PROJECTIVE GENERATION OF IDEALS IN POLYNOMIAL EXTENSIONS

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ABSTRACT. Let R be an affine domain of dimension $n \geq 3$ over a field of characteristic 0. Let L be a projective $R[T]$-module of rank 1 and $I \subset R[T]$ a local complete intersection ideal of height n. Assume that I/I^2 is a surjective image of $L \oplus R[T]^{n-1}$. This paper examines under what conditions I is a surjective image of a projective $R[T]$-module P of rank n with determinant L.

1. INTRODUCTION

Assumptions: *In this paper, k will denote a field of characteristic 0, all rings are commutative Noetherian containing $\mathbb{Q}$ and projective modules are finitely generated of constant rank. For a ring R, $\mathcal{P}_n(R)$ will denote the set consisting of isomophism classes of projective R-modules of rank n.*

Let R be a ring and M a finitely generated R-module. We write $\mu_R(M)$ for the minimum number of generators of M as an R-module. Assume I is an ideal of R with $\mu_{R/I}(I/I^2) = n$. If $\mu_R(I) = n$, then I is called efficiently generated and if there exists $Q \in \mathcal{P}_n(R)$ such that I is a surjective image of Q, then I is called projectively generated.

Let R be a ring of dimension n and $I \subset R[T]$ an ideal of height n with $\mu_{R/I}(I/I^2) = n$. If I contains a monic polynomial, then Mandal [M] proved that I is efficiently generated. This result is not true if I does not contain a monic polynomial (for an example, see [B-D], Introduction). However, if $I \subset R[T]$ is a maximal ideal not containing a monic polynomial, then Bhatwadekar [Bh 1] proved that I is projectively generated. For a non-maximal ideal I which does not contain a monic polynomial, Bhatwadekar and Das [B-D] proved the following result.

"Let R be an affine k-algebra of dimension $n \geq 3$. Let $I \subset R[T]$ be a local complete intersection ideal of height n such that $\mu_{R/I}(I/I^2) = n$ and $I(0) \subset R$ is an ideal of height $\geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ with trivial determinant and a surjection $Q[T] \twoheadrightarrow I/(I^2 \cap (T))$. Then I is projectively generated."

In terms of Euler class group of $R[T]$, they proved the following result [B-D]. "Let $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I and $\omega_I(0) : (R/I(0))^n \twoheadrightarrow I(0)/I(0)^2$

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be the induced local orientation of $I(0)$. Let (I, ω_I) and $(I(0), \omega_I(0))$ be elements of Euler class groups $E(R[T])$ and $E(R)$ respectively. Assume that $(I(0), \omega_I(0))$ is obtained as the Euler class of a projective R-module. Then (I, ω_I) is also obtained as the Euler class of a projective $R[T]$-module."

Let R be an affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R[T])$. Das [D 1] has developed the theory of Euler class group $E(R[T], R[T])$ which is used in [B-D]. Das and Zinna [D-Z 2] extended results of Das [D 1] to $E(R[T], L)$. So it is natural to ask the following generalization of results of [B-D].

Question 1.1. *Let R be an affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R[T])$. Let $I \subset R[T]$ be a local complete intersection ideal of height n such that $ht(I(0)) \geq n$. Let $Q \in \mathcal{P}_n(R)$ with determinant L/TL.*

(1) *Let $(I, \omega_I) \in E(R[T], L)$ be such that $(I(0), \omega_{I(0)}) = e(Q, \widetilde{\chi}) \in E(R, L/TL)$, where $\widetilde{\chi} : L/TL \stackrel{\sim}{\to} \wedge^n(Q)$ is an isomorphism. Does there exist $P \in \mathcal{P}_n(R[T])$ with determinant L and an isomorphism $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$ in $E(R[T], L)$?*
(2) *Assume there is a surjection $Q[T] \twoheadrightarrow I/(I^2 \cap (T))$. Is I* **projectively generated**? *In other words, does there exist $P \in \mathcal{P}_n(R[T])$ with determinant L such that I is a surjective image of P?*

We answer question 1.1(2) in case L is extended from R (see 4.7).

Theorem 1.2. *Let R be an affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be a local complete intersection ideal of height n such that $ht(I(0)) \geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L and a surjection $Q[T] \twoheadrightarrow I/I^2 \cap (T)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and a surjection $P \twoheadrightarrow I$. In other words, I is projectively generated.*

We answer question 1.1(1) for reduced ring R (see 5.1).

Theorem 1.3. *Let R be a reduced affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R[T])$. Let $I \subset R[T]$ be an ideal of height n such that $ht(I(0)) \geq n$. Assume that $(I, \omega_I) \in E(R[T], L)$ when $n \geq 4$ and $(I, \omega_I) \in \widetilde{E}(R[T], L)$, the restricted Euler class group of $R[T]$ when $n = 3$ (see (2.12)). Assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L/TL and an isomorphism $\chi : L/TL \stackrel{\sim}{\to} \wedge^n(Q)$ such that $e(Q, \chi) = (I(0), \omega_{I(0)})$ in $E(R, L/TL)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant L and an isomorphism $\chi_1 : L \stackrel{\sim}{\to} \wedge^n(P)$ such that $e(P, \chi_1) = (I, \omega_I)$ in $E(R[T], L)$.*

Steps of proof of (1.3): First we prove the result when L is extended from R. For arbitrary L, there exists a finite subintegral extension S of R such that $L \otimes S[T]$ is extended from S. Now we know the result in $S[T]$ by extended case. Finally we descend from $S[T]$ to $R[T]$ by proving that for $(I,\omega_I) \in E(R[T],L)$, if its image $(IS[T],\omega_I^*)$ in $E(S[T],L\otimes S[T])$ is obtained as the Euler class of a projective $S[T]$-module, then (I,ω_I) is also obtained as the Euler class of a projective $R[T]$-module.

The following result (see 4.11) is an application. It improves [B-RS 2, Theorem 2.7] and [B-D, Corollary 3.11], where it is proved for $L = R[T]$.

Corollary 1.4. *Let R be an affine k-algebra of dimension $n \geq 2$ with k an algebraically closed field. Let $L \in \mathcal{P}_1(R)$ and $I \subset R[T]$ an ideal of height n. Assume that I/I^2 is a surjective image of $L[T] \oplus R[T]^{n-1}$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ such that I is a surjective image of P.*

2. Preliminaries

In this section, we recall some results for later use.

Lemma 2.1. [B-RS 3, Lemma 5.4] *Let R be a ring of dimension $n \geq 2$ and $L \in \mathcal{P}_1(R)$. Let $J \subset R$ be an ideal of height n and $\omega_J : (L\oplus R^{n-1})/J(L\oplus R^{n-1}) \twoheadrightarrow J/J^2$ be a local L-orientation of J. If $\overline{u} \in R/J$ is a unit, then $(J,\omega_J) = (J,\overline{u}^2\omega_J)$ in the Euler class group $E(R,L)$.*

Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let "bar" denote reduction modulo $N[T]$, where N is the nilradical of R. So $\overline{R} = R_{red}$ and $\overline{L} = L/NL$. Let $I \subset R[T]$ be an ideal of height n such that $Spec(R[T]/I)$ is connected and I/I^2 is generated by n elements. We have $\overline{I} = (I + N[T])/N[T]$. Note that $Spec(\overline{R}[T]/\overline{I})$ is also connected. Further, if we write $\mathcal{L} = L\oplus R^{n-1}$, then any surjection $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ induces a surjection $\omega_{\overline{I}} : \overline{\mathcal{L}}[T]/\overline{I\mathcal{L}}[T] \twoheadrightarrow \overline{I}/\overline{I}^2$.

Let $J \subset R[T]$ be an ideal of height n and ω_J be a local orientation of J. Now J can be decomposed uniquely as $J = J_1 \cap \cdots \cap J_k$, where J_i's are pairwise comaximal ideals of $R[T]$ of height n such that $Spec(R[T]/J_i)$ is connected for each i. Clearly $\overline{J} = \overline{J}_1 \cap \cdots \cap \overline{J}_k$ is a similar decomposition for $\overline{J}$. Now ω_J induces a local orientation $\omega_{\overline{J}}$ in a natural way. Therefore, we have a group homomorphism $\Phi : E(R[T],L[T]) \longrightarrow E(\overline{R}[T],\overline{L}[T])$ which takes (J,ω_J) to $(\overline{J},\omega_{\overline{J}})$.

Proposition 2.2. *Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Then*

(1) the group homomorphism $\Phi : E(R[T],L[T]) \longrightarrow E(\overline{R}[T],\overline{L}[T])$ is an isomorphism.

(2) Let $(I,\omega_I) \in E(R[T],L[T])$. If $\Phi((I,\omega_I))$ is the Euler class of a projective module, then so is (I,ω_I). More precisely, assume that $\Phi((I,\omega_I)) = e(P',\chi')$, where $P' \in \mathcal{P}_n(\overline{R}[T])$ with determinant $\overline{L}[T]$ and $\chi' : \overline{L}[T] \xrightarrow{\sim} \wedge^n(P')$ an isomorphism. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and an isomorphism $\chi : L[T] \xrightarrow{\sim} \wedge^n(P)$ such that $e(P,\chi) = (I,\omega_I)$ in $E(R[T],L[T])$.

Proof. (1) is due to Das-Zinna [D-Z 2, Proposition 6.8].

For (2), follow the proof of [B-D, Proposition 2.15, Remark 2.16] where it is proved for $L = R$ and use [D-Z 2, Corollary 4.14] which says that $(\overline{I}, \omega_{\overline{I}}) = e(P', \chi')$ in $E(\overline{R}[T], \overline{L}[T])$ implies that there is a surjection $\alpha : P' \twoheadrightarrow \overline{I}$ such that $(\overline{I}, \omega_{\overline{I}})$ is obtained from the pair (α, χ'). $\square$

Remark 2.3. Note that we do not know [D-Z 2, Corollary 4.14] for arbitrary $L \in \mathcal{P}_1(R[T])$. Hence, we do not have (2.2(2)) for arbitrary L. That is why we are taking reduced ring in section 5 with arbitrary L. $\square$

The following result is proved in [B-D, Lemma 3.2] when $L = R$.

Lemma 2.4. *Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $Q \in \mathcal{P}_n(R)$ with determinant L and an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(Q)$. Let $(I, \omega_I) \in E(R[T], L[T])$ with $ht(I(0)) = n$. Consider $(I(0), \omega_{I(0)}) \in E(R, L)$, where $\omega_{I(0)}$ is the local orientation of $I(0)$ induced by ω_I. Assume that there is a surjection $\alpha : Q \twoheadrightarrow I(0)$ such that (α, χ) induces $e(Q, \chi) = (I(0), \omega_{I(0)})$. Then there is a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ such that $\theta(0) = \alpha$.*

Proof. As Q has determinant L and $\dim(R[T]/I) \leq 1$, by Serre's result [Se], we have $Q[T]/IQ[T] \simeq L[T]/IL[T] \oplus (R[T]/I)^{n-1}$. Choose an isomorphism $\sigma : Q[T]/IQ[T] \xrightarrow{\sim} L[T]/IL[T] \oplus (R[T]/I)^{n-1}$ such that $\wedge^n(\sigma) = (\chi \otimes R[T]/I)^{-1}$. The composite surjection $\overline{\theta} : Q[T] \twoheadrightarrow I/I^2$ given by

$$Q[T] \twoheadrightarrow Q[T]/IQ[T] \overset{\sigma}{\xrightarrow{\sim}} L[T]/IL[T] \oplus (R[T]/I)^{n-1} \overset{\omega_I}{\twoheadrightarrow} I/I^2$$

is such that $\overline{\theta}(0) \otimes R/I(0) = \alpha \otimes R/I(0)$. Applying [B-RS 1, Remark 3.9], we can lift $\overline{\theta}$ to a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ such that $\theta(0) = \alpha$. $\square$

Lemma 2.5. *Let R be a reduced ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a finite subintegral extension. Let $Q \in \mathcal{P}_n(S)$ be such that its determinant is extended from R, i.e. $\wedge^n(Q) \xrightarrow{\sim} L \otimes S$ for some $L \in \mathcal{P}_1(R)$. Then Q is extended from R, i.e. there exists $P \in \mathcal{P}_n(R)$ with determinant L such that $P \otimes S \simeq Q$.*

Proof. Since $R \hookrightarrow S$ is a finite subintegral extension, without loss of generality, we may assume that S is an elementary subintegral extension of R. Let C be the conductor ideal of $R \subset S$. Then ht$(C) \geq 1$ and $(R/C)_{red} = (S/C)_{red}$ [D-Z 1, Lemma 3.7]. Consider the conductor (fiber product) diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/C & \longrightarrow & S/C\,. \end{array}$$

Since every projective $(R/C)_{red}$-module comes from a projective R/C-module, there exists $\widetilde{P} \in \mathcal{P}_n(R/C)$ with an isomorphism $\widetilde{\theta} : \widetilde{P} \otimes (S/C)_{red} \simeq Q \otimes (S/C)_{red}$. Now we can lift $\widetilde{\theta}$ to an isomorphism $\theta : \widetilde{P} \otimes S/C \simeq Q/CQ$. Patching Q and $\widetilde{P}$ over θ, we get $P \in \mathcal{P}_n(R)$ such that $P \otimes S \simeq Q$. Since rank$(Q/CQ) = n > \dim R/C$, by Serre's result [Se], Q/CQ has a unimodular element. Hence, we can modify the patching automorphism θ such that $\wedge^n(P) \simeq L$. □

Lemma 2.6. *Let R be a reduced ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a finite subintegral extension. Let $Q \in \mathcal{P}_n(S[T])$ be such that its determinant is extended from $R[T]$, i.e. $\wedge^n(Q) \simeq L \otimes S[T]$ for some $L \in \mathcal{P}_1(R[T])$. Then Q is extended from $R[T]$, i.e. there exists $P \in \mathcal{P}_n(R[T])$ with determinant L such that $P \otimes S[T] \simeq Q$.*

Proof. Follow the proof of (2.5). By Plumstead's result [P], Q/CQ has a unimodular element, where C is the conductor ideal of $R \hookrightarrow S$. □

Definition 2.7. We recall some definitions from [D-Z 1]. Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a subintegral extension. Let $L \in \mathcal{P}_1(R)$ and write $\mathcal{L} = L \oplus R^{n-1}$. Let $J \subset R$ be an ideal of height n and $\omega_J : \mathcal{L}/J\mathcal{L} \twoheadrightarrow J/J^2$ a surjection. By [D-Z 1, Remark 3.8], we have ht$(JS) = n$. Tensoring w_J with S/JS over R/J, we obtain an induced surjection

$$\widetilde{\omega_J} : \frac{\mathcal{L} \otimes_R S}{JS(\mathcal{L} \otimes_R S)} \twoheadrightarrow \frac{J \otimes_R S}{JS(J \otimes_R S)}.$$

Define a local orientation ω_J^* of JS as the composition

$$\omega_J^* : \frac{\mathcal{L} \otimes_R S}{JS(\mathcal{L} \otimes_R S)} \overset{\widetilde{\omega_J}}{\twoheadrightarrow} \frac{J \otimes_R S}{JS(J \otimes_R S)} \overset{\widetilde{f}}{\twoheadrightarrow} \frac{JS}{J^2 S},$$

where $\widetilde{f}$ is induced by the natural surjection $f : J \otimes_R S \twoheadrightarrow JS$. Note that if ω_J can be lifted to a surjection $\theta : \mathcal{L} \twoheadrightarrow J$, then ω_J^* can be lifted to a surjection $f \circ (\theta \otimes S) : \mathcal{L} \otimes S \twoheadrightarrow JS$. Therefore, we have a well defined group homomorphism $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$ defined by $\Phi((J, \omega_J)) = (JS, \omega_J^*)$.

Similarly for $L \in \mathcal{P}_1(R[T])$, we have a group homomorphism $E(R[T], L) \to E(S[T], L \otimes S[T])$. □

The following three results are due to Das and Zinna.

Theorem 2.8. [D-Z 1, Theorem 3.12] *Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a subintegral extension. If $L \in \mathcal{P}_1(R)$, then the natural map $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$ is an isomorphism.*

Theorem 2.9. [D-Z 1, Theorem 3.16] *Let R be a ring of dimension $n \geq 3$ and $R \hookrightarrow S$ a subintegral extension. Then $E(R[T]) \simeq E(S[T])$.*

Theorem 2.10. [D-Z 3, Theorem 3.12] *Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a subintegral extension. Then weak Euler class groups $E_0(R)$ and $E_0(S)$ are isomorphic.*

Definition 2.11. Let R be a reduced ring of dimension $n \geq 3$. Let $L \in \mathcal{P}_1(R[T])$ and $\mathcal{L} = L \oplus R[T]^{n-1}$. We will define the restricted Euler class group $\widetilde{E}(R[T], L)$, see [D-Z 2, Section 7] when $n = 3$. Let $\widetilde{R}$ be the seminormalization of R and C the conductor ideal of $R \subset \widetilde{R}$. Let $\widetilde{G}$ be the free abelian group on pairs (I, ω_I), where $I \subset R[T]$ is an ideal of height n such that Spec$(R[T]/I)$ is connected, $I + C[T] = R[T]$ (this is the restriction) and $\omega_I : \mathcal{L}/I\mathcal{L} \twoheadrightarrow I/I^2$ is an equivalence class of local L-orientation of I. Here two local L-orientations ω_I and $\widetilde{\omega}_I$ are equivalent if there exists $\theta \in SL(\mathcal{L}/I\mathcal{L})$ such that $\omega_I \circ \theta = \widetilde{\omega}_I$. Take $\widetilde{H}$ to be the subgroup of $\widetilde{G}$ generated by those $(I, \omega_I) \in \widetilde{G}$ such that w_I is a global L-orientation of I, i.e. w_I can be lifted to a surjection $\mathcal{L} \twoheadrightarrow I$. Define the "restricted" Euler class group $\widetilde{E}(R[T], L) = \widetilde{G}/\widetilde{H}$.

Let $P \in \mathcal{P}_n(R[T])$ with determinant L and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ an isomorphism. Since $\dim(R/C) \leq n - 1$, by [P, Corollary 2 of Section 3], P/CP has a unimodular element. Applying ([B-RS 3, Lemma 2.13]), it is easy to see that there is an ideal $I \subset R[T]$ of height n such that $I + C[T] = R[T]$ and a surjection $\alpha : P \twoheadrightarrow I$. Choose an isomorphism $\overline{\gamma} : \mathcal{L}/I\mathcal{L} \xrightarrow{\sim} P/IP$ such that $\wedge^n \overline{\gamma} = \chi \otimes R[T]/I$. Let ω_I be the composite surjection

$$w_I : \mathcal{L}/I\mathcal{L} \overset{\overline{\gamma}}{\xrightarrow{\sim}} P/IP \overset{\overline{\alpha}}{\twoheadrightarrow} I/I^2.$$

We define the Euler class of the pair (P, χ) obtained from the pair (α, χ) as $e(P, \chi) = (I, \omega_I) \in \widetilde{E}(R[T], L)$. Following [D-Z 2, Lemma 6.11], it is easy to prove that the Euler class $e(P, \chi)$ is well defined and it does not depend on the choice of α and $\overline{\gamma}$. $\square$

Remark 2.12. For $n \geq 4$, there is a natural map $\widetilde{E}(R[T], L) \to E(R[T], L)$ which is an isomorphism. This can be seen using moving lemma [D-Z 2, Lemma 2.11] and the fact that ht$(C) \geq 1$.

The following result is from [D-Z 2, Corollary 7.3, Theorem 7.4].

Theorem 2.13. *Let R be a reduced ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R[T])$. Let $P \in \mathcal{P}_n(R[T])$ with determinant L and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ an isomorphism.*

(1) *If $(I, \omega_I) = 0$ in $\widetilde{E}(R[T], L)$, then ω_I is a global L-orientation of I.*
(2) *P has a unimodular element if and only if $e(P, \chi) = 0$ in $\widetilde{E}(R[T], L)$*

Remark 2.14. Let R be a ring of dimension $n \geq 2$ and $L \in \mathcal{P}_1(R[T])$. Let $(I, \omega_I) \in E(R[T], L)$ when $n \neq 3$ and $(I, \omega_I) \in \widetilde{E}(R[T], L)$ when $n = 3$. Let $\overline{f} \in R[T]/I$ be a unit. Composing ω_I with an automorphism of $(L \oplus R[T]^{n-1})/I(L \oplus R[T]^{n-1})$ with determinant $\overline{f}$, we obtain another local orientation of I which we denote by $\overline{f}\omega_I$. On the other hand, let ω_I and $\widetilde{\omega}_I$ be two local orientations of I. Then it follows from [Bh 2, Lemma 2.2] that $\omega_I = \overline{f}\widetilde{\omega}_I$ for some unit $\overline{f} \in R[T]/I$.

The next result follows from [B-RS 3, Lemmas 2.7, 2.8].

Lemma 2.15. *Let R be a ring of dimension $n \geq 2$ and $L \in \mathcal{P}_1(R[T])$. Let $P \in \mathcal{P}_n(R[T])$ with determinant L and $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ an isomorphism. Let $I \subset R[T]$ be an ideal of height n and $\alpha : P \twoheadrightarrow I$ a surjection. Let $e(P,\chi) = (I,\omega_I)$ be obtained from the pair (α,χ), where $e(P,\chi) \in E(R[T],L)$ when $n \neq 3$ and $e(P,\chi) \in \widetilde{E}(R[T],L)$ when $n = 3$. Let $\overline{f} \in R[T]/I$ be a unit. Then there exists $P_1 \in \mathcal{P}_n(R[T])$ with determinant L such that $P \oplus R[T] \stackrel{\sim}{\to} P_1 \oplus R[T]$, an isomorphism $\chi_1 : L \stackrel{\sim}{\to} \wedge^n(P_1)$ and a surjection $\beta : P_1 \twoheadrightarrow I$ such that $e(P_1,\chi_1) = (I, \overline{f}^{n-1}\omega_I)$ is obtained from the pair (β,χ_1).*

The following result extends [D 1, Lemma 5.2].

Lemma 2.16. *Let R be a ring of dimension $n \geq 3$. Let $L \in \mathcal{P}_1(R)$ and $\mathcal{L} = L \oplus R^{n-1}$. Let $I \subset R[T]$ be an ideal of height n and $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ a surjection. Let $\overline{f} \in R[T]/I$ be a unit and θ an automorphism of $\mathcal{L}[T]/I\mathcal{L}[T]$ with determinant $\overline{f}^2$. Assume that ω_I can be lifted to a surjection $\alpha : \mathcal{L}[T] \twoheadrightarrow I$. Then the surjection $\omega_I \circ \theta : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ can also be lifted to a surjection $\beta : \mathcal{L}[T] \twoheadrightarrow I$.*

Proof. Replacing T by $T - \lambda$ for some $\lambda \in \mathbb{Q}$, we may assume that $\mathrm{ht}(I(0)) \geq n$. If $\mathrm{ht}(I(0)) > n$, then $I(0) = R$. By [B-RS 1, Remark 3.9], we can lift $\omega_I \circ \theta$ to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. We now show that the same can be done if $\mathrm{ht}(I(0)) = n$. Now ω_I induces a surjection $\omega_I(0) : \mathcal{L}/I(0)\mathcal{L} \twoheadrightarrow I(0)/I(0)^2$, which can be lifted to $\alpha(0) : \mathcal{L} \twoheadrightarrow I(0)$. Note that $\overline{f(0)} \in R/I(0)$ is a unit and $\theta(0)$ is an automorphism of $\mathcal{L}/I(0)\mathcal{L}$ with determinant $\overline{f(0)}^2$. Therefore, by [B-RS 3, Lemma 5.3], $\omega_I(0) \circ \theta(0)$ can be lifted to a surjection $\phi : \mathcal{L} \twoheadrightarrow I(0)$. Consequently, we can lift $\omega_I \circ \theta$ to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$.

Now we move to the ring $R(T)$ which is obtained from $R[T]$ by inverting all monic polynomials in T. Applying [B-RS 3, Lemma 5.3] to $R(T)$, we get

$$(\omega_I \circ \theta) \otimes R(T) : \mathcal{L} \otimes R(T)/I\mathcal{L} \otimes R(T) \twoheadrightarrow IR(T)/I^2R(T)$$

can be lifted to a surjection $\psi : \mathcal{L} \otimes R(T) \twoheadrightarrow IR(T)$. By [D-Z 2, Theorem 4.1], we get $\omega_I \circ \theta$ can be lifted to a surjection $\beta : \mathcal{L}[T] \twoheadrightarrow I$. □

The following result extends [D 1, Lemma 5.3].

Lemma 2.17. *Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be an ideal of height n and ω_I be a local L-orientation of I. Let $\overline{f} \in R[T]/I$ be a unit. Then $(I,\omega_I) = (I, \overline{f}^2\omega_I)$ in $E(R[T], L[T])$.*

Proof. If $(I,\omega_I) = 0$ in $E(R[T], L[T])$, then it follows from [D-Z 2, Theorem 4.10] and (2.16) that $(I, \overline{f}^2\omega_I) = 0$ in $E(R[T], L[T])$. So assume that $(I,\omega_I) \neq 0$ in $E(R[T], L[T])$. By [D 1, Lemma 2.12], ω_I can be lifted to a surjection $\alpha : L[T]\oplus R[T]^{n-1} \twoheadrightarrow I \cap I'$, where $I' \subset R[T]$ is an ideal of height n with $I + I' = R[T]$. By Chinese remainder theorem, choose $g \in R[T]$ such that $g = f^2$ modulo I and $g = 1$ modulo I'. Applying (2.16), there exists a surjection $\gamma : L[T]\oplus R[T]^{n-1} \twoheadrightarrow I\cap I'$ such that $\gamma \otimes R[T]/I = \overline{f}^2\omega_I$ and $\gamma \otimes R[T]/I' = \alpha \otimes R[T]/I'$. From surjections α and γ, we get

$$(I,\omega_I) + (I',\omega_{I'}) = 0 \text{ and } (I, \overline{f}^2\omega_I) + (I',\omega_{I'}) = 0 \text{ in } E(R[T], L[T]).$$

Therefore, $(I,\omega_I) = (I, \overline{f}^2\omega_I)$ in $E(R[T], L[T])$. □

The next lemma extends (2.17) to arbitrary $L \in \mathcal{P}_1(R[T])$.

Lemma 2.18. *Let R be a ring of dimension $n \geq 4$ and $L \in \mathcal{P}_1(R[T])$. Let $I \subset R[T]$ be an ideal of height n and ω_I be a local L-orientation of I. Let $\overline{f} \in R[T]/I$ be a unit. Then $(I,\omega_I) = (I, \overline{f}^2\omega_I)$ in $E(R[T], L)$.*

Proof. By [D-Z 2, Proposition 6.8], there is a canonical isomorphism $E(R[T], L) \xrightarrow{\sim} E(R_{red}[T], L \otimes R_{red}[T])$. Hence, we may assume that R is reduced. Then there exists an extension $R \hookrightarrow S$ such that

(1) $R \hookrightarrow S \hookrightarrow Q(R)$, where $Q(R)$ is the total ring of fractions of R,
(2) S is a finite R-module,
(3) $R \hookrightarrow S$ is subintegral and
(4) $L \otimes_R S[T]$ is extended from S.

Using (4) and (2.17), we get $(IS[T], \omega_I^*) = (IS[T], \overline{f}^2\omega_I^*)$ in $E(S[T], L \otimes S[T])$. By [D-Z 2, Theorem 6.16], the natural group homomorphism $E(R[T], L) \to E(S[T], L \otimes S[T])$ defined by $(I,\omega_I) \mapsto (IS[T], \omega_I^*)$ is an isomorphism. Hence $(I,\omega_I) = (I, \overline{f}^2\omega_I)$ in $E(R[T], L)$. □

Following the proof of (2.18), we get the following result.

Lemma 2.19. *Let R be a ring of dimension $n = 3$ and $L \in \mathcal{P}_1(R[T])$. Let $(I,\omega_I) \in \widetilde{E}(R[T], L)$. Let $\overline{f} \in R[T]/I$ be a unit. Then $(I,\omega_I) = (I, \overline{f}^2\omega_I)$ in $\widetilde{E}(R[T], L)$.*

3. Subintegral extensions and projective generation of ideals

The following result is due to S. M. Bhatwadekar (personal communication).

Lemma 3.1. *Let R be a ring of* **odd** *dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $P \in \mathcal{P}_n(R)$ with determinant L and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ an isomorphism. Then the Euler class $e(P,\chi) \in E(R, L)$ is independent of the choice of χ.*

Proof. Let $\alpha : P \twoheadrightarrow J$ be a surjection, where $J \subset R$ is an ideal of height n. Then we get a surjection $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ induced by α. Write $\mathcal{L} = L \oplus R^{n-1}$. Let $\theta : \mathcal{L}/J\mathcal{L} \stackrel{\sim}{\to} P/JP$ be an isomorphism such that $\wedge^n(\theta) = \overline{\chi}$. If $\omega_J = \overline{\alpha} \circ \theta$, then $e(P, \chi) = (J, \omega_J)$ in $E(R, L)$.

Let $\chi' : L \stackrel{\sim}{\to} \wedge^n(P)$ be another isomorphism. Then $\chi' = u\chi$ for some unit $u \in R$. Let $\sigma \in Aut(P)$ be given by $\sigma(p) = up$. Then $\alpha \circ \sigma : P \twoheadrightarrow J$ is a surjection. If $\widetilde{\omega}_J = \overline{\alpha} \circ \overline{\sigma} \circ \theta$, then $e(P, \chi) = (J, \widetilde{\omega}_J) = (J, \overline{u}^n \omega_J) = (J, \overline{u}\omega_J)$ in $E(R, L)$, by (2.1) as n is odd.

Let $\Delta \in Aut(\mathcal{L}/J\mathcal{L})$ be the diagonal matrix $\Delta = diagonal(1, \ldots, 1, \overline{u})$. Since $\wedge^n(\Delta \circ \theta) = \overline{u\chi} = \overline{\chi'}$, we get $e(P, \chi') = (J, \overline{u}\omega_J) = e(P, \chi)$. $\square$

Lemma 3.2. *Let R be a ring of* **odd** *dimension $n \geq 3$ and $L \in \mathcal{P}_1(R[T])$. Let $P \in \mathcal{P}_n(R[T])$ with determinant L and $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ an isomorphism. Then the Euler class $e(P, \chi)$ of the pair (P, χ), which takes values in the Euler class group $E(R[T], L)$ when $n \geq 4$ and in the restricted Euler class group $\widetilde{E}(R[T], L)$ when $n = 3$, is independent of the choice of χ.*

Proof. Follow the proof of (3.1) and use (2.18, 2.19) in place of (2.1). $\square$

Theorem 3.3. *Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ a subintegral extension. Let $L \in \mathcal{P}_1(R)$ and $(J, \omega_J) \in E(R, L)$. Let (JS, ω_J^*) be the image of (J, w_J) in $E(S, L \otimes S)$. Let $Q \in \mathcal{P}_n(S)$ be such that its determinant is extended from R. Further assume that $\chi' : L \otimes S \stackrel{\sim}{\to} \wedge^n(Q)$ is an isomorphism such that $(JS, \omega_J^*) = e(Q, \chi')$ in $E(S, L \otimes S)$. Then there exists $P \in \mathcal{P}_n(R)$ with determinant L and $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ an isomorphism such that $e(P, \chi) = (J, \omega_J)$ in $E(R, L)$. Further, there exists a surjection $\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is obtained from (α, χ).*

Proof. By [B-D, Proposition 2.15], we may assume that R is reduced. Further, we may assume that $R \hookrightarrow S$ is finite. By (2.5), we can find $P_1 \in \mathcal{P}_n(R)$ with determinant L such that $P_1 \otimes S \simeq Q$.

Case I: Assume n is odd. Let $\chi : L \stackrel{\sim}{\to} \wedge^n(P_1)$ be an isomorphism. Consider the image $e(P_1 \otimes S, \chi \otimes S)$ of $e(P_1, \chi)$ in $E(S, L \otimes S)$. Since n is odd, by (3.1), $e(P_1 \otimes S, \chi \otimes S) = e(Q, \chi \otimes S) = e(Q, \chi')$. Therefore, by (2.8), $e(P_1, \chi) = (J, \omega_J)$. Take $P = P_1$.

Case II: Assume n is even. Since $e(Q, \chi') = (JS, \omega_J^*)$ in $E(S, L \otimes S)$, it follows that the weak Euler class $e(Q) = e(P_1 \otimes S) = (JS)$ in $E_0(S, L \otimes S)$. Therefore, by (2.10), $e(P_1) = (J)$ in $E_0(R, L)$. By [B-RS 3, Proposition 6.4], there exists $P_2 \in \mathcal{P}_n(R)$ such that $[P_2] = [P_1]$ in $K_0(R)$ and J is a surjective image of P_2. Let $\beta : P_2 \twoheadrightarrow J$ be a surjection and $\chi_2 : L \stackrel{\sim}{\to} \wedge^n(P_2)$ be an isomorphism. Suppose that $e(P_2, \chi_2) = (J, \omega_2)$ is obtained by (β, χ_2). Then $\omega_J = \overline{u}\omega_2$ for some unit $\overline{u} \in (R/J)^*$. By [B-RS 3, Lemma 5.1], there exists $P \in \mathcal{P}_n(R)$ with $[P] = [P_2]$ in $K_0(R)$ and an isomorphism $\chi : L \stackrel{\sim}{\to} \wedge^n P$ such that $e(P, \chi) = (J, \overline{u}^{n-1}\omega_2)$. Since n is even, by (2.1), $(J, \overline{u}^{n-1}\omega_2) = (J, \overline{u}\omega_2)$ and hence $e(P, \chi) = (J, \overline{u}\omega_2) = (J, \omega_J)$. By [B-RS 3, Corollary 4.3], there exists a surjection

$\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is obtained from the pair (α, χ). This completes the proof. □

Proposition 3.4. [D 2, Proposition 6.3] *Let R be a ring of* **even** *dimension $n \geq 4$ and $J \subset R[T]$ be an ideal of height n. Let $P \in \mathcal{P}_n(R[T])$ with trivial determinant. Assume that the weak Euler class $e(P) = (J)$ in $E_0(R[T])$. Then there exists $Q \in \mathcal{P}_n(R[T])$ such that $[P] = [Q]$ in $K_0(R[T])$ and J is a surjective image of Q.*

Theorem 3.5. *Let R be a ring of dimension $n \geq 3$ and $R \hookrightarrow S$ a subintegral extension. Let $(I, \omega_I) \in E(R[T])$ be such that its image $(IS[T], \omega_I^*) = e(Q, \chi')$ in $E(S[T])$, where $Q \in \mathcal{P}_n(S[T])$ with trivial determinant and $\chi' : S[T] \xrightarrow{\sim} \wedge^n(Q)$ an isomorphism. Then there exists $P \in \mathcal{P}_n(R[T])$ with trivial determinant and $\chi : R[T] \xrightarrow{\sim} \wedge^n(P)$ an isomorphism such that $e(P, \chi) = (I, \omega_I)$. Further, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from the pair (α, χ).*

Proof. Note that this is an extension of (3.3) from $R \hookrightarrow S$ case to $R[T] \hookrightarrow S[T]$ case when $L = R$. By (2.9), we already have $E(R[T]) \simeq E(S[T])$. We need to show that if the image of (I, ω_I) in $E(S[T])$ is the Euler class of a projective $S[T]$-module with trivial determinant, then (I, ω_I) in $E(R[T])$ is also the Euler class of a projective $R[T]$-module with trivial determinant.

By [B-D, Remark 2.16], we may assume that R is reduced. Further, we may assume that $R \hookrightarrow S$ is finite. By (2.6), we can find $P_1 \in \mathcal{P}_n(R[T])$ with trivial determinant such that $P_1 \otimes S[T] \simeq Q$.

Case 1. Assume n is odd. Let $\chi : R[T] \xrightarrow{\sim} \wedge^n(P_1)$ be an isomorphism. Consider the image $e(P_1 \otimes S[T], \chi \otimes S[T])$ of $e(P_1, \chi)$ in $E(S[T])$. Since n is odd, by (3.2), $e(P_1 \otimes S[T], \chi \otimes S[T]) = e(Q, \chi \otimes S[T]) = e(Q, \chi')$. Therefore, by (2.9), $e(P_1, \chi) = (I, \omega_I)$. Take $P = P_1$.

Case 2. Assume n is even. We note that $e(Q) = e(P_1 \otimes S[T]) = (IS[T])$ in $E_0(S[T])$. Therefore, by [D-Z 1, Remark 3.26], $e(P_1) = (I)$ in $E_0(R[T])$. Follow the proof of (3.3, Case II) and use [D 2, Proposition 6.3], [D 1, Lemma 6.1, Corollary 4.10] to complete the proof. □

4. Projective generation: Extended case

Next result is proved in [B-D, Lemma 3.1] when $L = R$.

Lemma 4.1. *Let R be a ring of dimension $n \geq 2$ and $J \subset R$ be an ideal of height $\geq n-1$. Let $Q \in \mathcal{P}_r(R)$ with determinant L. Then there exists $b \in J^2$ such that $ht(b) = 1$ and $Q_{1+b} \simeq {R_{1+b}}^{r-1} \oplus L_{1+b}$.*

Proof. As the determinant of Q is L and $\dim(R/J^2) \leq 1$, by Serre's result [Se], it follows that Q/J^2Q is isomorphic to $(R/J^2)^{r-1} \oplus L/J^2L$. Consequently, Q_{1+J^2} is isomorphis to ${R_{1+J^2}}^{r-1} \oplus L_{1+J^2}$. Therefore, there exists $b \in J^2$ such that Q_{1+b} is isomorphis to ${R_{1+b}}^{r-1} \oplus L_{1+b}$.

If $\text{ht}(b) = 0$, then we can find $c \in J^2$ such that $\text{ht}(b + bc + c) = 1$. Since $1 + b + bc + c = (1 + b)(1 + c)$, without loss of generality, we can assume that $\text{ht}(b) = 1$ and $Q_{1+b} \simeq {R_{1+b}}^{r-1} \oplus L_{1+b}$. □

The following result is from [B-D, Lemma 3.4]. Its proof is contained in [Bh 1, Proposition 3.1, 3.2].

Lemma 4.2. *Let $\widetilde{B}$ be a semilocal ring of dimension 1. Then $Pic(\widetilde{B}[T])$ is a divisible group. Let M be an invertible ideal of $\widetilde{B}[T]$ with $dim(\widetilde{B}[T]/M) = 0$. Let $\mathfrak{b} = M \cap \widetilde{B}$ and $(0) = \mathfrak{b} \cap \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\widetilde{B}$ with $M + \mathfrak{a}[T] = \widetilde{B}[T]$. Then given any positive integer d, there exists an invertible ideal N of $\widetilde{B}[T]$ such that*

(1) $N + M\mathfrak{a}[T] = \widetilde{B}[T]$,
(2) $N^d \cap M = (\widetilde{f})$ *for some non-zerodivisor* $\widetilde{f} \in \widetilde{B}[T]$,
(3) $dim(\widetilde{B}[T]/N) = 0$.

Proposition 4.3. *Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is a surjective image of $L[T] \oplus R[T]^{n-1}$. Further assume that $I = M_1 \cap \cdots \cap M_k$, where each M_i is a maximal ideal of $R[T]$ of height n. Let ω_1 and ω_2 be any two local orientations of I. Then $(I, \omega_1) = (I, \omega_2)$ in $E(R[T], L[T])$.*

Proof. Let $(I, \omega_1) = \sum_1^k (M_i, \omega_{M_i})$ in $E(R[T], L[T])$. It is enough to show that $(M_i, \omega_{M_i}) = (M_i, \omega'_{M_i})$ in $E(R[T], L[T])$ for any other local orientation ω'_{M_i} of M_i. Therefore, we may assume that I is a maximal ideal of height n.

If R is local, then $L \xrightarrow{\sim} R$ and we are done by [D 3, Proposition 3.12], where it is proved that if I is a maximal ideal of $R[T]$ of height n, then $(I, \omega_1) = (I, \omega_2)$ in $E(R[T])$ for any two local orientations ω_1, ω_2 of I.

Now we prove the result for general R. Rest of the proof is similar to [D 3]. First we consider the case when I contains a monic polynomial. Applying [B-RS 4, Proposition 3.3], $(I, \omega) = 0$ in $E(R[T], L[T])$ for any local orientation ω of I. Hence we are done in this case.

Now assume that I is a maximal ideal not containing a monic polynomial. Then $I + (T) = R[T]$ and hence $I(0) = R$. Consider the element $(I, \omega_1) - (I, \omega_2)$ in $E(R[T])$. For any maximal ideal $\mathcal{M}$ of R, the image of $(I, \omega_1) - (I, \omega_2)$ in $E(R_{\mathcal{M}}[T], L_{\mathcal{M}}[T])$ is zero. Use local global principle for Euler class groups [D-Z 2, Theorem 4.17] which says that the following sequence of groups

$$0 \to E(R,L) \to E(R[T],L[T]) \to \prod_{\mathcal{M}} E(R_{\mathcal{M}}[T],L_{\mathcal{M}}[T])$$

is exact. Here the product is over all maximal ideals of R. Hence there exists $(J,\omega_J) \in E(R,L)$ such that

$$\Phi((J,\omega_J)) = (I,\omega_1) - (I,\omega_2).$$

Here $\Phi : E(R,L) \to E(R[T],L[T])$ and $\Psi : E(R[T],L[T]) \to E(R,L)$ are group homomorphisms such that $\Psi \circ \Phi = Id$ [D-Z 2, Remark 4.9]. Since $I(0) = R$, $\Psi(I,\omega_1) = 0 = \Psi(I,\omega_2)$ in $E(R,L)$. Use $\Psi \circ \Phi = Id$, we get $(J,\omega) = 0$ in $E(R,L)$. Hence $(I,\omega_1) = (I,\omega_2)$. $\square$

Theorem 4.4. *Let R be a ring of dimension $n \geq 3$. Let $L \in \mathcal{P}_1(R)$ and $\mathcal{L} = R[T]^{n-1} \oplus L[T]$. Let $J \subset R[T]$ be a local complete intersection ideal of height n such that dim$(R[T]/J) = 0$ and $J = (f_1,\cdots,f_n) + J^2$. Let $I = (f_1,\cdots,f_{n-1}) + J^{(n-1)!}$. Let $\omega : \mathcal{L}/I\mathcal{L} \twoheadrightarrow I/I^2$ be a surjection. Then there exists $P \in \mathcal{P}_n(R[T])$ and an isomorphism $\chi : L[T] \xrightarrow{\sim} \wedge^n(P)$ such that*

1. *$[P] - [\mathcal{L}] = -[R[T]/J]$ in $K_0(R[T])$,*
2. *there is a surjection $P \twoheadrightarrow I$ and*
3. *$e(P,\chi) = (I,\omega)$ in $E(R[T],L[T])$.*

Proof. Das-Mandal [D-M, Theorem 3.2] proved the following result for $E(R,L)$. Let $\widetilde{J} \subset R$ be a local complete intersection ideal of height n such that $\widetilde{J} = (\widetilde{f}_1,\cdots,\widetilde{f}_n) + \widetilde{J}^2$. Let $\widetilde{I} = (\widetilde{f}_1,\cdots,\widetilde{f}_{n-1}) + \widetilde{J}^{(n-1)!}$. Write $\widetilde{\mathcal{L}} = R^{n-1} \oplus L$. Let $\widetilde{\omega} : \widetilde{\mathcal{L}}/\widetilde{I}\widetilde{\mathcal{L}} \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$ be a surjection. Then there exists $\widetilde{P} \in \mathcal{P}_n(R)$ with determinant L and $\widetilde{\chi} : L \xrightarrow{\sim} \wedge^n(\widetilde{P})$ an isomorphism such that

1. $[\widetilde{P}] - [\widetilde{\mathcal{L}}] = -[R/\widetilde{J}]$ in $K_0(R)$,
2. there is a surjection $\widetilde{P} \twoheadrightarrow \widetilde{I}$ and
3. $e(\widetilde{P},\widetilde{\chi}) = (\widetilde{I},\widetilde{\omega})$ in $E(R,L)$.

In our case, $\dim(R[T]/J) = 0$. Since whole proof of [D-M, Theorem 3.1, 3.2] works in our case, we are done. $\square$

The proof of the next result closely follow that of [B-D, Proposition 3.3] where it is proved for $L = R$.

Proposition 4.5. *Let $R \hookrightarrow B$ be a flat extension of rings such that dim$(R) =$ dim$(B) = n \geq 3$. Let $L \in \mathcal{P}_1(R)$ and write $\mathcal{L} = L \oplus R^{n-1}$. Let $Q \in \mathcal{P}_n(R)$ with determinant L and $P \in \mathcal{P}_n(B[T])$ with determinant $L \otimes B[T]$. Further, assume that $Q \otimes B \xrightarrow{\sim} \mathcal{L} \otimes B$ and $P/TP \xrightarrow{\sim} \mathcal{L} \otimes B$. Let $\chi : L \xrightarrow{\sim} \wedge^n(Q)$ and $\chi' : L \otimes B[T] \xrightarrow{\sim} \wedge^n(P)$ be isomorphisms. Let $I \subset R[T]$ be an ideal of height n such that ht$(I(0)) = n$ and both $IB[T]$ and $I(0)B$ are proper* *ideals. Assume that there are surjections* $\omega : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$, $\alpha : Q \twoheadrightarrow I(0)$ *and* $\beta : P \twoheadrightarrow IB[T]$ *such that*

(1) (α, χ) *induces* $e(Q, \chi) = (I(0), \omega(0))$ *in* $E(R, L)$, *where* $\omega(0)$ *is induced by* ω.
(2) (β, χ') *induces* $e(P, \chi') = (IB[T], \omega \otimes B[T])$ *in* $E(B[T], L \otimes B[T])$.

Then there exists an isomorphism $\psi : P/TP \xrightarrow{\sim} Q \otimes B$ *and a surjection* $\eta : P \twoheadrightarrow IB[T]$ *such that* $\eta(0) = (\alpha \otimes B) \circ \psi : P/TP \twoheadrightarrow I(0)B$.

Proof. Write $P/TP = P_0$. Let "tilde" denote reduction modulo $IB[T]$ and "bar" denote reduction modulo $I(0)B$. We have two surjections

$$\widetilde{\beta} : \widetilde{P} \twoheadrightarrow IB[T]/I^2B[T] \quad \text{and} \quad \widetilde{\omega} : \mathcal{L}[\widetilde{T] \otimes B}[T] \twoheadrightarrow IB[T]/I^2B[T]$$

induced from β and $\omega \otimes B$ respectively. Since the pair (β, χ') induces the Euler class $e(P, \chi') = (IB[T], \omega \otimes B)$ in $E(B[T], L \otimes B[T])$, by definition of $e(P, \chi')$, if $\sigma : \mathcal{L}[\widetilde{T] \otimes B}[T] \xrightarrow{\sim} \widetilde{P}$ is an isomorphism such that $\wedge^n(\sigma) = \widetilde{\chi'}$, then $\widetilde{\beta} \circ \sigma = \widetilde{\omega}$.

Let $\overline{\sigma} : \overline{\mathcal{L} \otimes B} \xrightarrow{\sim} \overline{P_0}$ be the isomorphism induced from σ. Since $P_0 \xrightarrow{\sim} \mathcal{L} \otimes B$, choose an isomorphism $\tau : \mathcal{L} \otimes B \xrightarrow{\sim} P_0$ such that $\wedge^n(\tau) = \chi'(0)$. Now we have two isomorphisms

$$\overline{\sigma}, \overline{\tau} : \overline{\mathcal{L} \otimes B} \xrightarrow{\sim} \overline{P_0} \quad \text{with} \quad \wedge^n(\overline{\sigma}) = \wedge^n(\overline{\tau}) = \widetilde{\chi'}(0).$$

Therefore, $\overline{\tau} = \overline{\sigma} \circ \Theta$ for some $\Theta \in SL(\overline{\mathcal{L} \otimes B})$. Since $\dim(\overline{B}) = 0$, $EL(\overline{\mathcal{L} \otimes B}) = SL(\overline{\mathcal{L} \otimes B})$. Hence $\Theta \in EL(\overline{\mathcal{L} \otimes B})$ can be lifted to an element $\theta \in EL(\mathcal{L} \otimes B)$. Therefore, we can lift $\overline{\sigma}$ to an isomorphism $\sigma_0 : \mathcal{L} \otimes B \xrightarrow{\sim} P_0$.

On the other hand, the pair (α, χ) induces the Euler class $e(Q, \chi) = (I(0), \omega(0))$ in $E(R, L)$. Hence if we choose an isomorphism $\delta : \mathcal{L} \otimes B \xrightarrow{\sim} Q \otimes B$ such that $\wedge^n(\delta) = \chi \otimes B$, then $\overline{(\alpha \otimes B)} \circ \overline{\delta} = \omega(0) = \overline{\omega}$. Let us define

$$\psi = \delta \circ {\sigma_0}^{-1} : P_0 \xrightarrow{\sim} Q \otimes B \quad \text{and} \quad \varphi = (\alpha \otimes B) \circ \psi : P_0 \twoheadrightarrow I(0)B$$

Then $\overline{\beta} = \overline{\omega} \circ \overline{\sigma}^{-1} = \overline{(\alpha \otimes B)} \circ \overline{\delta} \circ \overline{\sigma}^{-1} = \overline{\varphi}$. By [B-RS 1, Remark 3.9], there is a surjection $\rho : P \twoheadrightarrow IB[T]/(I^2T)B[T]$ such that $\widetilde{\beta} = \widetilde{\rho}$ and $\overline{\rho} = \varphi$.

Let $B(T)$ be the ring obtained from $B[T]$ by inverting all the monic polynomials in T. Then ρ induces the surjection

$$\rho \otimes B(T) : P \otimes B(T) \twoheadrightarrow IB(T)/I^2B(T)$$

and clearly $\beta \otimes B(T)$ is lift of $\rho \otimes B(T)$. Applying [D-Z 2, Theorem 4.11], we can find a surjection $\eta : P \twoheadrightarrow IB[T]$ such that η is a lift of ρ. Note that $\eta(0) = \varphi = (\alpha \otimes B) \circ \psi$. This completes the proof. $\square$

The proof of the next result closely follows [B-D, Theorem 3.5] where it is proved for $L = R$.

Theorem 4.6. *Let R be an affine k-algebra of dimension $n \geq 2$. Let $L \in \mathcal{P}_1(R)$ and $\mathcal{L} = L \oplus R^{n-1}$. Let $(I, \omega_I) \in E(R[T], L[T])$ and $\lambda \in k$ be such that $ht(I(\lambda)) \geq n$. When $ht(I(\lambda)) > n$, write $Q = \mathcal{L}$. When $ht(I(\lambda)) = n$, assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L and $\chi : L \overset{\sim}{\to} \wedge^n(Q)$ an isomorphism such that $e(Q, \chi) = (I(\lambda), \omega_{I(\lambda)})$ in $E(R, L)$, where $\omega_{I(\lambda)}$ is induced from ω_I. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and an isomorphism $\chi_1 : L[T] \overset{\sim}{\to} \wedge^n(P)$ such that $e(P, \chi_1) = (I, \omega_I)$ in $E(R[T], L[T])$. Moreover, $P/TP \simeq Q$.*

Proof. When $n = 2$, any (I, ω_I) is Euler class of a rank 2 projective $R[T]$-module, without the condition that $(I(\lambda), \omega_{I(\lambda)}) = e(P, \chi)$. To see this, note that projective modules of rank 1 are always cancellative. It follows easily using a standard patching argument that there exists $P_1 \in \mathcal{P}_2(R[T])$ with determinant $L[T]$ and a surjection $\zeta : P_1 \twoheadrightarrow I$. Fix an isomorphism $\chi' : L[T] \overset{\sim}{\to} \wedge^2(P_1)$. Let $e(P_1, \chi') = (I, \omega)$ in $E(R[T], L[T])$ be induced from (ζ, χ'). Then $\omega_I = \overline{u}\omega$ for some unit $\overline{u} \in R[T]/I$. By (2.15), there exists $P \in \mathcal{P}_2(R[T])$, an isomorphism $\chi_1 : L[T] \overset{\sim}{\to} \wedge^2(P)$ and a surjection $\beta : P \twoheadrightarrow I$ such that $e(P, \chi_1) = (I, \overline{u}\omega)$ is induced from (β, χ_1). Therefore, $e(P, \chi_1) = (I, \omega_I)$.

Assume $n \geq 3$. Replacing T by $T - \lambda$, we assume $\lambda = 0$. Using (2.2), we assume that R is a reduced affine algebra. Since $\mathbb{Q} \subset R$, we get R is a geometrically reduced affine algebra.

Given a surjection $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$. If $\mathrm{ht}(I(0)) > n$, then $I(0) = R[T]$ and we can lift ω_I to a surjection $\omega' : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. If $\mathrm{ht}(I(0)) = n$, then it is given that $e(Q, \chi) = (I(0), \omega_{I(0)})$ in $E(R, L)$. Hence by [B-RS 3, Corollary 4.3], there exists a surjection $\alpha : Q \twoheadrightarrow I(0)$ such that $(I(0), \omega_{I(0)})$ is obtained from the pair (α, χ). By (2.4), there is a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ such that $\theta(0) = \alpha$.

Step 1: If $J = I \cap R$, then $\mathrm{ht}(J) \geq n-1$. By (4.1), there exists a non-zerodivisor $b \in J^2$ such that $Q_{1+b} \overset{\sim}{\to} \mathcal{L}_{1+b}$. By [B-D, Lemma 2.5, Remark 2.6], the surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ can be lifted to a surjection $\gamma : Q[T] \twoheadrightarrow I'' = I \cap I_1$ such that

(1) $I = I'' + (bT)$,
(2) $I_1 + (bT) = R[T]$, hence $I + I_1 = R[T]$,
(3) $\mathrm{ht}(I_1) = n$ and $R[T]/I_1$ is reduced.

It follows that $e(Q[T], \chi \otimes R[T]) = (I, \omega_I) + (I_1, \omega_{I_1})$, where ω_{I_1} is induced by the pair $(\gamma, \chi \otimes R[T])$.

Step 2: Let $B = R_{1+bR}$. We first note that if $I_1B[T] = B[T]$, then the surjection $\gamma \otimes B[T] : Q \otimes B[T] \twoheadrightarrow IB[T]$ is a lift of $\theta \otimes B[T]$. By [D 1, Lemma 3.8], θ can be lifted to a surjection $\Theta : Q[T] \twoheadrightarrow I$. Further, from above, $e(Q[T], \chi \otimes R[T]) = (I, \omega_I)$ in

$E(R[T], L[T])$ and we are done in this case by taking $P = Q[T]$. Therefore, we assume that $\text{ht}(I_1B[T]) = n$.

Since bB is contained in the Jacobson radical of B, using $(2, 3)$, we conclude that $I_1B[T]$ is a zero dimensional radical ideal. Hence $I_1B[T] = \cap_1^r \mathcal{M}_i$, where $\mathcal{M}_i$'s are maximal ideals of $B[T]$ of height n and containing I_1. If $K = B \cap I_1B[T]$, then K is a reduced ideal of height $n-1$. Further, $K + bB$ is an ideal of B of height n. It is easy to see that $B[T]_{\mathcal{M}_i}$ are regular for $i = 1, \ldots, r$. By [B-H, Theorem 2.2.12], if $\mathfrak{p}_i = \mathcal{M}_i \cap B$, then $B_{\mathfrak{p}_i}$ is regular local.

Now B/K is a reduced ring of dimension 1 and the image of b belongs to the Jacobson radical of B/K. Hence $(B/K)_b$ is a product of fields. Therefore, we can find $a_1, \cdots, a_{n-1} \in K$ such that $\text{ht}(a_1, \cdots, a_{n-1}) = n-1$, $\text{ht}(a_1, \cdots, a_{n-1}, b) = n$ and $K_b = (a_1, \cdots, a_{n-1})_b + {K_b}^2$. Therefore, $K_{\mathfrak{p}} = (a_1, \cdots, a_{n-1})_{\mathfrak{p}}$ for all minimal prime ideals $\mathfrak{p}$ over K. Let $(a_1, \cdots, a_{n-1}) = K \cap K_1$ be a reduced primary decomposition.

Step 3: Let $\widetilde{B} = B/(a_1, \cdots, a_{n-1})$. Since b belongs to the Jacobson radical of B, $\widetilde{B}$ is a semilocal ring of dimension 1 and $\widetilde{K} \cap \widetilde{b}\widetilde{K}_1 = 0$ in $\widetilde{B}$. Moreover, $\widetilde{I}_1$ is an invertible ideal and $\widetilde{I}_1 + \widetilde{b}\widetilde{K}_1[T] = \widetilde{B}[T]$. Note that $\widetilde{B}$ is a subring of $\widetilde{B}/\widetilde{K} \oplus \widetilde{B}/\widetilde{b}\widetilde{K}_1$ with the conductor ideal $\widetilde{K} + \widetilde{b}\widetilde{K}_1$.

Applying (4.2) to the invertible ideal $\widetilde{I}_1$ with $\mathfrak{a} = \widetilde{b}\widetilde{K}_1$, we get an invertible ideal N of $\widetilde{B}[T]$ such that

(1) $N + \widetilde{I}_1\widetilde{b}\widetilde{K}_1[T] = \widetilde{B}[T]$,
(2) $N^d \cap \widetilde{I}_1\widetilde{B}[T] = (\widetilde{f})$ for some non-zerodivisor $\widetilde{f} \in \widetilde{B}[T]$,
(3) $\dim(\widetilde{B}[T]/N) = 0$.

Since $\widetilde{K}.\widetilde{K}_1 = (\widetilde{0})$ in $\widetilde{B}$ and $N + \widetilde{b}\widetilde{K}_1[T] = \widetilde{B}[T]$, it follows that any maximal ideal of $\widetilde{B}[T]$ containing N must contain $\widetilde{K}[T]$.

Let I_2 be the inverse image of N in $B[T]$ and $\mathcal{M}$ be a maximal ideal of $B[T]$ containing I_2. Then $\mathcal{M} \cap B = \mathfrak{q}$ is a prime ideal of B containing K and of height $n-1$, since $M + bB[T] = B[T]$. Hence $\mathfrak{q}$ is a minimal prime over K. Therefore, $B_{\mathfrak{q}}$ is a regular local ring and consequently $B[T]_{\mathcal{M}}$ is also regular. This shows that the ideal I_2 has finite projective dimension and it is locally generated by a regular sequence of length n.

Since $\widetilde{B}$ is semilocal, $L \otimes \widetilde{B} \stackrel{\sim}{\to} \widetilde{B}$. Therefore, we can write (2) of step 3 as a surjection $\phi' : L[T] \otimes \widetilde{B} \twoheadrightarrow N^d \cap \widetilde{I}_1\widetilde{B}[T]$. We get a surjection

$$\phi : \mathcal{L} \otimes B[T] \twoheadrightarrow I_2^{(d)} \cap I_1B[T]$$

such that $\phi|_{L \otimes B[T]}$ is a lift of ϕ' and $\phi(B[T]^{n-1}) = (a_1, \ldots, a_{n-1})$. Since $I_1B[T]$ is reduced, by (4.3), $I_1B[T]$ is independent of the local orientations. Therefore, we have

$$(I_1B[T], \omega_{I_1} \otimes B[T]) + (I_2^{(d)}, \omega) = 0$$

in $E(B[T], L\otimes B[T])$, where ω is induced by ϕ. By (4.4), there exists $P' \in \mathcal{P}_n(B[T])$ with determinant $L\otimes B[T]$ such that:

(1) There is a surjection $\delta : P' \twoheadrightarrow I_2^{(d)}$,
(2) $[P'] - [\mathcal{L}\otimes B[T]] = -[B[T]/I_2]$ in $K_0(B[T])$ and
(3) an orientation $\chi' : L\otimes B[T] \overset{\sim}{\rightarrow} \wedge^n(P')$ can be defined such that $e(P',\chi') = (I_2^{(d)},\omega) = -(I_1B[T],\omega_{I_1}\otimes B[T])$.

Since $Q\otimes B[T] \overset{\sim}{\rightarrow} \mathcal{L}\otimes B[T]$, we have

$$e(Q\otimes B[T],\chi\otimes B[T]) = (IB[T],\omega_I\otimes B[T]) + (I_1B[T],\omega_{I_1}\otimes B[T]) = 0$$

in $E(B[T], L\otimes B[T])$. Therefore, $e(P',\chi') = (IB[T],\omega_I\otimes B[T])$. By [D-Z 2, Corollary 4.14], there exists a surjection $\beta : P' \twoheadrightarrow IB[T]$ such that (β,χ') induces $e(P',\chi') = (IB[T],\omega_I\otimes B[T])$ in $E(B[T], L\otimes B[T])$.

Since $I_2+(b) = B[T]$ and b belongs to the Jacobson radical of B, we have $I_2+(T) = B[T]$. In other words, $I_2(0) = B$. Therefore,

$$[P'/TP'] - [\mathcal{L}\otimes B] = 0 \text{ in } K_0(B)$$

i.e. P'/TP' is a stably isomorphic to $\mathcal{L}\otimes B$. Since height of Jacobson radical of B is ≥ 1, we get $P'/TP' \simeq \mathcal{L}\otimes B$.

As $\dim(\widetilde{B}_{\widetilde{b}}) = 0$, we have $N_{\widetilde{b}} \overset{\sim}{\rightarrow} \widetilde{B}_{\widetilde{b}}$. Hence $(I_2)_b$ is a complete intersection ideal of $B_b[T]$. So

$$[P'_b] - [\mathcal{L}\otimes B_b[T]] = -[B_b[T]/(I_2)_b] = 0 \text{ in } K_0(B_b[T])$$

i.e. P'_b is stably isomorphic to $\mathcal{L}\otimes B_b[T]$ and as $\dim(B_b) \leq n-1$, we get $P'_b \simeq \mathcal{L}\otimes B_b[T]$.

Step 4: Applying (4.5) with $P = P'$, we obtain an isomorphism $\psi : P'/TP' \overset{\sim}{\rightarrow} Q\otimes B$ and a surjection $\eta : P' \twoheadrightarrow IB[T]$ such that $\eta(0) = (\alpha\otimes B)\circ\psi$. Now consider the following surjections.

$$\Phi = \alpha\otimes R_{b(1+bR)}[T] : Q_{b(1+bR)}[T] \twoheadrightarrow I_{b(1+bR)} = R_{b(1+bR)}[T]$$

$$\eta_b : P'_b \twoheadrightarrow I_{b(1+bR)} = R_{b(1+bR)}$$

Note that $Q_{b(1+bR)}[T] \overset{\sim}{\rightarrow} \mathcal{L}_{b(1+bR)}[T] \overset{\sim}{\rightarrow} P'_b$. Since $\dim(R_{b(1+bR)}) \leq n-1$ and $\mathbb{Q}\subset R$, by [Ra, Corollary 2.5], $ker(\Phi)$ and $ker(\eta_b)$ are locally free. Therefore, by Quillen's local-global principle [Q], $ker(\Phi)$ and $ker(\eta_b)$ are extended from $R_{b(1+bR)}$. Further, reducing modulo T, we observe that $\alpha_{b(1+bR)}\circ\psi_b = \eta_b(0)$. This implies that $ker(\Phi) \overset{\sim}{\rightarrow} ker(\eta_b)$ and there is an isomorphism $\Psi : P'_b \overset{\sim}{\rightarrow} Q_{b(1+bR)}[T]$ such that $\Psi(0) = \psi_b$. By a standard patching argument, the result follows. $\square$

In (4.6), we have essentially proved the following result.

Theorem 4.7. *Let R be an affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be a local complete intersection ideal of height n such that ht$(I(0)) \geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L and a surjection $Q[T] \twoheadrightarrow I/I^2 \cap (T)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and a surjection $P \twoheadrightarrow I$. In other words, I is projectively generated.*

Proof. Write $\mathcal{L} = L[T] \oplus R[T]^{n-1}$. We have a surjection $w_I : \mathcal{L}/I\mathcal{L} \twoheadrightarrow I/I^2$ as the composition of surjections $\mathcal{L}/I\mathcal{L} \overset{\sim}{\to} Q[T]/IQ[T] \twoheadrightarrow I/I^2$, where the last map is induced from a given surjection $\theta_1 : Q[T] \twoheadrightarrow I/I^2 \cap (T)$. Take $(I, \omega_I) \in E(R[T], L[T])$. Let $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ be the surjection induced from θ_1. Now follow the proof of (4.6). $\square$

For even dimensional ring, we have the following stronger result. In case $L = R$, it is proved in [B-D, Corollary 3.7].

Corollary 4.8. *Let R be an affine k-algebra of* **even** *dimension $n \geq 2$ and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be an ideal of height n. Write $\mathcal{L} = L \oplus R^{n-1}$ and assume that there is a surjection $\mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$. Let $\lambda \in k$ be such that ht$(I(\lambda)) \geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L and a surjection $Q \twoheadrightarrow I(\lambda)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and a surjection $P \twoheadrightarrow I$.*

Proof. Changing T to $T - \lambda$, we assume $\lambda = 0$. Let $\omega : \mathcal{L}[T] \twoheadrightarrow I/I^2$ be a given surjection and consider $(I, \omega) \in E(R[T], L[T])$. If $I(0) = R$, then ω can be lifted to a surjection $\mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. Now we are done by (4.7).

Assume ht$(I(0)) = n$ and consider $(I(0), \omega(0)) \in E(R, L)$. Given a surjection $\alpha : Q \twoheadrightarrow I(0)$ with $\chi : L \overset{\sim}{\to} \wedge^n(Q)$. Let $e(Q, \chi) = (I(0), \sigma)$ in $E(R, L)$ be induced from α. By [B-RS 3, Remark 5.0], any two local orientations of $I(0)$ differ by a unit. Hence there exists a unit $a \in R/I(0)$ such that $a\sigma = \omega(0)$. By [B-RS 3, Lemma 5.1], there exists $Q' \in \mathcal{P}_n(R)$ stably isomorphic to Q and $\chi' : L \overset{\sim}{\to} \wedge^n(Q')$ such that $e(Q', \chi') = (I(0), a^{n-1}\sigma)$. Since n is even, by [B-RS 3, Lemma 5.4], $(I(0), a^{n-1}\sigma) = (I(0), a\sigma) = (I(0), \omega(0))$. By [B-RS 3, Corollary 4.3], there is a surjection $\beta : Q' \twoheadrightarrow I(0)$ such that (β, χ') induces $e(Q', \chi')$. By (2.4), there is a surjection $Q'[T] \twoheadrightarrow I/(I^2T)$. Now we are done by (4.7). $\square$

The following result is proved in [D 2, Proposition 5.1, Corollary 5.2] when $L = R$.

Proposition 4.9. *Let R be an affine k-algebra of dimension $n \geq 3$ over a C_1 field k. Let $L \in \mathcal{P}_1(R)$ and $\mathcal{L} = L \oplus R^{n-1}$. Let $I \subset R$ be an ideal of height n. Assume that I is a surjective image of $\mathcal{L}$. Then*

(1) *any surjection $\phi : \mathcal{L}/I\mathcal{L} \twoheadrightarrow I/I^2$ can be lifted to a surjection $\psi : \mathcal{L} \twoheadrightarrow I$.*
(2) *$E(R, L) \simeq E_0(R, L)$.*
(3) *$E(R, L) \simeq E(R)$.*

Proof. (1). Let $\theta : \mathcal{L} \twoheadrightarrow I$ be a given surjection and write $P = L \oplus R^{n-3}$. Let ω be the trivial orientation of I induced from θ. By [B-RS 3, Remark 5.0], we have $\widetilde{u}\,\omega = \phi$ for some unit $\widetilde{u} \in R/I$.

Write $\theta = (\theta_1, a_1, a_2) : P \oplus R^2 \twoheadrightarrow I$. Without loss of generality, we may assume that height of $\theta(P) = J$ is $n-2$. Let "bar" denote reduction modulo J. Since $\overline{R}$ is an affine k-algebra of dimension 2 with k a C_1-field of charateristic 0, by Suslin's cancellation result, the unimodular row $(\overline{u}, \overline{a}_1, \overline{a}_2)$ is completable to a matrix in $\text{SL}_3(\overline{R})$. By [RS, Lemma 2.3], there exists $\overline{\sigma} \in \text{GL}_2(\overline{R})$ with $\det(\overline{\sigma}) = \overline{u}^{-1}$ and $\overline{\sigma}(\overline{a}_1, \overline{a}_2) = (\overline{b}_1, \overline{b}_2)$. Consider the surjection $\psi = (\theta_1, b_1, b_2) : P \oplus R^2 \twoheadrightarrow I$. Since $\wedge^n(\phi) = \wedge^n(\psi \otimes R/I)$, there exists $\delta \in \text{SL}(\mathcal{L}/I\mathcal{L})$ such that $\phi = (\psi \otimes R/I) \circ \delta$. Since $\dim R/I = 0$, we have $\text{SL}(\mathcal{L}/I\mathcal{L}) = EL(\mathcal{L}/I\mathcal{L})$. Let $\Delta \in EL(\mathcal{L})$ be a lift of δ. Then the surjection $\psi \circ \Delta : \mathcal{L} \twoheadrightarrow I$ is a lift of ϕ. This completes the proof of (1).

(2). It follows from (1).

(3). We have $E(R) \simeq E_0(R)$, by [D 2, Corollary 5.2] and $E_0(R, L) \simeq E_0(R)$, by [B-RS 3, Theorem 6.8]. Now (3) follows from (2). $\square$

The next result extends (4.8) when k is a C_1-field. In case $L = R$, it is proved in [B-D, Corollary 3.10].

Corollary 4.10. *Let R be an affine k-algebra of dimension $n \geq 2$ over a C_1 field k. Let $L \in \mathcal{P}_1(R)$ and write $\mathcal{L} = L[T] \oplus R[T]^{n-1}$. Let $I \subset R[T]$ be an ideal of height n and assume that there is a surjection $\mathcal{L}[T] \twoheadrightarrow I/I^2$. Let $\lambda \in k$ be such that ht$(I(\lambda)) \geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ with determinant L and a surjection $Q \twoheadrightarrow I(\lambda)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and a surjection $P \twoheadrightarrow I$.*

Proof. We may assume $\lambda = 0$. Let $\omega_I : \mathcal{L}[T] \twoheadrightarrow I/I^2$ be a given surjection and $(I, \omega_I) \in E(R[T], L[T])$. If $I(0) = R$, then ω_I can be lifted to a surjection $\mathcal{L}[T] \twoheadrightarrow I/(I^2T)$ and we are done by (4.7). Assume ht$(I(0)) = n$. Then $(I(0), \omega_{I(0)}) \in E(R, L)$. By assumption, there is a surjection $\alpha : Q \twoheadrightarrow I(0)$. Let $\chi : L \xrightarrow{\sim} \wedge^n(Q)$ be an isomorphism. Since weak Euler class $e(Q) = I(0)$ in $E_0(R)$ and by (4.9), $E(R, L) \xrightarrow{\sim} E_0(R, L)$, it follows that the Euler class of Q induced by (α, χ) is $e(Q, \chi) = (I(0), w_{I(0)})$. By (2.4), there exists a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ with $\theta(0) = \alpha$. Applying (4.7), we are done. $\square$

The following result extends [B-RS 2, Theorem 2.7] and [B-D, Corollary 3.11].

Corollary 4.11. *Let R be an affine k-algebra of dimension $n \geq 2$ over an algebraically closed field k and $L \in \mathcal{P}_1(R)$. Let $I \subset R[T]$ be an ideal of height n and there is a surjection $L[T] \oplus R[T]^{n-1} \twoheadrightarrow I/I^2$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and a surjection $P \twoheadrightarrow I$.*

Proof. Replacing T by $T-\lambda$ for some $\lambda \in k$, we may assume that $\text{ht}(I(0)) \geq n$. Write $I(0) = J$ and $L \oplus R^{n-1} = \mathcal{L}$. By hypothesis, we have a surjection $\alpha : \mathcal{L} \twoheadrightarrow J/J^2$. By [B-RS 3, Lemma 2.11], there exists $e \in J$ such that $(\alpha(\mathcal{L}), e) = J$ with $e(1-e) \in \alpha(\mathcal{L})$. If we write $f = 1-e$, then $\alpha_f : \mathcal{L}_f \twoheadrightarrow J_f$ is a surjection. Define $\pi : \mathcal{L}_{1+fk[f]} \twoheadrightarrow J_{1+fk[f]} = R_{1+fk[f]}$ to be the projection onto the last factor. We have two unimodular elements $\alpha_{f(1+fk[f])}$ and π_f in $\mathcal{L}_{f(1+fk[f])}{}^*$. Note that $R_{f(1+fk[f])}$ is an affine algebra of dimension $n-1$ over a C_1-field $k(f)$. By [Bh 3, Theorem 4.1], projective $R_{f(1+fk[f])}$-modules of rank n are cancellative. Hence there exists an automorphism σ of $\mathcal{L}_{f(1+fR)}$ such that $\alpha_{f(1+fk[f])} \circ \sigma = \pi_f$. By standard patching argument there exists $Q \in \mathcal{P}_n(R)$ with determinant L and a surjection $Q \twoheadrightarrow J$. Now the result follows from (4.10). □

Let R be a ring of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Consider the following sets

$$H = \{e(Q,\chi) \in E(R,L) \,|\, Q \in \mathcal{P}_n(R),\ \chi : L \xrightarrow{\sim} \wedge^n(Q)\}$$
$$K = \{e(P,\chi) \in E(R[T],L[T]) \,|\, P \in \mathcal{P}_n(R[T]),\ \chi : L[T] \xrightarrow{\sim} \wedge^n(P)\}$$

It is a natural question whether H and K are subgroups of $E(R,L)$ and $E(R[T],L[T])$ respectively?

The following result extends [B-D, Proposition 3.14] where it is proved for $L = R$.

Corollary 4.12. *Let R be an affine k-algebra of dimension $n \geq 3$ and $L \in \mathcal{P}_1(R)$. Then H is a subgroup of $E(R,L)$ if and only if K is a subgroup of $E(R[T],L[T])$.*

Proof. If K is a subgroup of $E(R[T],L[T])$, then it is easy to see that H is also a subgroup of $E(R,L)$.

Now suppose that H is a subgroup of $E(R,L)$. Let (J_1,ω_{J_1}), $(J_2,\omega_{J_2}) \in K$. By moving lemma [D-Z 2, Lemma 2.11], there exists an ideal $J_3 \subset R[T]$ of height n and a local orientation ω_{J_3} such that $(J_2,\omega_{J_2}) + (J_3,\omega_{J_3}) = 0$ in $E(R[T],L[T])$ and $(J_1 \cap J_2) + J_3 = R[T]$. Let $J_4 = J_1 \cap J_3$. Then we have

$$(J_4,\omega_{J_4}) = (J_1,\omega_{J_1}) + (J_3,\omega_{J_3}) = (J_1,\omega_{J_1}) - (J_2,\omega_{J_2})$$

where ω_{J_4} is the local orientation of J_4 induced by ω_{J_1} and ω_{J_3}. Now there is group homomorphism $\Psi : E(R[T],L[T]) \longrightarrow E(R,L)$ which takes (J,ω_J) to $(J(0),\omega_{J(0)})$, where $\omega_{J(0)}$ is the local orientation of $J(0)$ induced by ω_J (if $J(0) = R$, then $\Psi((J,\omega_J)) = 0$) (see [D-Z 2, Remark 4.9]). Therefore, we have

$$(J_4(0),\omega_{J_4(0)}) = (J_1(0),\omega_{J_1(0)}) - (J_2(0),\omega_{J_2(0)}).$$

Since $(J_1(0),\omega_{J_1(0)})$ and $(J_2(0),\omega_{J_2(0)})$ are in H and H is a subgroup of $E(R,L)$, we get $(J_4(0),\omega_{J_4(0)}) \in H$. Therefore, there exists $Q \in \mathcal{P}_n(R)$ with determinant L and an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(Q)$ such that $e(Q,\chi) = (J_4(0),\omega_{J_4(0)})$. By (4.6), there exists

$P \in \mathcal{P}_n(R[T])$ with determinant $L[T]$ and an isomorphism $\chi_1 : L[T] \xrightarrow{\sim} \wedge^n(P)$ such that $e(P, \chi_1) = (J_4, \omega_{J_4})$. This completes the proof.

□

5. Projective generation: General case

The following result extends (4.6) where it is proved when L is extended from R.

Theorem 5.1. *Let R be a reduced affine k-algebra of dimension $n \geq 2$ and $L \in \mathcal{P}_1(R[T])$. Let (I, ω_I) be an element of $E(R[T], L)$ when $n \neq 3$ and $\widetilde{E}(R[T], L)$ when $n = 3$. Let $\lambda \in k$ be such that $ht(I(\lambda)) \geq n$. Assume that there exists $Q \in \mathcal{P}_n(R)$ and an isomorphism $\chi : L/TL \xrightarrow{\sim} \wedge^n(Q)$ such that $e(Q, \chi) = (I(\lambda), \omega_{I(\lambda)})$ in $E(R, L/TL)$. Then there exists $P \in \mathcal{P}_n(R[T])$ and an isomorphism $\chi_1 : L \xrightarrow{\sim} \wedge^n(P)$ such that $e(P, \chi_1) = (I, \omega_I)$ in $E(R[T], L)$.*

Proof. The case $n = 2$ is same as (4.6). Consider $n \geq 4$. We may assume $\lambda = 0$. Since R is reduced, there exists an extension $R \hookrightarrow S$ such that

(1) $R \hookrightarrow S \hookrightarrow Q(R)$,
(2) S is a finite R-module,
(3) $R \hookrightarrow S$ is subintegral and
(4) $L \otimes_R S[T]$ is extended from S.

Note that $L \otimes S[T]$ is extended from S and $(I(0)S, \omega^*_{I(0)}) = e(Q \otimes S, \chi \otimes S)$ in $E(S, L/TL \otimes S)$, where $(I(0)S, \omega^*_{I(0)})$ is the image of $(I(0), \omega_{I(0)})$. Applying (4.6), there exists $P' \in \mathcal{P}_n(S[T])$ with determinant $L \otimes S[T]$ and an isomorphism $\chi' : L \otimes S[T] \xrightarrow{\sim} \wedge^n(P')$ such that $e(P', \chi') = (IS[T], \omega^*_I)$ in $E(S[T], L \otimes S[T])$.

Since $R \hookrightarrow S$ is a finite subintegral extension and $\text{rank}(P') = n = \dim(R)$, by (2.6), there exists $P \in \mathcal{P}_n(R[T])$ with determinant L such that $P \otimes S[T] \simeq P'$. Choose an isomorphism $\chi_1 : L \xrightarrow{\sim} \wedge^n(P)$

Case I: Assume n is odd. By (3.2), $e(P', \chi') = e(P', \chi_1 \otimes S[T]) = e(P \otimes S[T], \chi_1 \otimes S[T])$. By [D-Z 2, Theorem 6.16], we have $E(R[T], L) \xrightarrow{\sim} E(S[T], L \otimes S[T])$. Therefore, $e(P, \chi_1) = (I, \omega_I)$.

Case II: Assume n is even. We may assume that $R \hookrightarrow S$ is an elementary subintegral extension. If C denotes the conductor ideal of $R \subset S$, then $\text{ht}(C) \geq 1$. Write $\mathcal{L} = L \oplus R[T]^{n-1}$. If $J = I^2 \cap C$, then $\text{ht}(J) \geq 1$. We can choose $b \in J$ such that $\text{ht}(b) = 1$. The surjection $\omega_I : \mathcal{L}/I \twoheadrightarrow I/I^2$ induces a surjection $\overline{\omega}_I : \overline{\mathcal{L}}/\overline{I\mathcal{L}} \twoheadrightarrow \overline{I}/\overline{I}^2$, where bar denotes reduction modulo the ideal (b).

Since $\dim(R/bR) < \dim(R)$, by [D-Z 2, Proposition 2.13], $\overline{\omega}_I$ can be lifted to a surjection $\eta' : \overline{\mathcal{L}} \twoheadrightarrow \overline{I}$. If $\eta : \mathcal{L} \longrightarrow I$ is a lift of η' and hence a lift of ω_I as $b \in I^2$, then $(\eta(\mathcal{L}), b) = I$. Applying [B-RS 3, Corollary 2.13] to the element (η, b) of $\mathcal{L}^* \oplus R[T]$, there exists $\Psi \in \mathcal{L}^*$ such that $\text{ht}(K_b) \geq n$, where $K = (\eta + b\Psi)(\mathcal{L})$. As the ideal

$(\eta(\mathfrak{L}), b) = I$ has height n, we further get that $\text{ht}(K) = n$. Replacing η by $\eta + b\Psi$, we assume $\eta(\mathcal{L}) = K$ has height n.

Applying [B-RS 3, Lemma 2.11] to $(K, b) = I$ and $b \in I^2$, we get an ideal $I_1 \subset R[T]$ such that

(1) $\eta(\mathcal{L}) = I \cap I_1$;
(2) $\eta \otimes R[T]/I = \omega_I$;
(3) $\text{ht}(I_1) \geq n$;
(4) $I_1 + bR[T] = R[T]$ and hence $I_1 + C[T] = R[T]$.

If $\text{ht}(I_1) > n$, then $I_1 = R[T]$. Hence $(I, w_I) = 0$ in $E(R[T], L)$ and we are done. Assume $ht(I_1) = n$. From (1), we have $(I, \omega_I) + (I_1, \omega_{I_1}) = 0$ in $E(R[T], L)$, where ω_{I_1} induced by η. Proceeding as above with (I_1, ω_{I_1}), we get an ideal $I_2 \subset R[T]$ of height n with $I_2 + CR[T] = R[T]$ and an local orientation ω_{I_2} of I_2 such that

$$(I, \omega_I) = -(I_1, \omega_{I_1}) = (I_2, \omega_{I_2}) \text{ in } E(R[T], L).$$

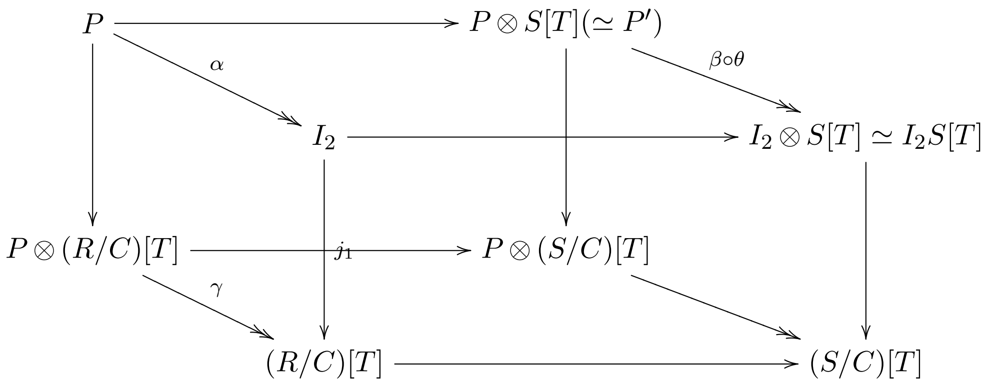
Recall that we have $e(P', \chi') = (IS[T], \omega_I^*) = (I_2S[T], \omega_{I_2}^*)$ in $E(S[T], L \otimes S[T])$. Since $L \otimes S[T]$ is extended from S, by [D-Z 2, Corollary 4.14], there exists a surjection $\beta : P' \twoheadrightarrow I_2S[T]$ such that $(I_2S[T], \omega_{I_2}^*)$ is obtained from (β, χ').

Since $I_2 + C[T] = R[T]$, we have the following:

(1) $I_2 \otimes (R/C)[T] \simeq (R/C)[T]$.
(2) $I_2 \otimes (S/C)[T] \simeq (S/C)[T]$.
(3) $R[T]/I_2 \simeq S[T]/I_2S[T]$.

Therefore, $\beta_1 := \beta \otimes (S/C)[T]$ is a unimodular element of $(P' \otimes (S/C)[T])^*$. So $\beta_1 \otimes (S/C)_{\text{red}}[T]$ is a unimodular element of $(P' \otimes (S/C)_{\text{red}}[T])^*$. Since $(R/C)_{red} = (S/C)_{red}$ and $P \otimes S[T] \simeq P'$, it is easy to see that we have a lift of $\beta_1 \otimes (S/C)_{\text{red}}[T]$ to a surjection $\gamma : P \otimes (R/C)[T] \twoheadrightarrow (R/C)[T]$. It is clear that $\gamma \otimes (S/C)[T] = \beta_1$ modulo the nil radical of $((S/C)[T])$. So, two unimodular elements β_1 and $\gamma \otimes (S/C)[T]$ of $(P \otimes (S/C)[T])^*$ are same modulo the nil radical of $((S/C)[T])$. By [D-Z 2, Proposition 2.8], there exists a transvection τ of $P \otimes (S/C)[T]$ such that $\beta_1 \circ \tau = \gamma \otimes (S/C)[T]$. By [B-R, Proposition 4.1], τ can be lifted to an automorphism θ of $P \otimes S[T] (\simeq P')$.

Consider the following Milnor square



As $\beta \circ \theta$ and γ agree over $S/C[T]$, they will patch to yield a surjection $\alpha : P \twoheadrightarrow I_2$.

Let $e(P, \chi_1) = (I_2, \omega'_{I_2})$ be obtained from the pair (α, χ_1). By (2.14), $(I_2, \omega_{I_2}) = (I_2, \overline{f}\omega'_{I_2})$ for some unit $\overline{f} \in R[T]/I_2$. By (2.15), there exists $P_2 \in \mathcal{P}_n(R[T])$ which is stably isomorphic to P, an isomorphism $\chi_2 : L \xrightarrow{\sim} \wedge^n(P_2)$ and a surjection $\upsilon : P_2 \twoheadrightarrow I_2$ such that $e(P_2, \chi_2) = (I_2, \overline{f}^{n-1}\omega'_{I_2})$ is obtained from (υ, χ_2). Since n is even, by (2.18), $(I_2, \overline{f}^{n-1}\omega'_{I_2}) = (I_2, \overline{f}\omega'_{I_2})$. Therefore, $e(P_2, \chi_2) = (I_2, \overline{f}\omega'_{I_2}) = (I_2, \omega_{I_2})$. Since $(I_2, \omega_{I_2}) = (I, \omega_I)$, we get $e(P_2, \chi_2) = (I, \omega_I)$. This completes the proof in the case $n \geq 4$.

For $n = 3$ case, we follow the steps of case I and use [D-Z 2, Theorem 7.2] which says that the natural group homomorphism $\widetilde{E}(R[T], L) \to \widetilde{E}(S[T], L \otimes S[T])$ is injective. $\square$

The proof of the following theorem is essentially contained in (5.1).

Theorem 5.2. *Let R be a reduced affine k-algebra of dimension $n \geq 2$ and $L \in \mathcal{P}_1(R[T])$. Let $(I, \omega_I) \in \widetilde{E}(R[T], L)$. Let $\lambda \in k$ be such that $ht(I(\lambda)) \geq n$ and there exists $Q \in \mathcal{P}_n(R)$ and an isomorphism $\chi : L/TL \xrightarrow{\sim} \wedge^n(Q)$ such that $e(Q, \chi) = (I(\lambda), \omega_{I(\lambda)})$ in $E(R, L/TL)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant L, an isomorphism $\chi_1 : L \xrightarrow{\sim} \wedge^n(P)$ and a surjection $\alpha : P \twoheadrightarrow I$ such that $e(P, \chi_1) = (I, \omega_I)$ in $\widetilde{E}(R[T], L)$. In particular, I is projectively generated.*

The following result generalizes (5.2) in case n is even.

Corollary 5.3. *Let R be a reduced affine k-algebra of* **even** *dimension $n \geq 2$ and $L \in \mathcal{P}_1(R[T])$. Let $(I, \omega_I) \in \widetilde{E}(R[T], L)$. Let $\lambda \in k$ be such that $ht(I(\lambda)) \geq n$ and there exists $Q \in \mathcal{P}_n(R)$ with determinant L/TL and a surjection $Q \twoheadrightarrow I(\lambda)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant L, an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(P)$ and a surjection $\alpha : P \twoheadrightarrow I$ such that $e(P, \chi) = (I, \omega_I)$ in $\widetilde{E}(R[T], L)$ is obtained from the pair (α, χ). In particular, I is projectively generated.*

Proof. Since n is even and there is a surjection $Q \twoheadrightarrow I(\lambda)$, by [B-RS 3, Lemma 5.1], there exists $\widetilde{Q} \in \mathcal{P}_n(R)$ with an isomorphism $\widetilde{\chi} : L/TL \stackrel{\sim}{\to} \wedge^n(\widetilde{Q})$ such that $e(\widetilde{Q}, \widetilde{\chi}) = (I(\lambda), w_{I(\lambda)})$ in $E(R, L/TL)$. By (5.2), there exists $P_1 \in \mathcal{P}_n(R[T])$, an isomorphism $\chi_1 : L \stackrel{\sim}{\to} \wedge^n(P_1)$ and a surjection $\alpha_1 : P_1 \twoheadrightarrow I$ such that $e(P_1, \chi_1) = (I, \omega_I)$ in $\widetilde{E}(R[T], L)$. Note that (I, ω_I) may not be obtained from the pair (α_1, χ_1). Let $e(P_1, \chi_1) = (I, \widetilde{\omega}_I)$ be obtained from the pair (α_1, χ_1). Then there is a unit $\overline{f} \in R[T]/I$ such that $\omega_I = \overline{f}\widetilde{\omega}_I$. Since n is even, by (2.15), there exists $P \in \mathcal{P}_n(R[T])$ with $P \oplus R[T] \stackrel{\sim}{\to} P_1 \oplus R[T]$, an isomorphism $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ and a surjection $\alpha : P \twoheadrightarrow I$ such that $e(P, \chi) = (I, \omega_I)$ is obtained from the pair (α, χ). $\square$

The following result extends (5.2).

Corollary 5.4. *Let R be a reduced affine k-algebra of dimension $n \geq 3$ over a C_1 field k and $L \in \mathcal{P}_1(R[T])$. Let $(I, \omega_I) \in \widetilde{E}(R[T], L)$. Let $\lambda \in k$ be such that ht$(I(\lambda)) \geq n$ and there exists $Q \in \mathcal{P}_n(R)$ with determinant L/TL and a surjection $Q \twoheadrightarrow I(\lambda)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant L, an isomorphism $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ and a surjection $\alpha : P \twoheadrightarrow I$ such that $e(P, \chi) = (I, \omega_I)$ in $\widetilde{E}(R[T], L)$. In particular, I is projectively generated.*

Proof. Let $\theta : Q \twoheadrightarrow I(\lambda)$ be a surjection and $\chi_1 : L/TL \stackrel{\sim}{\to} \wedge^n(Q)$ be an isomorphism. Let $e(Q, \chi_1) = (I(\lambda), \omega) \in E(R, L/TL)$ be obtained from the pair (θ, χ_1). By (4.9), $(I(\lambda), \omega) = (I(\lambda), \omega_{I(\lambda)})$ in $R(R, L/TL)$. Using (5.2), we are done. $\square$

Corollary 5.5. *Let R be a reduced affine k-algebra of dimension $n \geq 3$ over an algebraically closed field k. Let $L \in \mathcal{P}_1(R[T])$ and $(I, \omega_I) \in \widetilde{E}(R[T], L)$. Then there exists $P \in \mathcal{P}_n(R[T])$ with determinant L, an isomorphism $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ and a surjection $\alpha : P \twoheadrightarrow I$ such that $e(P, \chi) = (I, \omega_I)$ in $\widetilde{E}(R[T], L)$.*

Proof. We can find $\lambda \in k$ such that ht$(I(\lambda)) \geq n$. Following the proof of (4.11), we get a projective R-module Q of rank n with determinant L and a surjection $Q \twoheadrightarrow I(\lambda)$. Finally using (5.4), we are done. $\square$

The following result is immediate from (5.5).

Corollary 5.6. *Let R be a reduced affine k-algebra of dimension $n \geq 3$ over an algebraically closed field k. Let (I, ω_I) be an element of $\widetilde{E}(R[T], L)$ when $n = 3$ and and $E(R[T], L)$ when $n > 3$. Then $(I, \omega_I) = e(P, \chi)$ for some $P \in \mathcal{P}_n(R[T])$ with determinant L and $\chi : L \stackrel{\sim}{\to} \wedge^n(P)$ an isomorphism.*

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REFERENCES

[Bh 1] S. M. Bhatwadekar, Projective generation of maximal ideals in polynomial rings, *Journal of Algebra* **91** (1984), 75-81.

[Bh 2] S. M. Bhatwadekar, Cancellation theorems for projective modules over a two dimensional ring and its polynomial extensions, *Compositio Math.* **128** (2001), 339-359.

[Bh 3] S. M. Bhatwadekar, A cancellation theorem for projective modules over a affine algebras over C_1-fields, *J.Pure and Applied Algebra* **183** (2003), 17-26.

[B-D] S. M. Bhatwadekar and M. K. Das, Projective generation of curves (III), *IMRN* **Vol. 2015, No. 4** (2015), 960-980.

[B-R] S. M. Bhatwadekar and Amit Roy, Some theorems about projective modules over polynomial rings, *J. Algebra* **86** (1984), 150-158.

[B-RS 1] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.

[B-RS 2] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain (II), *K-Theory* **15** (1998), 293-300.

[B-RS 3] S. M. Bhatwadekar and Raja Sridharan, Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.

[B-RS 4] S. M. Bhatwadekar and Raja Sridharan, On a question of Roitman, *J. Ramanujan Math.Soc.* **16** (2001), 45-61.

[B-H] W. Bruns and J. Herzog, Cohen Macaulay rings, Cambridge university press, 1993.

[D 1] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[D 2] M. K. Das, The Euler class group of a polynomial algebra II, *J. Algebra* **299** (2006), 94-114.

[D 3] M. K. Das, Revisiting Nori's question and homotopy invariance of Euler class groups. *Journal of K-Theory* **8** (2011), 451-480.

[D-M] M. K. Das and Satya Mandal, Euler class construction, *J. Pure Appl. Algebra* **198** (2005), 93-104.

[D-Z 1] M. K. Das and Md. Ali Zinna, On invariance of the Euler class group under a subintegral base change, *J. Algebra* **398** (2014), 131-155.

[D-Z 2] M. K. Das and Md. Ali Zinna, The Euler class group of a polynomial algebra with coefficients in a line bundle, *Math. Z.* **276** (2014) 257-283.

[D-Z 3] M. K. Das and Md. Ali Zinna, "Strong" Euler class of a stably free module of odd rank, *J. Algebra* **432** (2015), 185-204.

[M] Satya Mandal, On efficient generation of ideals, *Invent. Math.* **75** (1984), 59-67.

[Q] D. Quillen, Projective modules over polynomial rings, *Inventiones Mathematicae* **36** (1976), 167-171.

[P] B. R. Plumstead, The conjectures of Eisenbud and Evans, *Am. J. Math.* **105(6)** (1983), 1417-1433.

[Ra] Ravi A. Rao, The Bass-Quillen conjecture in dimension three but characteristic $\neq 2, 3$ via a question of A. Suslin, *Inventiones Mathematicae* **93** (1988), 609-618.

[RS] Raja Sridharan, Projective modules and complete intersections, *K-Theory* **13** (1998), 269-278.

[Se] Serre, J-P, Sur les modules projectifs, *Semin. Dubreil-Pisot* **14** (1960-61).

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