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Journal of Algebra 294 (2005) 226–238

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

Stability results for projective modules over blowup rings

Manoj Kumar Keshari

School of Mathematics, Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211019, India

Received 23 October 2004

Available online 22 April 2005

Communicated by Craig Huneke

Keywords: Affine algebra; Projective modules; Cancellation theorem

1. Introduction

Let R be a normal affine domain of dimension $n \geqslant 3$ over an algebraically closed field k. Suppose $\operatorname{char} k = 0$ or $\operatorname{char} k = p \geqslant n$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Let P be a stably free A-module of rank $n-1$. Then, Murthy proved that there exists a projective R-module Q such that $Q \otimes_R A = P$ and $\bigwedge^{n-1} Q = R$ [9, Theorem 2.10]. As a consequence of Murthy's result, if $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ with $g \neq 0$, then all stably free modules over $\mathbb{C}[X_1, \ldots, X_n, f/g]$ of rank $\geqslant n-1$ are free [9, Corollary 2.11].

In this paper, we prove the following result (3.5), which generalizes the above result of Murthy.

Theorem 1.1. *Let R be an affine algebra of dimension $n \geqslant 3$ over an algebraically closed field k. Suppose $\operatorname{char} k = 0$ or $\operatorname{char} k = p \geqslant n$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Let P' be a projective A-module of rank $n-1$ which is extended from R. Let $(a, p) \in \operatorname{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then, P is extended from R.*

E-mail address: keshari@mri.ernet.in.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.03.014

Now, we will describe our next result. Let R be an affine algebra over $\mathbb{R}$ of dimension n. Let $g \in R$ be an element not belonging to any real maximal ideal. Let Q be a projective R-module of rank $\geqslant n-1$. Let $(a, p) \in \mathrm{Um}(R_g \oplus Q_g)$ and $P = R_g \oplus Q_g/(a, p)R_g$. Then, P is extended from R [5, Theorem 3.10]. This result was proved earlier by Ojanguren and Parimala in case Q is free [11, Theorem]).

In this paper, we prove the following result (4.4), which is similar to 1.1.

Theorem 1.2. *Let R be an affine algebra of dimension $n \geqslant 3$ over $\mathbb{R}$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Assume that g does not belong to any real maximal ideal of R. Let P' be a projective A-module of rank $\geqslant n-1$ which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then, P is extended from R. In particular, every stably free A-module of rank n is extended from R.*

As a consequence of above result, if $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ with g not belonging to any real maximal ideal, then all stably free modules of rank $\geqslant n-1$ over $\mathbb{R}[X_1, \ldots, X_n, f/g]$ are free (4.6).

The proof of the main theorem makes crucial use of results and techniques of [2].

2. Preliminaries

In this paper, all the rings are assumed to be commutative Noetherian and all the projective modules are finitely generated. We denote the Jacobson radical of A by $\mathcal{J}(A)$.

Let B be a ring and let P be a projective B-module. Recall that $p \in P$ is called a *unimodular element* if there exists a $\psi \in P^* = \mathrm{Hom}_B(P, B)$ such that $\psi(p) = 1$. We denote by $\mathrm{Um}(P)$, the set of all unimodular elements of P. We write $O(p)$ for the ideal of B generated by $\psi(p)$, for all $\psi \in P^*$. Note that, if $p \in \mathrm{Um}(P)$, then $O(p) = B$.

Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and, hence, $1 + \varphi_p$ is a unipotent automorphism of P.

By a *transvection*, we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either $\varphi \in \mathrm{Um}(P^*)$ or $p \in \mathrm{Um}(P)$. We denote by $E(P)$, the subgroup of $\mathrm{Aut}(P)$ generated by all transvections of P. Note that, $E(P)$ is a normal subgroup of $\mathrm{Aut}(P)$.

An existence of a transvection of P pre-supposes that P has a unimodular element. Let $P = B \oplus Q$, $q \in Q$, $\alpha \in Q^*$. Then, the automorphisms Δ_q and Γ_α of P defined by $\Delta_q(b, q') = (b, q' + bq)$ and $\Gamma_\alpha(b, q') = (b + \alpha(q'), q')$ are transvections of P. Conversely, any transvection Θ of P gives rise to a decomposition $P = B \oplus Q$ in such a way that $\Theta = \Delta_q$ or $\Theta = \Gamma_\alpha$.

Definition 2.1. Let A be a ring and let P be a projective A-module. We say that P is *cancellative* if $P \oplus A^r \simeq Q \oplus A^r$ for some positive integer r and some projective A-module Q implies that $P \simeq Q$.

We begin by stating two classical results due to Serre [14] and Bass [1] respectively.

Theorem 2.2. *Let A be a ring with* $\dim A/\mathcal{J}(A) = d$. *Then, any projective A-module P of rank $> d$ has a unimodular element.*

Theorem 2.3. *Let A be a ring of dimension d and let P be a projective A-module of rank $> d$. Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$. *In particular, P is cancellative.*

The above result of Bass is best possible in general. But, in case of affine algebras over algebraically closed fields, we have the following result due to Suslin [17].

Theorem 2.4. *Let A be an affine algebra of dimension n over an algebraically closed field. Then, all projective A-modules of rank $\geqslant n$ are cancellative.*

Remark 2.5. Let P be a finitely generated projective A-module of rank d. Let t be a non-zero divisor of A such that P_t is free. Then, it is easy to see that there exits a free submodule $F = A^d$ of P and a positive integer l such that, if $s = t^l$, then $sP \subset F$. Therefore, $sF^* \subset P^* \subset F^*$. If $p \in F$, then $\Delta_p \in E(A \oplus F) \cap E(A \oplus P)$ and if $\alpha \in F^*$, then $\Gamma_{s\alpha} \in E(A \oplus F) \cap E(A \oplus P)$.

The following result is due to Bhatwadekar and Roy [3, Proposition 4.1].

Proposition 2.6. *Let A be a ring and let J be an ideal of A. Let P be a projective A-module of rank n. Then, any transvection $\tilde{\Theta}$ of P/JP, i.e., $\tilde{\Theta} \in E(P/JP)$, can be lifted to a* (*unipotent*) *automorphism Θ of P. In particular, if P/JP is free of rank n, then any element $\bar{\Psi}$ of $E((A/J)^n)$ can be lifted to $\Psi \in$* $\mathrm{Aut}(P)$. *If, in addition, the natural map* $\mathrm{Um}(P) \to \mathrm{Um}(P/JP)$ *is surjective, then the natural map $E(P) \to E(P/JP)$ is surjective.*

Definition 2.7. For a ring A, we say that projective stable range of A is $\leqslant r$ (notation: $\mathrm{psr}(A) \leqslant r$) if for all projective A-modules P of rank $\geqslant r$ and $(a, p) \in \mathrm{Um}(A \oplus P)$, we can find $q \in P$ such that $p + aq \in \mathrm{Um}(P)$. Similarly, A has stable range $\leqslant r$ (notation: $\mathrm{sr}(A) \leqslant r$) is defined the same way as $\mathrm{psr}(A)$ but with P required to be free.

The following result is due to Bhatwadekar [2, Corollary 3.3] and is a generalization of a result of Suslin [15, Lemma 2.1]. See [5], for the definition of $ESp_4(B)$.

Proposition 2.8. *Let B be a ring with* $\mathrm{psr}(B) \leqslant 3$ *and let I be an ideal of B. Let P be a projective B-module of rank 2 such that P/IP is free. Then, any element of $SL_2(B/I) \cap ESp_4(B/I)$ can be lifted to an element of $SL(P)$.*

Remark 2.9. In [2, Corollary 3.3], Proposition 2.8 is stated with the assumption that $\dim B = 2$. However, the proof works equally well in above case.

The following result is due to Mohan Kumar, Murthy and Roy [8, Theorem 3.7] and is used in 3.6.

Theorem 2.10. *Let A be an affine algebra of dimension $d \geqslant 2$ over $\bar{\mathbb{F}}_p$. Suppose that A is regular when $d = 2$. Then* $\operatorname{psr}(A) \leqslant d$.

The following two results are due to Lindel ([7, Theorem] and [6, Theorem 2.6]). Recall that a ring A is called essentially of finite type over a field k, if A is the localisation of an affine algebra over k.

Theorem 2.11. *Let A be a regular ring which is essentially of finite type over a field k. Then, every projective $A[X]$-module is extended from A.*

Theorem 2.12. *Let B be a ring of dimension d and let $R = B[T_1, \dots, T_n]$. Let P be a projective R-module of rank $\geqslant \max(2, d + 1)$. Then $E(P \oplus R)$ acts transitively on* $\operatorname{Um}(P \oplus R)$.

The following result is due to Suslin [16, Theorem 2]. The special case, namely $n = 2$ was proved earlier by Swan and Towber [20].

Theorem 2.13. *Let A be a ring and $[a_0, a_1, \dots, a_n] \in \operatorname{Um}_{n+1}(A)$. Then, there exists $\Gamma \in SL_{n+1}(A)$ with $[a_0^{n!}, a_1, \dots, a_n]$ as the first row.*

The next three results are due to Suslin [15, Propositions 1.4, 1.7 and Corollary 2.3] and are very crucial for the proof of our main theorem (see also [9, Remark 2.2]). Here "cd" stands for cohomological dimension (see [13] for definition).

Proposition 2.14. *Let X be a regular affine curve over a field k and let l be a prime with $l \neq \operatorname{char} k$. Suppose that $\operatorname{cd}_l k \leqslant 1$. Then, the group $SK_1(X)$ is l-divisible.*

Proposition 2.15. *Let X be a regular affine curve over a field k of characteristic $\neq 2$ and $\operatorname{cd}_2 k \leqslant 1$. Then, the canonical homomorphism $K_1Sp(X) \to SK_1(X)$ is an isomorphism.*

Proposition 2.16. *Let A be a ring and $[a_1, \dots, a_n] \in \operatorname{Um}_n(A)$ $(n \geqslant 3)$. Let $I = \sum_{i \geqslant 3} Aa_i$ and $J = \sum_{i \geqslant 4} Aa_i$ be ideals of A. Let $b_1, b_2 \in A$ be such that $Ab_1 + Ab_2 + I = A$. Let "bar" denotes reduction mod I. Suppose that*

(i) $\dim A/I \leqslant 1$ *and* $\operatorname{sr}(A/J) \leqslant 3$,
(ii) *there exists an $\bar{\alpha} \in SL_2(\bar{A}) \cap ESp(\bar{A})$, such that $[\bar{a}_1, \bar{a}_2]\bar{\alpha} = [\bar{b}_1, \bar{b}_2]$.*

Then, there exists a $\gamma \in E_n(A)$ such that $[a_1, \dots, a_n]\gamma = [b_1, b_2, a_3 \dots, a_n]$.

Using above results, Suslin proved the following cancellation theorem [15, Theorem 2.4].

Theorem 2.17. *Let A be an affine algebra of dimension $d \geqslant 2$ over an infinite perfect field k. Suppose $\operatorname{cd} k \leqslant 1$ and $d! \in k^*$. Let $[a_0, a_1, \dots, a_d] \in \operatorname{Um}_{d+1}(A)$ and let r be a positive integer. Then, there exists $\Gamma \in E_{d+1}(A)$ such that $[a_0, a_1, \dots, a_d]\Gamma = [c_0^r, c_1, \dots, c_d]$. As a consequence, every stably free A-module of rank d is free* (*Theorem* 2.13).

The following result is due to Bhatwadekar [2, Theorem 4.1] and is a generalisation of above result of Suslin.

Theorem 2.18. *Let A be an affine algebra of dimension $d \geqslant 2$ over an infinite perfect field k. Suppose $\operatorname{cd} k \leqslant 1$ and $d! \in k^*$. Then, every projective A-module P of rank d is cancellative.*

The following result is very crucial for our main theorem and the proof of it is contained in [2, Theorem 4.1].

Proposition 2.19. *Let A be a ring and let P be a projective A-module of rank d. Let $s \in A$ be a non-zero-divisor such that P_s is free. Let $F = A^d$ be a free submodule of P with $F_s = P_s$ and $sP \subset F$. Let $e_1, \ldots, e_d$ denote the standard basis of F. Let $(a, p) \in \mathrm{Um}(A \oplus P)$ be such that*

(1) $a = 1 \mod As$,
(2) *$p \in F \subset P$ with $p = c_1^d e_1 + c_2 e_2 + \cdots + c_d e_d$, for some $c_i \in A$,*
(3) *every stably free A/Aa-module of rank $\geqslant d - 1$ is free.*

Then, there exists $\Delta \in \mathrm{Aut}(A \oplus P)$ such that $(a, p)\Delta = (1, 0)$.

The following result is used to prove our second result (4.4) and is due to Ojanguren and Parimala [11, Propositions 3 and 4].

Proposition 2.20. *Let $\mathcal{C} = \operatorname{Spec} C$ be a smooth affine curve over a field k of characteristic 0. Suppose that every residue field of $\mathcal{C}$ at a closed point has cohomological dimension $\leqslant 1$. Then, $SK_1(C)$ is divisible and the natural homomorphism $K_1Sp(C) \to SK_1(C)$ is an isomorphism.*

3. Main theorem 1

In this section, we will prove our first result (3.5). We begin with the following result, the proof of which is similar to [9, Corollary 2.8].

Lemma 3.1. *Let R be an affine algebra of dimension $n \geqslant 3$ over a field k. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Let P be a projective A-module of rank $\geqslant n - 1$ and $(a, p) \in \mathrm{Um}(A \oplus P)$. Then, there exists $\Psi \in SL(A \oplus P)$ such that $(a, p)\Psi = (1, 0) \mod Ag$.*

Proof. Since $g, f_1, \ldots, f_r$ is a R-regular sequence, $A = R[X_1, \ldots, X_r]/I$, where $I = (gX_1 - f_1, \ldots, gX_r - f_r)$. Let "bar" denote reduction modulo Ag. Then $\bar{A} = \bar{R}[X_1, \ldots, X_r]$, where $\bar{R} = R/(g, f_1, \ldots, f_r)$. Since $\dim \bar{R} \leqslant n - 2$, by 2.12, there exists $\bar{\Psi} \in E(\bar{A} \oplus \bar{P})$ such that $(\bar{a}, \bar{p})\bar{\Psi} = (1, 0)$. By 2.6, we can lift $\bar{\Psi}$ to $\Psi \in SL(A \oplus P)$. Hence, we have $(a, p)\Psi = (1, 0) \mod Ag$. $\square$

Lemma 3.2. *Let R be an affine algebra of dimension $n \geqslant 3$ over a field k. Let $g, f_1, \ldots, f_r \in R$ with g a non-zero-divisor and $A = R[f_1/g, \ldots, f_r/g]$. Let $S = 1 + gR$ and $B = A_S$. Let P be a projective B-module of rank $\geqslant n - 1$ which is extended from R_S. Let $(a, p) \in \mathrm{Um}(B \oplus P)$ with $(a, p) = (1, 0) \bmod Bg$. Then, we have the followings*:

(1) *there exists $s \in R$ such that P_s is free and*
(2) *there exists $\Delta \in \mathrm{Aut}(B \oplus P)$ such that $(a, p)\Delta = (1, 0) \bmod Bsg$.*

Further, given any ideal J of R of height $\geqslant 1$ with $\mathrm{ht}(g, J)R \geqslant 2$, we can choose s such that $s \in J$.

Proof. Choose $s_1 \in J$ such that $\mathrm{ht}\, s_1 R = 1$ and $\mathrm{ht}(s_1, g)R \geqslant 2$. By replacing (a, p) by $(a + \alpha(p), p)$ for some $\alpha \in P^*$, if necessary, we may assume that $\mathrm{ht}\, aB = 1$ and $\mathrm{ht}(a, s_1)B \geqslant 2$. Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are minimal primes of gR_S; $\tilde{\mathfrak{p}}_1, \ldots, \tilde{\mathfrak{p}}_{t'}$: minimal primes of R_S and $\mathfrak{q}_1, \ldots, \mathfrak{q}_{t''}$: minimal primes of aB. Since P is extended from R_S, P_Σ is free, where

$$\Sigma = R_S \Big\backslash \bigcup_{i=1}^{t} \mathfrak{p}_i \bigcup_{j=1}^{t'} \tilde{\mathfrak{p}}_j \bigcup_{l=1}^{t''} (\mathfrak{q}_l \cap R_S)$$

(any projective module over a semi local ring is free). Hence, there exists some $s_2 \in \Sigma$ such that P_{s_2} is free. We may assume $s_2 \in R$. Hence $\mathrm{ht}\, Rs_2 \geqslant 1$, $\mathrm{ht}(s_2, g)R_S \geqslant 2$ and $\mathrm{ht}(a, s_2)B \geqslant 2$.

Write $s = s_1 s_2$. Then $\mathrm{ht}(g, s)R_S \geqslant 2$ and $\mathrm{ht}(a, s)B \geqslant 2$. Since $\dim R_S/(\mathcal{J}(R_S), s) \leqslant n - 2$ and P is extended from R_S, by 2.2, P/sP has a unimodular element. Write $B_1 = B/Bs$, $P/sP = B_1 \oplus P_1$ and (b, p_1) as the image of p in P/sP.

Let "bar" denotes reduction modulo the ideal $B_1 a$. Since $a = 1 \bmod Bg$ and $\mathrm{ht}(s, a)B \geqslant 2$, $\dim \bar{B}_1 \leqslant n - 3$. Note that, $\bar{p} = (\bar{b}, \bar{p}_1) \in \mathrm{Um}(\bar{P} = \bar{B}_1 \oplus \bar{P}_1)$. Since rank of $\bar{P} \geqslant n - 1$, by 2.3, there exists $\bar{\Phi} \in E(\bar{P})$ such that $(\bar{b}, \bar{p}_1)\bar{\Phi} = (1, 0)$. In general, the natural map $\mathrm{Um}(B_1 \oplus P_1) \to \mathrm{Um}(\bar{B}_1 \oplus \bar{P}_1)$ is surjective. Hence, by 2.6, we can lift $\bar{\Phi}$ to some element $\Phi \in E(B_1 \oplus P_1)$. Let $(b, p_1)\Phi = (c, p_2)$. Then $(c, p_2) = (1, 0) \bmod B_1 a$.

Let $(c, p_2) = (1, 0) - a(c_1, p_3)$ for some $(c_1, p_3) \in B_1 \oplus P_1$. Then $(a, c, p_2)\Delta_{(c_1, p_3)} = (a, 1, 0)$, where $\Delta_{(c_1, p_3)} \in E(B_1 \oplus P_1)$. Recall that $P/sP = B_1 \oplus P_1$. By 2.6, we can lift $(1, \Phi)\Delta_{(c_1, p_3)} \in E(B \oplus P/s(B \oplus P))$ to some element $\Psi \in \mathrm{Aut}(B \oplus P)$ such that $(a, p)\Psi = (a, q)$ with $O(q) = B \bmod Bs$. Since $a = 1 \bmod Bg$, there exists $\Psi_1 \in E(B \oplus P)$ such that $(a, q)\Psi_1 = (1, 0) \bmod Bsg$. Let $\Delta = \Psi\Psi_1$. Then $(a, p)\Delta = (1, 0) \bmod Bsg$. This proves the lemma. □

Lemma 3.3. *Let R be an affine algebra of dimension $n \geqslant 3$ over an algebraically closed field k. Suppose $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p \geqslant n$. Let $g, f_1, \ldots, f_r \in R$ with g a non-zero-divisor and $A = R[f_1/g, \ldots, f_r/g]$. Let $S = 1 + Rg$ and $B = A_S$. Let P be a projective B-module of rank $n - 1$ which is extended from R_S. Let $(a, p) \in \mathrm{Um}(B \oplus P)$ with $(a, p) = (1, 0) \bmod Bg$. Then, there exists $\Delta \in \mathrm{Aut}(B \oplus P)$ such that $(a, p)\Delta = (1, 0)$.*

Proof. Without loss of generality, we can assume that R is reduced. Let J_1 be the ideal of R_g defining the singular locus $\mathrm{Sing}\, R_g$. Since R_g is reduced, $\mathrm{ht}\, J_1 \geqslant 1$. Note that $\sqrt{J_1} = J_1$.

Let $J = J_1 \cap R$. Then, we may assume that g does not belong to any minimal primes of J and $\operatorname{ht} J \geqslant 1$. Hence $\operatorname{ht}(g, J)R \geqslant 2$.

Since $(a, p) = (1, 0) \mod Bg$, by 3.2, there exists some $s \in J$ and $\Phi \in \operatorname{Aut}(B \oplus P)$ such that P_s is free and $(a, p)\Phi = (1, 0) \mod Bsg$. Hence, replacing (a, p) by $(a, p)\Phi$, we can assume that $(a, p) = (1, 0) \mod Bsg$. It is easy to see that we can replace B by $C = A_T$, where $T = (1 + gk[g])h$ for some $h \in 1 + gR$. Note that, $B = C_S$.

Since P_s is free of rank $n - 1$, there exists a free submodule $F = C^{n-1}$ of P such that $F_s = P_s$. By replacing s by a power of s, we may assume that $sP \subset F$. Let $e_1, \ldots, e_{n-1}$ denote the standard basis of C^{n-1}. Since $(a, p) = (1, 0) \mod Csg$, $p \in sgP \subset gF$. Let $p = b_1e_1 + \cdots + b_{n-1}e_{n-1}$, for some $b_i \in gC$. Then $[a, b_1, \ldots, b_{n-1}] \in \mathrm{Um}_n(C)$. As $1 - a \in Csg$, $[a, sb_1, \ldots, sb_{n-1}] \in \mathrm{Um}_n(C)$.

For $n = 3$, by Swan's Bertini theorem [19, Theorem 1.3] as quoted in [10, Theorem 2.3], there exists $c_1, c_2 \in C$ such that, if $a' = a + sb_1c_1 + sb_2c_2$, then C/Ca' is a reduced regular (since $a' = 1 \mod Csg$ and $s \in J$) affine $k(g)$-algebra of dimension 1. For $n \geqslant 4$, by prime avoidance, there exists $c_1, \ldots, c_{n-1} \in C$ such that, if $a' = a + sb_1c_1 + \cdots + sb_{n-1}c_{n-1}$, then C/Ca' is affine $k(g)$-algebra of dimension $\leqslant n - 2$. Note that, $a' = 1 \mod Csg$.

Let $e_1^*, \ldots, e_{n-1}^*$ be a dual basis of F^* and let $\theta_i = c_ie_i^* \in F^*$. Then, by 2.5, $\Gamma_{s\theta_i} \in E(C \oplus P)$ and $\Gamma_{s\theta_i}(a, p) = (a + sb_ic_i, p)$. Hence, it follows that there exists $\Psi_1 \in E(C \oplus P)$ such that $\Psi_1(a, p) = (a', p)$.

Let "bar" denote reduction modulo Ca'. Since $Ca' + Cs = C$ and $P_s = F_s$ is free, the inclusion $F \subset P$ gives rise to equality $\bar{F} = \bar{P}$. In particular, $\bar{P}$ is free of rank $n - 1 \geqslant 2$ with a basis $\bar{e}_1, \ldots, \bar{e}_{n-1}$ and $\bar{p} \in \mathrm{Um}(\bar{P})$. Recall that $\bar{C}$ is an affine algebra of dimension $n - 2$ over a C_1-field $k(g)$. Hence, by 2.18, every projective $\bar{C}$-module of rank $n - 2$ is cancellative.

If $n = 3$, then $\bar{C}$ is a regular affine algebra of dimension 1 over a C_1-field $k(g)$. Hence, by 2.14, 2.15, $SK_1(\bar{C})$ is a divisible group and the canonical homomorphism $K_1Sp(\bar{C}) \to SK_1(\bar{C})$ is an isomorphism. Hence, there exists $\Theta' \in SL_2(\bar{C}) \cap ESp(\bar{C})$ and $t_1, t_2 \in C$ such that, if $p_1 = t_1^2e_1 + t_2e_2 \in F$, then $\Theta'(\bar{p}) = \bar{p}_1$. We have $\dim B/Bg = \dim A/Ag = 2$ and B_g is an $k(g)$-algebra of dimension two. Thus $\operatorname{Spec} B = \operatorname{Spec} B/Bg \cup \operatorname{Spec} B_g$ with $\dim B/Bg = 2 = \dim B_g$. Hence $\operatorname{psr}(B) \leqslant 3$. Hence, by 2.8, $\Theta' \otimes \bar{B}$ has a lift $\Theta \in SL(P \otimes B)$.

For $n \geqslant 4$. Since $\bar{P}$ is free of rank $n - 1$, $E_{n-1}(\bar{C}) = E(\bar{P})$. Hence, by 2.17, there exists $\tilde{\Theta} \in E(\bar{P})$ and $t_i \in C$, $1 \leqslant i \leqslant n - 1$ such that, if $p_1 = t_1^{n-1}e_1 + t_2e_2 + \cdots + t_{n-1}e_{n-1} \in F$, then $\tilde{\Theta}(\bar{p}) = \bar{p}_1$. By 2.6, $\tilde{\Theta}$ can be lifted to an element $\Theta \in SL(P)$.

Write P for $P \otimes B$. Thus, in either case, there exists $q \in P$ such that

$$\Theta(p) = p_1 - a'q, \quad \text{where} \quad p_1 = t_1^{n-1}e_1 + t_2e_2 + \cdots + t_{n-1}e_{n-1}.$$

The automorphism Θ of P induces an automorphism $\Lambda_1 = (\mathrm{Id}_B, \Theta)$ of $B \oplus P$. Let Λ_2 be the transvection Δ_q of $B \oplus P$. Then $(a', p)\Lambda_1\Lambda_2 = (a', p_1)$.

By 2.19, there exists $\Lambda_3 \in \operatorname{Aut}(B \oplus P)$ such that $(a', p_1)\Lambda_3 = (1, 0)$. Let $\Delta = \Psi_1\Lambda_1\Lambda_2\Lambda_3$. Then $\Delta \in \operatorname{Aut}(B \oplus P)$ and $(a, p)\Delta = (1, 0)$. This proves the result. $\square$

Remark 3.4. Let A be a ring and $g, h \in A$ with $Ag + Ah = A$. Then, any projective A-module E is given by a triple (Q, α, P), where Q, P are projective modules over A_h and A_g, respectively, and α is a prescribed A_{gh}-isomorphism $\alpha : Q_g \xrightarrow{\sim} P_h$.

Let $g, h \in A$ with $Ag + Ah = A$ and let P be a projective A-module. Let $(a, p) \in \mathrm{Um}(A_g \oplus P_g)$ and $Q = A_g \oplus P_g/(a, p)A_g$. If $\varphi : Q_h \xrightarrow{\sim} P_{gh}$ is an isomorphism, then the triple (P_h, φ, Q) yields a projective A-module E such that $Q = E \otimes A_g$.

Now, we prove the main result of this section. In case P' is free (i.e., P is stably free), it is proved in [9, Theorem 2.10].

Theorem 3.5. *Let R be an affine algebra of dimension $n \geqslant 3$ over an algebraically closed field k. Suppose $\operatorname{char} k = 0$ or $\operatorname{char} k = p \geqslant n$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Let P' be a projective A-module of rank $n-1$ which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then, P is extended from R.*

Proof. By 3.1, there exists $\Psi \in SL(A \oplus P')$ such that $(a, p)\Psi = (1, 0) \mod Ag$. Let $S = 1 + Rg$ and $B = A_S$. Applying 3.3, there exists $\Psi_1 \in \mathrm{Aut}(B \oplus (P' \otimes B))$ such that $(a, p)\Psi\Psi_1 = (1, 0)$. Let $\Delta = \Psi\Psi_1$. Then, there exists some $h \in 1 + Rg$ such that $\Delta \in \mathrm{Aut}(A_h \oplus P'_h)$ and $(a, p)\Delta = (1, 0)$. We have the isomorphism $\Gamma : P_h \xrightarrow{\sim} P'_h$ induced from Δ. The module P is given by the triple (P'_h, Γ_g, P_g). Since $Rg + Rh = R$, $R_g = A_g$, $R_{gh} = A_{gh}$ and $\Gamma_g : P_{gh} \xrightarrow{\sim} P'_{gh}$ is an isomorphism of R_{gh} module, the triple (P'_h, Γ_g, P_g) defines a projective R-module Q of rank $n-1$ such that $P = Q \otimes A$. This proves the theorem. □

The following result is a generalisations of [9, Theorem 2.12], where it is proved for stably free modules.

Theorem 3.6. *Let R be an affine domain of dimension $n \geqslant 4$ over $\bar{\mathbb{F}}_p$. Suppose $p \geqslant n$. Let K be the field of fractions of R and let A be a subring of K with $R \subset A \subset K$. Let P' be a projective A-module of rank $n-1$ which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then, P is extended from R.*

Proof. We may assume that A is finitely generated over R, i.e., there exist $g, f_1, \ldots, f_r \in R$ such that $A = R[f_1/g, \ldots, f_r/g]$. Since P' is extended from R, we can choose an element $s \in R$ such that ${P'}_s$ is free. Let "bar" denote reduction modulo Asg. Then $\bar{A} = A/Asg$ is an affine algebra of dimension $\leqslant n-1$ over $\bar{\mathbb{F}}_p$. Since $n - 1 \geqslant 3$, by 2.10, $\mathrm{psr}(\bar{A}) \leqslant n-1$. Hence, there exists $\bar{\Psi} \in E(\bar{A} \oplus \bar{P}')$ such that $(\bar{a}, \bar{p})\bar{\Psi} = (1, 0)$. By 2.6, $\bar{\Psi}$ can be lifted to $\Psi \in SL(A \oplus P')$. Replacing (a, p) by $(a, p)\Psi$, we can assume that $(a, p) = (1, 0) \mod Asg$. Let $B = A_{1+gR}$. Then, by 3.3 there exists $\Gamma \in \mathrm{Aut}(B \oplus (P' \otimes B))$ such that $(a, p)\Gamma = (1, 0)$. Rest of the argument is same as in 3.5. □

The following result is a generalisations of [9, Theorem 2.14], where it is proved for stably free modules.

Theorem 3.7. *Let R be a regular affine algebra of dimension $n-1 \geqslant 2$ over an algebraically closed field k. Let $A = R[X, f/g]$, where g, f is a $R[X]$-regular sequence. Suppose*

(1) $\operatorname{char} k = 0$ *or* $\operatorname{char} k = p \geqslant n$,
(2) *either g is a monic polynomial or $g(0) \in R^*$.*

Let P' be a projective A-module of rank $n-1$ which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then $P \xrightarrow{\sim} P'$.

Proof. By 3.5, there exists a projective $R[X]$-module Q' of rank $n-1$ such that $P = Q' \otimes A$. By 2.11, $Q' = Q \otimes R[X]$ with Q a projective R-module of rank $n-1$. Hence $P = Q \otimes_R A$. From [9, Theorem 2.14], we have that $K_0(R) \to K_0(A)$ is injective. Since P' is extended from R and P is stably isomorphic to P', hence Q is stably isomorphic to P' as R-modules. By 2.4, $Q \xrightarrow{\sim} P'$ as R-modules and hence $P \xrightarrow{\sim} P'$. This proves the result. □

4. Main theorem 2

In this section we prove our second result (4.4). Given an affine algebra A over $\mathbb{R}$ and a subset $I \subset A$, we denote by $Z(I)$, the closed subset of $X = \operatorname{Spec} A$ defined by I and by $Z_{\mathbb{R}}(I)$, the set $Z(I) \cap X(\mathbb{R})$, where $X(\mathbb{R})$ is the set of all real maximal ideals $\mathfrak{m}$ of A (i.e., $A/\mathfrak{m} \xrightarrow{\sim} \mathbb{R}$).

We begin by stating the following result of Ojanguren and Parimala [11, Lemma 2].

Lemma 4.1. *Let A be a reduced affine algebra of dimension n over $\mathbb{R}$ and $X = \operatorname{Spec} A$. Let $[a_1, \dots, a_d] \in \mathrm{Um}_d(A)$. Suppose $a_1 > 0$ on $X(\mathbb{R})$. Then, there exists $b_2, \dots, b_d \in A$ such that $\tilde{a} = a_1 + b_2a_2 + \cdots + b_da_d > 0$ on $X(\mathbb{R})$ and $Z(\tilde{a})$ is smooth on $X \setminus \operatorname{Sing} X$ of dimension $\leqslant n-1$.*

The following result is analogous to [11, Proposition 1] and [5, Lemma 3.8].

Lemma 4.2. *Let R be a reduced affine algebra of dimension $n \geqslant 3$ over $\mathbb{R}$ and let $g, f_1, \dots, f_r \in R$ with g not belonging to any real maximal ideal of R. Let $A = R[f_1/g, \dots, f_r/g]$ and $X = \operatorname{Spec} A$. Let P be a projective A-module and let $(a, p) \in \mathrm{Um}(A \oplus P)$ with $a - 1 \in sgA$ for some $s \in R$. Then, there exists $h \in 1 + gR$ and $\Delta \in \mathrm{Aut}(A_h \oplus P_h)$ such that if $(a, p)\Delta = (\tilde{a}, \tilde{p})$, then*

(1) $\tilde{a} > 0$ *on* $X(\mathbb{R}) \cap \operatorname{Spec} A_h$,
(2) $Z(\tilde{a})$ *is smooth on* $\operatorname{Spec} A_h \setminus \operatorname{Sing} X$ *of dimension* $\leqslant n-1$, *and*
(3) $(\tilde{a}, \tilde{p}) = (1, 0) \pmod{sgA_h}$.

Proof. By replacing g by g^2, we may assume that $g > 0$ on $X(\mathbb{R})$. Since $a = 1 \bmod sgA$, $(a, sp) \in \mathrm{Um}(A \oplus P)$. Therefore, a has no zero on $Z_{\mathbb{R}}(O(sp))$. Let r be a positive integer such that $g^r a \in gR$. Let $Y = \operatorname{Spec} R$. Then $g^r a$ has no zero on $Z_{\mathbb{R}}(O(sp)) \cap Y(\mathbb{R})$. By Łojasiewicz's inequality [4, Proposition 2.6.2], there exists $c \in R$ with $c > 0$ on $Y(\mathbb{R})$ such that $1/|a|g^r < c$ on $Z_{\mathbb{R}}(O(sp)) \cap Y(\mathbb{R})$. Let $(1 + ag^r c)a = a'$. Then $g^r a' > 0$ on $Z_{\mathbb{R}}(O(sp)) \cap Y(\mathbb{R})$ and hence $a' > 0$ on $Z_{\mathbb{R}}(O(sp))$. Write $h = 1 + ag^r c \in 1 + gR$. Then $a' = ha$.

Let W be the closed semi-algebraic subset of $X(\mathbb{R})$ defined by $a' \leqslant 0$. Since $Z_{\mathbb{R}}(O(sp)) \cap W = \emptyset$, if $O(p) = (b_1, \ldots, b_d)$ then $s^2(b_1^2 + \cdots + b_d^2) > 0$ on W. Hence, by Łojasiewicz's inequality, there exists $c_1 \in A$ with $c_1 > 0$ on $X(\mathbb{R})$ such that $|a'|/gs^2(b_1^2 + \cdots + b_d^2) < c_1$. Hence $a'' = a' + c_1 gs^2(b_1^2 + \cdots + b_d^2) > 0$ on W and hence $a'' > 0$ on $X(\mathbb{R})$.

We still have $a'' = 1 \mod sgA_h$. Since $[a'', gs^2 b_1^2, \ldots, gs^2 b_d^2] \in \operatorname{Um}_{d+1}(A_h)$, by 4.1, there exists $h_i \in A_h$ such that $\tilde{a} = a'' + \sum_{i=1}^d gs^2 b_i^2 h_i > 0$ on $X(\mathbb{R}) \cap \operatorname{Spec} A_h$ and $Z(\tilde{a})$ is smooth on $\operatorname{Spec} A_h \setminus \operatorname{Sing} X$ of dimension $\leqslant n - 1$. It is clear from the proof that there exists $\Delta_1 \in \operatorname{Aut}(A_h \oplus P_h)$ such that $(a, p)\Delta_1 = (\tilde{a}, p)$. Since $\tilde{a} = 1 \mod sgA_h$, there exists $\Delta_2 \in E(A_h \oplus P_h)$ such that $(\tilde{a}, p)\Delta_2 = (\tilde{a}, \tilde{p})$ with $\tilde{p} \in sgP_h$. Take $\Delta = \Delta_1 \Delta_2$. This proves the result. □

Lemma 4.3. *Let R be an affine algebra of dimension $n \geqslant 3$ over $\mathbb{R}$. Let $g, f_1, \ldots, f_r \in R$ with g a non-zero-divisor and $A = R[f_1/g, \ldots, f_r/g]$. Assume that g does not belong to any real maximal ideal of R. Let $S = 1 + gR$ and $B = A_S$. Let P be a projective B-module of rank $\geqslant n - 1$ which is extended from R_S. Let $(a, p) \in \operatorname{Um}(B \oplus P)$ with $(a, p) = (1, 0) \mod Bg$. Then, there exists $\tilde{\Delta} \in \operatorname{Aut}(B \oplus P)$ such that $(a, p)\tilde{\Delta} = (1, 0)$.*

Proof. In view of 2.3, it is enough to prove the result when rank of P is $\leqslant n$. For the sake of simplicity, we assume that rank of $P = n - 1$. The same proof goes through when rank $P = n$.

Without loss of generality, we may assume that R is reduced. Let J_1 be the ideal of R_g defining the singular locus $\operatorname{Sing} R_g$. Since R_g is reduced, $\operatorname{ht} J_1 \geqslant 1$. Note that $\sqrt{J_1} = J_1$. Let $J = J_1 \cap R$. Then, we may assume that g does not belong to any minimal primes of J and $\operatorname{ht} J \geqslant 1$. Hence $\operatorname{ht}(g, J)R \geqslant 2$.

Since $(a, p) = (1, 0) \mod Bg$, by 3.2, there exists some $s \in J$ and $\Phi \in \operatorname{Aut}(B \oplus P)$ such that P_s is free and $(a, p)\Phi = (1, 0) \mod Bsg$. Hence, replacing (a, p) by $(a, p)\Phi$, we can assume that $(a, p) = (1, 0) \mod Bsg$.

There exists some $h \in S$ such that P is a projective A_h-module with P_s free and $(a, p) \in \operatorname{Um}(A_h \oplus P)$ with $(a, p) = (1, 0) \mod sgA_h$. Applying 4.2, there exists some $h' \in 1 + gR_h$ and $\Delta \in \operatorname{Aut}(A_{hh'} \oplus P_{hh'})$ such that $(a, p)\Delta = (a', p')$ with

(1′) $a' > 0$ on $X(\mathbb{R}) \cap \operatorname{Spec} A_{hh'}$, where $X = \operatorname{Spec} A_h$,
(2′) $(a', p') = (1, 0) \mod sgA_{hh'}$, and
(3′) $Z(a')$ is smooth (since $a' = 1 \mod sgA_{hh'}$ and $s \in J_1$) on $\operatorname{Spec} A_{hh'}$ of dimension $\leqslant n - 1$.

Note that, since $h^r h' \in 1 + Rg$ for some positive integer r, $A_{hh'} \subset B$. Hence, replacing $A_{hh'}$ by A and (a', p') by (a, p), we assume that the above properties (1′)–(3′) holds for (a, p) in the ring A, i.e., we have

(1) $a > 0$ on $X(\mathbb{R})$, where $X = \operatorname{Spec} A$,
(2) $(a, p) = (1, 0)$ mod sgA, and
(3) $Z(a)$ is smooth on $\operatorname{Spec} A$ of dimension $\leqslant n - 1$.

Since P_s is free of rank $n - 1$, there exists a free submodule $F = A^{n-1}$ of P such that $F_s = P_s$. Replacing s by a suitable power of s, we may assume that $sP \subset F$. Let $e_1, \dots, e_{n-1}$ denote the standard basis of A^{n-1}.

Since $p \in sgP \subset gF$, $p = b_1e_1 + \cdots + b_{n-1}e_{n-1}$ for some $b_i \in gA$. Then $[a, b_1, \dots, b_{n-1}] \in \mathrm{Um}_n(A)$. Let $T = 1 + g\mathbb{R}[g]$ and $C = A_T$. Note that $B = A_S = C \otimes C_S$. Let "bar" denotes reduction modulo Ca. Since $a - 1 \in Csg$ and $s \in J$, $\bar{C}$ is a smooth affine algebra over $\mathbb{R}(g)$ of dimension $n - 2$. Since $P_s = F_s$ is free, the inclusion $F \subset P$ gives rise to equality $\bar{F} = \bar{P}$. In particular, $\bar{P}$ is free of rank $n - 1 \geqslant 2$ with a basis $\bar{e}_1, \dots, \bar{e}_{n-1}$ and $\bar{p} \in \mathrm{Um}(\bar{P})$.

Assume $n \geqslant 4$. We have $[\bar{b}_1, \dots, \bar{b}_{n-1}] \in \mathrm{Um}_{n-1}(\bar{C})$. As in [9, Lemma 2.6], by Swan's Bertini theorem [19, Theorem 1.3], there exists an $\Theta \in E_{n-1}(\bar{C})$ such that $[\bar{b}_1, \dots, \bar{b}_{n-1}]\Theta = [\bar{b}_1, \bar{b}_2, \bar{c}_3, \dots, \bar{c}_{n-1}]$ with the following properties:

(1) $\bar{C}/\bar{J}$ is smooth affine $\mathbb{R}(g)$-algebra of dimension 2, where J denotes the ideal of C generated by $(c_4, \dots, c_{n-1})$,
(2) $\bar{C}/\bar{I}$ is smooth affine $\mathbb{R}(g)$-algebra of dimension 1, where I denotes the ideal of C generated by $(c_3, \dots, c_{n-1})$.

Every maximal ideal $\mathfrak{m}$ of $\bar{C}/\bar{I}$ is the image in $\operatorname{Spec} \bar{C}/\bar{I}$ of a prime ideal $\mathfrak{p}$ of C of height $n - 1$ containing a. Since a does not belongs to any real maximal ideal of C, by Serre's result [13], the residue field $\mathbb{R}(\mathfrak{p}) = k(\mathfrak{m})$ of $\mathfrak{m}$ has cohomological dimension $\leqslant 1$. By 2.20, $SK_1(\bar{C}/\bar{I})$ is divisible and the natural map $K_1Sp(\bar{C}/\bar{I}) \to SK_1(\bar{C}/\bar{I})$ is an isomorphism.

Let "tilde" denotes reduction modulo $\bar{I}$. Write $D = \bar{C}$, $\tilde{D} = D/\bar{I}$. Then, there exists $\Theta' \in SL_2(\tilde{D}) \cap ESp(\tilde{D})$ and $t_1, t_2 \in D$ such that

$$\big[\tilde{b}_1, \tilde{b}_2\big]\Theta' = \big[\tilde{t}_1^{\,n-1}, \tilde{t}_2\big].$$

Since $B = C_S$, $\bar{B} = \bar{C}_S$. We have $\Theta' \in SL_2(\bar{B}/\bar{I}) \cap ESp(\bar{B}/\bar{I})$ and $t_1, t_2 \in \bar{B}$ such that $[\tilde{b}_1, \tilde{b}_2]\Theta' = [\tilde{t}_1^{\,n-1}, \tilde{t}_2]$.

If $n = 3$, then $\bar{I} = 0$ and hence $\bar{B}/\bar{I} = \bar{B} = B/Ba$. We have $\dim B/Bg = \dim A/Ag = 2$ and B_g is an $\mathbb{R}(g)$-algebra of dimension two. Thus $\operatorname{Spec} B = \operatorname{Spec} B/Bg \cup \operatorname{Spec} B_g$ with $\dim B/Bg = 2 = \dim B_g$. Hence $\operatorname{psr}(B) \leqslant 3$. Therefore, by 2.8, Θ' has a lift $\Theta_1 \in SL(P \otimes B)$.

For $n \geqslant 4$. Since $\dim \bar{B}/\bar{I} \leqslant 1$ and $\dim \bar{B}/\bar{J} \leqslant 2$, by 2.16, there exists $\Theta'' \in E_{n-1}(B/Ba)$ such that

$$\big[\bar{b}_1, \bar{b}_2, \bar{c}_3, \dots, \bar{c}_{n-1}\big]\Theta'' = \big[\bar{t}_1^{\,n-1}, \bar{t}_2, \bar{c}_3, \dots, \bar{c}_{n-1}\big].$$

Recall that, there exists $\Theta \in E_{n-1}(B/Ba)$ such that $[\bar{b}_1, \dots, \bar{b}_{n-1}]\Theta = [\bar{b}_1, \bar{b}_2, \bar{c}_3, \dots, \bar{c}_{n-1}]$. Since $\bar{P}$ is free of rank $n - 1 \geqslant 3$, $E_{n-1}(\bar{A}) = E(\bar{P})$. By 2.6, $\Theta\Theta'' \in E_{n-1}(B/Ba)$ can be lifted to an element $\Theta_1 \in SL(P \otimes B)$. (In particular, the above argument shows that every stably free B/Ba-module of rank $\geqslant n-2$ is cancellative.)

Write P for $P \otimes B$. Thus, in either case ($n \geqslant 3$), there exists $q \in P$ such that

$$\Theta_1(p) = p_1 - aq, \quad \text{where} \quad p_1 = t_1^{n-1} e_1 + t_2 e_2 + c_3 e_3 + \cdots + c_{n-1} e_{n-1}.$$

The automorphism Θ_1 of P induces an automorphism $\Lambda_1 = (\mathrm{Id}_B, \Theta_1)$ of $B \oplus P$. Let Λ_2 be the transvection Δ_q of $B \oplus P$. Then $(a, p)\Lambda_1\Lambda_2 = (a, p_1)$.

By 2.19, there exists $\Lambda_3 \in \mathrm{Aut}(B \oplus P)$ such that $(a, p_1)\Lambda_3 = (1, 0)$. Let $\tilde{\Delta} = \Lambda_1\Lambda_2\Lambda_3$. Then $\tilde{\Delta} \in \mathrm{Aut}(B \oplus P)$ and $(a, p)\tilde{\Delta} = (1, 0)$. This proves the result. $\square$

Now, we prove the main theorem of this section.

Theorem 4.4. *Let R be an affine algebra of dimension $n \geqslant 3$ over $\mathbb{R}$. Let $g, f_1, \ldots, f_r$ be a R-regular sequence and $A = R[f_1/g, \ldots, f_r/g]$. Assume that g does not belong to any real maximal ideal of R. Let P' be a projective A-module of rank $\geqslant n-1$ which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then, P is extended from R.*

Proof. By 3.1, there exists $\Delta \in \mathrm{Aut}(A \oplus P')$ such that $(a, p)\Delta = (1, 0) \mod Ag$. Let $S = 1 + Rg$ and $B = A_S$. Applying 4.3, there exists $\Delta_1 \in \mathrm{Aut}(B \oplus (P' \otimes B))$ such that $(a, p)\Delta\Delta_1 = (1, 0)$. Let $\Psi = \Delta\Delta_1$. Then, there exists $h \in 1 + Rg$ such that $\Psi \in \mathrm{Aut}(A_h \oplus {P'}_h)$ and $(a, p)\Psi = (1, 0)$. Rest of the argument is same as in 3.5. $\square$

Remark 4.5. The proof of 4.4 works for any real closed field k. For simplicity, we have taken $k = \mathbb{R}$.

Corollary 4.6. *Let $R = \mathbb{R}[X_1, \ldots, X_n]$ and $f, g \in R$ with g not belonging to any real maximal ideal. Then, every stably free $R[f/g]$-modules P of rank $\geqslant n-1$ is free.*

Proof. Write $A = R[f/g]$. We may assume that f, g have no common factors so that g, f is a regular sequence in R. Since rank $P \geqslant n-1$, $P \oplus A^2$ is free. Applying 4.4, we get that $P \oplus A$ is extended from R. By Quillen–Suslin theorem [12,18], every projective R-module is free. Hence $P \oplus A$ is free. Again, by 4.4, P is extended from R and hence is free. $\square$

The proof of the following result is similar to 3.7, hence we omit it.

Theorem 4.7. *Let R be a regular affine algebra of dimension $n-1 \geqslant 2$ over $\mathbb{R}$. Let $A = R[X, f/g]$, where g, f is a $R[X]$-regular sequence. Suppose that*

(1) *g does not belongs to any real maximal ideal,*
(2) *g is a monic polynomial or $g(0) \in R^*$.*

Let P' be a projective A-module of rank n which is extended from R. Let $(a, p) \in \mathrm{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then $P \xrightarrow{\sim} P'$.

In particular, every stably free A-module of rank n is free.

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