, f^{-1}]$-automorphism ϕ of A defined by $\phi(X_i) = X_i + Y^{t_i} + f^{-s_i}$ and $\phi(Y_i) = Y_i f^{l_i}$ satisfies the following:*

(i) $\phi(I)$ contains a monic polynomial in Y with coefficients from B and

(ii) $\phi(I)$ contains a polynomial of the form $1 + fh$ for some $h \in B[Y]$.

Proof Assume $n = 0$. Then $B = R[X_1, \ldots, X_m]$. By (2.14), $\operatorname{ht} I \cap B[f^{-1}] = \operatorname{ht} I > d$. By (2.15), there exists $N_1 > 0$ such that for any $s_i > N_1$, if ϕ_1 is the $R[Y, f^{-1}]$-automorphism of $B[f^{-1}]$ defined by $\phi_1(X_i) = X_i + f^{-s_i}$ for $1 \leq i \leq m$, then $\phi_1(I \cap B[f^{-1}])$ contains $1 + fg$ for some $g \in B[Y]$.

By (2.13), $\operatorname{ht} \phi_1(I) \cap B[Y] = \operatorname{ht} I > d$. Applying (2.5) to $B[Y]$, there exists $N_2 > 0$ such that for any $t_i > N_2$, if ϕ_2 is the $R[Y]$-automorphism of $B[Y]$ defined by $\phi_2(X_i) = X_i + Y^{t_i}$ for $1 \leq i \leq m$, then $\phi_2(\phi_1(I) \cap B[Y])$ contains a monic polynomial, say G, in the variable Y. Since $\phi_2\phi_1$ naturally extends to an $R[Y, f^{-1}]$-automorphism of A, we get that $\phi_2\phi_1(I)$ contains

(i) a monic polynomial G in the variable Y with coefficients from B, and

(ii) an element $1 + fh$, where $h = \phi_2(g) \in B[Y]$.

Note that $\phi_2\phi_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by $X_i \mapsto X_i + Y^{t_i} + f^{-s_i}$. This proves the result in case $n = 0$, by taking $N = max\{N_1, N_2\}$.

Assume $n > 0$ and use induction on n. Defining $L_{Y_n}(I)$, $L_{Y_n^{-1}}(I)$ and L as in (2.15), we get that L is an ideal of $C[Y, f^{-1}]$ of height $\geq \operatorname{ht} I > d$, where $C = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}]$. By inductive hypothesis, there exist an integer N_3 such that for any set of integers t_i, s_i, l_i all bigger than N_3, if θ_1 is an $R[Y, f^{-1}]$-automorphism of $C[Y, f^{-1}]$ defined by $\theta_1(X_i) = X_i + Y^{t_i} + f^{-s_i}$ and $\theta_1(Y_j) = Y_j f^{l_j}$ for all i, j, then $\theta_1(L)$ contains

(a) a monic polynomial, say $\widetilde{g}$, in Y with coefficients from C and

(b) a polynomial $\widetilde{h}$ of the form $1 + fh'$ for some $h' \in C[Y]$.

We extend θ_1 to an $R[Y_n^{\pm 1}, Y, f^{-1}]$-automorphism of A. We can find F and G in $\theta_1(I)$ of the form $F = \widetilde{g}Y_n^s + g_{n-1}Y_n^{s-1} + \cdots + g_0$ and $G = \widetilde{h} + h_1Y_n + \cdots + h_tY_n^t$ for some $s, t \in \mathbb{N}$, $g_i, h_i \in C[Y, f^{-1}]$ and $\widetilde{g}, \widetilde{h}$ as in (a), (b).

Choose $N_4 = \max$ { power of f^{-1} occurring in G and degrees of $\widetilde{g}$, g_i in Y } such that for any integer $l_n > N_4$, if θ_2 is an $C[Y, f^{-1}]$-automorphism of A defined by $\theta_2(Y_n) = Y_n f^{l_n}$, then

(i) $Y_n^{-s}\theta_2(F)$ is a monic polynomial in Y with coefficients from $C[Y_n^{\pm 1}] = B$ (here we are using f to be monic) and

(ii) $\theta_2(G) = 1 + fh$ for some $h \in B[Y]$.

We note that $\theta_2\theta_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by $X_i \mapsto X_i + Y^{t_i} + f^{-s_i}$ and $Y_j \mapsto Y_j f^{l_j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Taking $N = max\{N_3, N_4\}$ completes the proof. ■

3 Main Theorems

In this section, we prove the results stated in the introduction.

Theorem 3.1 *Let R be a ring of dimension d, $B = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ and $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial. Then Serre dimension of A is $\leq d$.*

Proof Without loss of generality, we may assume that R is reduced. If $m = 0$, then replacing A by $A[X_1]$, we will assume that $m > 0$. Let P be a projective A-module of rank $r > d = \dim R$. We need to show that P has a unimodular element. If S denote the set of all non-zerodivisors of R, then $S^{-1}R$ is a zero dimensional ring. Therefore, by Dhorajia-Keshari ([8], Lemma 3.9), we can find some $s \in S$ such that P_s is a free A_s-module of rank r. By Lindel (2.1), there exist an integer $t > 0$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in P^*$ such that the submodules $F = \sum_{i=1}^r Ap_i$ of P and $G = \sum_{i=1}^r A\phi_i$ of P^* satisfies the followings: $s^tP \subset F$, $s^tP^* \subset G$ and the matrix $(\phi_i(p_j)) = diag(s^t, \ldots, s^t)$. The submodules F and G are called s^t-dual submodules of P and P^* respectively. Replacing s by s^t, we assume that F and G satisfies $sP \subset F$, $sP^* \subset G$ and $(\phi_i(p_j)) = diag(s, \ldots, s)$.

Since $A/(s(Y-1)) = \widetilde{R}[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ is a Laurent polynomial ring over a d dimensional ring $\widetilde{R} := R[Y, f^{-1}]/(s(Y-1))$, by Bhatwadekar-Lindel-Rao (2.7), $P/(s(Y-1))$ has a unimodular element. Let $p \in P$ be such that its image $\overline{p}$ in $P/s(Y-1)P$ is a unimodular element.

Let us write $\phi_i(p) = a_i \in A$ for $1 \leq i \leq r$ and define $b := (1-Y)\prod_{i=1}^m X_i \prod_{j=1}^n Y_j$. Then sb is a non-zerodivisor in A. We can find an integer $l > deg(a_1)$ such that $a_1' := a_1 + s^2b^l$ is a non-zerodivisor in A, where $deg(a_1)$ is the total degree of a_1 as a polynomial in $X_1, \ldots, X_m$ with coefficients from $R[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}, Y, f^{-1}]$. Hence height of the ideal $a_1'A$ is ≥ 1.

Since $\overline{p}$ is a unimodular element in $P/s(Y-1)P$ and $\phi_1, \ldots, \phi_r$ is a basis of the free module P_s^*, we get that $(a_1, a_2, \ldots, a_r, s^2(Y-1)) \in \text{Um}_{r+1}(A_s)$. Since $a_1' \in a_1 + s^2(Y-1)A$, we get $(a_1', a_2, \ldots, a_r, s^2(Y-1)) \in \text{Um}_{r+1}(A_s)$. Hence by prime avoidance argument, we can choose $c_2, \ldots, c_r$ in A such that if $a_i' = a_i + s^2(Y-1)c_i$ for $2 \leq i \leq r$, then height of the ideal $(a_1', \ldots, a_r')A_{s(Y-1)}$ is $\geq r$. Let $l' > 2\widetilde{d}$ be an integer, where $\widetilde{d}$ is the maximum of total degrees of $a_1', \ldots, a_r'$ as a polynomial in

$X_1,\dots,X_m$. If we write $a''_r := a'_r + s^2(Y-1)(a'_1)^{l'}$, then degree of a''_r as a polynomial in $X_1,\dots,X_m$ is $e' := mll'$.

Let $q = c_2p_2 + \cdots + c_{r-1}p_{r-1} + (c_r + (a'_1)^{l'})p_r$. Then $\widetilde{p} := p + sb^lp_1 + s(Y-1)q$ is also a lift of $\overline{p}$. Further we have $\phi_i(\widetilde{p}) = a'_i$ for $1 \le i \le r-1$ and $\phi_r(\widetilde{p}) = a''_r$. Hence replacing p by $\widetilde{p}$, we see that height of the ideal $O_P(p)A_{s(Y-1)} = (a'_1,\dots,a'_{r-1},a''_r)A_{s(Y-1)}$ is $\ge r$.

Since $\overline{p}$ is a unimodular element in $P/s(Y-1)P$ and $p \in P$ is a lift of $\overline{p}$, we get $O_P(p)+s(Y-1)A = A$. Further height of the ideal $O_P(p)A_{s(Y-1)}$ is $\ge r$. Therefore we get that height of the ideal $O_P(p)$ is $\ge r$. By (2.16), there exist an integer $N > 0$ such that for any integers t', s', l'' all bigger than N, if Θ is the $R[Y, f^{-1}]$-automorphism of A defined by $\Theta(X_i) = X_i + Y^{t'} + f^{-s'}$ and $\Theta(Y_j) = Y_j f^{l''}$ for $1 \le i \le m$ and $1 \le j \le n$, then the following holds:

(a) $\Theta(O_P(p))$ contains a monic polynomial in Y with coefficients from B.

(b) $\Theta(O_P(p))$ contains a polynomial $g \in B[Y]$ of the form $1 + fh$ for some $h \in B[Y]$.

Further if we choose s' and l'' in the automorphism Θ such that $s' > \frac{nl}{(ml-1)}l''$, then with $e := (ms' - nl'')ll'$, the following holds:

(c) $f^e\Theta(a'_i) \in B[Y]$ for $1 \le i \le r-1$.

(d) $f^e\Theta(a''_r) \in s^{2l'+2}\prod_1^n Y_i^{ll'} + fB[Y]$.

Parts (a) and (b) follows from (2.16). For (c), recall $l' >$ the maximum of total degrees of $a'_1,\dots,a'_r$, hence we only have to ensure $e > l's'$. This is indeed the case because of our choice of s'. Part (d) is a direct consequence of the choice of e and s'.

Replacing A by $\Theta(A)$, we assume that

(a') $O_P(p)$ contains a monic polynomial in Y with coefficients from B.

(b') $O_P(p)$ contains a polynomial $g \in B[Y]$ of the form $1 + fh$ for some $h \in B[Y]$.

(c') $f^e a'_i \in B[Y]$ for $1 \le i \le r-1$.

(d') $f^e a''_r \in s^{2l'+2}\prod_1^n Y_i^{ll'} + fB[Y]$.

We have $g = 1 + fh \in O_P(p)$ for some $h \in B[Y]$, hence $A = B[Y] + gA$. To see this, write $a \in A$ as $a = b/f^r$ for some $b \in B[Y]$. Then $f^ta \in B[Y]$ and $(g-1)^ta = (fh)^ta \in B[Y]$, since $h \in B[Y]$. Hence $a = (g-1)^ta + gg_0 \in B[Y] + gA$.

Since $A = O_P(p) + s(Y-1)A$, using previous relation, we get $A = O_P(p) + s(Y-1)B[Y]$. Therefore $B[Y] = A \cap B[Y] = O_P(p) \cap B[Y] + s(Y-1)B[Y]$. Using (a'), ([11], page 100, Lemma 1.1) and $B[Y] = O_P(p) \cap B[Y] + sB[Y]$, we get $B = (O_P(p) \cap B) + sB$. Hence we get that

(i) $O_P(p)$ contains an elements $1 + b's$ for some $b' \in B$.

(ii) $O_P(p)$ contains an element $1 + s(Y-1)a$ for some $a \in B[Y]$.

Let ψ'_1 and ψ'_2 in P^* be such that $\psi'_1(p) = 1 + b's$ and $\psi'_2(p) = 1 + s(Y-1)a$. We can choose an integer $l_0 > 0$ such that $f^{l_0}\psi'_j(p_i) \in B[Y]$ for $j = 1,2$ and $1 \le i \le r$. Write $\phi_{r+j} = f^{l_0}\psi'_j \in P^*$ for $j = 1,2$ and $p_{r+1} = f^ep \in P$.

Consider the $B[Y]$-modules $M := \sum_{i=1}^{r+1} B[Y]p_i$ and $H := \sum_{i=1}^{r+2} B[Y]\phi_i$. We have $\phi_i(p_j) \in B[Y]$ for $1 \le i \le r+2$ and $1 \le j \le r+1$. Further we have $M \otimes_{B[Y]} A \subset P$ and $H \otimes_{B[Y]} A \subset P^*$.

Since $sP \subset F$, we get $sp_{r+1} = \sum_{i=1}^r b_ip_i$ for some $b_i \in A$ and hence $\phi_i(sp_{r+1}) = sb_i$ for $1 \le i \le r$. Since s is a non-zerodivisor in A, we get $b_i = \phi_i(p_{r+1}) \in B[Y]$. Therefore $sp_{r+1} =$

$\sum_1^r \phi_i(p_{r+1})p_i$. Hence if we write $F' := \sum_{i=1}^r B[Y]p_i$, then we have $sp_{r+1} \in F'$. Similarly if we write $G' := \sum_{i=1}^r B[Y]\phi_i$, then we get $s\phi_{r+j} \in G'$ for $j = 1, 2$. Therefore M_s and H_s are free modules over $B_s[Y]$ with $M_s = F'_s$ and $H_s = G'_s$. Further F' and G' are s-dual submodules of M and M^* respectively, i.e. $sM \subset F'$, $sH \subset G'$ and the matrix $(\phi_i(p_j)) = diag(s, \dots, s)$.

Let us define a B-algebra endomorphism $\delta : B[Y] \to B[Y]$ by $\delta|B = id$ and $\delta(Y) = 1 + (Y-1)(1 - b'^2 s^2) = Y + s^2 b'^2(1 - Y)$, where $b' \in B$ is chosen earlier as $\phi_{r+1}(p) = f^{l_0}(1 + b's)$. Since $\delta(Y^t) - Y^t \in s^2 B[Y]$ for all integers $t \geq 0$, we get that $\delta(\alpha) - \alpha \in s^2 B[Y]$ for any $\alpha \in B[Y]$. Such an endomorphism δ of $B[Y]$ is called s^2-analytic. Recall that $M = \sum_1^{r+1} B[Y]p_i$ is a $B[Y]$-module.

Applying (2.3) to above data, we get δ-semi-linear maps $\xi : M \to M$ and $\xi^* : M^* \to M^*$ such that $\xi^*(\phi)(\xi(p)) = \delta(\phi(p))$ for any $\phi \in M^*$ and $p \in M$. Further $\xi^*(H) \subset H$. Therefore $A \otimes_{B[Y]} \xi^*(H) \subset P^*$.

We have the followings:

(i') $\phi_r(p_{r+1}) = \phi_r(f^e p) = s^{2l'+2} \prod_1^n Y_i^{ll'} + f\widetilde{b}$ for some $\widetilde{b} \in B[Y]$, using (d').

(ii') $\phi_{r+1}(p_{r+1}) = f^{l_0}\psi'_1(f^e p) = f^{l_0+e}(1 + b's)$ for $b' \in B$, using (i).

(iii') $\phi_{r+2}(p_{r+1}) = f^{l_0}\psi'_2(f^e p) = f^{l_0+e}(1 + s(Y-1)a)$ for some $a \in B[Y]$, using (ii).

Using the relation $\xi^*(\phi)(\xi(p_{r+1})) = \delta(\phi(p_{r+1}))$, we see that the δ-images of elements in $(i') - (iii')$ belong to $O_M(\xi(p_{r+1}))$. Further using $A \otimes_{B[Y]} \xi(M) \subset P$ and $A \otimes_{B[Y]} \xi^*(H) \subset P^*$, we see that δ-images of above three elements belong to $O_P(1 \otimes \xi(p_{r+1}))$. We will show that $1 \otimes \xi(p_{r+1})$ is a unimodular element of P by showing that the δ-images of above three elements generate the unit ideal. Suppose not, then there exist a maximal ideal $\mathfrak{m}$ containing elements

(i'') $\delta(s^{2l'+2} \prod_1^n Y_i^{ll'} + f\widetilde{b}) = s^{2l'+2} \prod_1^n Y_i^{ll'} + \delta(f)\delta(\widetilde{b})$,

(ii'') $\delta(f^{l_0+e}(1 + b's)) = \delta(f)^{l_0+e}(1 + b's)$ and

(iii'') $\delta(f^{l_0+e}(1 + s(Y-1)a)) = \delta(f)^{l_0+e}(1 + s\delta(Y-1)\delta(a))$.

Assume $\delta(f) \in \mathfrak{m}$. Then by (i''), we get that $s \in \mathfrak{m}$ as Y_i's are units in A. Since $\delta(f) - f \in (s^2)$, we get $f \in \mathfrak{m}$. This is a contradiction, since f is a unit in A. In the other case, assume $\delta(f) \notin \mathfrak{m}$. Then $1 + s\delta(Y-1)\delta(a) \in \mathfrak{m}$ and $1 + b's \in \mathfrak{m}$. Since $\delta(Y-1) = (Y-1)(1 - b'^2 s^2) \in (1 + b's)A$, we get $\delta(Y-1) \in \mathfrak{m}$. This shows that $1 \in \mathfrak{m}$, a contradiction. Therefore we get that $1 \otimes \xi(p_{r+1})$ is a unimodular element. This completes the proof. ■

Theorem 3.2 *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$ and p an integer such that $p \geq max\{\dim A - p + 2, d + 1\}$. Let $P = Q \oplus R$ be a projective R-module of rank p and I a proper ideal of A of height $\geq d + 1$. Assume there is a surjection $\phi : P \otimes A/I(P \otimes A) \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\Phi : P \otimes A \twoheadrightarrow I$. As a consequence, taking P to be free, we get that any p generators of I/I^2 can be lifted to p generators of I.*

Proof We assume $n \geq 1$. If $n = 0$, then we can use retraction from $n = 1$ case. If $C := R[X_1, \dots, X_m, Y_1^{\pm 1}, \dots, Y_{n-1}^{\pm 1}]$, then $A = C[Y_n^{\pm 1}, Y, f^{-1}]$ with $f \in R[Y]$. We are given a surjection $\phi : P \otimes (A/I) \twoheadrightarrow I/I^2$, where $P = Q \oplus R$. We want to show that ϕ can be lifted to a surjection $\Phi : P \otimes A \twoheadrightarrow I$.

Let $\Phi_1 : P \otimes A \to I$ be a lift of ϕ. We can find an integer $k > 0$ such that $f^k\Phi_1$ maps $P \otimes C[Y_n^{\pm 1}, Y]$ into $J := I \cap C[Y_n^{\pm 1}, Y]$. Now $f^k\Phi_1$ induces a map $\psi : P \otimes (C[Y_n^{\pm 1}, Y]/J) \to J/J^2$. Note that $\psi_f = f^k\phi : P \otimes (A/I) \twoheadrightarrow (J/J^2)_f$ is a surjection.

Using height of $I > d$ and applying (2.15), we get an $R[Y, f^{-1}]$-automorphism Θ of A such that $\Theta(I)$ contains $1+fh$ for some $h \in C[Y_n^{\pm 1}, Y]$. Replacing A by $\Theta(A)$ and I by $\Theta(I)$, we can assume that $1 + fh \in I$. Since $1 + fh \in J$, we get $(J/J^2)_{1+fh}$ is the zero module. Hence ψ_{1+fh} is a surjections. Applying (2.4), we get $\psi : P \otimes (C[Y_n^{\pm 1}, Y]/J) \to J/J^2$ is a surjection. If ψ has a surjective lift $\Psi : P \otimes C[Y_n^{\pm 1}, Y] \twoheadrightarrow J$, then $f^{-k}\Psi_f : P \otimes A \twoheadrightarrow I$ will be our required surjective lift of ϕ. Therefore it is enough to show that ψ has a surjective lift from $P \otimes C[Y_n^{\pm 1}, Y]$ onto J.

Note that $C[Y_n^{\pm 1}, Y] = B[Y]$ is a Laurent polynomial ring over R and J is an ideal of $C[Y_n^{\pm 1}, Y]$ of height $> d = \dim R$. By (2.6), there exist a $R[Y_n^{\pm 1}]$-automorphism Θ of $C[Y_n^{\pm 1}, Y]$ such that $\Theta(J)$ contains a monic polynomial in Y_n of the form $1 + Y_n h'$ for some $h' \in C[Y, Y_n]$. Replacing J by $\Theta(J)$, we can assume that J contains a monic polynomial $1 + Y_n h'$ in the variable Y_n.

Lift ψ to a map $\Psi_1 : P \otimes C[Y, Y_n^{\pm 1}] \to J$. If we write $K := J \cap C[Y, Y_n]$, then $Y_n^l\Psi_1$ will map $P \otimes C[Y, Y_n]$ into K for some integer $l > 0$. Now $Y_n^l\Psi_1$ will induce a map $\delta : P \otimes (C[Y, Y_n]/K) \to K/K^2$ such that $\delta_{Y_n} = Y_n^l\psi$ is a surjection from $P \otimes (C[Y, Y_n^{\pm 1}]/J)$ onto J/J^2. Since K contains a monic polynomial $1 + Y_n h'$ in Y_n, we get $(K/K^2)_{1+Y_n h'} = 0$. Applying (2.4), we get that $\delta : P \otimes (C[Y, Y_n]/K) \twoheadrightarrow K/K^2$ is a surjection. Applying Bhatwadekar-Raja Sridharan (2.8), we get that δ can be lifted to a surjection $\Delta : P \otimes C[Y, Y_n] \twoheadrightarrow K$. Therefore $Y_n^{-l}\Delta$ is a surjective lift of ψ. This completes the proof. ■

Theorem 3.3 *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial with $f(1)$ a unit in R. Let I be an ideal of A of height $\geq d+1$ and P a projective B-module of rank $\geq max\{d+1, \dim(A/I)+2\}$. Let $\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow I/I^2$ and $\delta : P \twoheadrightarrow I(1)(:= \{g(Y = 1) | g \in I\})$ be two surjections such that $\delta = \phi \otimes A/(Y-1)$, where $I(1)$ is an ideal of B. Then there exists a surjection $\Psi : P[Y, f^{-1}] \twoheadrightarrow I$ such that $\Psi \otimes A/I = \phi$ and $\Psi(1) = \delta$.*

Proof Without loss of generality we assume that $f \in R[Y] - R$. Let $\Phi_1 : P[Y, f^{-1}] \to I$ be any lift of ϕ. Then $\Phi_1(1) = \delta$ modulo $I(1)^2$. Hence $\Phi_1(1) - \delta \in I(1)^2 Hom(P, B)$. Write $\Phi_1(1) - \delta = f_1(1)g_1(1)\alpha_1 + \cdots + f_r(1)g_r(1)\alpha_r$ for some $f_i, g_i \in I$ and $\alpha_i \in Hom(P, B)$. If we write $\Phi_2 := \Phi_1 - (f_1g_1\widetilde{\alpha}_1 + \cdots + f_rg_r\widetilde{\alpha}_r)$, where $\widetilde{\alpha}_i = \alpha_i \otimes id : P \otimes_B A \to A$, then $\Phi_2 : P[Y, f^{-1}] \to I$ is also a lift of ϕ with $\Phi_2(1) = \delta$.

Let $J := I \cap B[Y]$. Then there exist $k > 0$ such that $f^k\Phi_2$ maps $P[Y]$ into J. Now $f^k\Phi_2$ induces a map $\psi : P[Y]/JP[Y] \to J/J^2$. Note that $\psi_f = f^k\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow (J/J^2)_f$ is a surjection.

Since $\operatorname{ht} I > d$, by (2.16), applying some $R[Y, f^{-1}]$-automorphism of A, we may assume that I contains (i) a monic polynomial g in Y with coefficients from B and (ii) an element $1 + fh$ for some $h \in B[Y]$. Since $1 + fh \in J$, we get $(J/J^2)_{1+fh} = 0$. Therefore ψ_{1+fh} is the zero map. By (2.4), we get $\psi : P[Y]/JP[Y] \twoheadrightarrow J/J^2$ is a surjection. Further $f(1)^k\delta : P \twoheadrightarrow J(1)$ is a surjection with $\psi(1) = f(1)^k\delta \otimes B/J(1)$. Since rank of $P \geq \dim B[Y]/J + 2$ holds and J contains monic polynomial

g, using Mandal ([15], Theorem 2.1), there exist a surjection $\Psi : P[Y] \twoheadrightarrow J$ which is a lift of ψ and $\Psi(1) = f(1)^k\delta$. Therefore $\Phi = f^{-k}\Psi_f : P[Y, f^{-1}] \twoheadrightarrow I$ is a surjection which is a lift of $f^{-k}\psi = \phi$ with $\Phi(1) = \delta$. This completes the proof. $\blacksquare$

4 Applications

Let M be a finitely generated R-module. If we write $\mu(M)$ for the minimum number of generators of M as an R-module, then Forster [9] and Swan [26] proved that $\mu(M) \leq max\{\mu(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) | \mathfrak{p} \in \text{Spec}\,(R), M_\mathfrak{p} \neq 0\}$. In particular, if P is a projective R-module of rank r, then $\mu(P) \leq r + d$. As a consequence of our result (1.2), we prove the following result.

Theorem 4.1 *Let $A = B[Y, f^{-1}]$ for some monic polynomial $f \in R[Y]$ and P a projective A-module of rank r. Then $\mu(P) \leq r + d$.*

Proof Assume P is generated by s elements, where $s > r + d$. Then we will show that P is also generated by $s - 1$ elements. By Forster-Swan, we have $s \leq \dim A + r = d + m + n + 1 + r$. Let $\phi : A^s \twoheadrightarrow P$ be a surjection. If Q is the kernel of ϕ, then rank of Q is $s - r > d$. Hence by (1.2), Q has a unimodular element, say $q \in \text{Um}(Q)$. Since $A^s \xrightarrow{\sim} P \oplus Q$, we get $q \in \text{Um}(A^s)$. Since $\phi(q) = 0$, ϕ induces a surjection $\overline{\phi} : A^s/qA \twoheadrightarrow P$. Since $s - 1 > d$, by (2.9), A^{s-1} is cancellative. Hence $A^s/qA \xrightarrow{\sim} A^{s-1}$. Therefore P is generated by $s - 1$ elements. This completes the proof. $\blacksquare$

Proposition 4.2 *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$. Let $J \subset I$ be two ideals of A such that $I = (f_1, \ldots, f_n) + I^2$ and $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Assume that I contains (i) a monic polynomial $F \in C[Y]$ in the variable Y and (ii) an element of the form $1 + fh$ for some $h \in C[Y]$. Then J is generated by n elements. As a consequence, I is set-theoretically generated by n elements.*

Proof Replacing f_i by $f^N f_i$ for some integer $N > 0$, we may assume that $f_i \in B[Y]$ for all i. Let $K = I \cap B[Y]$ be an ideal of $B[Y]$. Let $\phi : (B[Y]/K)^n \to K/K^2$ be the map defined by $e_i \mapsto \overline{f}_i$. Then ϕ_f is surjective and ϕ_{1+fh} is zero map, since $1 + fh \in K$. Hence by (2.4), ϕ is a surjection. Therefore, we get $K = (f_1, \ldots, f_n) + K^2$. If $L := (f_1, \ldots, f_{n-1}) + K^{(n-1)!}$, then $L_f = J$. Since K contains a monic polynomial F, using (2.11), we get that L is generated by n elements. Therefore J is generated by n elements. $\blacksquare$

Theorem 4.3 *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$. Let $J \subset I$ be two ideals of A such that $I = (f_1, \ldots, f_n) + I^2$ and $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Assume that height of $I > d$. Then J is generated by n elements. In particular, I is set-theoretically generated by n elements.*

Proof Applying an automorphism as in (2.15), we may assume that I contains an element $1+fh$ for some $h \in B[Y]$. Now as in (4.2), replacing f_i by $f^N f_i$, we may assume that $f_i \in B[Y]$. Then if $K = I \cap B[Y]$, then $K = (f_1, \ldots, f_n) + K^2$ as in (4.2). Since height of $K > d$, using some automorphism of

$B[Y]$, we may assume that K contains a monic polynomial in Y. Now, if $L = (f_1, \ldots, f_{n-1}) + K^{(n-1)!}$, then by (2.11), L is generated by n elements. Hence $J = K_f$ is generated by n elements. ■

Theorem 4.4 *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$ with further condition that $m+n \geq 1$. Let $I \subset A$ be a locally complete intersection ideal of height $r \geq max\{\dim A - 1, \dim A - r + 2\}$ with $\dim A/I \leq 1$. Then I is set theoretically generated by r elements.*

Proof By Ferrand-Szpiro (2.12), there is a locally complete intersection ideal J of height r such that (i) $\sqrt{J} = \sqrt{I}$ and (ii) J/J^2 is a free A/J-module of rank r. Since $m + n \geq 1$, we get $r \geq d + 1$. By (1.3), the r generators of free module J/J^2 can be lifted to r generators of J. Hence I is set theoretically generated by r elements. ■

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References

[1] S.M. Bhatwadekar, *Inversion of monic polynomials and existence unimodular elements (II)*, Math. Z. **200** (1989) 233-238.

[2] S.M. Bhatwadekar, H. Lindel and R.A. Rao, *The Bass-Murthy question: Serre dimension of Laurent polynomial extensions*, Invent. Math. **81** (1985) 189-203.

[3] S.M. Bhatwadekar and Raja Sridharan, *The Euler class group of a Noetherian ring*, Compositio Math. **122** (2000) no. 2, 183-222.

[4] S.M. Bhatwadekar and Raja Sridharan, *On a question of Roitman*, J. Ramanujan Math. Soc. **16** (2001) no. 1, 45-61.

[5] S.M. Bhatwadekar and Raja Sridharan, *On Euler classes and stably free projective modules*, Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 139-158, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002.

[6] S.M. Bhatwadekar and A. Roy, *Some theorems about projective modules over polynomial rings*, J. Algebra **86** (1984) 150-158.

[7] M.K. Das, *The Euler class group of a polynomial algebra*, J. Algebra **264** (2003), no. 2, 582-612.

[8] A.M. Dhorajia and M.K. Keshari, *Projective modules over overrings of polynomial rings.*, J. Algebra **323** (2010) 551-559.

[9] Otto Forster, *Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring*, Math. Z. **84** (1964) 80-87.

[10] S.D. Kumar and S. Mandal, *Some results on generators of ideals* J. Pure Appl. Algebra **169** (2002) no. 1, 29-32.

[11]

[12] T.Y. Lam, *Serre's Problem on Projective Modules*, Springer-Verlag, Berlin, Heidelberg, 2006.

[13] H. Lindel, *Unimodular elements in projective modules*, J. Algebra **172** (1995) no-2, 301-319.

[14] S. Mandal, *On efficient generation of ideals*, Invent. Math. **75** (1984) no. 1, 59-67.

[15] S. Mandal, *Homotopy of sections of projective modules*, J. Algebraic Geom. **1** (1992), no. 4, 639-646.

[16] S. Mandal, *Projective modules and complete intersections*, Lecture Notes in Mathematics, **1672**, Springer-Verlag, Berlin, 1997.

[17] S. Mandal and K. Parker, *Vanishing of Euler class groups*, J. Algebra **308** (2007), no. 1, 107-117.

[18] S. Mandal and A. Roy, *Generating ideals in polynomial ring*, Math. Z. **195** (1987) 15-20.

[19] M.P. Murthy, *Generators for certain ideals in regular rings of dimension three*, Comment. Math. Helv. **47** (1972), 179-184.

[20] B. Plumstead, *The conjectures of Eisenbud and Evans*, Amer. J. Math. **105** (1983) 1417-1433.

[21] D. Quillen. *Projective modules over polynomial rings,* Invent. Math. **36** (1976), 167-171.

[22] R.A. Rao, *Stability theorems for overrings of polynomial rings II*, J. Algebra **78** (1982), 437-444.

[23] J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, 1958 Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23 18 pp.

[24] A.A. Suslin, *Projecte modules over polynomial rings are free*, Sov. Math. Dokl. **17** (1976), 1160-1164.

[25] A.A. Suslin, *On structure of special linear group over polynomial rings*, Math. of USSR, Isvestija **11** (No. 1-3), (1977) 221-238 (English translation).

[26] R.G. Swan, *The number of generators of a module*, Math. Z. **102** (1967), 318-322.

[27] L. Szpiro, *Equations defining space curves*, Published for Tata Institute of Fundamental Research by Springer-Verlag (1979).