Complete Intersection Ideals and a Question of Nori

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Words cannot express my gratitude to my parents. I dedicate this thesis to them.

Contents

Chapter 0

Introduction

Let k be a field and let V be a closed irreducible sub-variety of $\mathbb{A}_k^n$. Let $\mathfrak{p} \subset k[X_1, \ldots, X_n]$ be the prime ideal defining the variety V. We say that V is *ideal theoretically* generated by d elements if the ideal $\mathfrak{p}$ is generated by d elements and V is *set-theoretically* generated by d elements if there exist $f_1, \ldots, f_d \in \mathfrak{p}$ such that $\sqrt{(f_1, \ldots, f_d)} = \mathfrak{p}$, i.e. the variety V is an intersection of d hyper-surfaces.

By a classical result of Kronecker [17], any variety in $\mathbb{A}_k^n$ is an intersection of $n+1$ hyper-surfaces. Eisenbud and Evans [13] and Storch [33] independently showed that any variety in $\mathbb{A}_k^n$ is an intersection of n hyper-surfaces, thus improving the above result of Kronecker. In other words, given a prime ideal $\mathfrak{p}$ of $k[X_1, \ldots, X_n]$, there exist $f_1, \ldots, f_n \in \mathfrak{p}$ such that $\sqrt{(f_1, \ldots, f_n)} = \mathfrak{p}$, i.e. any prime ideal p in $k[X_1, \ldots, X_n]$ is set-theoretically generated by n elements.

In view of the above result of Eisenbud-Evans and Storch, it is natural to ask:

Question: Does there exists a positive integer d such that $\mu(\mathfrak{p}) \leq d$ for all prime ideals $\mathfrak{p}$ in $k[X_1, \ldots, X_n]$ (k: field), where $\mu(\mathfrak{p})$ denotes the minimal number of generators of $\mathfrak{p}$?

Since $k[X_1, \ldots, X_n]$ is a UFD, every prime ideal $\mathfrak{p}$ of height 1 is principle. Also, it is well known that every maximal ideal of $k[X_1, \ldots, X_n]$ is generated by n elements (see [23], Theorem 5.3). Therefore, it is enough to consider prime ideals of height $1 < \operatorname{ht}\mathfrak{p} < n$.

A classical example of Macaulay (see [2] for details) shows that given any positive integer $r \geq 4$, there exists a height 2 prime ideal $\mathfrak{p}$ in $\mathbb{C}[X_1, X_2, X_3]$ such that $\mu(\mathfrak{p}) \geq r$. Therefore, the set $\{\mu(\mathfrak{p}) \mid \mathfrak{p} : \text{prime ideal in } \mathbb{C}[X_1, X_2, X_3]\}$ is not bounded above. The ring $\mathbb{C}[X_1, X_2, X_3]/\mathfrak{p}$ is not regular for all prime ideals $\mathfrak{p}$ in Macaulay's example. Therefore, one can ask the following modified question:

Question: Does there exists a positive integer d such that $\mu(\mathfrak{p}) \leq d$ for all prime ideal $\mathfrak{p} \subset k[X_1, \dots, X_n]$ such that $k[X_1, \dots, X_n]/\mathfrak{p}$ is regular?

Forster [14] gave an affirmative answer to the above question. More precisely, he proved the following:

Theorem 0.1 *Let $\mathfrak{p} \subset k[X_1, \dots, X_n]$ be a prime ideal, where k is a field. Assume that $k[X_1, \dots, X_n]/\mathfrak{p}$ is regular. Then $\mathfrak{p}/\mathfrak{p}^2$ as $k[X_1, \dots, X_n]/\mathfrak{p}$-module is generated by n elements and hence $\mathfrak{p}$ is generated by $n+1$ elements.*

Since $k[X_1, \dots, X_n]$ is regular, if $\mathfrak{p} \subset k[X_1, \dots, X_n]$ is a prime ideal such that $k[X_1, \dots, X_n]/\mathfrak{p}$ is regular, then $\mathfrak{p}$ is locally generated by $\operatorname{ht}\mathfrak{p}$ elements. Now, it follows from the following theorem (0.2) of Forster [14] (with $A = k[X_1, \dots, X_n]$ and $M = \mathfrak{p}/\mathfrak{p}^2$) that $\mathfrak{p}/\mathfrak{p}^2$ is generated by n elements.

Theorem 0.2 *Let A be a Noetherian ring and let M be a finitely generated A-module. Then M is generated by sup $\{\mu(M_\mathfrak{q}) + \dim(A/\mathfrak{q}) : \mathfrak{q} \in \operatorname{Spec} A\}$ elements.*

Moreover, after proving above result, Forster conjectured:

Conjecture 0.3 *Let $\mathfrak{p} \subset k[X_1, \dots, X_n]$ be a prime ideal such that $k[X_1, \dots, X_n]/\mathfrak{p}$ is regular. Then $\mathfrak{p}$ is generated by n elements, i.e. $\mu(\mathfrak{p}) \leq n$.*

Note that to prove Forster's conjecture, we can assume that $\operatorname{ht}\mathfrak{p} \geq 2$.

Abhyankar [1] (in the case k is algebraically closed) and Murthy [25] independently settled the case $n = 3$ (the first non-trivial case) of Forster's conjecture. More precisely, they proved that if $\mathfrak{p}$ is a prime ideal of $k[X_1, X_2, X_3]$ such that

$k[X_1, X_2, X_3]/\mathfrak{p}$ is regular, then $\mathfrak{p}$ is generated by 3 elements. General case of Forster's conjecture was settled by Sathaye [31] (in the case k is an infinite field) and Mohan Kumar [24] independently.

To prove Forster's conjecture, Mohan Kumar proved the following more general result (proof of this is implicit in ([24], Theorem 5)).

Theorem 0.4 *Let A be a commutative Noetherian ring and let I be an ideal of $A[T]$ such that I/I^2 is generated by n elements. Assume that $n \geq \dim(A[T]/I)+2$. If I contains a monic polynomial, then I is a surjective image of a projective $A[T]$-module of rank n with trivial determinant.*

Forster's conjecture follows from above result of Mohan Kumar. For, suppose $\mathfrak{p}$ is a prime ideal of $k[X_1, \ldots, X_n]$ of height ≥ 2 such that $\mathfrak{p}/\mathfrak{p}^2$ is generated by n elements. Since $\operatorname{ht}\mathfrak{p} \geq 2$, $n \geq \dim(k[X_1, \ldots, X_n]/\mathfrak{p}) + 2$. Further, after a change of variable, we can assume that $\mathfrak{p}$ contains a monic polynomial in the variable X_n. Hence, by Mohan Kumar's result (0.4), $\mathfrak{p}$ is a surjective image of a projective $k[X_1, \ldots, X_n]$-module of rank n. Since, by Quillen-Suslin result [29, 34], every projective $k[X_1, \ldots, X_n]$-module is free. Hence $\mathfrak{p}$ is generated by n elements.

Subsequently, Mandal improved Mohan Kumar's result by showing that I is generated by n elements ([20], Theorem 1.2). More precisely, he proved the following result.

Theorem 0.5 *Let A be a commutative Noetherian ring and let I be an ideal of $A[T]$ such that I/I^2 is generated by n elements. Assume that $n \geq \dim(A[T]/I)+2$. If I contains a monic polynomial, then I is generated by n elements. In-fact, he proved that any n generators of I/I^2 can be lifted to n generators of I.*

It is interesting to investigate the following:
Question: In what generality the above result of Mandal is valid?

Suppose that A is the coordinate ring of the real 3-sphere and $\mathfrak{m}$ is a real maximal ideal. Let $I = \mathfrak{m}A[T]$. Then, it is easy to see that $\mu(I/I^2) = 3 = \dim(A[T]/I) + 2$. Since $\mathfrak{m}$ is not generated by 3 elements [11], I can not be generated by 3 elements. Such examples show that the above result of Mandal is not valid for an ideal I not containing a monic polynomial without further assumptions.

Obviously, one such natural assumption would be that $I(0)$ is generated by n elements, where $I(0)$ denotes the ideal $\{f(0) : f(T) \in I\}$ of A. Even then, as shown in ([6], Example 5.2) I may not be generated by n elements. Therefore, it is natural to ask: *what further conditions are needed to conclude that I is generated by n elements?* Towards this goal, motivated by a result from topology (see Appendix by M. Nori in [21]), Nori posed the following general question:

Question 0.6 *Let A be a regular affine domain of dimension d over an infinite perfect field k and let n be an integer such that $2n \geq d+3$. Let I be a prime ideal of $A[T]$ of height n such that $A[T]/I$ and $A/I(0)$ are regular k-algebras. Let P be a projective A-module of rank n and let $\phi : P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. Then, can we lift ϕ to a surjection from $P[T]$ to I?*

Note that, giving a surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$ is equivalent to giving two surjections $\psi : P[T] \twoheadrightarrow I/I^2$ and $\alpha : P \twoheadrightarrow I(0)$ such that $\psi \otimes A/I(0) = \alpha \otimes A/I(0) : P \twoheadrightarrow I(0)/I(0)^2$ ([6], Remark 3.9).

The **main result** of this thesis (3.15) gives an affirmative answer to the above question of Nori. More precisely, we prove the following ([5], Theorem 4.13):

Theorem 0.7 *Let k be an infinite perfect field and let A be a regular domain of dimension d which is essentially of finite type over k. Let n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n and let P be a projective A-module of rank n. Assume that we are given a surjection*

$$\phi : P[T] \twoheadrightarrow I/(I^2T).$$

Then, there exists a surjection

$$\Phi : P[T] \twoheadrightarrow I$$

such that Φ is a lift of ϕ.

In particular, suppose $I/(I^2T)$ is generated by n elements. Then I is generated by n elements.

Prior to our theorem, the following partial results were obtained:

Mandal ([21], Theorem 2.1) answered the question in affirmative in the case I contains a monic polynomial even without any smoothness condition. An example is given in the case $d = n = 3$ (see [6], Example 6.4) which shows that the question does not have an affirmative answer if we do not assume that I contains a monic polynomial and drop the assumption that A is smooth.

Mandal and Varma ([22], Theorem 4) settled the question, where A is a regular k-spot (i.e. a local ring of a regular affine k-algebra). Subsequently, Bhatwadekar and Raja Sridharan ([6], Theorem 3.8) answered the question in the case $\dim(A[T]/I) = 1$.

Using the techniques developed to prove Theorem 0.7, we prove the following result (4.6).

Theorem 0.8 *Let A be a commutative Noetherian ring containing an infinite field and let P be a projective $A[T]$-module of rank $r \geq (\dim A + 3)/2$ which is extended from A. Assume that $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then P has a unimodular element.*

The above result gives a partial answer to the following question of Roitman [30].

Question 0.9 *Let A be a commutative Noetherian ring and let P be a projective $A[T]$-module such that $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T)$. Then, does P have a unimodular element?*

The layout of the thesis is as follows: In chapter 1, we recall some basic definitions and state some well known results for later use. In chapter 2, we prove some basic results and Subtraction principle which is the main ingredient for our main result. In chapter 3, we prove our main result. Chapter 4 contains some applications of the results proved in previous chapters.

Chapter 1

Preliminaries

All rings considered in this thesis are commutative and Noetherian with unity and all modules are finitely generated. For a ring A, the Jacobson radical of A is denoted by $\mathcal{J}(A)$. We begin with a few definitions and subsequently state some basic and useful results without proof. For all the terms not defined here, we refer to [23].

Definition 1.1 Let A be a ring. The supremum of the lengths r, taken over all strictly increasing chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals of A, is called the *Krull dimension* of A or simply the dimension of A, denoted by $\dim A$.

For a prime ideal $\mathfrak{p}$ of A, the supremum of the lengths r, taken over all strictly increasing chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$ of prime ideals of A, is called the the height of $\mathfrak{p}$, denoted by $\operatorname{ht} \mathfrak{p}$. Note that for a Noetherian ring A, $\operatorname{ht} \mathfrak{p} < \infty$.

For an ideal $I \subset A$, the infimum of the heights of $\mathfrak{p}$, taken over all prime ideals $\mathfrak{p} \subset A$ such that $I \subset \mathfrak{p}$, is defined to be *height* of I, denoted by $\operatorname{ht} I$.

Remark 1.2 Let I be an ideal of A. Then, it is clear from the definition that $\dim(A/I) + \operatorname{ht} I \leq \dim A$.

Definition 1.3 An A-module P is said to be *projective* if it satisfies one of the following equivalent conditions:

(i) Given A-modules M, N and an A-linear surjective map $\alpha : M \twoheadrightarrow N$, the canonical map from $\text{Hom}_A(P, M)$ to $\text{Hom}_A(P, N)$ sending θ to $\alpha\theta$ is surjective.

(ii) Given an A-module M and a surjective A-linear map $\alpha : M \twoheadrightarrow P$, there exists an A-linear map $\beta : P \to M$ such that $\alpha\beta = 1_P$.

(iii) There exists an A-module Q such that $P \oplus Q \simeq A^n$ for some positive integer n, i.e. $P \oplus Q$ is free.

Now, we state the well known Nakayama Lemma.

Lemma 1.4 *Let A be a ring and let M be a finitely generated A-module. Let $I \subset A$ be an ideal such that $IM = M$. Then, there exists $a \in I$ such that $(1 + a)M = 0$. In particular, if $I \subset \mathcal{J}(A)$, then $(1 + a)$ is a unit and hence $M = 0$.*

Lemma 1.5 *Let I be an ideal of A which is contained in the Jacobson radical of A. Let P, Q be projective A-modules. If projective A/I-modules P/IP and Q/IQ are isomorphic, then P and Q are isomorphic as A-modules.*

Proof. Let $\overline{\alpha} : P/IP \xrightarrow{\sim} Q/IQ$ be an isomorphism. Since P is projective, $\overline{\alpha}$ can be lifted to an A-linear map $\alpha : P \to Q$. We claim that α is an isomorphism.

Since $\overline{\alpha}$ is surjective, $Q = \alpha(P) + IQ$. Hence, as $I \subset \mathcal{J}(A)$, by Nakayama lemma (1.4), we get $Q = \alpha(P)$. Hence α is surjective.

Since Q is projective, there exists an A-linear map $\beta : Q \to P$ such that $\alpha\beta = \text{Id}_Q$. Let $\overline{\beta} : Q/IQ \to P/IP$ be the map induced by β. Then, we have $\overline{\alpha}\,\overline{\beta} = \text{Id}_{Q/IQ}$. As $\overline{\alpha}$ is an isomorphism, we get that $\overline{\beta}$ is also an isomorphism and in particular $\overline{\beta}$ is surjective. Therefore $P = \beta(Q) + IP$. Hence, as before, we see that β is surjective. Now, injectivity of α follows from the fact that $\alpha\beta = \text{Id}$. $\square$

The following result is an immediate consequence of the above result.

Corollary 1.6 *Let A be a local ring. Then, every projective A-module is free.*

Definition 1.7 For a ring A, $\operatorname{Spec} A$ denotes the set of all prime ideals of A. For an ideal $I \subset A$, we denote by $V(I)$, the set of all prime ideals of A containing I. For $f \in A$, we denote by $D(f)$, the set of all prime ideals of A not containing the element f. The *Zariski topology* on Spec (A) is the topology for which all the closed sets are of the form $V(I)$, for some ideal I of A or equivalently the basic open sets are of the form $D(f),\ f \in A$.

Definition 1.8 Let P be a projective A-module. In view of (1.6), we define the rank function, $\text{rank}_P : \operatorname{Spec} A \to \mathbb{Z}$ by $\text{rank}_P(\mathfrak{q})$ = rank of the free $A_{\mathfrak{q}}$-module $P \otimes_A A_{\mathfrak{q}}$. If rank_P is a constant function taking the value n, then, we define the rank of P to be n and denote it by rk P.

Remark 1.9 rank_P is a continuous function (with the discrete topology on $\mathbb{Z}$ and Zariski topology on Spec A). Moreover, rank_P is a constant function for every finitely generated projective A-module P if A has no non trivial idempotent elements.

Definition 1.10 Given a projective A-module P and an element $p \in P$, we define $\mathcal{O}_P(p) = \{\alpha(p) \mid \alpha \in P^*\}$. We say that p is *unimodular* if $\mathcal{O}_P(p) = A$. The set of all unimodular elements of P is denoted by $\text{Um}(P)$. If $P = A^n$, then we write $\text{Um}_n(A)$ for $\text{Um}(A^n)$. Note that $\mathcal{O}_P(p)$ is an ideal of A and $p \in P$ is a unimodular element if and only if there exists $\alpha \in P^* = \text{Hom}_A(P, A)$ such that $\alpha(p) = 1$.

Let P be a projective A-module of rank n. Let $\wedge^n(P)$ denote the n^{th} exterior power of P. Then $\wedge^n(P)$ is a projective A-module of rank 1 and is called the *determinant of* P. We say determinant of P is trivial if $\wedge^n(P) = A$.

Now, we state a classical result of Serre [32].

Theorem 1.11 *Let A be a ring with $\dim(A/\mathcal{J}(A)) = d$. Then, any projective A-module P of rank $> d$ has a unimodular element.*

The following is a classical result of Bass [3].

Theorem 1.12 *Let A be a ring of dimension d and let P be a projective A-module of rank $> d$. Let $(p, a) \in \mathrm{Um}(P \oplus A)$. Then, there exists $q \in P$ such that $p + aq \in \mathrm{Um}(P)$. In particular, $E(P \oplus A)$ acts transitively on $\mathrm{Um}(P \oplus A)$.*

Notation 1.13 Let A be a ring and let $A[T]$ be the polynomial algebra in one variable T. We denote, by $A(T)$, the ring obtained from $A[T]$ by inverting all monic polynomials. For an ideal I of $A[T]$ and $a \in A$, $I(a)$ denotes the ideal $\{f(a) : f(T) \in I\}$ of A.

Let P be a projective A-module. Then $P[T]$ denotes the projective $A[T]$-module $P \otimes_A A[T]$ and $P(T)$ denotes the projective $A(T)$-module $P[T] \otimes_{A[T]} A(T)$.

Definition 1.14 Let B be a ring and let P be a projective B-module. Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$. If $\varphi(p) = 0$, then ${\varphi_p}^2 = 0$ and hence $1 + \varphi_p$ is a unipotent automorphism of P.

By a *transvection*, we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. We denote by $E(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by all transvections of P. Note that $E(P)$ is a normal subgroup of $\mathrm{Aut}(P)$. Also, an existence of a transvection of P pre-supposes that P has a unimodular element.

Definition 1.15 Let B be a ring and let P be a projective B-module. An automorphism σ of P is said to be *isotopic to identity*, if there exists an automorphism $\Phi(W)$ of the projective $B[W]$-module $P[W] = P \otimes B[W]$ such that $\Phi(0)$ is the identity automorphism of P and $\Phi(1) = \sigma$. Two elements $p_1, p_2 \in P$ are said to be *isotopically connected* if there exists an automorphism σ of P such that σ is isotopic to identity and $\sigma(p_1) = p_2$.

Remark 1.16 Let B be a ring and let P be a projective B-module. Let σ be an automorphism of P and let σ^* be the induced automorphism of P^* defined by $\sigma^*(\alpha) = \alpha\sigma$ for $\alpha \in P^*$.

If $\sigma \in E(P)$, then $\sigma^* \in E(P^*)$. If σ is isotopic to identity, then, so is σ^*.

If σ is unipotent, then it is isotopic to identity. Therefore, any element of $E(P)$ is also isotopic to identity.

Now, suppose that $B = A[T]$ and $P = Q[T] = Q \otimes_A A[T]$. Then, since $\text{End}_B(P) = \text{End}_A(Q)[T]$, we regard σ as polynomial in T with coefficients in $\text{End}_A(Q)$, say $\sigma = \theta(T)$. If $\theta(0)$ is the identity automorphism of Q, then, since $\Phi(W) = \theta(WT)$ is an automorphism of $Q[T,W] = Q \otimes_A A[T,W] = P \otimes_B B[W]$, it follows that σ is isotopic to identity.

The following lemma follows from the well known Quillen's Splitting lemma ([29], Lemma 1) and its proof is essentially contained in ([29], Theorem 1).

Lemma 1.17 *Let B be a ring and let P be a projective B-module. Let $a, b \in B$ be such that $Ba + Bb = B$. Let σ be a B_{ab}-automorphism of P_{ab} which is isotopic to identity. Then $\sigma = \tau_a \, \theta_b$, where τ is a B_b-automorphism of P_b such that $\tau = Id$ modulo the ideal aB_b and θ is a B_a-automorphism of P_a such that $\theta = Id$ modulo the ideal bB_a.*

The following result is due to Bhatwadekar and Roy ([10], Proposition 4.1) and is about lifting an automorphism of a projective module.

Proposition 1.18 *Let B be a ring and let P be a projective B-module. Let $I \subset B$ be an ideal. Then, any transvection $\overline{\Phi}$ of P/IP (i.e. $\overline{\Phi} \in E(P/IP)$) can be lifted to an automorphism Φ of P.*

The following result is a consequence of a theorem of Eisenbud and Evans as stated in ([27], p. 1420).

Theorem 1.19 *Let A be a ring and let P be a projective A-module of rank r. Let $(\alpha, a) \in (P^* \oplus A)$. Then, there exists an element $\beta \in P^*$ such that* $\operatorname{ht} I_a \geq r$*,* *where* $I = (\alpha + a\beta)(P)$. *In particular, if the ideal* $(\alpha(P), a)$ *has height* $\geq r$*, then* $\operatorname{ht} I \geq r$. *Further, if* $(\alpha(P), a)$ *is an ideal of height* $\geq r$ *and* I *is a proper ideal of* A*, then* $\operatorname{ht} I = r$.

The following result is due to Lindel ([19], Theorem 2.6).

Theorem 1.20 *Let* B *be a ring of dimension* d *and* $A = B[T_1, \ldots, T_n]$. *Let* P *be a projective* A*-module of rank* $\geq$ *max* $(2, d+1)$. *Then* $E(P \oplus A)$ *acts transitively on the set of unimodular elements of* $P \oplus A$.

Now, we quote a result of Mandal ([21], Theorem 2.1).

Theorem 1.21 *Let* A *be a ring and let* $I \subset A[T]$ *be an ideal containing a monic polynomial. Let* P *be a projective* A*-module of rank* $n \geq \dim(A[T]/I) + 2$. *Let* $\phi : P[T] \twoheadrightarrow I/(I^2T)$ *be a surjection. Then* ϕ *can be lifted to a surjection* $\Phi : P[T] \twoheadrightarrow I$.

Definition 1.22 Let A be a local ring of dimension d with unique maximal ideal $\mathfrak{m}$. If $\mathfrak{m}$ is generated by d elements, then A is said to be a *regular local ring.* A ring B is called *regular* if $B_{\mathfrak{m}}$ is a regular local ring for every maximal ideal $\mathfrak{m}$ of B. A local ring A is called a k*-spot* if it is a localization of an affine k-algebra.

The following result is due to Mandal and Varma ([22], Theorem 4).

Theorem 1.23 *Let* A *be a regular* k*-spot, where* k *is an infinite perfect field. Let* $I \subset A[T]$ *be an ideal of height* ≥ 4 *and let* n *be an integer such that* $n \geq \dim(A[T]/I) + 2$. *Let* $f_1, \ldots, f_n \in I$ *be such that* $I = (f_1, \ldots, f_n) + (I^2T)$. *Assume that* $I(0)$ *is a complete intersection ideal of* A *of height* n *or* $I(0) = A$. *Then* $I = (F_1, \ldots, F_n)$ *with* $F_i - f_i \in (I^2T)$.

The following result is a variant of ([4], Proposition 3.1). We give a proof for the sake of completeness.

Proposition 1.24 *Let B be a ring and let $I \subset B$ be an ideal of height n. Let $f \in B$ be such that it is not a zero divisor modulo I. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\alpha : P \to I$ be a linear map such that the induced map $\alpha_f : P_f \twoheadrightarrow I_f$ is a surjection. Then, there exists $\Psi \in E({P_f}^*)$ such that*

(1) $\beta = \Psi(\alpha) \in P^*$ and

(2) $\beta(P)$ is an ideal of B of height n contained in I.

Proof. Note that, since f is not a zero divisor modulo I and $\alpha_f(P_f) = I_f$, if Δ is an automorphism of ${P_f}^*$ such that $\delta = \Delta(\alpha) \in P^*$, then $\delta(P) \subset I$.

Let $\mathcal{S}$ be the set $\{\, \Gamma \in E({P_f}^*) : \Gamma(\alpha) \in P^* \,\}$. Then $\mathcal{S} \neq \varnothing$, since the identity automorphism of ${P_f}^*$ is an element of $\mathcal{S}$. For $\Gamma \in \mathcal{S}$, let $N(\Gamma)$ denote height of the ideal $\Gamma(\alpha)(P)$. Then, in view of the above observation, it is enough to prove that there exists $\Psi \in \mathcal{S}$ such that $N(\Psi) = n$. This is proved by showing that for any $\Gamma \in \mathcal{S}$ with $N(\Gamma) < n$, there exists $\Gamma_1 \in \mathcal{S}$ such that $N(\Gamma) < N(\Gamma_1)$.

Since $P = P_1 \oplus B$, we write $\alpha = (\theta, a)$, where $\theta \in {P_1}^*$ and $a \in B$. Let $\Gamma \in \mathcal{S}$ be such that $N(\Gamma) < n$. Let $\Gamma((\theta, a)) = (\beta, b) \in {P_1}^* \oplus B$. Applying Eisenbud-Evans theorem (1.19), there exists $\phi \in {P_1}^*$ such that $\operatorname{ht} L_b \geq n-1$, where $L = (\beta + b\phi)(P_1)$. It is easy to see that the automorphism Λ of ${P_1}^* \oplus B$ defined by $\Lambda((\delta, c)) = (\delta + c\phi, c)$ is a transvection of ${P_1}^* \oplus B$ and $\Lambda(\beta, b) = (\beta + b\phi, b)$. Hence $\Lambda\,\Gamma \in \mathcal{S}$ and moreover $N(\Gamma) = N(\Lambda\,\Gamma)$. Therefore, if necessary, we can replace Γ by $\Lambda\,\Gamma$ and assume that if a prime ideal $\mathfrak{p}$ of B contains $\beta(P_1)$ and does not contain b, then $\operatorname{ht}\mathfrak{p} \geq n-1$. Now, we claim that $N(\Gamma) = \operatorname{ht}\beta(P_1)$.

We have $N(\Gamma) \leq n-1$. Since $N(\Gamma) = \operatorname{ht}(\beta(P_1), b)$, we have $\operatorname{ht}\beta(P_1) \leq N(\Gamma) \leq n-1$. Let $\mathfrak{p}$ be a minimal prime ideal of $\beta(P_1)$ such that $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\beta(P_1)$. If $b \notin \mathfrak{p}$, then $\operatorname{ht}\mathfrak{p} \geq n-1$. Hence, we have the inequalities $n-1 \leq \operatorname{ht}\beta(P_1) \leq N(\Gamma) \leq n-1$. This implies that $N(\Gamma) = \operatorname{ht}\beta(P_1) = n-1$. If $b \in \mathfrak{p}$, then $\operatorname{ht}\beta(P_1) = \operatorname{ht}\mathfrak{p} \geq \operatorname{ht}(\beta(P_1), b) = N(\Gamma) \geq \operatorname{ht}\beta(P_1)$. This proves the claim.

Let $\mathcal{K}$ denote the set of minimal prime ideals of $\beta(P_1)$. Since P_1 is a projective B-module of rank $n-1$, if $\mathfrak{p} \in \mathcal{K}$, then $\operatorname{ht}\mathfrak{p} \leq n-1$.

Let $\mathcal{K}_1 = \{\mathfrak{p} \in \mathcal{K} : b \in \mathfrak{p}\,\}$ and let $\mathcal{K}_2 = \mathcal{K} - \mathcal{K}_1$. Note that, since $\operatorname{ht}\beta(P_1) = \operatorname{ht}(\beta(P_1), b)$, $\mathcal{K}_1 \neq \varnothing$. Moreover, every member $\mathfrak{p}$ of $\mathcal{K}_1$ is a prime ideal of height

$< n$ which contains $I_1 = (\beta(P_1), b)$. Therefore, since $(I_1)_f = I_f$ and $\operatorname{ht} I = n$, it follows that $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{K}_1$.

Since $\bigcap_{\mathfrak{p}\in\mathcal{K}_2} \mathfrak{p} \not\subset \bigcup_{\mathfrak{p}\in\mathcal{K}_1} \mathfrak{p}$, there exists $x \in \bigcap_{\mathfrak{p}\in\mathcal{K}_2} \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{p}\in\mathcal{K}_1} \mathfrak{p}$. Since $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{K}_1$, we have $xf \in \bigcap_{\mathfrak{p}\in\mathcal{K}} \mathfrak{p}$. This implies that $(xf)^r \in \beta(P_1)$ for some positive integer r.

Let $(xf)^r = \beta(q)$. As before, it is easy to see that the automorphism Φ of ${P_1}^*\oplus B$ defined by $\Phi((\tau, d)) = (\tau, d + \tau(q))$ is a transvection of ${P_1}^*\oplus B$. Let Δ be an automorphism of ${(P_1)_f}^* \oplus B_f$ defined by $\Delta(\eta, c) = (\eta, f^r c)$. Then, since $E({(P_1)_f}^*\oplus B_f)$ is a normal subgroup of $GL({(P_1)_f}^*\oplus B_f)$, $\Phi_1 = \Delta^{-1}\,\Phi\,\Delta$ is an element of $E({(P_1)_f}^*\oplus B_f)$. Moreover, $\Phi_1((\beta, b)) = (\beta, b + x^r)$.

Let $\Gamma_1 = \Phi_1\,\Gamma$. Then $\Gamma_1(\alpha) = \Gamma_1((\theta, a)) = \Phi_1((\beta, b)) = (\beta, b + x^r)$. Therefore $\Gamma_1 \in \mathcal{S}$. Moreover, since $b + x^r$ does not belong to any minimal prime ideal of $\beta(P_1)$, we have $N(\Gamma) = \operatorname{ht} \beta(P_1) < N(\Gamma_1)$. This proves the result. $\square$

Chapter 2

Subtraction Principle

In this chapter, we prove "Subtraction principle" (2.8) together with some other results for later use. Though these results are technical in nature, they are the backbone for our main result (3.15) proved in this thesis. We begin with the following easy lemma.

Lemma 2.1 *Let B be a ring and let I be an ideal of B. Let $K \subset I$ be an ideal such that $I = K + I^2$. Then $I = K$ if and only if any maximal ideal of B containing K contains I.*

Proof. Since I/K is an idempotent ideal of a Noetherian ring B/K and I^2 maps surjectively onto I/K, there exists an element $a \in I^2$ such that $K + (a) = I$ and $a(1-a) \in K$. Therefore, $(1-a)I \subset K$ and hence $I_{\mathfrak{m}} = K_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of B, since any maximal ideal of B containing K contains I. Hence $I = K$. $\square$

Lemma 2.2 *Let B be a ring and let $I \subset B$ be an ideal. Let I_1 and I_2 be ideals of B contained in I such that $I_2 \subset I^2$ and $I_1 + I_2 = I$. Then $I = I_1 + (e)$ for some $e \in I_2$ and $I_1 = I \cap I'$, where $I_2 + I' = B$.*

Proof. Since I/I_1 is an idempotent ideal of a Noetherian ring B/I_1 and I_2 maps surjectively onto I/I_1, there exists an element $a \in I_2$ such that $I_1 + (a) = I$ and $a(1-a) \in I_1$. The result follow by taking $I' = I_1 + (1-a)$. $\square$

The proof of the following result uses the explicit completion of the unimodular row $[a^2, b, c]$ given by Krusemeyer [18].

Lemma 2.3 *Let B be a ring and let $I = (c_1, c_2)$ be an ideal of B. Let $b \in B$ be such that $I + (b) = B$ and let r be a positive even integer. Then $I = (e_1, e_2)$ with $c_1 - e_1 \in I^2$ and $b^r c_2 - e_2 \in I^2$.*

Proof. Replacing b by $b^{r/2}$, we can assume that $r = 2$. Since b is a unit modulo $I = (c_1, c_2)$, it is unit modulo $({c_1}^2, {c_2}^2)$. Let $1 - bz = x'{c_1}^2 + y'{c_2}^2 = xc_1 + yc_2$, where $x = x'c_1 \in I$ and $y = y'c_2 \in I$. The unimodular row (z^2, c_1, c_2) has the following Krusemeyer completion (see [18]) to an invertible matrix Γ given by

$$\begin{pmatrix} z^2 & c_1 & c_2 \\ -c_1 - 2zy & y^2 & b - xy \\ -c_2 + 2zx & -b - xy & x^2 \end{pmatrix}.$$

Let $\Theta : B^3 \twoheadrightarrow I$ be a surjective map defined by $\Theta(1,0,0) = 0$, $\Theta(0,1,0) = -c_2$ and $\Theta(0,0,1) = c_1$. Then, since Γ is invertible and $\Theta(z^2, c_1, c_2) = 0$, it follows that $I = (d_1, d_2)$, where $d_1 = -y^2 c_2 + c_1(b - xy)$ and $d_2 = c_2(b + xy) + c_1 x^2$. From the construction of elements d_1 and d_2, it follows that $d_1 - c_1 b \in I^2$ and $d_2 - c_2 b \in I^2$. Let $\Delta = \text{diag}\,(z, b) \in M_2(B)$. Since diagonal matrices of determinant 1 are elementary, $\Delta \otimes B/I \in E_2(B/I)$. Since the canonical map $E_2(B) \to E_2(B/I)$ is surjective, there exists $\Phi \in E_2(B)$ such that $\Delta \otimes B/I = \Phi \otimes B/I$. Let $[d_1, d_2]\,\Phi = [e_1, e_2]$. From the construction of Φ, it follows that $I = (e_1, e_2)$ with $e_1 - c_1 \in I^2$ and $e_2 - b^2 c_2 \in I^2$. This proves the lemma. $\square$

Lemma 2.4 *Let A be a ring and let I be an ideal of A. Let $s \in A$ be such that $I + (s) = A$. Let Q be a projective A-module such that Q/IQ is free and let $P = Q \oplus A^2$. Let $\Phi : P \twoheadrightarrow I$ be a surjection. Let r be a positive integer.*

Then, the map $\Phi' = s^r\,\Phi : P \to I$ induces a surjection $\Phi' \otimes A/I : P/IP \twoheadrightarrow I/I^2$. Moreover if r is even, then, the surjection $\Phi' \otimes A/I$ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.

Proof. Since $I + (s) = A$ and $\Phi : P \twoheadrightarrow I$ is a surjection, it is easy to see that $\Phi' \otimes A/I$ is a surjection from P/IP to I/I^2. Now, we assume that $r = 2l$.

Since $P = Q \oplus A^2$, we write $\Phi = (\phi, f_1, f_2)$. Let rank $Q/IQ = n-2$. Let "tilde" denote reduction modulo I. Then, since Q/IQ is free of rank $n-2$, fixing a basis of Q/IQ, we can write $\widetilde{\Phi} = (\widetilde{k_1}, \ldots, \widetilde{k_{n-2}}, \widetilde{f_1}, \widetilde{f_2})$. Let $\beta = \text{diag}\,(s^r, \ldots, s^r)$. Then $\widetilde{\beta} \in \text{Aut}(P/IP)$ and $\widetilde{\Phi'} = \widetilde{\Phi}\,\widetilde{\beta}$. Since diagonal matrices of determinant 1 are elementary, we get $\widetilde{\beta} = \text{diag}\,(1, \ldots, 1, \widetilde{s^{nr}})\,\widetilde{\beta'}$, where $\widetilde{\beta'} \in E(P/IP)$. By (1.18), $\widetilde{\beta'}$ can be lifted to an automorphism of P. Therefore, to prove the lemma, it is enough to show that the surjection $(\phi, f_1, s^{nr} f_2) \otimes A/I : P/IP \twoheadrightarrow I/I^2$ can be lifted to a surjection $(\phi, g_1, g_2) : P \twoheadrightarrow I$. Since nr is even, $s^{nr} = {s_1}^2$. Therefore, replacing s by s_1, we can assume that $nr = 2$.

Let $K = \phi(Q)$ and let "bar" denote reduction modulo K. Then $\overline{I} = (\overline{f_1}, \overline{f_2})$. Applying (2.3), we get $\overline{I} = (\overline{h_1}, \overline{h_2})$ with $\overline{f_1} - \overline{h_1} \in \overline{I^2}$ and $\overline{s^2 f_2} - \overline{h_2} \in \overline{I^2}$. Therefore, $I = (h_1, h_2) + K$, where $f_1 - h_1 = f_1' + h_1'$ and $s^2 f_2 - h_2 = f_2' + h_2'$ for some $f_1', f_2' \in I^2$ and $h_1', h_2' \in K$. Let $g_i = h_i + h_i'$ for $i = 1, 2$. Then, we have $I = (g_1, g_2) + K$ with $f_1 - g_1 \in I^2$ and $s^2 f_2 - g_2 \in I^2$. This proves the result. $\square$

Remark 2.5 It will be interesting to know if the above result is valid without the assumption that Q/IQ is free.

The following result is very crucial for our main result (3.15).

Lemma 2.6 *Let B be a ring and let $s, t \in B$ be such that $Bs + Bt = B$. Let I, L be ideals of B such that $L \subset I^2$. Let P be a projective B-module and let $\phi : P \twoheadrightarrow I/L$ be a surjection. If $\phi \otimes B_t$ can be lifted to a surjection $\Phi : P_t \twoheadrightarrow I_t$. Then ϕ can be lifted to a surjection $\Psi : P \twoheadrightarrow I/(sL)$.*

Proof. Without loss of generality, we can assume that $t = 1$ modulo the ideal (s). Let l be a positive integer such that $t^l \Phi(P) \subset I$. Let $\Phi' : P \to I$ be a lift of ϕ. Then, since Φ is a lift of ϕ_t, there exists an integer $r \geq l$ such that $(t^r\Phi - t^r\Phi')(P) \subset L$. Let $\Gamma = t^r\Phi$ and $K = \Gamma(P)$. Then, since $r \geq l$, $K \subset I$. Since $K_t = I_t$, we have $t^n I \subset K$ for some positive integer n. Since $1 - t \in (s)$, $t^n = 1 - sx$ for some $x \in B$. Hence $(1-sx)I \subset K$. Therefore, we have $K + sI = I$.

Let $t^r = 1 - sa$ and let $\Theta = \Gamma + sa\Phi'$. Then $\Theta - \Phi' = \Gamma - t^r\Phi'$. Therefore $(\Theta - \Phi')(P) \subset L$ and hence Θ is also lift of ϕ. Therefore, $\Theta(P) + L = I$. Moreover, $\Theta(P) + sI = \Gamma(P) + sI = K + sI = I$. Write $I_1 = \Theta(P) + sL$. Any maximal ideal of B containing I_1 contains s or L and hence contains I. Moreover, since $L \subset I^2$, $I = I_1 + I^2$. Therefore, by (2.1), $I = I_1$, i.e. $\Theta(P) + sL = I$. If $\Gamma' : I \twoheadrightarrow I/sL$ is a canonical surjection, then putting $\Psi = \Gamma'\Theta$, we are through. $\square$

The following technical lemma is used in the proof of (3.11) which is very crucial for our main result (3.15).

Lemma 2.7 *Let B be a ring and let I_1, I_2 be two comaximal ideals of B. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Phi : P \twoheadrightarrow I_1$ and $\Psi : P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Phi \otimes B/I_1 = \Psi \otimes B/I_1$. Assume that*

(1) $a = \Phi(0,1)$ is a non zero divisor modulo the ideal $\sqrt{\Phi(P_1)}$.

(2) $n - 1 > \dim(\overline{B}/\mathcal{J}(\overline{B}))$, where $\overline{B} = B/\Phi(P_1)$.

Let $L \subset {I_2}^2$ be an ideal such that $\Phi(P_1) + L = B$. Then, the surjection $\Psi : P \twoheadrightarrow I_1 \cap I_2$ induces a surjection $\overline{\Psi} : P \twoheadrightarrow I_2/L$. Moreover, $\overline{\Psi}$ can be lifted to a surjection $\Lambda : P \twoheadrightarrow I_2$.

Proof. Since $L + I_1 = B$ (in fact $L + \Phi(P_1) = B$) , it is easy to see that Ψ induces a surjection $\overline{\Psi} : P \twoheadrightarrow I_2/L$.

Let $K = \Phi(P_1)$ and $S = 1 + K$. Then $S \cap L \neq \varnothing$. Therefore, we have surjections Φ_S and Ψ_S from P_S to $(I_1)_S$.

Claim: There exists an automorphism Δ of P_S such that $\Delta^*(\Psi_S) = \Psi_S\,\Delta = \Phi_S$, where Δ^* is an automorphism of ${P_S}^*$ induced from Δ.

Assume the claim. Then, there exists $s = 1 + t \in S$, $t \in K$ such that $\Delta \in \mathrm{Aut}(P_s)$ and $\Psi_s\,\Delta = \Phi_s$. Since $S \cap L \neq \varnothing$, we can assume that $s \in S \cap L$.

With respect to the decomposition $P = P_1 \oplus B$, we write $\Phi \in P^*$ as (Φ_1, a), where $\Phi_1 \in {P_1}^*$ and $a \in B$. Similarly, we write $\Psi = (\Psi_1, b)$, where $\Psi_1 \in {P_1}^*$ and $b \in B$. Let $pr : P_1 \oplus B (= P) \twoheadrightarrow B$ be the map defined by $pr(p_1, \widetilde{b}) = \widetilde{b}$, where $p_1 \in P_1$ and $\widetilde{b} \in B$.

Since $s \in L$, $(I_2)_s = B_s$ and therefore, we can regard pr_s as a surjection from $(P_1)_s \oplus B_s$ to $(I_2)_s$. Since $t \in K = \Phi_1(P_1)$, the element $(\Phi_1)_t \in {(P_1)_t}^*$ is a unimodular element. Hence, there exists an element $\Gamma \in E((P_1)_{st} \oplus B_{st})$ such that $\Gamma^*((\Phi_1, a)_{st}) = pr_{st}$ i.e. $(\Phi_t)_s \, \Gamma = (pr_s)_t$. Note that Ψ_t is a surjection from P_t to $(I_2)_t$.

We also have $\Psi_s \, \Delta = \Phi_s$. Hence $(\Psi_s \, \Delta)_t \, \Gamma = (pr_s)_t$. Let $\widetilde{\Delta} = \Delta_t \, \Gamma \, {\Delta_t}^{-1}$. Then, we have $(\Psi_s)_t \, \widetilde{\Delta} = (\Psi_t)_s \, \widetilde{\Delta} = (pr_s)_t \, {\Delta_t}^{-1}$. Since Γ is an element of $E(P_{st})$ which is a normal subgroup of $\text{Aut}(P_{st})$, $\widetilde{\Delta} \in E(P_{st})$ and hence is isotopic to identity, by (1.16). Therefore, by (1.17), $\widetilde{\Delta} = {\Delta''}_s \, {\Delta'}_t$, where Δ' is an automorphism of P_s such that $\Delta' = \text{Id}$ modulo (t) and Δ'' is an automorphism of P_t such that $\Delta'' = \text{Id}$ modulo (s).

Thus, we have surjections $(\Psi_t \, \Delta'') : P_t \twoheadrightarrow (I_2)_t$ and $(pr_s \, \Delta^{-1} \, (\Delta')^{-1}) : P_s \twoheadrightarrow (I_2)_s$ such that $(\Psi_t \, \Delta'')_s = (pr_s \, \Delta^{-1} \, (\Delta')^{-1})_t$. Therefore, they patch up to yield a surjection $\Lambda : P \twoheadrightarrow I_2$. Since $s = 1 + t \in L$, the map $B \to B/(s)$ factors through B_t. Since $\Delta'' = \text{Id}$ modulo (s), we have $\Lambda \otimes B/L = \Psi \otimes B/L$.

Proof of the claim: Since $S = 1 + K$, $\overline{B} = B/K = B_S/K_S$ and $K_S \subset \mathcal{J}(B_S)$. Therefore, $\overline{B}/\mathcal{J}(\overline{B}) = B_S/\mathcal{J}(B_S)$. Hence $\dim B_S/\mathcal{J}(B_S) < n - 1$.

To simplify the notation, we denote B_S by B, $(P_1)_S$ by P_1 and $(I_1)_S$ by I. Then, we have two surjections $\Phi = (\Phi_1, a)$ and $\Psi = (\Psi_1, b)$ from $P_1 \oplus B$ to I such that $\Phi \otimes B/I = \Psi \otimes B/I$. Moreover, $\Phi_1(P_1) = K \subset \mathcal{J}(B)$ and rank $P_1 (= n - 1) > \dim(\overline{B}/\mathcal{J}(\overline{B}))$, where $\overline{B} = B/K$. Our aim is to show that there exists an automorphism Δ of $P = P_1 \oplus B$ such that $\Psi \, \Delta = \Phi$.

Hence onward, we write an element $\sigma \in \text{End}(P_1 \oplus B)$ in the following matrix form

$$\sigma = \begin{pmatrix} \alpha & p \\ \eta & d \end{pmatrix}, \text{ where } \alpha \in \text{End}(P_1),\ p \in P_1,\ \eta \in {P_1}^* \text{ and } d \in B.$$

Note that, with this presentation of $\sigma \in \text{End}(P)$, if $\Theta = (\Theta_1, e) \in {P_1}^* \oplus B$, then $\sigma^*(\Theta) = \Theta\sigma = (\Theta_1\alpha + e\eta, \Theta_1(p) + ed)$. Moreover, if $\sigma' \in \text{End}(P)$ has a matrix representation $\sigma' = \left(\begin{smallmatrix} \beta & p_1 \\ \mu & f \end{smallmatrix}\right)$, then the endomorphism $\sigma'\sigma$ of P has the matrix representation

$$\sigma'\sigma = \begin{pmatrix} \beta & p_1 \\ \mu & f \end{pmatrix} \begin{pmatrix} \alpha & p \\ \eta & d \end{pmatrix} = \begin{pmatrix} \beta\alpha + \eta_{p_1} & \beta(p) + dp_1 \\ \mu\alpha + f\eta & \mu(p) + fd \end{pmatrix},$$

where $\eta_{p_1} \in \text{End}(P_1)$ is the composite map $P_1 \xrightarrow{\eta} B \xrightarrow{p_1} P_1$.

Since $\Phi \otimes B/I = \Psi \otimes B/I : P \twoheadrightarrow I/I^2$, $\Phi - \Psi : P \to I^2$. Since $\Phi : P \twoheadrightarrow I$, $\Phi(IP) = I^2$. Hence, there exists $\eta : P \to IP$ such that $\Phi\eta = \Phi - \Psi$ (since P is projective). Write $\Gamma = Id - \eta$. Then $\Gamma \in \text{End}(P)$ is identity modulo the ideal I and $\Phi\Gamma = \Psi$. Similarly, there exist $\Gamma' \in \text{End}(P)$ which is identity modulo the ideal I such that $\Psi\,\Gamma' = \Phi$. Let

$$\Gamma = \begin{pmatrix} \gamma & q \\ \zeta & c \end{pmatrix}, \; \Gamma' = \begin{pmatrix} \gamma' & q' \\ \zeta' & c' \end{pmatrix}$$

be the matrix representation of Γ and Γ', where $\gamma, \gamma' \in \text{End}(P_1)$, $q, q' \in P_1$, $\zeta, \zeta' \in {P_1}^*$ and $c, c' \in B$. Then

$$\Gamma\,\Gamma' = \begin{pmatrix} \gamma\gamma' + {\zeta'}_q & \gamma(q') + c'q \\ \zeta\gamma' + c\zeta' & \zeta(q') + cc' \end{pmatrix}.$$

Since $\Phi\,\Gamma\,\Gamma' = \Phi$ ($\Phi\Gamma = \Psi$ and $\Psi\Gamma' = \Phi$) and $\Phi = (\Phi_1, a)$, we get $\Phi_1(\gamma(q') + c'q) + a(\zeta(q') + cc') = a$. Hence $a(1 - \zeta(q') - cc') \in K$. Since, by hypothesis, no minimal prime ideal of K contains a, we have $(1 - \zeta(q') - cc') \in \sqrt{K}$, i.e. $(\zeta(q') + cc') + \sqrt{K} = B$. But $K \subset \mathcal{J}(B)$ and hence $(\zeta(q') + cc') = B$, i.e. the element $\zeta(q') + cc' \in B^*$. Therefore $(\zeta, c) \in P^*$ is a unimodular element. Note that, since Γ is an endomorphism of P which is identity modulo I, $(\zeta, c) = (0, 1)$ modulo I. Now, we show that there exists an automorphism Δ_1 of P such that (1) $(\zeta, c)\,\Delta_1 = (0, 1)$ and (2) Δ_1 is an identity automorphism of P modulo I.

Let "bar" denote reduction modulo K. Since $\dim(\overline{B}/\mathcal{J}(\overline{B})) < n - 1$, by a classical result of Bass (1.12), there exists $\zeta_1 \in {P_1}^*$ such that $(\overline{\zeta + c\,\zeta_1})$ is a unimodular element of $\overline{{P_1}^*}$. But then, since $K \subset \mathcal{J}(B)$, $\zeta + c\,\zeta_1$ is a unimodular element of ${P_1}^*$. Let $q_1 \in P_1$ be such that $(\zeta + c\,\zeta_1)(q_1) = 1$. Let

$$\varphi_1 = \begin{pmatrix} 1 & 0 \\ \zeta_1 & 1 \end{pmatrix},\ \varphi_2 = \begin{pmatrix} 1 & (1-c)\,q_1 \\ 0 & 1 \end{pmatrix},\ \varphi_3 = \begin{pmatrix} 1 & 0 \\ -(\zeta + c\zeta_1) & 1 \end{pmatrix}.$$

Write $\Delta_1 = \varphi_1\,\varphi_2\,\varphi_3$. Since $(\zeta, c) = (0,1)$ modulo I, from the construction, it follows that Δ_1 is an automorphism of $P = P_1 \oplus B$ which is identity modulo I. Moreover, it is easy to see that $(\zeta, c)\Delta_1 = (0,1)$. Therefore, we have

$$\Gamma\,\Delta_1 = \begin{pmatrix} \gamma_1 & q_2 \\ 0 & 1 \end{pmatrix},$$

for some $\gamma_1 \in \mathrm{End}(P_1)$ and $q_2 \in P_1$. Since both Γ and Δ_1 are identity modulo I, γ_1 is an endomorphism of P_1 which is identity modulo I and $q_2 \in IP_1$. Let $\Delta_2 = \begin{pmatrix} 1 & -q_2 \\ 0 & 1 \end{pmatrix}$. Then, Δ_2 is an automorphism of $P_1 \oplus B$ which is identity modulo I. Moreover,

$$\Delta = \Delta_2\,\Gamma\,\Delta_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\widetilde{a} = \Phi_1(q_2) + a$. Then $\Phi\,{\Delta_2}^{-1} = (\Phi_1, \widetilde{a})$ and hence $K + (\widetilde{a}) = I$. Moreover, $(\Phi_1, \widetilde{a})\,\Delta = (\Phi_1\gamma_1, \widetilde{a}) = \Psi\,\Delta_1$ (since $(\Phi_1, \widetilde{a})\Delta = (\Phi_1, \widetilde{a})\Delta_2\Gamma\Delta_1 = \Phi\Gamma\Delta_1 = \Psi\Delta_1$). Let $\widetilde{\Psi}_1 = \Phi_1\,\gamma_1$. Therefore, to complete the proof (of the claim), it is enough to show that the surjections $\widetilde{\Phi} = (\Phi_1, \widetilde{a})$ and $\widetilde{\Psi} = (\widetilde{\Psi}_1, \widetilde{a})$ from P to I are connected by an automorphism of P. Note that $\Delta \in \mathrm{End}(P)$.

Since $\gamma_1 \in \mathrm{End}(P_1)$ is identity modulo I, $(1-\gamma_1)(P_1) \subset IP_1$. Since P_1 is a projective B-module, we have $\mathrm{Hom}(P_1, IP_1) = I\,\mathrm{Hom}(P_1, P_1)$. Hence $1 - \gamma_1 = \sum b_i\beta_i$, where $\beta_i \in \mathrm{End}(P_1)$ and $b_i \in I$. Let $b_i = c_i + d_i\widetilde{a}$, where $c_i \in K$ and $d_i \in B$. Then $1 - \gamma_1 = \sum c_i\beta_i + \widetilde{a}\sum d_i\beta_i$. Hence $\gamma_1 = \theta + \widetilde{a}\theta'$, where $\theta = 1 - \sum c_i\beta_i$ and $\theta' = -\sum d_i\beta_i$. Since determinant of θ is $1 + x$ for some $x \in K \subset \mathcal{J}(B)$, θ is an automorphism of P_1.

We have $\widetilde{\Psi}_1 = \Phi_1\gamma_1 = \Phi_1\theta + \widetilde{a}\Phi_1\theta'$. Let $\Lambda = \begin{pmatrix} \theta & 0 \\ \Phi_1\theta' & 1 \end{pmatrix}$. Then $(\Phi_1, \widetilde{a})\Lambda = (\widetilde{\Psi}_1, \widetilde{a})$ and Λ is an automorphism of P. This proves the result. $\square$

Now, we will prove the main result of this chapter which is labeled as "Subtraction Principle". This result is very important for the proof of our main result and is also used crucially to prove other results of the next chapter.

Theorem 2.8 *Let B be a ring of dimension d and let $I_1, I_2 \subset B$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Gamma : P \twoheadrightarrow I_1$ and $\Theta : P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Gamma \otimes B/I_1 = \Theta \otimes B/I_1$. Then, there exists a surjection $\Psi : P \twoheadrightarrow I_2$ such that $\Psi \otimes B/I_2 = \Theta \otimes B/I_2$.*

Proof. Let $\Gamma = (\Gamma_1, a)$. Let "bar" denote reduction modulo I_2. Then $\overline{\Gamma} = (\overline{\Gamma_1}, \overline{a})$ is a unimodular element of $\overline{P^*}$. Since, by (1.2), $\dim(B/I_2) \leq \dim B - \operatorname{ht} I_2 = d - n < n = \operatorname{rank} \overline{P_1}$, by Bass' result (1.12), there exists $\Theta_1 \in {P_1}^*$ such that $\overline{\Gamma_1} + \overline{a^2\Theta_1}$ is a unimodular element of $\overline{{P_1}^*}$. Therefore, replacing Γ_1 by $\Gamma_1 + a^2\Theta_1$, we can assume that $\Gamma_1(P_1) = K$ is comaximal with I_2. Moreover, using similar arguments, one can assume that height of K is $n-1$ and therefore, $\dim(B/K) \leq d-(n-1) \leq n-2$. Since K is a surjective image of P_1 (a projective B-module of rank $n-1$), every minimal prime ideal of K has height $n-1$. Hence, since $I_1 = K + (a)$ is an ideal of height n, a is a non-zero divisor modulo the ideal $\sqrt{K}$. Therefore, by (2.7), there exists a surjection $\Psi : P \twoheadrightarrow I_2$ which is a lift of $\Theta \otimes B/I_2$. This proves the result. $\square$

Remark 2.9 The above theorem has been already proved in the following cases.

(1) In the case P is free ([8], Proposition 3.2).

(2) For $n = d$ ([7], Theorem 3.3).

Our approach is different from that of ([8] and [7]) and we believe is of some independent interest.

Chapter 3

Main Theorem

In this chapter, we prove our main result (Theorem 3.15). We begin with the following lemma which is easy to prove (see [15], Proposition 1, p. 206).

Lemma 3.1 *Let A be a ring and $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \mathfrak{p}_3$ be a chain of prime ideals of $A[T]$. Then, we can not have $(\mathfrak{p}_1 \cap A) = (\mathfrak{p}_2 \cap A) = (\mathfrak{p}_3 \cap A)$.*

Lemma 3.2 *Let A be a ring and let $I \subset A[T]$ be an ideal of height k. Then* $\mathrm{ht}\,(I \cap A) \geq k - 1$.

Proof. First, we assume that $I = \mathfrak{p}$ is a prime ideal. Then, we claim that

$$\mathrm{ht}\,\mathfrak{p} = \begin{cases} \mathrm{ht}\,(\mathfrak{p} \cap A) & \text{if } \mathfrak{p} = (\mathfrak{p} \cap A)[T] \\ \mathrm{ht}\,(\mathfrak{p} \cap A) + 1 & \text{if } \mathfrak{p} \supsetneqq (\mathfrak{p} \cap A)[T] \end{cases}$$

Any prime chain $\mathfrak{q}_0 \subsetneqq \ldots \subsetneqq \mathfrak{q}_r \subsetneqq (\mathfrak{p} \cap A)$ in A extends to a prime chain $\mathfrak{q}_0[T] \subsetneqq \ldots \subsetneqq \mathfrak{q}_r[T] \subsetneqq (\mathfrak{p} \cap A)[T] \subset \mathfrak{p}$ in $A[T]$. Hence $\mathrm{ht}\,\mathfrak{p} \geq \mathrm{ht}\,(\mathfrak{p} \cap A)$ when $\mathfrak{p} = (\mathfrak{p} \cap A)[T]$ and $\mathrm{ht}\,\mathfrak{p} \geq \mathrm{ht}\,(\mathfrak{p} \cap A) + 1$ when $\mathfrak{p} \supsetneqq (\mathfrak{p} \cap A)[T]$. Now, let $\mathrm{ht}\,(\mathfrak{p} \cap A) = r$. Then, by the dimension theorem (see [23] Theorem 13.6), $\mathfrak{p} \cap A$ is minimal over an ideal $\mathfrak{a} = (a_1, \ldots, a_r)$. Then $(\mathfrak{p} \cap A)[T]$ is minimal over $\mathfrak{a}[T]$, so $\mathrm{ht}\,(\mathfrak{p} \cap A)[T] \leq r$. Thus, we have $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,(\mathfrak{p} \cap A)$ in the case 1.

Now, assume that $(\mathfrak{p}\cap A)[T] \subsetneqq \mathfrak{p}$, say $f \in \mathfrak{p} - (\mathfrak{p}\cap A)[T]$. We will be done if we can show that $\mathfrak{p}$ is a minimal prime over $\mathfrak{a}[T] + fA[T]$, for then $\operatorname{ht}\mathfrak{p} \leq r+1$. Let $\mathfrak{p}'$ be a prime between these. Then $\mathfrak{a} \subset (\mathfrak{p}'\cap A) \subset (\mathfrak{p}\cap A)$, so $(\mathfrak{p}'\cap A) = (\mathfrak{p}\cap A)$, since $(\mathfrak{p}\cap A)$ is minimal prime over $\mathfrak{a}$. In particular, $(\mathfrak{p}\cap A)[T] \subsetneqq \mathfrak{p}' \subset \mathfrak{p}$. By (3.1), we have $\mathfrak{p} = \mathfrak{p}'$. Thus, we are done in case 2.

Now, we prove the lemma for any ideal $I \subset A[T]$. Let $\sqrt{I} = \bigcap_1^r \mathfrak{p}_i$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are minimal primes over I. Then $\sqrt{I\cap A} = \bigcap_1^r (\mathfrak{p}_i \cap A)$. The prime ideals minimal over $(I\cap A)$ occur among $(\mathfrak{p}_1\cap A), \ldots, (\mathfrak{p}_r\cap A)$. Choose $\mathfrak{p}_i$ such that $\operatorname{ht}(I\cap A) = \operatorname{ht}(\mathfrak{p}_i\cap A)$. Then $\operatorname{ht}(I\cap A) = \operatorname{ht}(\mathfrak{p}_i\cap A) \geq \operatorname{ht}\mathfrak{p}_i - 1 \geq \operatorname{ht} I - 1$. This proves the lemma. $\square$

Lemma 3.3 *Let A be a ring of dimension d. Suppose $K \subset A[T]$ is an ideal such that $K + \mathcal{J}(A)A[T] = A[T]$. Then, any maximal ideal of $A[T]$ containing K has height $\leq d$.*

Proof. Suppose $\mathfrak{m} \subset A[T]$ is a maximal ideal of height $d+1$. Then $\mathfrak{m}\cap A$ is a maximal ideal of A. Hence $\mathfrak{m}$ contains $\mathcal{J}(A)$. Since $K + \mathcal{J}(A)A[T] = A[T]$, it follows that K is not contained in $\mathfrak{m}$. This proves the lemma. $\square$

The following result is labeled as "Moving lemma". Its proof is similar to ([6], Lemma 3.6).

Lemma 3.4 (Moving Lemma) *Let A be a ring of dimension d and let n be an integer such that $2n \geq d+3$. Let I be an ideal of $A[T]$ of height n and let $J = I\cap A$. Let $\widetilde{P}$ be a projective $A[T]$-module of rank n and $f \in A[T]$. Suppose $\phi : \widetilde{P} \twoheadrightarrow I/(I^2 f)$ be a surjection. Then, we can find a lift $\Delta \in \operatorname{Hom}_{A[T]}(\widetilde{P}, I)$ of ϕ such that the ideal $\Delta(\widetilde{P}) = I''$ satisfies the following properties:*

(i) $I = I'' + (J^2 f)$.

(ii) $I'' = I \cap I'$, where $\operatorname{ht} I' \geq n$.

(iii) $I' + (J^2 f) = A[T]$.

Proof. Let Φ be a lift of ϕ. Then $I = \Phi(\widetilde{P}) + (I^2 f)$. By (2.2), there exists $b \in (I^2 f)$ such that $I = \Phi(\widetilde{P}) + (b)$. Let "bar" denote reduction modulo $(J^2 f)$. Applying Eisenbud-Evans theorem (1.19), there exist $\Phi_1 \in \widetilde{P}^*$ such that if $N = (\Phi + b\Phi_1)(\widetilde{P})$, then $\operatorname{ht} \overline{N}_{\overline{b}} \geq n$. Since $I = N + (b)$ and $b \in I^2$, by (2.2), we get $N = I \cap K$ with $K + (b) = A[T]$. We claim that $\overline{K} = \overline{A[T]}$.

Assume otherwise, i.e. $\overline{K}$ is a proper ideal of $\overline{A[T]}$. Since b is a multiple of f, $K + (f) = A[T]$. Hence $\operatorname{ht} \overline{K}_{\overline{f}} = \operatorname{ht} \overline{K} = \operatorname{ht} \overline{K}_{\overline{b}} = \operatorname{ht} \overline{N}_{\overline{b}} \geq n$. Therefore,

$$\begin{aligned}
& n \leq \operatorname{ht} \overline{K}_{\overline{f}} \leq \dim(\overline{A[T]}_{\overline{f}}) = \dim((A/J^2)[T, f^{-1}]) \\
& \leq \dim(A/J) + 1 \leq \dim A - \operatorname{ht} J + 1 \text{ (by (1.2))} \\
& \leq d - (n-1) + 1 \text{ (since } \operatorname{ht} J \geq n-1, \text{ by (3.2))} \\
& \leq n - 1 \text{ (since } 2n \geq d+3).
\end{aligned}$$

This is a contradiction. Hence $\overline{K} = \overline{A[T]}$ and $\overline{N} = \overline{I}$, i.e. $K + (J^2 f) = A[T]$ and $I = N + (J^2 f)$. This proves the claim.

Write $\Psi = \Phi + b\Phi_1$. Then Ψ is also a lift of ϕ. We have $I = \Psi(\widetilde{P}) + (J^2 f)$. There exists $c \in (J^2 f)$ such that $I = \Psi(\widetilde{P}) + (c)$. Again, applying (1.19), there exists $\Psi_1 \in \widetilde{P}^*$ such that if $I'' = (\Psi + c\Psi_1)(\widetilde{P})$, then $\operatorname{ht} I''_c \geq n$.

Write $\Delta = \Psi + c\Psi_1$. Then Δ is also a lift of ϕ. We have $I'' = \Delta(\widetilde{P})$ and $I = \Delta(\widetilde{P}) + (c)$. By (2.2), we get $I'' = I \cap I'$ with $I' + (c) = A[T]$ and $\operatorname{ht} I' \geq n$.

Thus, we have (1) $I = I'' + (J^2 f)$, (2) $I'' = I \cap I'$, where $\operatorname{ht} I' \geq n$ and (3) $I' + (J^2 f) = A[T]$. This proves the result. $\square$

Lemma 3.5 *Let C be a ring with $\dim(C/\mathcal{J}(C)) = r$ and let P be a projective C-module of rank $m \geq r+1$. Let I and L be ideals of C such that $L \subset I^2$. Let $\phi : P \twoheadrightarrow I/L$ be a surjection. Then ϕ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.*

Proof. Let $\Psi : P \to I$ be a lift of ϕ. Then $\Psi(P) + L = I$. Since $L \subset I^2$, by (2.2), there exists $e \in L$ such that $\Psi(P) + (e) = I$.

Let "tilde" denote reduction modulo $\mathcal{J}(C)$. Then $\widetilde{\Psi}(\widetilde{P}) + (\widetilde{e}) = \widetilde{I}$. Applying Eisenbud-Evans theorem (1.19) to the element $(\widetilde{\Psi}, \widetilde{e})$ of $\widetilde{P^*} \oplus \widetilde{C}$, we see that there exists $\Theta \in P^*$ such that if $K = (\Psi + e\,\Theta)(P)$, then $\operatorname{ht} \widetilde{K}_{\widetilde{e}} \geq m$. As $\dim \widetilde{C} = r \leq m-1$, we have $\widetilde{K}_{\widetilde{e}} = \widetilde{C}_{\widetilde{e}}$. Hence $\widetilde{e}^{\,l} \in \widetilde{K}$ for some positive integer l. Since

$\widetilde{K} + (\widetilde{e}) = \widetilde{I}$ and $e \in L \subset I^2$, by (2.1), $\widetilde{K} = \widetilde{I}$. Since $e \in L$, the element $\Psi + e\,\Theta$ is also a lift of ϕ. Hence, replacing Ψ by $\Psi + e\Theta$, we can assume that $\widetilde{\Psi(P)} = \widetilde{I}$ i.e. $\widetilde{\Psi} : \widetilde{P} \twoheadrightarrow \widetilde{I}$ is a surjection. Therefore, since $\widetilde{I} = (I + \mathcal{J}(C))/\mathcal{J}(C) = I/(I \cap \mathcal{J}(C))$, we have $\Psi(P) + (I \cap \mathcal{J}(C)) = I$. We also have $\Psi(P) + L = I$. Therefore, since $L \subset I^2$, by (2.1), $\Psi(P) = I$. $\square$

As a consequence of (3.5), we have the following result.

Lemma 3.6 *Let A be a ring with $\dim(A/\mathcal{J}(A)) = r$. Let I and L be ideals of $A[T]$ such that $L \subset I^2$ and L contains a monic polynomial. Let P' be a projective $A[T]$-module of rank $m \geq r+1$. Let $\phi : P' \oplus A[T] \twoheadrightarrow I/L$ be a surjection. Then, we can lift ϕ to a surjection $\Phi : P' \oplus A[T] \twoheadrightarrow I$ with $\Phi(0,1)$ a monic polynomial.*

Proof. Let $\Phi' = (\Theta, g(T))$ be a lift of ϕ. Let $f(T) \in L$ be a monic polynomial. By adding some large power of $f(T)$ to $g(T)$, we can assume that the lift $\Phi' = (\Theta, g(T))$ of ϕ is such that $g(T)$ is a monic polynomial. Let $C = A[T]/(g(T))$. Since $A \hookrightarrow C$ is an integral extension, we have $\mathcal{J}(A) = \mathcal{J}(C) \cap A$ and hence $A/\mathcal{J}(A) \hookrightarrow C/\mathcal{J}(C)$ is also an integral extension. Therefore, $\dim(C/\mathcal{J}(C)) = r$.

Let "bar" denote reduction modulo $(g(T))$. Then, Θ induces a surjection $\alpha : \overline{P'} \twoheadrightarrow \overline{I}/\overline{L}$, which, by (3.5), can be lifted to a surjection from $\overline{P'}$ to $\overline{I}$. Therefore, there exists a map $\Gamma : P' \to I$ such that $\Gamma(P') + (g(T)) = I$ and $(\Theta - \Gamma)(P') = K \subset L + (g(T))$. Hence $\Theta - \Gamma \in KP'^*$. This shows that $\Theta - \Gamma = \Theta_1 + g(T)\,\Gamma_1$ for some $\Theta_1 \in LP'^*$ and $\Gamma_1 \in P'^*$.

Let $\Phi_1 = \Gamma + g(T)\,\Gamma_1$ and let $\Phi = (\Phi_1, g(T))$. Then, $\Phi(P' \oplus A[T]) = \Phi_1(P') + (g(T)) = \Gamma(P') + (g(T)) = I$. Thus $\Phi : P' \oplus A[T] \twoheadrightarrow I$ is a surjection. Moreover, $\Phi(0,1) = g(T)$ is a monic polynomial. Since $\Phi' - \Phi = (\Theta - \Phi_1, 0) = (\Theta_1, 0)$, where $\Theta_1 \in LP'^*$ and Φ' is a lift of ϕ, we see that Φ is a (surjective) lift of ϕ. $\square$

Lemma 3.7 *Let A be a ring of dimension d and let $I, I_1 \subset A[T]$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Assume $J = I \cap A \subset \mathcal{J}(A)$ and $I_1 + (J^2T) = A[T]$. Let $\Phi : P[T] \twoheadrightarrow I \cap I_1$ and $\Psi : P[T] \twoheadrightarrow I_1$ be two surjections with $\Phi \otimes A[T]/I_1 = \Psi \otimes A[T]/I_1$. Then, there exists a surjection $\Lambda : P[T] \twoheadrightarrow I$ such that $(\Phi - \Lambda)(P[T]) \subset (I^2T)$.*

Proof. We first note that, to prove the lemma, we can replace Φ and Ψ by $\Phi\,\Delta$ and $\Psi\,\Delta$, where Δ is an automorphism of $P[T]$.

Let $\Psi = (\Psi_1, f)$. Let "bar" denote reduction modulo (J^2T) and let $D = A[T]/(J^2T)$. Since $I_1 + (J^2T) = A[T]$, it follows that $(\overline{\Psi}_1, \overline{f}) \in \mathrm{Um}(\overline{P_1[T]^*}\oplus D)$. Since $J \subset \mathcal{J}(A)$, we have $JD \subset \mathcal{J}(D)$. Moreover, $D/JD = (A/J)[T]$ and $\mathrm{ht}\, J \geq n-1$, by (3.2). Hence $\dim(A/J) \leq \dim A - \mathrm{ht}\, J \leq d-(n-1) \leq n-2$. Therefore, since rank $P_1 = n-1$, by ([27], Corollary 2), $\overline{P_1[T]}$ has a unimodular element. By Lindel's result (1.20), $E(\overline{P_1[T]^*}\oplus D)$ acts transitively on $\mathrm{Um}(\overline{P_1[T]^*}\oplus D)$ and by (1.18), any element of $E(\overline{P_1[T]^*}\oplus D)$ can be lifted to an automorphism of $P_1[T]\oplus A[T]$. Putting above facts together, we can assume, replacing (Ψ_1, f) by $(\Psi_1, f)\Delta$ (Δ: suitable automorphism of $P[T]$) if necessary, that $\Psi_1(P_1[T]) + (J^2T)A[T] = A[T]$ and $f \in (J^2T)$. Moreover, applying Eisenbud-Evans theorem (1.19), we can assume, that $\mathrm{ht}\,(\Psi_1(P_1[T])) = n-1$.

Since $J \subset \mathcal{J}(A)$ and $\Psi_1(P_1[T]) + (J^2T) = A[T]$, we have $\Psi_1(P_1[T]) + \mathcal{J}(A)A[T] = A[T]$ and therefore, by (3.3), any maximal ideal of $A[T]$ containing $\Psi_1(P_1[T])$ is of height $\leq d$. Hence, by (1.2), $\dim(A[T]/\Psi_1(P_1[T])) \leq d - \mathrm{ht}\,(\Psi_1(P_1[T])) \leq d-(n-1) \leq n-2$. Hence, applying (2.7), we get a surjection $\Lambda : P[T] \twoheadrightarrow I$ such that $(\Phi - \Lambda)(P[T]) \subset (I^2T)$. $\square$

Lemma 3.8 *Let A be a ring of dimension d and let n be an integer such that $2n \geq d+3$. Let I be an ideal of $A[T]$ of height n such that $I + \mathcal{J}(A)A[T] = A[T]$. Assume that* $\mathrm{ht}\,\mathcal{J}(A) \geq n-1$. *Let P be a projective A-module of rank n and let $\phi : P[T] \twoheadrightarrow I/I^2$ be a surjection. If the surjection $\phi\otimes A(T) : P(T) \twoheadrightarrow IA(T)/I^2A(T)$ can be lifted to a surjection from $P(T)$ to $IA(T)$, then ϕ can be lifted to a surjection $\Phi : P[T] \twoheadrightarrow I$.*

Proof. Recall that $A(T)$ denote the ring obtained from $A[T]$ by inverting all monic polynomials and $P(T) = P[T]\otimes A(T)$. It is easy to see that, under the hypothesis of the lemma, there exists a monic polynomial $f(T) \in A[T]$ and a surjection $\Phi' : P[T]_f \twoheadrightarrow I_f$ such that Φ' is a lift of ϕ_f. Since $I + \mathcal{J}(A)A[T] = A[T]$, I is not contained in any maximal ideal of $A[T]$ which contains a monic polynomial and hence $f(T)$ is a unit modulo I.

Since $\dim(A/\mathcal{J}(A)) \leq \dim A - \operatorname{ht} \mathcal{J}(A) \leq d - (n-1) \leq n-2$, by Serre's result (1.11), P has a free direct summand of rank 2, i.e. $P = Q \oplus A^2$.

For the sake of simplicity of notation, we write R for $A[T]$, $\widetilde{Q}$ for $Q[T]$ and $\widetilde{P}$ for $P[T]$. Since $\Phi' \in \operatorname{Hom}_{R_f}(\widetilde{P}_f, I_f)$, there exists a positive even integer N such that $\Phi'' = f^N \Phi' \in \operatorname{Hom}_R(\widetilde{P}, I)$. It is easy to see, by the very construction of Φ'', that the induced map ${\Phi''}_f$ from $\widetilde{P}_f$ to I_f is a surjection. Since f is a unit modulo I, the canonical map $R/I \to R_f/I_f$ is an isomorphism and hence $I/I^2 = I_f/{I_f}^2$. Putting these facts together, we see that $\phi'' = \Phi'' \otimes R/I : \widetilde{P} \twoheadrightarrow I/I^2$ is a surjection. Moreover, $\phi'' = f^N \phi$.

Claim: $\phi'' : \widetilde{P} \twoheadrightarrow I/I^2$ can be lifted to a surjection from $\widetilde{P}$ to I.

Assume the claim. Let $\Lambda : \widetilde{P} \twoheadrightarrow I$ be a lift of ϕ''. Write $D = R/(f(T))$. Since $(f(T)) + I = R$ and $\Lambda(\widetilde{P}) = I$, $\Lambda \otimes D$ is a unimodular element of $\widetilde{P}^* \otimes D$. Let $\Lambda = (\lambda, d_1, d_2)$, where $\lambda \in \operatorname{Hom}_R(\widetilde{Q}, R)$ and $d_1, d_2 \in R$.

Since $f(T)$ is monic, $A \hookrightarrow D$ is an integral extension and hence $A/\mathcal{J}(A) \hookrightarrow D/\mathcal{J}(D)$ is also an integral extension. Hence $\dim(D/\mathcal{J}(D)) = \dim(A/\mathcal{J}(A)) \leq n-2$. Therefore, in view of Bass' result (1.12), the unimodular element $(\lambda, d_1, d_2) \otimes D$ can be taken to $(0, 0, 1)$ by an element of $E(\widetilde{P}^* \otimes D)$. By (1.18), every element of $E(\widetilde{P}^* \otimes D)$ can be lifted to an automorphism of $\widetilde{P}^*$. Moreover, since $I + (f) = R$, a lift can be chosen to be an automorphism of $\widetilde{P}^*$ which is identity modulo I.

The upshot of the above discussion is that there exists an automorphism Ω of $\widetilde{P}$ such that Ω is identity modulo I and $\Omega^*(\Lambda) = \Lambda\,\Omega = (0, 0, 1)$ modulo $(f(T))$. Therefore, replacing Λ by $\Lambda\,\Omega$, we can assume that $\Lambda = (\lambda, d_1, d_2)$ with $1 - d_2 \in (f(T))$.

Recall that our aim is to lift the surjection $\phi : \widetilde{P} \twoheadrightarrow I/I^2$ to a surjection $\Phi : \widetilde{P} \twoheadrightarrow I$. Recall also that the surjection $\Lambda : \widetilde{P} \twoheadrightarrow I$ is a lift of $f^N \phi : \widetilde{P} \twoheadrightarrow I/I^2$.

Let $g \in R$ be such that $fg = 1$ modulo (d_2) and hence modulo I. Let $\mathfrak{a} = (g^N d_1, d_2)$. Then, since N is even, by (2.3), $\mathfrak{a} = (e_1, e_2)$ with $e_1 - g^N d_1 \in \mathfrak{a}^2$ and $e_2 - g^N d_2 \in \mathfrak{a}^2$. Since $\Lambda = (\lambda, d_1, d_2)$, $\Lambda(\widetilde{P}) = I$ and $Rg + Rd_2 = R$, we see that

$$I = \lambda(\widetilde{Q}) + (d_1, d_2) = g^N\lambda(\widetilde{Q}) + (g^N d_1, d_2) = g^N\lambda(\widetilde{Q}) + (e_1, e_2).$$

Let $\Phi = (g^N\lambda, e_1, e_2) \in \text{Hom}_R(\widetilde{P}, I)$. From the above equality, we see that $\Phi : \widetilde{P} \twoheadrightarrow I$ is a surjection. Moreover, since $1 - fg \in I$, $\Phi \otimes R/I = g^N \Lambda \otimes R/I$ and $\Lambda \otimes R/I = f^N \phi \otimes R/I$, Φ is a (surjective) lift of ϕ. This proves the lemma.

Proof of the claim: Recall that $\Phi'' : \widetilde{P} \to I$ such that the induced map ${\Phi''}_f : \widetilde{P}_f \twoheadrightarrow I_f$ is a surjection and $\phi'' = \Phi'' \otimes R/I : \widetilde{P} \twoheadrightarrow I/I^2$.

We first note that if Δ is an automorphism of $\widetilde{P}$ and if the surjection $\phi''\Delta : \widetilde{P} \twoheadrightarrow I/I^2$ has a surjective lift from $\widetilde{P}$ to I, then so also has ϕ''. We also note that, by (1.18), any element of $E(\widetilde{P}/I\widetilde{P})$ can be lifted to an automorphism of $\widetilde{P}$. Keeping these facts in mind, we proceed to prove the claim.

By (1.24), there exists $\Delta_1 \in E(\widetilde{P}_f)$ such that (1) $\Psi = {\Delta_1}^*(\Phi'') \in \text{Hom}_R(\widetilde{P}, I)$ and (2) $\Psi(\widetilde{P})$ is an ideal of R of height n, where ${\Delta_1}^*$ is an element of $E(\widetilde{P}_f^*)$ induced from Δ_1.

Since $\Psi_f(\widetilde{P}_f) = I_f$ and f is a unit modulo I, we have $I = \Psi(\widetilde{P}) + I^2$. Hence, by (2.2), $\Psi(\widetilde{P}) = I_1 = I \cap I'$, where $I' + I = R$. Since $(I_1)_f = I_f$, ${I'}_f = R_f$ and hence I' contains a monic polynomial f^r for some positive integer r.

Since $\Delta_1 \in E(\widetilde{P}_f)$, $\overline{\Delta} = \Delta_1 \otimes R_f/I_f \in E(\widetilde{P}_f/I_f\widetilde{P}_f)$. Since $\widetilde{P}/I\widetilde{P} = \widetilde{P}_f/I_f\widetilde{P}_f$, we can regard $\overline{\Delta}$ as an element of $E(\widetilde{P}/I\widetilde{P})$. By (1.18), $\overline{\Delta}$ can be lifted to an automorphism Δ of $\widetilde{P}$.

The map $\Psi : \widetilde{P} \twoheadrightarrow I \cap I'$ induces a surjection $\psi : \widetilde{P} \twoheadrightarrow I/I^2$ and it is easy to see that $\psi = \phi''\Delta$. Therefore, to prove the claim, it is enough to show that ψ can be lifted to a surjection from $\widetilde{P}$ to I. If $I' = R$, then obviously Ψ is a required surjective lift of ψ. Hence, we assume that I' is an ideal of height n.

The map $\Psi : \widetilde{P} \twoheadrightarrow I \cap I'$ induces a surjection $\psi' : \widetilde{P} \twoheadrightarrow I'/I'^2$. Recall that $\widetilde{P} = \widetilde{Q} \oplus R^2$. Therefore, since I' contains f^r; a monic polynomial and $\dim(A/\mathcal{J}(A)) \leq n-2$, by (3.6), ψ' can be lifted to a surjection $\Psi'(= (\Gamma, h_1, h_2)) : \widetilde{P} \twoheadrightarrow I'$, where $\Gamma \in \widetilde{Q}^*$, $h_1, h_2 \in R = A[T]$ and h_1 is monic. Moreover, if necessary, by (1.19), we can replace Γ by $\Gamma + {h_2}^2\,\Gamma_1$ for suitable $\Gamma_1 \in \widetilde{Q}^*$ and assume that $\operatorname{ht} K = n-1$, where $K = \Gamma(\widetilde{Q}) + Rh_1$. Let $\overline{R} = R/K$ and $\overline{A} = A/(K \cap A)$. Then $\overline{A} \hookrightarrow \overline{R}$ is an integral extension and hence $\dim(\overline{R}/\mathcal{J}(\overline{R})) = \dim(\overline{A}/\mathcal{J}(\overline{A})) \le \dim(A/\mathcal{J}(A)) \le n-2$.

Let $P_1 = \widetilde{Q} \oplus R$. Then $\widetilde{P} = P_1 \oplus R$ and $K = \Psi'(P_1)$. Since K contains a monic polynomial h_1, $K + I^2 = R$. Moreover, surjections $\Psi : \widetilde{P} \twoheadrightarrow I \cap I'$ and $\Psi' : \widetilde{P} \twoheadrightarrow I'$ are such that $\Psi \otimes R/I' = \Psi' \otimes R/I'$. Therefore, since $\overline{R} = R/K$ and $\dim(\overline{R}/\mathcal{J}(\overline{R})) < n-1$, by (2.7), there exists a surjection $\Lambda_1 : \widetilde{P} \twoheadrightarrow I$ with $\Lambda_1 \otimes R/I = \Psi \otimes R/I = \psi$. Therefore, $\Lambda = \Lambda_1 \, \Delta^{-1} : \widetilde{P} \twoheadrightarrow I$ is a lift of ϕ''. Thus, the proof of the claim is complete. $\square$

Remark 3.9 The above result has been proved in ([12], Lemma 3.6) in case A is semi-local and $n = d \ge 3$.

The following result is due to Bhatwadekar and Raja Sridharan ([6], Lemma 3.5).

Lemma 3.10 *Let A be a regular domain containing a field k, $I \subset A[T]$ an ideal, $J = A \cap I$ and $B = A_{1+J}$. Let P be a projective A-module and let $\overline{\phi} : P[T] \to\to I/(I^2T)$ be a surjective map. Suppose there exists a surjection $\theta : P_{1+J}[T] \twoheadrightarrow I_{1+J}$ such that θ is a lift of $\overline{\phi} \otimes B$. Then, there exists a surjection $\Phi : P[T] \twoheadrightarrow I$ such that Φ is a lift of $\overline{\phi}$.*

The following result is very crucial for the proof of our main result (3.15).

Proposition 3.11 *Let A be a regular domain of dimension d containing a field k and let n be an integer such that $2n \ge d+3$. Let I be an ideal of $A[T]$ of height n. Let P be a projective A-module of rank n and let $\psi : P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. Suppose there exists a surjection $\Psi' : P[T] \otimes A(T) \twoheadrightarrow IA(T)$ which is a lift of $\psi \otimes A(T)$. Then, we can lift ψ to a surjection $\Psi : P[T] \twoheadrightarrow I$.*

Proof. In view of (3.10), we can assume that $J = I \cap A \subset \mathcal{J}(A)$. Hence $\operatorname{ht} \mathcal{J}(A) \ge n-1$, by (3.2) and $\dim(A/\mathcal{J}(A)) \le d-(n-1) \le n-2$. Therefore, by Serre's result (1.11), we can assume that P has a unimodular element i.e. $P = P_1 \oplus A$.

Applying Moving lemma (3.4) for the surjection $\psi : P[T] \twoheadrightarrow I/(I^2T)$, we get a lift $\Theta \in \text{Hom}_{A[T]}(P[T], I)$ of ψ such that the ideal $\Theta(P[T]) = I''$ satisfies the following properties:

(i) $I = I'' + (J^2T)$.

(ii) $I'' = I \cap I'$, where I' is an ideal of height n.

(iii) $I' + (J^2T) = A[T]$.

The surjection $\Theta : P[T] \twoheadrightarrow I \cap I'$ induces surjections $\phi : P[T]/I'P[T] \twoheadrightarrow I'/I'^2$ and $\Theta \otimes A(T) : P(T) \twoheadrightarrow (I \cap I')A(T)$ such that $\Psi' \otimes A(T)/IA(T) = (\Theta \otimes A(T)) \otimes A(T)/IA(T)$.

Since $\dim A(T) = d$ and I, I' are two comaximal ideals of height n, where $2n \geq d + 3$, applying Subtraction principle (2.8) to surjections Ψ' and $\Theta \otimes A(T)$, we get a surjection $\Phi' : P(T) \twoheadrightarrow I'A(T)$ such that $\Phi' \otimes A(T)/I'A(T) = \phi \otimes A(T)$.

Since $I' + \mathcal{J}(A) = A[T]$ and $\phi \otimes A(T)$ has a surjective lift, namely, $\Phi' : P(T) \twoheadrightarrow I'A(T)$, by (3.8), there exists a surjection $\Phi : P[T] \twoheadrightarrow I'$ which is a lift of ϕ.

Thus, we have surjections $\Phi : P[T] \twoheadrightarrow I'$ and $\Theta : P[T] \twoheadrightarrow I \cap I'$ such that $\Phi \otimes A[T]/I' = \phi = \Theta \otimes A[T]/I'$. Hence, as $I' + (J^2T) = A[T]$ and $J \subset \mathcal{J}(A)$, by (3.7), there exists a surjection $\Psi : P[T] \twoheadrightarrow I$ such that $(\Psi - \Theta)(P[T]) \subset (I^2T)$. Since Θ is a lift of ψ, we are through. $\square$

Remark 3.12 For $n = d$, the above proposition has been already proved in ([12], Theorem 4.7) in the case A is an arbitrary ring containing a field of characteristic 0.

As an application of (3.11), we prove the following "Subtraction principle" for polynomial algebra.

Corollary 3.13 *Let A be a regular domain of dimension d containing an infinite field k and let n be an integer such that $2n \geq d + 3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n and let $I, I' \subset A[T]$ be two comaximal ideals of height n. Let $\Gamma : P[T] \twoheadrightarrow I$ and $\Theta : P[T] \twoheadrightarrow I \cap I'$ be surjections such that*

$\Gamma\otimes A[T]/I = \Theta\otimes A[T]/I$. *Then, there exists a surjection* $\Psi : P[T] \twoheadrightarrow I'$ *such that* $\Psi\otimes A[T]/I' = \Theta\otimes A[T]/I'$.

Remark 3.14 Since $\dim A[T] = d+1$, if $2n \geq d+4$, then, we can appeal to (2.8) for the proof. So, we need to prove the result only in the case $2n = d+3$. However, the proof given below in this case works equally well for $2n > d+3$ and hence, allows us to give a unified treatment.

Proof. Let $K = I \cap I'$. Then, since k is infinite, there exists a $\lambda \in k$ such that $K(\lambda) = A$ or $K(\lambda)$ has height n ([6], Lemma 3.3). Therefore, replacing T by $T-\lambda$, if necessary, we assume that $K(0) = A$ or $\operatorname{ht} K(0) = n$.

Note that Θ induces a surjection $\overline{\theta} : P[T] \twoheadrightarrow I'/I'^2$. We first show that $\overline{\theta}$ can be lifted to a surjection from $P[T]$ to $I'/(I'^2T)$.

Case 1. If $I'(0) = A$, then, since $P = P_1\oplus A$, we can lift $\overline{\theta}$ to a surjection $\phi : P[T] \twoheadrightarrow I'/(I'^2T)$.

Case 2. Assume that $\operatorname{ht} I'(0) = n$. The map Θ induces a surjection $\Theta(0) : P \twoheadrightarrow K(0)(= I(0)\cap I'(0))$. If $I(0) = A$, then $K(0) = I'(0)$ and therefore it is easy to see that $\Theta(0)$ and $\overline{\theta}$ will patch up to give a surjection $\psi : P[T] \twoheadrightarrow I'/(I'^2T)$ which is a lift of $\overline{\theta}$. Now, if $\operatorname{ht} I(0) = n$, then, since $\Gamma\otimes A[T]/I = \Theta\otimes A[T]/I$, we can apply the Subtraction principle (2.8) to the surjections $\Gamma(0) : P \twoheadrightarrow I(0)$ and $\Theta(0) : P \twoheadrightarrow I(0)\cap I'(0)$ to conclude that there is a surjection $\varphi : P \twoheadrightarrow I'(0)$ such that $\varphi\otimes A/I'(0) = \Theta(0)\otimes A/I'(0)$. Hence, as before, we see that $\overline{\theta}$ and φ will patch up to give a surjection $\psi : P[T] \twoheadrightarrow I'/(I'^2T)$ which is a lift of $\overline{\theta}$.

In view of (3.11), to show that there exists a surjection $\Psi : P[T] \twoheadrightarrow I'$ such that $\Psi\otimes A[T]/I' = \overline{\theta} = \Theta\otimes A[T]/I'$, it is enough to show that $\psi\otimes A(T)$ has a surjective lift from $P(T)$ to $I'A(T)$.

The surjections Γ and Θ induces surjections $\Gamma\otimes A(T) : P(T) \twoheadrightarrow IA(T)$ and $\Theta\otimes A(T) : P(T) \twoheadrightarrow (I\cap I')A(T)$ respectively with the property

$$(\Gamma\otimes A(T))\otimes A(T)/IA(T) = (\Theta\otimes A(T))\otimes A(T)/IA(T).$$

Therefore, by Subtraction principle (2.8), there exists a surjection $\Psi' : P(T) \twoheadrightarrow I'A(T)$ with the property $\Psi' \otimes A(T)/I'A(T) = (\Theta \otimes A(T)) \otimes A(T)/I'A(T)$.

Since, $(\Theta \otimes A(T)) \otimes A(T)/I'A(T) = \psi \otimes A(T)$, we are through. $\square$

Recall that ring A is called *essentially of finite type over a field* k, if A is a localization of an affine algebra over k.

Now, we prove our main result of this thesis.

Theorem 3.15 *Let k be an infinite perfect field and let A be a regular domain of dimension d which is essentially of finite type over k. Let n be an integer such that $2n \geq d + 3$. Let $I \subset A[T]$ be an ideal of height n and let P be a projective A-module of rank n. Assume that we are given a surjection*

$$\phi : P[T] \twoheadrightarrow I/(I^2T).$$

Then, there exists a surjection

$$\Phi : P[T] \twoheadrightarrow I$$

such that Φ is a lift of ϕ.

Remark 3.16 We first say a few words about the method of the proof. The essential ideas are contained in the case where $P = A^n$ is free. To simplify the notation, we denote the ring $A[T]$ by R.

Following an idea of Quillen (see [29]), we show that the collection of elements $s \in A$ such that the surjection $\phi_s = \phi \otimes R_s$ can be lifted to a surjection $\Psi : {R_s}^n \twoheadrightarrow I_s$ is an ideal of A. This ideal, in view of the result of Mandal and Varma (the local case), is not contained in any maximal ideal of A and hence contains 1. Therefore, we are through.

Denote this collection by $\mathcal{S}$. It is easy to see that if $s \in \mathcal{S}$ and $a \in A$, then $as \in \mathcal{S}$. Hence $\mathcal{S}$ will be an ideal if we show that for $s, t \in \mathcal{S}$, $s+t \in \mathcal{S}$. As in [29], by replacing A by A_{s+t}, we may assume that $s+t = 1$. Since A is regular, if some power of s is in I, then, by using Quillen's splitting lemma for an automorphism of ${R_{st}}^n$ which is isotopic to identity, one can easily show that $1 = s + t \in \mathcal{S}$ (for example see [6], Lemma 3.5). The crux of the proof is to reduce the problem to this case. We indicate in brief how this reduction is achieved. First we digress a bit.

The surjection $\phi : R^n \twoheadrightarrow I/(I^2T)$ can be lifted to $\Phi' : R^n \twoheadrightarrow I \cap I'$, where I' is an ideal of R of height n comaximal with I (we say I' is residual to I with respect to ϕ). A "Subtraction principle" (see Theorem 2.8 and Corollary 3.13) says that if the surjection (induced by Φ') $\phi_1 : R^n \twoheadrightarrow I'/({I'}^2T)$ has a surjective lift from R^n to I', then ϕ can be lifted to a surjection $\Phi : R^n \twoheadrightarrow I$.

Now, using the fact that $t = 1 - s \in \mathcal{S}$, we first show the existence of an ideal I_1 which is residual to I with respect to ϕ and satisfying the additional property that I_1 is comaximal with Rs. Then, using the fact that $s \in \mathcal{S}$, we show that there exists an ideal I_2 which contains a power of s and is residual to I_1. Thus, the desired reduction is achieved.

Since the problem is solved for I_2, applying "Subtraction principle", the problem is solved for I_1. Applying Subtraction principle once again, the problem is solved for I. This completes the proof of Theorem 3.15.

Proof of Theorem 3.15. If I has height $d + 1$, then I contains a monic polynomial in T. Hence, by Mandal's theorem (1.21), we are through. Therefore, we always assume that $n \leq d$ and hence, the inequality $2n \geq d + 3$ would imply that $d \geq 3$.

We first assume that A is local. In this case, if $n \geq 4$ and $I(0) = A$ or $I(0)$ is a complete intersection ideal of height n, then, by Mandal-Varma theorem (1.23), we are through. It is easy to see that in the case $I(0) = A$, (1.23) is valid even if $\operatorname{ht} I = \dim A = 3$. To complete the proof in the case A is local we proceed as follows.

Let $J = I \cap A$. By Moving lemma (3.4), the surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$ has a lift $\Phi' \in \operatorname{Hom}_{A[T]}(P[T], I)$ such that the ideal $\Phi'(P[T]) = I''$ satisfies the following properties:

(i) $I'' + (J^2T) = I$.

(*ii*) $I'' = I \cap I'$, where I' is an ideal of height $\geq n$.
(*iii*) $I' + (J^2T) = A[T]$.

Since I' is locally generated by n elements, if $\operatorname{ht} I' > n$, then $I' = A[T]$ and we are through. So assume that $\operatorname{ht} I' = n$. The surjection $\Phi' : P[T] \twoheadrightarrow I''(= I \cap I')$ induces a surjection $\psi' : P[T] \twoheadrightarrow I'/I'^2$. Since $I' + (J^2T) = A[T]$, $I'(0) = A$. Hence, as P is free, ψ' can be lifted to a surjection $\psi : P[T] \twoheadrightarrow I'/(I'^2T)$. Now, as $I'(0) = A$, by (1.23), the surjection ψ can be lifted to a surjection $\Psi : P[T] \twoheadrightarrow I'$. Thus, we have surjections $\Phi' : P[T] \twoheadrightarrow I \cap I'$ and $\Psi : P[T] \twoheadrightarrow I'$ such that $\Phi' \otimes A[T]/I' = \Psi \otimes A[T]/I'$. Therefore, since $I' + (J^2T) = A[T]$ and A is local, by (3.7), there exists a surjection $\Phi : P[T] \twoheadrightarrow I$ such that $(\Phi - \Phi')(P[T]) \subset (I^2T)$. Since Φ' is a lift of ϕ, we are through.

Now, we prove the theorem in the general case. Let

$$S = \{s \in A \mid \exists\, \Lambda : P_s[T] \twoheadrightarrow I_s\ ;\ \Lambda \text{ is a lift of } \phi \otimes A_s[T]\,\}.$$

Our aim is to prove that $1 \in S$. Note that if $t \in S$ and $a \in A$, then $at \in S$. Moreover, since the theorem is proved in the local case, it is easy to see that for every maximal ideal $\mathfrak{m}$ of A, there exists $s \in A - \mathfrak{m}$ such that P_s is free and $s \in S$. Hence, we can find $s_1, \ldots, s_r \in S$ such that P_{s_i} is free and $s_1 + \cdots + s_r = 1$. Therefore, by inducting on r, it is enough to show that if $s, t \in S$ and P_s is free, then $s + t \in S$. Since, in the ring $B = A_{s+t}$, $x + y = 1$, where $x = s/s + t$ and $y = t/s + t$, replacing A by B if necessary, we are reduced to prove that if $s, 1 - s = t \in S$ and P_s is free, then $1 \in S$.

The rest of the argument is devoted to the proof of this assertion. The proof is given in steps.

Step 1: Let $J = I \cap A$. In view of (3.10), replacing A by A_{1+J} if necessary, we assume that $J \subset \mathcal{J}(A)$. If s or t is a unit in A, then obviously $1 \in S$. So, without loss of generality, we can assume that s and t are not invertible elements of A. Therefore, as $J \subset \mathcal{J}(A)$, $s \notin \sqrt{J}$ and $t \notin \sqrt{J}$.

Since $\operatorname{ht} I = n$, $\operatorname{ht} J \geq n-1$ by (3.2). Therefore $\dim(A/\mathcal{J}(A)) \leq d-(n-1) \leq n-2$. Hence, since rank $P = n$, by Serre's result (1.11), $P \xrightarrow{\sim} Q \oplus A^2$.

Let $\Gamma_2 : P_t[T] \twoheadrightarrow I_t$ be a surjection which is a lift of $\phi \otimes A_t[T]$. Since $As + At = A$, applying (2.6) (with $L = (I^2T)$ and $B = A[T]$), we get a surjection $\gamma' : P[T] \twoheadrightarrow I/(I^2Ts)$ which is a lift of ϕ. Applying (3.4) to the surjection γ', we get a lift $\Gamma' \in \mathrm{Hom}_{A[T]}(P[T], I)$ of γ' such that the ideal $\Gamma'(P[T]) = \widetilde{I}$ satisfies the following properties:

(*i*) $\widetilde{I} + (J^2Ts) = I$.

(*ii*) $\widetilde{I} = I \cap I_1$, where $\mathrm{ht}\, I_1 \geq n$.

(*iii*) $I_1 + (J^2Ts) = A[T]$.

As before, if $\mathrm{ht}\, I_1 > n$, then $I_1 = A[T]$ and we are through. So we assume that $\mathrm{ht}\, I_1 = n$. The surjection $\Gamma' : P[T] \twoheadrightarrow I \cap I_1$ induces a surjection $\theta : P[T] \twoheadrightarrow I_1/{I_1}^2$. Recall that $J \subset \mathcal{J}(A)$ and $P \xrightarrow{\sim} Q \oplus A^2$. Moreover, $I_1 + (J^2T) = A[T]$. Therefore, if θ can be lifted to a surjection $\Theta : P[T] \twoheadrightarrow I_1$, then, by (3.7), ϕ can be lifted to a surjection $\Phi : P[T] \twoheadrightarrow I$.

In subsequent steps, we will show that θ has a surjective lift $\Theta : P[T] \twoheadrightarrow I_1$.

Step 2: Let $\Gamma_1 : P_s[T] \twoheadrightarrow I_s$ be a surjection which is a lift of $\phi \otimes A_s[T]$. Since the map $\Gamma' : P[T] \twoheadrightarrow I \cap I_1$ is a lift of ϕ, $\Gamma' \otimes A_s[T]/I_s = \Gamma_1 \otimes A_s[T]/I_s$. Therefore, applying Subtraction principle (3.13), we get a surjection $\Theta_1 : P_s[T] \twoheadrightarrow (I_1)_s$ which is a lift of $\theta \otimes A_s[T]$.

Since $I_1 + (J^2Ts) = A[T]$, there exists an element $g \in A[T]$ such that $1 - sg \in I_1$ and the canonical map $A[T]/I_1 \to A_s[T]/(I_1)_s$ is an isomorphism. Therefore, as $P[T] = Q[T] \oplus A^2[T]$ and $P_s[T]$ is a free $A_s[T]$-module, $Q[T]/I_1Q[T]$ is a stably free $A[T]/I_1$-module of rank $n-2$. Since $J \subset \mathcal{J}(A)$, $I_1 + JA[T] = A[T]$ and $\mathrm{ht}\, I_1 = n$, by (3.3), any maximal ideal of $A[T]$ containing I_1 has height $\leq d$. Hence $\dim(A[T]/I_1) \leq d - n \leq n - 3$. Hence, by Bass' result (1.12), $Q[T]/I_1Q[T]$ is a free $A[T]/I_1$-module.

Let N be a positive even integer such that $(s^N\Theta_1)(P[T]) \subset I_1$ and let $\widetilde{\Theta} = s^N\Theta_1 \in \mathrm{Hom}_{A[T]}(P[T], I_1)$. Then, as $1 - sg \in I_1$, $\widetilde{\Theta}$ induces a surjection $\widetilde{\theta} : P[T] \twoheadrightarrow I_1/{I_1}^2$. Since N is even, if $\widetilde{\theta}$ can be lifted to a surjection $\Theta_2 : P[T] \twoheadrightarrow I_1$, then, by (2.4), there would exist a surjection $\Theta : P[T] \twoheadrightarrow I_1$ such that $\Theta \otimes A[T]/I_1 = g^N\Theta_2 \otimes A[T]/I_1$. In that case, since $1 - s^Ng^N \in I_1$, $A[T]/I_1 =$

$A_s[T]/(I_1)_s$, $\Theta_2 \otimes A[T]/I_1 = s^N \Theta_1 \otimes A[T]/I_1$ and Θ_1 is a lift of θ, Θ would be a lift of θ.

Thus, it is enough to show that the surjection $\widetilde{\theta} : P[T] \twoheadrightarrow I_1/{I_1}^2$ can be lifted to a surjection $\Theta_2 : P[T] \twoheadrightarrow I_1$.

Step 3: Recall that $\Theta_1 : P_s[T] \twoheadrightarrow (I_1)_s$ is a surjection and $\widetilde{\Theta} = s^N \Theta_1 : P[T] \to I_1$ is a lift of $\widetilde{\theta}$. Therefore, the induced map $\widetilde{\Theta}_s : P_s[T] \twoheadrightarrow (I_1)_s$ is also a surjection. Hence, by (1.24), there exists $\Delta \in E(P_s[T])$ such that if $\Delta^*(\widetilde{\Theta}) = \Lambda$, then (1) $\Lambda \in P[T]^*$ and (2) $\Lambda_1(P[T]) = K \subset I_1$ is an ideal of $A[T]$ of height n, where Δ^* is an element of $E(P[T]^*)$ induced by Δ. Since $K_s = (I_1)_s$ and $A[T] \cap (I_1)_s = I_1$ (as the ideals I_1 and $sA[T]$ are comaximal), we get $K = I_1 \cap I_2$ with $(I_2)_s = A_s[T]$. Therefore, $s^r \in I_2$ and hence $I_1 + I_2 = A[T]$, since $I_1 + (s) = A[T]$. Since K is an ideal of $A[T]$ of height n which is a surjective image of $P[T]$, either $I_2 = A[T]$ or I_2 is an ideal of height n.

Since $A[T]/I_1 = A_s[T]/(I_1)_s$, $P[T]/I_1P[T] = P_s[T]/I_1P_s[T]$. Hence, the element Δ of $E(P_s[T])$ gives rise to an element $\overline{\Delta}$ of $E(P[T]/I_1P[T])$. By (1.18), there exists an automorphism Δ_0 of $P[T]$ which is a lift of $\overline{\Delta}$. Let $\widetilde{\theta}\,\overline{\Delta} = \lambda_1 : P[T]/I_1P[T] \twoheadrightarrow I_1/{I_1}^2$ be a surjection. Then, it is obvious that if λ_1 can be lifted to a surjection $\Lambda_1 : P[T] \twoheadrightarrow I_1$, then $\widetilde{\theta}$ also has a surjective lift $\Theta_2 : P[T] \twoheadrightarrow I_1$.

Step 4: Note that $\Lambda : P[T] \twoheadrightarrow I_1 \cap I_2$ is a surjection such that $\Lambda \otimes A[T]/I_1 = \lambda_1$. Therefore, if $I_2 = A[T]$, then we are through. Now, we assume that I_2 is an ideal of $A[T]$ of height n.

Since $I_1(0) = A$, Λ gives rise to a surjection $\lambda_2 : P[T] \twoheadrightarrow I_2/({I_2}^2 T)$. If λ_2 has a surjective lift from $P[T]$ to I_2, then, by Subtraction principle (3.13), λ_1 would have a surjective lift $\Lambda_1 : P[T] \twoheadrightarrow I_1$. Therefore, it is enough to show that λ_2 can be lifted to a surjection $\Lambda_2 : P[T] \twoheadrightarrow I_2$.

Since $s^r \in I_2 \cap A$ and $t = 1 - s$, by (3.10), it is enough to show that $\lambda_2 \otimes A_t[T] : P_t[T] \twoheadrightarrow (I_2)_t/({I_2}^2 T)_t$ has a surjective lift. In view of (3.11), it is sufficient to prove that the surjection $\lambda_2 \otimes A_t(T) : P_t(T) \twoheadrightarrow I_2 A_t(T)/{I_2}^2 A_t(T)$ can be lifted to a surjection $\widetilde{\Lambda}_2 : P_t(T) \twoheadrightarrow I_2 A_t(T)$.

Recall that we have a surjection $\Gamma_2 : P_t[T] \twoheadrightarrow I_t$ which is a lift of $\phi \otimes A_t[T]$. Moreover, we also have surjections $\Gamma' : P[T] \twoheadrightarrow I \cap I_1$, $\Lambda : P[T] \twoheadrightarrow I_1 \cap I_2$, where I_1 and I_2 are ideals of $A[T]$ of height n and an automorphism Δ_0 of $P[T]$ such that

(1) $\Gamma' \otimes A[T]/I = \phi$.

(2) $I_1 + (J^2 Ts) = A[T]$, where $J = I \cap A \subset \mathcal{J}(A)$.

(3) $I_1 + I_2 = A[T]$.

(4) $s^N \, \Gamma' \otimes A[T]/I_1 = \Lambda \, {\Delta_0}^{-1} \otimes A[T]/I_1$, where N is an even integer.

Let $R_1 = A_t(T)$. Then, by Subtraction principle (2.8), there exists a surjection $\Phi_1 : P[T] \otimes R_1 \twoheadrightarrow I_1 R_1$ such that $\Phi_1 \otimes R_1/I_1 R_1 = \Gamma' \otimes R_1/I_1 R_1$. Since $P[T] = Q[T] \oplus A[T]^2$ and $Q[T]/I_1 Q[T]$ is free, by (2.4), there exists a surjection $\Phi_2 : P[T] \otimes R_1 \twoheadrightarrow I_1 R_1$ such that $\Phi_2 \otimes R_1/I_1 R_1 = s^N \, \Gamma' \otimes R_1/I_1 R_1 = \Lambda \, {\Delta_0}^{-1} \otimes R_1/I_1 R_1$. Since Δ_0 is an automorphism of $P[T]$, there exists a surjection $\Phi_3 : P[T] \otimes R_1 \twoheadrightarrow I_1 R_1$ such that $\Phi_3 \otimes R_1/I_1 R_1 = \Lambda \otimes R_1/I_1 R_1$. Therefore, by (2.8), there exists a surjection $\widetilde{\Lambda}_2 : P[T] \otimes R_1 \twoheadrightarrow I_2 R_1$ such that $\widetilde{\Lambda}_2 \otimes R_1/I_2 R_1 = \lambda_2 \otimes R_1$.

Thus, the proof of the theorem is complete. $\square$

Chapter 4

Some Auxiliary results

In this section we prove two results. Though these results do not have any direct bearing on the main theorem (proved in the last section), we think that they are interesting off shoots of (3.6) and (2.8) and are of independent interest.

First result gives a partial answer to the following question of Roitman:

Question 4.1 *Let A be a ring and let P be a projective $A[T]$-module such that $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T)$. Then, does P have a unimodular element?*

Roitman in ([30], Lemma 10) answered this question affirmatively in the case A is local. If rank $P > \dim A$, then, by Plumstead's result ([27], Theorem 2), P has a unimodular element. In ([9], Theorem 3.4) an affirmative answer is given to the above question in the case rank $P = \dim A$ under the additional assumption that A contains an infinite field. In this section we settle the case (affirmatively): P is extended from A, rank $P \geq (\dim A + 3)/2$ and A contains an infinite field.

For the proof we need the following two lemmas which are proved in ([9], Lemma 3.1 and Lemma 3.2 respectively).

Lemma 4.2 *Let A be a ring containing an infinite field k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic* *polynomial $f(T) \in A[T]$. Then, there exists a surjection from $\widetilde{P}$ to I, where $I \subset A[T]$ is an ideal of height $\geq n$ containing a monic polynomial.*

Lemma 4.3 *Let R be a ring and let Q be a projective R-module. Let $(\alpha(T), f(T)) : Q[T] \oplus R[T] \twoheadrightarrow R[T]$ be a surjective map with $f(T)$ monic. Let $pr_2 : Q[T] \oplus R[T] \rightarrow\rightarrow R[T]$ be the projection onto the second factor. Then, there exists an automorphism $\sigma(T)$ of $Q[T] \oplus R[T]$ which is isotopic to identity and $pr_2\, \sigma(T) = (\alpha(T), f(T))$.*

The following two results are easy to prove. We give the proof for the sake of completeness.

Lemma 4.4 *Let A be a ring and let P be a projective A-module. Let $\Phi = (\phi, f(T)) : P[T] \oplus A[T] \twoheadrightarrow A[T]$ be a surjection. Suppose $f(T) \in A[T]$ is a monic polynomial. Then, kernel of Φ is extended from A.*

Proof. Let $Q = \ker(\Phi)$. By ([29], Theorem 1), it is enough to show that $Q_{\mathfrak{m}}$ is free for every maximal ideal $\mathfrak{m}$ of A. Hence, we can assume that A is local and hence P is free. Applying Horrock's theorem (see [16]) which says that "if A is local ring and $\widetilde{P}$ is a projective $A[T]$-module, then $\widetilde{P}_f$ free for $f \in A[T]$ monic implies that $\widetilde{P}$ is free", it is enough to show that Q_f is free. But $Q_f \xrightarrow{\sim} P[T]_f$ which is free. Hence, we are through. $\square$

Lemma 4.5 *Let A be a ring and let P be a projective A-module. Let $\Phi : P[T] \rightarrow\rightarrow A[T]$ and $\Psi : P[T] \twoheadrightarrow A[T]$ be two surjections such that $\Phi(0) = \Psi(0)$. Further, assume that the projective $A[T]$-modules kernel of Φ and kernel of Ψ are extended from A. Then, there exists an automorphism Δ of $P[T]$ such that $\Psi\Delta = \Phi$ and $\Delta(0) = Id$*

Proof. We first show that there exists an automorphism Θ of $P[T]$ such that $\Theta(0) = Id$ and $\Phi\,\Theta = \Phi(0) \otimes A[T]$. Let $Q = \ker(\Phi)$ and $L = \ker(\Phi(0))$. Since Q

is extended from A, there exists an isomorphism $\Gamma : L[T] \xrightarrow{\sim} Q$. Now, since the rows of the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L[T] & \longrightarrow & P[T] & \xrightarrow{\Phi(0)\otimes A[T]} & A[T] & \longrightarrow & 0 \\
 & & \downarrow \Gamma & & \downarrow \Lambda & & \downarrow Id & & \\
0 & \longrightarrow & Q & \longrightarrow & P[T] & \xrightarrow[\Phi]{} & A[T] & \longrightarrow & 0
\end{array}$$

are split, we can find an automorphism Λ of $P[T]$ such that the above diagram is commutative. We have $\Phi\Lambda = \Phi(0)\otimes A[T]$ and hence $\Phi(0)\Lambda(0) = \Phi(0)$. Consider an automorphism $\Theta = \Lambda(\Lambda(0)\otimes A[T])^{-1}$ of $P[T]$. Then $\Phi\Theta = (\Phi(0)\otimes A[T])(\Lambda(0)\otimes A[T])^{-1} = (\Phi(0)\otimes A[T])$ and $\Theta(0) = Id$.

Similarly, we have an automorphism Θ_1 of $P[T]$ such that $\Psi\Theta_1 = \Psi(0)\otimes A[T]$ and $\Theta_1(0) = Id$. Consider the automorphism $\Delta = \Theta_1\Theta^{-1}$ of $P[T]$. As $\Phi(0) = \Psi(0)$, we have $\Psi\Delta = (\Psi(0)\otimes A[T])\Theta^{-1} = (\Phi(0)\otimes A[T])\Theta^{-1} = \Phi$ and $\Delta(0) = Id$. This proves the lemma. $\square$

Theorem 4.6 *Let A be a ring of dimension d containing an infinite field k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n which is extended from A, where $2n \geq d + 3$. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then $\widetilde{P}$ has a unimodular element.*

Proof. By (4.2), we get a surjection $\Phi : \widetilde{P} \twoheadrightarrow I$, where I is an ideal of height $\geq n$ containing a monic polynomial. Since I is locally generated by n elements, if $\operatorname{ht} I > n$, then $I = A[T]$ and hence $\widetilde{P}$ has a unimodular element. Hence, we assume that $\operatorname{ht} I = n$.

Since $\widetilde{P}$ is extended from A, we write $\widetilde{P} = P[T]$, where P is a projective A-module of rank n. Then Φ induces a surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$ which in its turn induces a surjection $\Phi(0) : P \twoheadrightarrow I(0)$.

Let $J = A \cap I$. Then $\operatorname{ht} J \geq n - 1$, by (3.2). Since $\dim(A/J) \leq d - (n - 1) \leq n - 2$ and $J \subset \mathcal{J}(A_{1+J})$, by Serre's result (1.11), P_{1+J} has a free direct summand. Let $P_{1+J} = Q \oplus A_{1+J}$ for some projective A_{1+J}-module Q of rank

$n-1$. Since $\dim(A/J) \leq n-2$, by (3.6), the surjection $\phi \otimes A_{1+J}[T]$ can be lifted to a surjection $\Psi(= (\psi, h(T))) : P_{1+J}[T](= Q[T] \oplus A_{1+J}[T]) \twoheadrightarrow I_{1+J}$ with $h(T)$ a monic polynomial. Hence $\Phi(0) \otimes A_{1+J} = \Psi(0)$.

It is easy to see that there exists $a \in J$ such that if $b = 1+a$, then, there exists a projective A_b-module Q_1 with the properties (i) $Q_1 \otimes A_{1+J} = Q$, (ii) $P_b = Q_1 \oplus A_b$, (iii) $\Psi : P_b[T] \twoheadrightarrow IA_b[T]$ and (iv) $\Phi(0)_b = \Psi(0)$. Let $pr_2 : Q_1[T] \oplus A_b[T] \twoheadrightarrow A_b[T]$ be the surjection defined by $pr_2(q, x) = x$ for $q \in Q_1[T]$ and $x \in A_b[T]$. We have the followings:

(1) Since $a \in J$, $I(0)_a = A_a$ and hence $\Phi(0)_a \otimes A_a[T]$ is a surjection from $P_a[T]$ to $A_a[T]$. Since $\Psi_a = (\psi, h(T))_a$ is a unimodular element of $P_{ab}[T]^*$ with $h(T)$ monic, by (4.3), unimodular elements $(pr_2)_a$ and Ψ_a of $P_{ab}[T]^*$ are isotopically connected.

(2) Since $h(T)$ is monic, by (4.4), kernel of Ψ_a is a projective $A_{ab}[T]$-module which is extended from A_{ab}. Therefore, applying (4.5) for the surjections Ψ_a and $\Psi(0)_a \otimes A_{ab}[T]$, there exists an automorphism Θ of $P_{ab}[T]$ such that $\Theta(0)$ is identity automorphism of P_{ab} and $\Psi_a \Theta = \Psi(0)_a \otimes A_{ab}[T] = \Phi(0)_{ab} \otimes A_{ab}[T]$. By (1.16), Θ is isotopic to identity. Hence, by (1.15), Ψ_a and $\Phi(0)_{ab} \otimes A_{ab}[T]$ are isotopically connected.

Thus, combining (1) and (2), the unimodular elements $(pr_2)_a$ and $\Phi(0)_{ab} \otimes A_{ab}[T]$ are isotopically connected. Therefore, there exists an automorphism Γ of $P_{ab}[T]$ such that Γ is isotopic to identity and $\Phi(0) \otimes A_{ab}[T]\, \Gamma = (pr_2)_a$.

Applying (1.17), we get $\Gamma = \Omega'_b\, \Omega_a$, where Ω is an $A_b[T]$-automorphism of $P_b[T]$ and Ω' is an $A_a[T]$-automorphism of $P_a[T]$. Hence, we have surjections $\Delta_1 = pr_2\, \Omega^{-1} : P_b[T] \twoheadrightarrow A_b[T]$ and $\Delta_2 = \Phi(0) \otimes A_a[T]\, \Omega' : P_a[T] \twoheadrightarrow A_a[T]$ such that $(\Delta_1)_a = (\Delta_2)_b$. Therefore, they patch up to yield a surjection $\Delta : P[T] \twoheadrightarrow A[T]$. Hence $\widetilde{P} = P[T]$ has a unimodular element. This proves the result. $\square$

Since every projective $A[T]$-module is extended from A, when A is a regular ring containing a field [28]. Hence, the following corollary is immediate from the above result.

Corollary 4.7 *Let A be a regular ring of dimension d containing an infinite field* *k and let $\widetilde{P}$ be a projective $A[T]$-module of rank n, where $2n \geq d+3$. Suppose $\widetilde{P}_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then $\widetilde{P}$ has a unimodular element.*

Now, we prove our second result which is a complement of the "Subtraction principle" (2.8) and is labeled as "Addition principle". For this result we need the following lemma which is proved in ([7], Corollary 2.14) for $n = d$ and in ([8], Corollary 2.4) in the case P is free. The idea of the proof here is same as in ([7, 8]). We give the proof for the sake of completeness.

Lemma 4.8 *Let A be a ring of dimension d and let P be a projective A-module of rank n, where $2n \geq d+1$. Let $J \subset A$ be an ideal of height n and let $\phi : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then, there exists an ideal $J' \subset A$ of height $\geq n$, comaximal with J and a surjection $\Phi : P \twoheadrightarrow J \cap J'$ such that $\Phi \otimes A/J = \phi$. Further, given finitely many ideals $J_1, \ldots, J_r$ of height n, J' can be chosen to be comaximal with $\cap_1^r J_i$.*

Proof. Let $K = J^2 \cap J_1 \cap \ldots \cap J_r$. Then, by assumption, $\operatorname{ht} K = n$. First, we show that the surjection ϕ can be lifted to a surjection from P/KP to J/K.

Since $K \subset J^2$, $(J/K)^2 = J^2/K$. Let $\Psi \in \operatorname{Hom}_{A/K}(P/KP, J/K)$ be a lift of ϕ. Then $\Psi(P/KP) + J^2/K = J/K$ and hence, by (2.2), there exists $c \in J^2/K$ such that $\Psi(P/KP) + (c) = J/K$. Now, applying Eisenbud-Evans theorem (1.19), there exists $\Psi' \in (P/KP)^*$ such that $\operatorname{ht} N_c \geq n$, where $N = (\Psi + c\Psi')(P/KP)$. Since $\operatorname{ht} K \geq n$, $\dim(A/K) \leq d - n \leq n - 1$. This implies that $N_c = (A/K)_c$. Hence $c^s \in N$ for some positive integer s. Therefore, as $N + (c) = J/K$ and $c \in (J/K)^2$, we have $N = J/K$, by (2.1). Thus, as $\Psi'' = \Psi + c\Psi'$ is also a lift of ϕ, the claim is proved.

Let $\Theta \in \operatorname{Hom}_A(P, J)$ be a lift of Ψ''. Then, as $J/K = \Psi''(P/KP)$, we have $\Theta(P) + K = J$. By (2.2), we get $a \in K$ such that $\Theta(P) + (a) = J$. Again, applying (1.19) to the element $(\Theta, a) \in P^* \oplus A$, there exists $\Theta' \in P^*$ such that $\operatorname{ht} J_1 = n$, where $J_1 = (\Theta + a\Theta')(P)$.

Since $J_1 + (a) = J$ and $a \in J^2$, by (2.2), $J_1 = J \cap J'$ and $J' + (a) = A$. Now, setting $\Phi = \Theta + a\Theta'$, we are through. $\square$

Theorem 4.9 (Addition Principle) *Let A be a ring of dimension d. Let $J_1, J_2 \subset A$ be two comaximal ideals of height n, where $2n \geq d + 3$. Let $P = Q \oplus A$ be a projective A-module of rank n. Let $\Phi : P \twoheadrightarrow J_1$ and $\Psi : P \twoheadrightarrow J_2$ be two surjections. Then, there exists a surjection $\Theta : P \twoheadrightarrow J_1 \cap J_2$ such that $\Phi \otimes A/J_1 = \Theta \otimes A/J_1$ and $\Psi \otimes A/J_2 = \Theta \otimes A/J_2$.*

Proof. Let $J = J_1 \cap J_2$. Since $J_1 + J_2 = A$, we have $J/J^2 = J_1/{J_1}^2 \oplus J_2/{J_2}^2$. Hence Φ and Ψ induces a surjection $\gamma : P \twoheadrightarrow J/J^2$ such that $\gamma \otimes A/J_1 = \Phi \otimes A/J_1$ and $\gamma \otimes A/J_2 = \Psi \otimes A/J_2$.

Applying (4.8), we get an ideal K of height n which is comaximal with J and a surjection $\Gamma : P \twoheadrightarrow J \cap K$ such that $\Gamma \otimes A/J = \gamma \otimes A/J$. Hence $\Gamma \otimes A/J_1 = \Phi \otimes A/J_1$ and $\Gamma \otimes A/J_2 = \Psi \otimes A/J_2$.

Applying Subtraction principle (2.8) for the surjections Φ and Γ, we get a surjection $\Lambda : P \twoheadrightarrow J_2 \cap K$ such that $\Lambda \otimes A/(J_2 \cap K) = \Gamma \otimes A/(J_2 \cap K)$. Hence $\Lambda \otimes A/J_2 = \Psi \otimes A/J_2$.

Again, applying (2.8) for the surjections Ψ and Λ, we get a surjection $\Delta : P \twoheadrightarrow K$ such that $\Delta \otimes A/K = \Lambda \otimes A/K$. Since $\Lambda \otimes A/K = \Gamma \otimes A/K$, we have $\Delta \otimes A/K = \Gamma \otimes A/K$.

Applying (2.8) for the surjections Δ and Γ, we get a surjection $\Theta : P \twoheadrightarrow J$ such that $\Theta \otimes A/J = \Gamma \otimes A/J$. Hence $\Theta \otimes A/J_1 = \Phi \otimes A/J_1$ and $\Theta \otimes A/J_2 = \Psi \otimes A/J_2$. This proves the result. $\square$

In a similar manner, using (3.13), we have the following "Addition principle" for polynomial algebra.

Theorem 4.10 *Let A be a regular domain of dimension d containing an infinite field k and let n be an integer such that $2n \geq d + 3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n and let $I, I' \subset A[T]$ be two comaximal ideals of height n. Let $\Gamma : P[T] \twoheadrightarrow I$ and $\Theta : P[T] \twoheadrightarrow I'$ be two surjections. Then, there* *exists a surjection* $\Psi : P[T] \twoheadrightarrow I \cap I'$ *such that* $\Psi \otimes A[T]/I = \Gamma \otimes A[T]/I$ *and* $\Psi \otimes A[T]/I' = \Theta \otimes A[T]/I'$.

Application 4.11 We will end this chapter by discussing some possible applications of Theorem 3.15. Let A be a regular affine domain of dimension d over an infinite perfect field k. Let P be a projective A-module of rank n. It is interesting to know when P has a unimodular element. By a classical result of Serre ([32]), if $n > d$, then P has a unimodular element. It is well known that this result is not true in general if $n = d$. So one can ask, if one can find the obstruction for a projective module P of rank $= \dim A$ to have a unimodular element.

In ([26], Theorem 3.8), Murthy proved that if P is a projective A-module of rank $n = d$ and k is algebraically closed, then, a necessary and sufficient condition for P to have a unimodular element is the vanishing of its "top Chern class" $C_n(P)$ in the Chow group $CH_0(A)$ of zero cycles modulo rational equivalence. However, this result of Murthy is not true if k is not algebraically closed, as is evidenced by the example of the tangent bundle of the real 2-sphere.

To tackle the above question when k is not necessarily algebraically closed, Nori defined the notion of *Euler class group* of A (see [6]) and to any projective A-module P of rank $= \dim A$ with trivial determinant, he attached an element of this group, called the Euler class of P. Then, he asked whether non-vanishing of Euler class of P is the only obstruction for P to have a unimodular element. Proving (3.15) in the case $\dim(A[T]/I) = 1$, Bhatwadekar and Raja Sridharan answered Nori's question in affirmative (see [6]). More precisely, they proved that a necessary and sufficient condition for P to have a unimodular element is the vanishing of the Euler class of P.

Now, let A be as above and $2n \geq d + 3$. Then, we can define the notion of n^{th} Euler class group of A, denoted by $E^n(A)$ (see [8]). Let P be a projective A-module of rank n with trivial determinant. We believe that using (3.15), one can attach an element of $E^n(A)$ corresponding to P (the Euler class of P) which will detect an obstruction for P to have a unimodular element.

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