Projective modules over overrings of polynomial rings and a question of Quillen

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Abstract

Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let $g_1, \ldots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. Let $A = R_{g_1 \cdots g_t}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(n_n)^{-1}]$, where $f_i(T) \in k[T]$ and $l_i = a_{i1}Y_1 + \ldots + a_{im}Y_m$ with $(a_{i1}, \ldots, a_{im}) \in k^m - (0)$. Then every projective A-module of rank $\geq t$ is free. Laurent polynomial case $f_i(l_i) = Y_i$ of this result is due to Popescu.

1 Introduction

In this paper, we will assume that rings are commutative Noetherian, modules are finitely generated, projective modules are of constant rank and k will denote a field.

Let R be a ring and P a projective R-module. We say that P is *cancellative* if $P \oplus R^m \xrightarrow{\sim} Q \oplus R^m$ for some projective R-module Q implies $P \xrightarrow{\sim} Q$. For simplicity of notations, we begin with a definition.

Definition 1.1 A ring $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is said to be **of type** $R[d, m, n]$ if R is a ring of dimension d, $Y_1, \ldots, Y_m$ are variables over R, each $f_i(T) \in R[T]$ and either each $l_i = Y_{i_j}$ for some i_j, or R contains a field k and $l_i = \sum_{j=1}^m a_{ij}Y_j - b_i$ with $b_i \in R$ and $(a_{i1}, \ldots, a_{im}) \in k^m - (0)$.

Let A be a ring of the type $R[d, m, n]$. We say that A is **of type** $R[d, m, n]^*$ if $f_i(T) \in k[T]$ and $b_i \in k$ for all i.

Let $A = R[Y_1, \ldots, Y_m, f_1(Y_1)^{-1}, \ldots, f_n(Y_n)^{-1}]$ be a ring of type $R[d, m, n]$ with $n \leq m$ and $l_i = Y_i$. If P is a projective A-module of rank $\geq$ max $\{2, d+1\}$, then Dhorajia-Keshari ([5], Theorem 3.12), proved that $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$ and hence P is cancellative. This result was proved by Bass [2] in case $n = m = 0$; Plumstead [12] in case $m = 1$, $n = 0$; Rao [16] in case $n = 0$; Lindel [8] in case $f_i = Y_i$. Gabber [6] proved the following result: *Let k be a field and A a ring of type $k[0, m, n]$. Then every projective A-module is free.* We prove the following result (3.4) which generalizes ([5], Theorem 3.12) and is motivated by Gabber's result.

Theorem 1.2 *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]$ and P a projective A-module of rank $\geq$ max $\{2, d+1\}$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular, P is cancellative.*

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The Bass-Quillen conjecture [3, 15] says: *If R is a regular ring, then every projective module over $R[X_1, \dots, X_r]$ is extended from R.* In B-Q conjecture, we may assume that R is a regular local ring, due to Quillen's local-global principal [15]: *For a ring B, projective module P over $B[X_1, \dots, X_r]$ is extended from B if and only if $P_{\mathfrak{m}}$ is free for every maximal ideal $\mathfrak{m}$ of B.* We remark that Quillen's local global principal is also true for projective modules over positive graded rings ([19], Theorem 3.1), whereas it is not true for Laurent polynomial rings ([4], Example 2, p. 809).

Lindel [9] gave an affirmative answer to B-Q conjecture when R is a *regular k-spot*, i.e. $R = R'_{\mathfrak{p}}$, where R' is some affine k-algebra and $\mathfrak{p}$ is a regular prime ideal of R'. Using Lindel's result, Popescu [13] proved B-Q conjecture when R is any regular local ring containing a field k.

Quillen [15] had asked the following question whose affirmative answer would imply that B-Q conjecture is true: *Assume $(R, \mathfrak{m})$ is a regular local ring and $f \in \mathfrak{m} - \mathfrak{m}^2$ a regular parameter of R. Is every projective R_f-module free?*

Bhatwadekar-Rao [4] answered Quillen's question when R is a regular k-spot. More generally, they proved: *Let $(R, \mathfrak{m})$ be a regular k-spot with infinite residue field and f a regular parameter of R. If B is one of R, $R(T)$ or R_f, then projective modules over $B[X_1, \dots, X_r, Y_1^{\pm 1}, \dots, Y_s^{\pm 1}]$ are free.*

Rao [17] generalized above result as follows: *Let $(R, \mathfrak{m})$ be a regular k-spot with infinite residue field. Let $g_1, \dots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \dots g_t}[X_1, \dots, X_r, Y_1^{\pm 1}, \dots, Y_s^{\pm 1}]$, then projective A-modules of rank $\geq$ min $\{t, d/2\}$ are free.*

Popescu [14] generalized Rao's result as follows: *Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let $g_1, \dots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \dots g_t}[X_1, \dots, X_r, Y_1^{\pm 1}, \dots, Y_s^{\pm 1}]$, then projective A-modules of rank $\geq t$ are free.*

We generalize Popescu's result as follows (5.8):

Theorem 1.3 *Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let $g_1, \dots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \dots g_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$ is a ring of type $R_{g_1 \dots g_t}[d-1, m, n]^*$, then every projective A-module of rank $\geq t$ is free.*

Note that we can not expect (1.3) for rings of type $R[d, m, n]$. For example, let R be either $\mathbb{R}[X, Y]_{(X,Y)}$ or $\mathbb{R}[[X, Y]]$ and $A = R[Z, f(Z)^{-1}]$ a ring of type $R[2, 1, 1]$, where $f(T) = T^2 + X^2 + Y^2$. Then stably free A-module P of rank 2 given by the kernel of the surjection $(X, Y, Z) : A^3 \to A$ is not free. This will follow from the fact that P over the rings $\mathbb{R}[X, Y, Z]_{(X,Y,Z)}[f(Z)^{-1}]$ or $\mathbb{R}[[X, Y, Z]][f(Z)^{-1}]$ is not free ([4], p. 808) and ([11], p. 366).

2 Preliminaries

Let A be a ring and M an A-module. We say $m \in M$ is *unimodular* if there exist $\phi \in M^* = \mathrm{Hom}_A(M, A)$ such that $\phi(m) = 1$. The set of all unimodular elements of M is denoted by $\mathrm{Um}(M)$. For an ideal $J \subset A$, we denote by $E^1(A \oplus M, J)$, the subgroup of $Aut_A(A \oplus M)$ generated by all the automorphisms

$$\Delta_{a\varphi} = \begin{pmatrix} 1 & a\varphi \\ 0 & id_M \end{pmatrix} \quad \text{and} \quad \Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix}$$

with $a \in J, \varphi \in M^*$ and $m \in M$. In particular, if $E_{r+1}(A)$ is the group generated by elementary matrices over A, then $E^1_{r+1}(A, J)$ denotes the subgroup of $E_{r+1}(A)$ generated by

$$\Delta_{\mathbf{a}} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{pmatrix} \quad \text{and} \quad \Gamma_{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{pmatrix},$$

where $F = A^r$, $\mathbf{a} \in JF$ and $\mathbf{b} \in F$. We write $E^1(A \oplus M)$ for $E^1(A \oplus M, A)$.

By $\mathrm{Um}^1(A \oplus M, J)$, we denote the set of all $(a, m) \in \mathrm{Um}(A \oplus M)$ with $a \in 1+J$, and $\mathrm{Um}(A \oplus M, J)$ denotes the set of all $(a, m) \in \mathrm{Um}^1(A \oplus M)$ with $m \in JM$. We write $\mathrm{Um}_r(A, J)$ for $\mathrm{Um}(A \oplus A^{r-1}, J)$ and $\mathrm{Um}^1_r(A, J)$ for $\mathrm{Um}^1(A \oplus A^{r-1}, J)$.

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(p) = 0$. Let $\varphi_p \in End(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a (unipotent) automorphism of M. An automorphism of M of the form $1 + \varphi_p$ is called a *transvection* of M if either $p \in \mathrm{Um}(M)$ or $\varphi \in \mathrm{Um}(M^*)$. We denote by $E(M)$, the subgroup of $Aut(M)$ generated by all transvections of M.

The following result is due to Bak-Basu-Rao ([1], Theorem 3.10). In [5], we proved results for $E^1(A \oplus P)$. Due to this result, we can interchange $E(A \oplus P)$ and $E^1(A \oplus P)$.

Theorem 2.1 *Let A be a ring and P a projective A-module of rank ≥ 2. Then $E^1(A \oplus P) = E(A \oplus P)$.*

The following result follows from the definition.

Lemma 2.2 *Let $I \subset J$ be ideals of a ring A and P a projective A-module. Then the natural map $E^1(A \oplus P, J) \to E^1(\frac{A}{I} \oplus \frac{P}{IP}, \frac{J}{I})$ is surjective.*

Recall that a ring R is *essentially of finite type over a ring B* if R is localization of an affine B-algebra C at some multiplicative closed subset of C. We state two results due to Gabber ([6], Theorem 2.1) and Popescu ([13], Theorem 3.1) respectively.

Theorem 2.3 *Let k be a field and $A = k[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ a ring of type $k[0, m, n]$. Then every projective A-module is free.*

Theorem 2.4 *Let R be a regular local ring containing a field. Then R is a filtered inductive limit of regular local rings essentially of finite type over $\mathbb{Z}$.*

We state two results due to Wiemers ([20], Proposition 2.5) and Lindel ([8], Lemma 1.1) respectively.

Proposition 2.5 *Let R be a ring of dimension d and $A = R[X_1, \ldots, X_r, Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$. Let $c \in \{1, X_r, Y_s - 1\}$. If $s \in R$ and $r \geq max\{3, d+2\}$, then $E_r^1(A, scA)$ acts transitively on $\mathrm{Um}_r^1(A, scA)$.*

Lemma 2.6 *Let A be a ring and P a projective A-module of rank r. Then there exist $s \in A$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in Hom(P, A)$ such that following holds: P_s is free, $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$, $sP \subset p_1 A + \ldots + p_r A$, the image of s in A_{red} is a non-zerodivisor and $(0 : sA) = (0 : s^2 A)$.*

Definition 2.7 Let $R \subset S$ be rings and $h \in R$ be a non-zerodivisor in R and S both. If the natural map $R/hR \to S/hS$ is an isomorphism, then we say $R \to S$ is an *analytic isomorphism* along h. In this case, we get the following fiber product diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R_h & \longrightarrow & S_h. \end{array}$$

In particular, if P is a projective S-module such that P_h is free, then P is extended from R.

The following result is due to Nashier ([10], Theorem 2.8). See also ([4], Proposition, p. 803).

Proposition 2.8 *Let $(R, \mathfrak{m})$ be a regular k-spot over a perfect field k. Let $g \in \mathfrak{m}$ and f be any regular parameter of R with (g, f) a regular sequence. Then there exist a field K/k and a regular K-spot R' such that*

(i) $R' = K[Z_1, \ldots, Z_d]_{(\phi(Z_1), Z_2, \ldots, Z_d)}$, where $\phi(Z_1) \in K[Z_1]$ is an irreducible monic polynomial. Moreover, we may assume $Z_d = f$.

(ii) $R' \subset R$ is an analytic isomorphism along h for some $h \in gR \cap R'$.

(iii) If $R/\mathfrak{m}$ is infinite, then K is also infinite.

We end this section by stating two results ([7], Lemma 3.3) and ([4], Proposition 3.7) respectively.

Lemma 2.9 *Let A be a ring and P a projective A-module of rank r. Choose $s \in A$ satisfying the properties of (2.6). Assume that R^r is cancellative, where $R = A[X]/(X^2 - s^2 X)$. Then every element of $\mathrm{Um}^1(A \oplus P, s^2 A)$ can be taken to $(1, 0)$ by some element of $Aut(A \oplus P)$.*

Proposition 2.10 *Let B be a reduced ring of dimension d and R an overring of $B[X]$ contained in its total quotient ring. Let $A = R[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$. Then A^r is cancellative for $r \geq d+1$.*

3 Cancellation over overrings of polynomial rings

In this section, we prove our first result (3.4). We begin with the following:

Proposition 3.1 *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]$. If $s \in R$ and $r \geq \max\{3, d+2\}$, then $E_r^1(A, sA)$ acts transitively on $\mathrm{Um}_r^1(A, sA)$.*

Proof By ([5], Lemma 3.1), we may assume that R is reduced. The case $n = 0$ follows from (2.5). Assume $n > 0$ and use induction on n. The case each $l_j = Y_{i_j}$ is proved in ([5], Proposition 3.5). We will prove the other case.

Let $(a_1, \ldots, a_r) \in \mathrm{Um}_r^1(A, sA)$. Recall that $l_n = a_{n1}Y_1 + \ldots + a_{nm}Y_m - b_n$ with $(a_{n1}, \ldots, a_{nm}) \in k^m - (0)$ and $b_n \in R$. We can find $\theta \in E_m(k)$ such that $\theta(a_{n1}, \ldots, a_{nm}) = (0, \ldots, 0, 1)$. Replacing the variables $(Y_1, \ldots, Y_m)$ by $\theta(Y_1, \ldots, Y_m)$, we may assume that $l_n = Y_m - b_n$. Further replacing Y_m by $Y_m + b_n$, we may assume that $l_n = Y_m$.

Let $S = 1 + f_n(Y_m)R[Y_m]$. Then $A_S = B[Y_1, \ldots, Y_{m-1}, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$, where $B = R[Y_m]_{f_n(Y_m)S}$ is a ring of dimension d, $l_i = \sum_{j=1}^{m-1} a_{ij}Y_j + \widetilde{b}_i$ with $\widetilde{b}_i = a_{im}Y_m - b_i \in B$. Hence A_S is of type $B[d, m-1, n-1]$. By induction on n, we can find $\sigma \in E_r^1(A_S, sA_S)$ such that $\sigma(a_1, \ldots, a_r) = (1, 0, \ldots, 0)$. We can find $g = 1 + f_n(Y_m)h(Y_m) \in S$ and $\sigma' \in E_r^1(A_g, sA_g)$ such that $\sigma'(a_1, \ldots, a_r) = (1, 0, \ldots, 0)$. Rest of the proof is similar to ([5], Proposition 3.5). Hence we only give a sketch.

Let $C = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ be a ring of type $R[d, m, n-1]$. Consider the fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & A = C_{f_n(Y_m)} \\ \downarrow & & \downarrow \\ C_{g(Y_m)} & \longrightarrow & A_{g(Y_m)} = C_{g(Y_m)f_n(Y_m)}. \end{array}$$

By ([5], Lemma 3.2), there exist $\sigma_1 \in E_r^1(C_{f_n}, s)$ and $\sigma_2 \in \mathrm{SL}_r^1(C_g, s)$ such that σ' has a splitting $\sigma' = (\sigma_2)_{f_n} \circ (\sigma_1)_g$. Patching unimodular elements $\sigma_1(a_1, \ldots, a_r) \in \mathrm{Um}_r^1(C_{f_n}, s)$ and $(\sigma_2)^{-1}(1, 0, \ldots, 0) \in \mathrm{Um}_r^1(C_g, s)$, we get $(c_1, \ldots, c_r) \in \mathrm{Um}_r^1(C, s)$. By induction on n, there exist $\phi \in E_r^1(C, s)$ such that $\phi(c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Taking projection of ϕ in A, we get $\Phi \in E_r^1(A, s)$ such that $\Phi\sigma_1(a_1, \ldots, a_r) = (1, 0, \ldots, 0)$. This completes the proof. ■

As a consequence of (3.1), we get the following:

Proposition 3.2 *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]$. Then*

(i) the canonical map $\Phi_r : \mathrm{GL}_r(A)/E_r(A) \to K_1(A)$ is surjective for $r \geq \max\{2, d+1\}$.

(ii) Further assume $f_i(T) \in R[T]$ is monic polynomial, $n \leq m$ and $l_i \in k[Y_1, \ldots, Y_i]$ with $a_{ii} \neq 0$ (see 1.1). Then for $r \geq \max\{3, d+2\}$, any stably elementary matrix in $\mathrm{GL}_r(A)$ is in $E_r(A)$. In particular, Φ_{d+2} is an isomorphism.

Proof The proof of (i) is same as ([5], Theorem 3.8). For (ii), let $M \in \mathrm{GL}_r(A)$ be a stably elementary matrix. In case $n = 0$ or each $l_i = Y_i$, the proof follows from ([5], Theorem 3.8). Assume $n > 0$ and use induction on n. Recall that $l_n = a_{n1}Y_1 + \ldots + a_{nn}Y_n - b_n$ with $a_{nn} \neq 0$. Changing $Y_n \mapsto a_{nn}^{-1}(Y_n - a_{n1}Y_1 - \ldots - a_{n-1,n-1}Y_{n-1}) + b_n$, we may assume that $l_n = Y_n$. Let $S = 1 + f_n(Y_n)R[Y_n]$ and $B = R[Y_n]_{f_nS}$. Then $A_S = B[Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ is a ring of type $B[d, m-1, n-1]$ with $l_i \in k[Y_1, \ldots, Y_i]$ and $a_{ii} \neq 0$. By induction on n, $M_S \in E_r(A_S)$. Hence we can choose $g \in S$ such that $M_g \in E_r(A_g)$. The remaining proof is same as ([5], Theorem 3.8), hence we omit it. ∎

In the following result, (1) will follow from (2.3, 2.6) and (2) will follow from ([5], Lemma 3.10).

Lemma 3.3 *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of the type $R[d, m, n]$ and P a projective A-module of rank r. Then there exist an $s \in R$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in$* $\hom(P, A)$ *such that*

(1) P_s is free; $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$; $sP \subset p_1A + \ldots + p_rA$; the image of s in R_{red} is a non-zerodivisor; and $(0 : sR) = (0 : s^2R)$.

(2) Let $(a, p) \in \mathrm{Um}(A \oplus P, sA)$ with $p = c_1p_1 + \ldots + c_rp_r$, where $c_i \in sA$ for all i. Assume there exist $\phi \in E^1_{r+1}(A, s)$ such that $\phi(a, c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Then there exist $\Phi \in E(A \oplus P)$ such that $\Phi(a, p) = (1, 0)$.

Following is the main result of this section which generalizes ([5], Theorem 3.12).

Theorem 3.4 *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]$ and P a projective A-module of rank $r \geq \max\{2, d+1\}$. Then $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$. In particular, P is cancellative.*

Proof Using ([5], Lemma 3.1), we may assume that R is reduced. If $d = 0$, then R is a direct product of fields. Hence P is free by (2.3) and the result follows from (3.1) with $s = 1$. Assume $d > 0$ and use induction on d.

By (3.3), there exist a non-zerodivisor $s \in R$, $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in P^*$ satisfying the properties of (3.3(1)). We may assume that s is not a unit, otherwise P is free and we are done by (3.1). Rest of the proof is similar to ([5], Theorem 3.12) with $J = R$, we only give a sketch.

Let $(a, p) \in \mathrm{Um}(A \oplus P)$. Using (2.2) and induction on d, we may assume that $(a, p) = (1, 0)$ modulo s^2A. By (3.3), $p = a_1p_1 + \ldots + a_rp_r$ with $a_i \in sA$ and $(a, a_1, \ldots, a_r) \in \mathrm{Um}_{r+1}(A, sA)$. By (3.1), there exist $\phi \in E^1(A, sA)$ such that $\phi(a, a_1, \ldots, a_r) = (1, 0, \ldots, 0)$. By (3.3(2)), we get $\Psi \in E(A \oplus P)$ such that $\Psi(a, p) = (1, 0)$. This completes the proof. ∎

Following result generalizes (2.10).

Proposition 3.5 *Let B be a reduced ring of dimension d containing a field k and R an overring of $B[X]$ contained in its total quotient ring. Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[\dim R, m, n]^*$ with $n \leq m$, $l_i \in k[Y_1, \ldots Y_i]$ and $a_{ii} \neq 0$. Then every projective A-module of rank $r \geq d+1$ is cancellative.*

Proof If $\dim R \leq d$ or $r \geq d+2$, then result follows from (3.4). Hence we assume dim $R = d+1$ and $r = d+1$.

Step 1: We first prove that A^{d+1} is cancellative. When $n = 0$, we are done by (2.10). Assume $n > 0$ and use induction on n.

Recall that $l_n = a_{n1}Y_1 + \ldots + a_{nn}Y_n - b_n$ with $a_{nn} \neq 0$. Changing $Y_n \mapsto a_{nn}^{-1}(Y_n - a_{n1}Y_1 - \ldots - a_{n,n-1}Y_{n-1}) + b_n$, we can assume that $l_n = Y_n$. Let P be a stably free A-module of rank $d+1$. If $S = 1 + f_n(Y_n)k[Y_n]$, then $\dim B[Y_n]_{f_n(Y_n)S} = d$. If $R' = R[Y_n]_{f_nS}$, then $A_S = R'[Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ is a ring of type $R'[d+1, m-1, n-1]^*$ with $l_i \in k[Y_1, \ldots Y_i]$ and $a_{ii} \neq 0$. By induction on n, P_S is free. Hence we can find $g \in k[Y_n]$ such that P_{1+f_ng} is free. If $C' = R[Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ and $C = C'[Y_n]$, then we have following fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & A = C_{f_n} \\ \downarrow & & \downarrow \\ C_{1+gf_n} & \longrightarrow & A_{1+gf_n} = C_{f_n(1+gf_n)}. \end{array}$$

Since P_{1+f_ng} is free, by (2.7), P is extended from C, say $P'_{f_n} = P$ for some projective C-module P'. Since $P \oplus A \stackrel{\sim}{\to} A^{d+2}$, we get $(P' \oplus C)_{f_n} \stackrel{\sim}{\to} {C_{f_n}}^{d+2}$. Since $f_n \in C'[Y_n]$ is a monic polynomial, using Suslin's monic inversion theorem ([18], Theorem 1), we get $P' \oplus C \stackrel{\sim}{\to} C^{d+2}$. But C is a ring of type $R[d+1, m, n-1]^*$ with $l_i \in k[Y_1, \ldots Y_i]$ and $a_{ii} \neq 0$. Hence by induction on n, C^{d+1} is cancellative. Therefore, P' is free and so P is free. This proves that A^r is cancellative.

Step 2: We will prove the general case. Let P be a projective A-module of rank $d+1$. If $d = 0$, then we may assume that B is a field. It is easy to see that $R = B[X, f(X)^{-1}]$ for some $f(X) \in B[X]$. Hence A is a ring of type $B[0, m+1, n+1]$, so P is free, by (2.3). Assume $d \geq 1$.

If S is the set of non-zerodivisors of B, then as above, projective modules over $S^{-1}A$ are free. Hence we can choose $s \in S$ such that P_s is free and (3.3 (1)) holds. Note that if $B' = B[T]/(T^2 - s^2T)$, $R' = R[T]/(T^2 - s^2T)$ and $A' = R'[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$, then $(A')^{d+1}$ is cancellative, by step 1. By (2.9), every element of $\text{Um}^1(A \oplus P, s^2A)$ can be taken to $(1, 0)$ by some element of $Aut(A \oplus P)$. To complete the proof, it is enough to show that if $(a, p) \in \text{Um}(A \oplus P)$, then there exist $\sigma \in Aut(A \oplus P)$ such that $\sigma(a, p) \in \text{Um}^1(A \oplus P, s^2A)$.

Let "bar" denote reduction modulo s^2A. Then $\overline{A} = \overline{R}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ and $\dim \overline{R} \leq d$. By (3.4), there exist $\overline{\sigma} \in E(\overline{A} \oplus \overline{P})$ such that $\overline{\sigma}(\overline{a}, \overline{p}) = (1, 0)$. Lifting $\overline{\sigma}$ to an element $\sigma \in E(A \oplus P)$, we get $\sigma(a, p) \in \text{Um}^1(A \oplus P, s^2A)$. This completes the proof. ■

4 Quillen's question and Bhatwadekar-Rao's results

In this section we will generalize some results from [4] regarding Quillen's question mentioned in the introduction. We begin with the following:

Lemma 4.1 *Let R be a UFD of dimension 1 and $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ a ring of type $R[1, m, n]$. Then every projective A-module is free.*

Proof If $n = 0$, we are done by ([4], Proposition 3.1). Assume $n > 0$ and use induction on n. Let P be a projective A-module of rank r. Using same arguments as in the proof of (3.1), after changing variables $(Y_1, \ldots, Y_m)$ by $\theta(Y_1, \ldots, Y_m)$ for some $\theta \in E_m(k)$, we may assume that $l_n = Y_m$.

Let $S = 1 + f_n(Y_m)R[Y_m]$ and $R' = R[Y_m]_{f_n S}$. Then R' is a UFD of dimension 1 and $A_S = R'[Y_1, \ldots, Y_{m-1}, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ is a ring of type $R'[1, m-1, n-1]$, where $l_i = a_{i1}Y_1 + \ldots + a_{i,m-1}Y_{m-1} - \widetilde{b}_i$ with $\widetilde{b}_i = b_i - a_{im}Y_m \in R'$ for $i = 1, \ldots, n-1$. By induction on n, every projective A_S-module is free. In particular, P_S is free. Find $1 + f_n g \in S$ such that $P_{1+f_n g}$ is free. The ring $C = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ is of type $R[1, m, n-1]$. Hence by induction on n, projective C-modules are free. Consider the following fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & A = C_{f_n} \\ \downarrow & & \downarrow \\ C_{1+gf_n} & \longrightarrow & A_{1+f_n g} = C_{f_n(1+gf_n)}. \end{array}$$

Since $P_{1+f_n g}$ is free, patching projective modules P and $(C_{1+f_n g})^r$ over $C_{f_n(1+f_n g)}$, we get that P is extended from C and hence P is free. $\blacksquare$

4.1 Infinite residue-field case

The following result generalizes Bhatwadekar-Rao's Laurent polynomial case ([4], Theorem 3.2).

Proposition 4.2 *Let R be a regular k-spot of dimension d with infinite residue field, f a regular parameter of R and $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ a ring of type $R[d, m, n]^*$. Then every projective A_f-module is free.*

Proof Let P be a projective A_f-module. If $T = R - \{0\}$, then $T^{-1}P$ is free, by (2.3). Find $g \in T$ such that P_g is free. We may assume that (g, f) is a regular sequence in R. By (2.8), there exist an infinite field K/k, a regular K-spot $R' = K[Z_1, \ldots, Z_d]_{(\phi(Z_1), Z_2, \ldots, Z_d)}$ such that $R' \subset R$ is an analytic isomorphism along $h \in gR \cap R'$ and $f = Z_d$. Therefore, $A' = R'[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is a ring of type $R'[d, m, n]^*$ and $A' \subset A$ is an analytic isomorphism along h. Since P_h is free, by (2.7), P is extended from A'_{Z_d}.

Enough to show that projective A'_{Z_d}-modules are free. Replace R' by R and A' by A. If $d \leq 2$, then R_{Z_d} is a UFD of dimension ≤ 1. Hence P is free, by (4.1, 2.3). Assume $d > 2$ and use induction on d. The proof is similar to ([4], Theorem 3.2), hence we only give a sketch.

Let S be multiplicative set of all non-zero homogeneous polynomials in $C = k[Z_2, \ldots, Z_d]$. Then $R_{Z_d S}$ is a localization of $C_S[Z_1]$. We can find $h \in C_S[Z_1]$ such that P_S is defined over the ring $D = C_S[Z_1, h(Z_1)^{-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. Note that C_S is a UFD of dimension ≤ 1, by ([10], Proposition 1.11). Since D is of type $C_S[1, m+1, n+1]$, by (4.1), P_S is free. Choose $F \in S$ such that P_F is free. Since K is infinite, by linear change of variables, we can assume that F is homogeneous and monic polynomial in Z_2 with coefficients in $k[Z_3, \ldots, Z_d]$.

If $\widetilde{R} = k[Z_1, Z_3, \ldots, Z_d]_{(\phi(Z_1), Z_3, \ldots, Z_d)}$, then $\widetilde{R}[Z_2] \subset R$ is an analytic isomorphism along F ([4], page 803). If $\widetilde{A} = \widetilde{R}[Z_2, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$, then $\widetilde{A}_{Z_d} \subset A_{Z_d}$ is an analytic isomorphism along F. Since P_F is free, P is extended from $\widetilde{A}_{Z_d}$, by (2.7). Observe that $\widetilde{A}_{Z_d}$ is a ring of type $\widetilde{R}_{Z_d}[d-2, m+1, n]$. Hence by induction on d, projective $\widetilde{A}_{Z_d}$-modules are free. In particular, P is free. ■

Recall that $R(T)$ denote the ring $S^{-1}R[T]$, where S is the multiplicative set consisting of all monic polynomials of $R[T]$.

Corollary 4.3 *Let $(R, \mathfrak{m})$ be a regular k-spot of dimension d with infinite residue field. If $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is a ring of type $R[d, m, n]^*$, then projective modules over A and $A \otimes_R R(T)$ are free.*

Proof (i) Assume P is a projective $A \otimes_R R(T)$-module. By ([4], Corollary 3.5), $R(T) = R[X]_{(\mathfrak{m}, X)}[1/X]$ with $X = T^{-1}$. Since $f = X \in R[X]_{(\mathfrak{m}, X)}$ is a regular parameter, we are done by (4.2).

(ii) Assume P is a projective A-module. Then, we are done by (i), using Suslin's monic inversion ([18], Theorem 1). ■

The laurent polynomial case of the following result is due to Popescu [14].

Theorem 4.4 *Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension d containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let f be a regular parameter of R and $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ a ring of type $R[d, m, n]^*$. Then projective modules over A, A_f and $A \otimes_R R(T)$ are free.*

Proof (i) Assume P is a projective A_f-module. By (2.4), R is a filtered inductive limit of some regular spots $(R_i)_{i \in I}$ over $\mathbb{Z}$, in particular over the prime subfield of R. Further, we may assume that f is an extension of $f' \in R_j$ for some j and that f' is a regular parameter of R_j (see [14]).

Choosing possibly a bigger index $j \in I$, we may assume that P is extended from $A'_{f'}$, where $A' = R_j[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is a ring of type $R_j[d, m, n]^*$. Since tr-deg $K/k \geq 1$, we can assume that the residue field of R_j is infinite. By (4.2), P' and hence P is free.

(ii) Following the proof of (4.3), projective modules over A and $A \otimes_R R(T)$ are free. ■

4.2 Finite residue-field case

The following result is an analogue of (4.2) in case residue field of R is finite and generalizes Bhatwadekar-Rao's ([4], Theorem 3.8).

Theorem 4.5 *Let R be a regular $\mathbb{F}_q$-spot of dimension d, f a regular parameter of R and $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ a ring of type $R[d, m, n]^*$. Then every projective A_f-module of rank $\geq d-1$ is free.*

Proof As in (4.2), using (2.8), we can assume $R = K[Z_1, \ldots, Z_d]_{(\phi(Z_1), Z_2, \ldots, Z_d)}$ and $f = Z_d$, where $K \supseteq \mathbb{F}_q$ may be a finite field. Let P be a projective A_f-module of rank $r \geq d-1$. Note that projective A_f-modules are stably free and $\dim R_f = d-1$. Hence if $r \geq d$, then P is free by (2.3, 3.4). Therefore, we need to prove the result in case $r = d-1$. We use induction on d.

If $d \leq 2$, then R_f is a UFD of dimension 1 and we are done by (4.1). Assume $d > 2$. If $\widetilde{R} = K[Z_1, \ldots, Z_{d-1}]_{(\phi(Z_1), Z_2, \ldots, Z_{d-1})}$, then $\widetilde{R}_{Z_{d-1}}$ is of dimension $d-2$. Since $R_{Z_d Z_{d-1}}$ is a localization of $\widetilde{R}_{Z_{d-1}}[Z_d]$, we can find $h(Z_d) \in \widetilde{R}_{Z_{d-1}}[Z_d]$ such that $P_{Z_{d-1}}$ is defined over $C = \widetilde{R}_{Z_{d-1}}[Z_d, h(Z_d)^{-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. Since C is of type $\widetilde{R}_{Z_{d-1}}[d-2, m+1, n+1]$ by (3.4), $P_{Z_{d-1}}$ being stably free is free.

If $R' = K[Z_1, \ldots, Z_{d-2}, Z_d]_{(\phi(Z_1), Z_2, \ldots, Z_{d-2}, Z_d)}$, then $R'_{Z_d}[Z_{d-1}] \subset R_{Z_d}$ is an analytic isomorphism along Z_{d-1} (see [4], page 803). If $A' = R'_{Z_d}[Z_{d-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ then $A'_{Z_d} \subset A_{Z_d}$ is also an analytic isomorphism along Z_{d-1}. Using $P_{Z_{d-1}}$ is free, P is extended from $D = R'_{Z_d}[Z_{d-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. By induction on d, projective D-modules of rank $\geq d-1$ are free. Hence P is free. $\blacksquare$

The following result is an analogue of (4.3) in case residue field of R is finite and follows from (4.5) by following the proof of (4.3).

Corollary 4.6 *Let R be a regular $\mathbb{F}_q$-spot of dimension d, and A a ring of type $R[d, m, n]^*$. Then projective modules of rank $\geq d$ over A and $A \otimes_R R(T)$ are free.*

The proof of following result is exactly same as ([4], Proposition 4.1, Theorem 4.2) using (2.3). Hence we omit the proof.

Theorem 4.7 *Let $R = \mathbb{F}_p[[Z_1, \ldots, Z_d]]$ and f be a regular parameter of R. If A is a ring of type $R[d, m, n]^*$, then projective modules over $A, A_f, A \otimes_R R(T)$ are free.*

5 Generalization of Rao's results

In this section, we will generalize some results from [17]. We begin with the following result. It's proof is exactly same as ([17], Theorem 2.1) by using (3.4) instead of Swan's result, hence we omit it. The case $t \leq 1$ is (4.7).

Proposition 5.1 *Let $(R, \mathfrak{m})$ be a formal (or convergent) power series ring of dimension d over a field k. Let $g_1, \ldots, g_t$, $0 \leq t \leq d$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. If A is a ring of type $R[d, m, n]^*$, then every projective $A_{g_1 \ldots g_t}$-module of rank $\geq t-1$ is free.*

Lemma 5.2 *Let $(R, \mathfrak{m})$ be a regular k-spot of dimension d and S a multiplicative closed subset of R which contains a regular parameter of R. Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]^*$. Then every projective $S^{-1}A$-module of rank $\geq d-1$ is free.*

Proof Since $\dim S^{-1}R \leq d-1$, if rank $P > d-1$, then we are done by (3.4). Assume that rank $P = d-1$. We will follow the notation and proof of ([17], Proposition 2.3). If we show that every stably free module P of rank $d-1$ over $R'_{Z_1 s}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is free, then remaining proof is exactly same as in [17].

Recall that $R' = K[Z_1, \ldots, Z_d]_{(Z_1, \ldots, Z_{d-1}, \phi(Z_d))}$. If $\widetilde{R} = K[Z_1, \ldots, Z_{d-2}, Z_d]_{(Z_1, \ldots, Z_{d-2}, \phi(Z_d))}$, then $R'_{Z_1 s}$ is a localization of $\widetilde{R}_{Z_1 s}[Z_{d-1}]$. We can find $f(Z_{d-1}) \in \widetilde{R}_{Z_1 s}[Z_{d-1}]$ such that P is defined over $C = \widetilde{R}_{s Z_1}[Z_{d-1}, f^{-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. Since C is a ring of type $\widetilde{R}_{Z_1}[d-2, m+1, n+1]$ and P is stably free of rank $d-1$, by (3.4), P is free. This completes the proof. ■

The following result is immediate from (5.2).

Corollary 5.3 *Let $(R, \mathfrak{m})$ be a regular k-spot of dimension 3 and f, g, h regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. Let A be a ring of type $R[3, m, n]^*$. Then projective modules over A, A_f, A_{fg} and A_{fgh} are free.*

Lemma 5.4 *Let k be an infinite field, $B = k[Z_1, \ldots, Z_d]$, $\mathfrak{m} = (Z_1, \ldots, Z_{d-1}, \phi(Z_d))$ a maximal ideal and $R = B_{\mathfrak{m}}$. Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]^*$ and $h \in k[Z_1, \ldots, Z_t]$. Then every projective A_h-module of rank $\geq t$ is free.*

Proof Assume $t = d$. If $h \in \mathfrak{m}$, then $\dim R_h = d-1$ and the result follows from (3.4). If $h \notin \mathfrak{m}$, then $R_h = R$ and we are done by (4.3). The proof in case $t < d$ is similar to ([17], Proposition 2.7), hence we only give a sketch.

We can find $f \in B - \mathfrak{m}$ such that P is defined over $B[(fh)^{-1}, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. If $S = k[Z_{t+1}, \ldots, Z_d] - (0)$, then P_S is defined over $\widetilde{R}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$, where $\widetilde{R} = K[Z_1, \ldots, Z_t]_{\mathfrak{m}_1}[(fh)^{-1}]$ with $K = k(Z_{t+1}, \ldots, Z_d)$ and $\mathfrak{m}_1 = (Z_1, \ldots, Z_t)$. Since rank $P \geq t$ and $\dim \widetilde{R} \leq t$, P_S is free ($t = d$ case).

Proceed as in [17], we get that if $B' = k[Z_1, \ldots, Z_{d-1}]_{(Z_1, \ldots, Z_{d-1})}$, then P is extended from C_h, where $C = B'[Z_d, Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. Since C_h is of type $B'_h[d', m+1, n]^*$, where $d' \leq d-1$, by induction on d, P is free. ■

Lemma 5.5 *Let K be an infinite field and $R = K[Z_1, \ldots, Z_d]_{\mathfrak{m}}$, where $\mathfrak{m} = (Z_1, \ldots, Z_{d-1}, \phi(Z_d))$ is a maximal ideal. Fix $q > 0$ an integer such that $d \geq 2q-1$. Let $B = R_{hg_1 \ldots g_k}$, where $h \in K[Z_1, \ldots, Z_p]$ with $1 \leq p < q$ and $g_1, \ldots, g_k$ are regular parameters of $\mathfrak{m}$ with $Z_1, \ldots, Z_p, g_1, \ldots, g_k$ linearly independent modulo $\mathfrak{m}^2$. Let $A = B[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $B[d-1, m, n]^*$ and P a projective A-module of rank $\geq d-q$. Then there exist $g \in k[Z_1, \ldots, Z_p]$ such that P_g is free.*

Proof We follow the proof and notations of ([17], Proposition 2.8) and indicate the necessary changes. If $k = 0$, then $B = R_h$ with $h \in K[Z_1, \dots, Z_p]$. In this case, using (5.4), P itself is free. Assume $k > 0$ and use induction on k. Proceed as in [17] using (5.2). Let $S = K[Z_1, \dots, Z_q] - (0)$. We only need to show that $S^{-1}P$ is free. Remaining arguments are same as in [17].

Recall that $g_1 = Z_{p+1}, \dots, g_{q-p} = Z_q$. Write $\widetilde{R} = K(Z_1, \dots, Z_q)[Z_{q+1}, \dots, Z_d]_{\mathfrak{m}'}$, where $\mathfrak{m}' = (Z_{q+1}, \dots, Z_{d-1}, \phi(Z_d))$. Then $S^{-1}P$ is defined over $C = \widetilde{R}_{g_{q-p+1} \dots g_k}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$.

Assume $k < q - p + 1$. Then $C = \widetilde{R}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$ and $S^{-1}P$ is free, by (4.3). If $k \geq q - p + 1$, use (5.2) to conclude that $S^{-1}P$ is free. ■

Theorem 5.6 *Let $(R, \mathfrak{m})$ be a regular k-spot of dimension d with infinite residue field. Let $A = R_{g_1 \dots g_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$ be a ring of type $R_{g_1 \dots g_t}[d-1, m, n]^*$, where $g_1, \dots, g_t \in R$ are regular parameters which are linearly independent modulo $\mathfrak{m}^2$. Then every projective A-module P of rank $r \geq min\{t, [d/2]\}$ is free.*

Proof We will follow the proof and notations of ([17], Theorem 2.9). If we show that $S^{-1}P$ is free, then rest of the argument is same as in [17]. Note $R' = k[Z_1, \dots, Z_d]_{(Z_1, \dots, Z_{d-1}, \phi(Z_d))}$ and P is defined over $R'_{Z_1 \dots Z_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$. Write $S = k[Z_1, \dots, Z_q] - (0)$ and $\widetilde{R} = K(Z_1, \dots, Z_q)[Z_{q+1}, \dots, Z_d]_{(Z_{q+1}, \dots, Z_{d-1}, \phi(Z_d))}$. Then R' is a localization of $\widetilde{R}$. We can find $h_1 \in K[Z_1, \dots, Z_d]$ such that $S^{-1}P$ is defined over $C = \widetilde{R}_{h_1 Z_{q+1} \dots Z_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$. Since $\dim \widetilde{R} = d - q$ which is q if d is even and $q+1$ when d is odd, $S^{-1}P$ is free, by (5.2). ■

The following result is an analog of (5.1) for regular k-spots in the geometric case. Recall that a local ring $(R, \mathfrak{m})$ is said to have a coefficient field if R contains a subfield K isomorphic to $R/\mathfrak{m}$. The proof is exactly same as of ([17], Theorem 2.12) using above results. Hence we omit the proof.

Theorem 5.7 *Let $(R, \mathfrak{m})$ be a regular k-spot with infinite residue field. Let $g_1, \dots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. Assume that $R/(g_1)$ contains a coefficient field. If $A = R_{g_1 \dots g_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$ is of type $R_{g_1 \dots g_t}[d-1, m, n]^*$, then every projective A-module P of rank $\geq t - 1$ is free.*

The following result generalizes Popescu's result [14]. For $t \leq 1$, this follows from (4.4).

Theorem 5.8 *Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension d containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let $g_1, \dots, g_t$ be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. Let $A = R_{g_1 \dots g_t}[Y_1, \dots, Y_m, f_1(l_1)^{-1}, \dots, f_n(l_n)^{-1}]$ be a ring of type $R_{g_1 \dots g_t}[d-1, m, n]^*$. Then every projective A-module of rank $\geq t$ is free.*

Proof We follow the proof of (4.4) and use same notations. As in [14], if $g = g_1 \dots g_t$, then g is an extension of $g' \in R_j$ for some j. Further, g' is a product of regular parameters $g'_1, \dots, g'_t$ of $(R_j, \mathfrak{m}_j)$ which are linearly independent modulo $\mathfrak{m}_j^2$. If P is a projective A_f-module, then by choosing possibly a bigger index $j \in I$, we may assume that P is an extension of a projective module P' over A', where $A' = (R_j)_{g'}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$. If $\dim R' = d'$, then A' is a ring of type $(R_j)_{g'}[d'-1, m, n]^*$. Now R_j is a regular k-spot. Since tr-deg $K/k \geq 1$, we can assume that the residue field of R_j is infinite. By (5.6), P' and hence P is free. ■

The following result is immediate from (5.8).

Corollary 5.9 *Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension d containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let f, g be regular parameters of R which are linearly independent modulo $\mathfrak{m}^2$. If A is a ring of type $R[d, m, n]^*$, then every projective module over A, A_f and A_{fg} are free.*

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