EXISTENCE OF UNIMODULAR ELEMENTS IN A PROJECTIVE MODULE

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ABSTRACT. Let R be an affine algebra over an algebraically closed field of characteristic 0 with $\dim(R) = n$. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose I is an ideal of A of height n such that there are two surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus A^{n-1} \twoheadrightarrow I$. Assume that either (a) $k = 1$ and $n \geq 3$ or (b) k is arbitrary but $n \geq 4$ is even. Then P has a unimodular element (see 4.1, 4.3).

1. INTRODUCTION

Let R be a commutative Noetherian ring of dimension n. A classical result of Serre [Se] asserts that if P is a projective R-module of rank $> n$, then P has a unimodular element. However, as is shown by the example of projective module corresponding to the tangent bundle of an even dimensional real sphere, this result is best possible in general. Therefore, it is natural to ask under what conditions P has a unimodular element when $rank(P) = n$. In [Ra 1], Raja Sridharan asked the following question.

Question 1.1. *Let R be a ring of dimension n and P be a projective R-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset R$ is an ideal of height n such that I is generated by n elements. Does P has a unimodular element?*

Raja Sridharan proved that the answer to Question 1.1 is affirmative in certain cases (see [Ra 1, Theorems, 3, 5]) and "negative" in general.

Plumstead [P] generalized Serre's result and proved that if P is a projective $R[T]$-module of rank $> n$, then P has a unimodular element. Bhatwadekar and Roy [B-R 2] extended Plumstead's result and proved that projective $R[T_1, \ldots, T_r]$-modules of rank $> n$ have a unimodular element. Mandal [Ma] proved analogue of Plumstead that projective $R[T, T^{-1}]$-modules of rank $> n$ have a unimodular element. In another direction, Bhatwadekar and Roy [B-R 1] proved that projective modules over $D = R[T_1, T_2]/(T_1T_2)$ of rank $> n$ have a unimodular element. Later Wiemers [Wi] extended this result and proved that if $D = R[T_1, \ldots, T_r]/\mathcal{I}$ is a discrete Hodge algebra over R (here $\mathcal{I}$ is a monomial ideal), then projective D-modules of rank $> n$ have a unimodular element.

In view of results mentioned above, it is natural to ask the following question. Let A be either a polynomial ring over R or a Laurent polynomial ring over R or a discrete Hodge

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algebra over R. Let P be a projective A-module of rank n. Under what conditions P has a unimodular element? We will mention two such results.

Bhatwadekar and Raja Sridharan [B-RS 2, Theorem 3.4] proved: *Let R be a ring of dimension n containing an infinite field. Let P be a projective $R[T]$-module of rank n. Assume P_f has a unimodular element for some monic polynomial $f \in R[T]$. Then P has a unimodular element.*

Das and Raja Sridharan [D-RS, Theorem 3.4] proved: *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $R[T]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of $R[T]$ of height n such that I is generated by n elements. Assume further that P/TP has a unimodular element. Then P has a unimodular element.*

Note that when n is odd, the above result is not known. Further, the requirement in the hypothesis that P/TP has a unimodular element is indeed necessary, in view of negative answer of Question 1.1. Motivated by Bhatwadekar-Sridharan and Das-Sridharan, we prove the following results.

Theorem 1.2. *(see 3.1) Let R be a ring of dimension n containing an infinite field. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n. Assume $P_{f(T_k)}$ has a unimodular element for some monic polynomial $f(T_k) \in A$. Then P has a unimodular element.*

Theorem 1.3. *(see 3.3) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of A of height n such that I is a surjective image of $L \oplus A^{n-1}$. Further assume that $P/(T_1, \cdots, T_k)P$ has a unimodular element. Then P has a unimodular element.*

Theorem 1.4. *(see 3.4) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Assume that height of the Jacobson radical of R is ≥ 2. Let P be a projective $R[T, T^{-1}]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of $R[T, T^{-1}]$ of height n such that I is generated by n elements. Then P has a unimodular element.*

Theorem 1.5. *(see 3.6) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$ and P be a projective $D = R[T_1, T_2]/(T_1T_2)$-module of rank n with determinant L. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of D of height n such that I is a surjective image of $L \oplus D^{n-1}$. Further, assume that $P/(T_1, T_2)P$ has a unimodular element. Then P has a unimodular element.*

In view of above results, we end this section with the following question.

Question 1.6. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$.*

(1) *Let P be a projective $A = R[T, T^{-1}]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset A$ is an ideal of height n such that I is generated by n elements. Assume further that $P/(T-1)P$ has a unimodular element. Does P has a unimodular element?*
(2) *Let $D = R[T_1, \ldots, T_k]/\mathcal{I}$ be a discrete Hodge algebra over R and P be a projective D-module of rank n with determinant L. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of D of height n such that I is a surjective image of $L \oplus D^{n-1}$. Further, assume that $P/(T_1, \ldots, T_k)P$ has a unimodular element. Does P has a unimodular element?*

(3) *Generalize question (1) replacing* A *by* $R[X_1,\dots,X_r,Y_1^{\pm 1},\dots,Y_s^{\pm 1}]$ *and* $P/(T-1)P$ *by* $P/(X_1,\dots,X_r,Y_1-1,\dots,Y_s-1)P$.

2. Preliminaries

Assumptions. Throughout this paper, rings are assumed to be commutative Noetherian and projective modules are finitely generated and of constant rank. For a ring A, $\dim(A)$ and $\mathcal{J}(A)$ will denote the Krull dimension of A and the Jacobson radical of A respectively. In this section we state some results for later use.

Definition 2.1. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\phi : P \twoheadrightarrow R$ such that $\phi(p) = 1$. In particular, a row $(a_1, \cdots, a_n) \in R^n$ is unimodular (of length n) if there exist $b_1, \cdots, b_n$ in R such that $a_1b_1 + \cdots + a_nb_n = 1$. We write $Um(P)$ for the set of unimodular elements of P.

Theorem 2.2. [K-Z 1, Corollary 3.1] *Let* R *be a ring of dimension* n, $A = R[X_1, \cdots, X_m]$ *a polynomial ring over* R *and* P *be a projective* $A[T]$*-module of rank* n. *Assume that* P/TP *and* P_f *both contain a unimodular element for some monic polynomial* $f(T) \in A[T]$. *Then* P *has a unimodular element.*

The following result is a consequence of a result of Ravi Rao [R, Corollary 2.5] and Quillen's local-global principle [Q, Theorem 1].

Proposition 2.3. *Let* R *be a ring of dimension* n. *Suppose* $n!$ *is invertible in* R. *Then all stably free* $R[T]$*-modules of rank* n *are extended from* R.

Lemma 2.4. [D-RS, Lemma 3.3] *Let* R *be a ring and* $I = (a_1, \cdots, a_n)$ *be an ideal of* R, *where* n *is even. Let* $u, v \in R$ *be such that* $uv = 1$ *modulo* I. *Assume further that the unimodular row* $(v, a_1, \cdots, a_n)$ *is completable. Then there exists* $\sigma \in M_n(R)$ *with* $det(\sigma) = u$ *modulo* I *such that if* $(a_1, \cdots, a_n)\sigma = (b_1, \cdots, b_n)$, *then* $b_1, \cdots, b_n$ *generate* I.

Proposition 2.5. [D-RS, Proposition 4.2] *Let* R *be a ring containing* $\mathbb{Q}$ *of dimension* n *with* $ht(\mathcal{J}(R)) \geq 1$. *Then any stably free* $R(T)$*-module of rank* n *is free, where* $A(T)$ *is obtained from* $A[T]$ *by inverting all monic polynomials in* T.

Lemma 2.6. [Bh 2, Lemma 2.2] *Let* R *be a ring and* $I \subset R$ *be an ideal of height* r. *Let* P *and* Q *be two projective* R/I*-modules of rank* r *and let* $\alpha : P \twoheadrightarrow I/I^2$ *and* $\beta : Q \twoheadrightarrow I/I^2$ *be surjections. Let* $\psi : P \longrightarrow Q$ *be a homomorphism such that* $\beta \circ \psi = \alpha$. *Then* ψ *is an isomorphism.*

Definition 2.7. Let R be a ring and $A = R[T, T^{-1}]$. We say $f(T) \in R[T]$ is special monic if $f(T)$ is a monic polynomial with $f(0) = 1$. Write $\mathcal{A}$ for the ring obtained from A by inverting all special monic polynomials of $R[T]$. Then it is easy to see that $\dim(\mathcal{A}) = \dim(R)$.

Theorem 2.8. [K, 4.4, 4.6, 4.8, 4.9] *Let* R *be a ring containing* $\mathbb{Q}$ *of dimension* $n \geq 3$ *and* $A = R[T, T^{-1}]$. *Assume that* $ht(\mathcal{J}(R)) \geq 2$. *Let* $I \subset A$ *be an ideal of height* n *such that* I/I^2 *is generated by* n *elements and* $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ *be a local orientation of* I. *Let* P *be a projective* A*-module of rank* n *and* $\chi : A \xrightarrow{\sim} \wedge^n(P)$ *an isomorphism. Then following holds.*

(1) *Suppose that the image of* (I, ω_I) *is zero in* $E(A)$. *Then* ω_I *is a global orientation of* I, *i.e.,* ω_I *can be lifted to a surjection from* A^n *to* I.
(2) *Suppose that* $e(P,\chi) = (I, \omega_I)$ *in* $E(A)$. *Then there exists a surjection* $\alpha : P \twoheadrightarrow I$ *such that* ω_I *is induced from* (α, χ).
(3) P *has a unimodular element if and only if* $e(P,\chi) = 0$ *in* $E(A)$.
(4) *The canonical map* $E(A) \longrightarrow E(\mathcal{A})$ *is injective.*

Remark 2.9. Let R be a ring containing $\mathbb{Q}$ of dimension n with ht$(\mathcal{J}(R)) \geq 2$. Let $I \subset A = R[T, T^{-1}]$ be an ideal of height $n \geq 3$ and let $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Let $\theta \in GL_n(A/I)$ be such that $det(\theta) = \overline{f}$. Then $\omega_I \circ \theta$ is another local orientation of I, which we denote by $\overline{f}\omega_I$. On the other hand, if ω_I and $\widetilde{\omega}_I$ are two local orientations of I, then by (2.6), it is easy to see that $\widetilde{\omega}_I = \overline{f}\omega_I$ for some unit $\overline{f} \in A/I$.

The following result is due to Bhatwadekar and Raja Sridharan [B-RS 2, Theorem 4.5] in domain case. Same proof works in general (See [D, Proposition 5.3]).

Theorem 2.10. *Let* R *be an affine algebra over an algebraically closed field of characteristic* 0 *with* $dim(R) = n$. *Let* P *be a projective* $R[T]$*-module of rank* n *with trivial determinant. Suppose there is a surjection* $\alpha : P \twoheadrightarrow I$, *where* I *is an ideal of* $R[T]$ *of height* n *such that* I *is generated by* n *elements. Then* P *has a unimodular element.*

One can obtain the following result from [Su 2] and [Ra 2, Lemma 2.4],

Proposition 2.11. *Let* R *be an affine algebra over an algebraically closed field of characteristic* 0 *with* $dim(R) = n \geq 3$. *Let* L *be a projective* R*-module of rank one. Then the canonical map* $E(R, L) \longrightarrow E_0(R, L)$ *is an isomorphism.*

The following result is a consequence of local-global principle for Euler class groups [D-Z 2, Theorem 4.17].

Proposition 2.12. *Let* R *be a ring containing* Q *with* $dim(R) = n \geq 3$. *Let* P *be a projective* $R[T]$*-module of rank* n *with determinant* $L[T]$. *Suppose that* P/TP *as well as* $P \otimes R_m[T]$ *have a unimodular element for every maximal ideal* $\mathfrak{m}$ *of* R. *Then* P *has a unimodular element.*

Proof. Let $\chi : L[T] \stackrel{\sim}{\rightarrow} \wedge^n(P)$ be an isomorphism and $\alpha : P \twoheadrightarrow I$ be a surjection, where $I \subset R[T]$ is an ideal of height n. Let $e(P,\chi) = (I, \omega_I)$ in $E(R[T], L[T])$, where (I, ω_I) is obtained from the pair (α, χ). As $P \otimes R_m[T]$ has a unimodular element for every maximal ideal $\mathfrak{m}$ of R, the image of $e(P,\chi)$ in $\prod_{\mathfrak{m}} E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$ is trivial. By [D-Z 2, Theorem 4.17] the following sequence of groups

$$0 \longrightarrow E(R, L) \longrightarrow E(R[T], L[T]) \longrightarrow \prod_{\mathfrak{m}} E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$$

is exact. Therefore, there exists $(J, \omega_J) \in E(R, L)$ such that $(JR[T], \omega_J \otimes R[T]) = e(P,\chi)$ in $E(R[T], L[T])$. Then we have $e(P/TP, \chi \otimes R[T]/(T)) = (J, \omega_J)$ in $E(R, L)$. Since P/TP has a unimodular element, by [B-RS 1, Corollary 4.4], $(J, \omega_J) = 0$ in $E(R, L)$. Consequently ,

$e(P,\chi) = 0$ in $E(R[T], L[T])$. Hence, by [D-Z 2, Corollary 4.15], P has a unimodular element. $\square$

The following result is due to Swan [Sw, Lemma 1.3]. We give a proof due to Murthy.

Lemma 2.13. *Let R be a ring and P be a projective $R[T, T^{-1}]$-module. Let $f(T) \in R[T]$ be monic such that $P_{f(T)}$ is extended from R. Then P is extended from $R[T^{-1}]$.*

Proof. If degree of $f(T)$ is m, then $f(T)T^{-m} = 1 + T^{-1}g_1(T^{-1}) := g(T^{-1})$. Consider the following Cartesian diagram.

$$\begin{array}{ccc} R[T^{-1}] & \longrightarrow & R[T^{-1}]_{T^{-1}} (= R[T, T^{-1}]) \\ \downarrow & & \downarrow \\ R[T^{-1}]_{g(T^{-1})} & \longrightarrow & R[T, T^{-1}]_{g(T^{-1})} = R[T, T^{-1}]_{f(T)} \end{array}$$

Since $P_{f(T)}$ is extended from R, $P_{g(T^{-1})}$ is extended from $R[T^{-1}]_{g(T^{-1})}$. Using a standard patching argument, there exists a projective $R[T^{-1}]$-module Q such that $Q \otimes_{R[T^{-1}]} R[T, T^{-1}] \simeq P$. This completes the proof. $\square$

3. Main results

In this section, we shall prove our main results stated in the introduction.

Theorem 3.1. *Let R be a ring of dimension n containing an infinite field. Let P be a projective $R[T_1, \cdots, T_k]$-module of rank n. Assume $P_{f(T_k)}$ has a unimodular element for some monic polynomial $f(T_k)$ in the variable T_k. Then P has a unimodular element.*

Proof. When $k = 1$, we are done by [B-RS 2, Theorem 3.4]. Assume $k \geq 2$ and use induction on k. Since $(P/T_1P)_f$ has a unimodular element, by induction on k, P/T_1P has a unimodular element. Also $(P \otimes R(T_1)[T_2, \ldots, T_k])_f$ has a unimodular element and dim $(R(T_1)) = n$. So again by induction on k, $P \otimes R(T_1)[T_2, \ldots, T_k]$ has a unimodular element. Since P is finitely generated, there exists a monic polynomial $g \in R[T_1]$ such that P_g has a unimodular element. Applying (2.2), we get that P has a unimodular element. $\square$

The next result is due to Das-Sridharan [D-RS, Theorem 3.4] when $L = R$.

Proposition 3.2. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $R[T]$-module of rank n with determinant L. Suppose there are surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus R[T]^{n-1} \twoheadrightarrow I$, where I is an ideal of $R[T]$ of height n. Assume further that P/TP has a unimodular element. Then P has a unimodular element.*

Proof. Let $R_{red} = R/\mathbf{n}(R)$, where $\mathbf{n}(R)$ is the nil radical of R. It is easy to derive that P has a unimodular element if and only if $P \otimes R_{red}$ has a unimodular element. Therefore, without loss of generality we may assume that R is reduced. We divide the proof into two steps.

Step 1: Assume that L is extended from R. Since P/TP has a unimodular element, in view of (2.12), it is enough to prove that if $\mathfrak{m}$ is a maximal ideal of R, then $P \otimes R_{\mathfrak{m}}[T]$ has a unimodular element. Note that $P \otimes R_{\mathfrak{m}}[T]$ has trivial determinant and $\alpha \otimes R_{\mathfrak{m}}[T] : P \otimes R_{\mathfrak{m}}[T] \twoheadrightarrow IR_{\mathfrak{m}}[T]$ is a surjection. Using surjection $\phi \otimes R_{\mathfrak{m}}[T]$, we get that $IR_{\mathfrak{m}}[T]$ is generated by n elements. Applying [D-RS, Theorem 3.4] to the ring $R_{\mathfrak{m}}[T]$, it follows that $P \otimes R_{\mathfrak{m}}[T]$ has a unimodular element. Hence we are done.

Step 2: Assume that L is not necessarily extended from R. Since R is reduced, we can find a ring $R \hookrightarrow S \hookrightarrow Q(R)$ such that

(1) The projective $S[T]$-module $L \otimes_{R[T]} S[T]$ is extended from S,
(2) S is a finite R-module,
(3) the canonical map Spec$(S) \hookrightarrow$ Spec(R) is bijective, and
(4) for every $\mathfrak{P} \in$ Spec(S), the inclusion $R/(\mathfrak{P} \cap R) \longrightarrow S/\mathfrak{P}$ is birational.

Since $L \otimes_{R[T]} S[T]$ is extended from S, by Step 1, $P \otimes_{R[T]} S[T]$ has a unimodular element. Applying [Bh 1, Lemma 3.2], we conclude that P has a unimodular element. $\square$

The following result extends (3.2) to polynomial ring $R[T_1, \cdots, T_k]$.

Theorem 3.3. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose there are surjection $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus A^{n-1} \twoheadrightarrow I$, where $I \subset A$ is an ideal of height n. Assume further that $P/(T_1, \cdots, T_k)P$ has a unimodular element. Then P has a unimodular element.*

Proof. When $k = 1$, we are done by (3.2). Assume $k \geq 2$ and use induction on k. We use "bar" when we move modulo (T_k). Since R contains $\mathbb{Q}$, replacing X_k by $X_k - \lambda$ for some $\lambda \in \mathbb{Q}$, we may assume that ht $(\overline{I}) \geq n$.

Consider the surjection $\overline{\alpha} : \overline{P} \twoheadrightarrow \overline{I}$ induced from α. If ht$(\overline{I}) > n$, then $\overline{I} = R[T_1, \cdots, T_{k-1}]$ and hence $\overline{P}$ contains a unimodular element. Assume ht$(\overline{I}) = n$. Since $\overline{P}/(T_1, \cdots, T_{k-1})\overline{P} = P/(T_1, \cdots, T_k)P$ has a unimodular element and ϕ induces a surjection $\overline{\phi} : \overline{L \oplus A^{n-1}} \twoheadrightarrow \overline{I}$, by induction hypothesis, $\overline{P}$ has a unimodular element. By similar arguments, we get that $P/T_{k-1}P$ has a unimodular element.

Write $\mathcal{A} = R(T_k)[T_1, \cdots, T_{k-1}]$ and $P \otimes \mathcal{A} = \widehat{P}$. We have surjections $\alpha \otimes \mathcal{A} : \widehat{P} \twoheadrightarrow I\mathcal{A}$ and $\phi \otimes \mathcal{A} : (L \otimes \mathcal{A}) \oplus \mathcal{A}^{n-1} \twoheadrightarrow I\mathcal{A}$. If $I\mathcal{A} = \mathcal{A}$, then $\widehat{P}$ has a unimodular element. Assume ht$(I\mathcal{A}) = n$. Since $\widehat{P}/T_{k-1}\widehat{P} = P/T_{k-1}P \otimes \mathcal{A}$, we get that $\widehat{P}/T_{k-1}\widehat{P}$ and hence $\widehat{P}/(T_1, \cdots, T_{k-1})\widehat{P}$ has a unimodular element. By induction on k, $\widehat{P}$ has a unimodular element. So, there exists a monic polynomial $f \in R[T_k]$ such that P_f has a unimodular element. Applying (2.2), we get that P has a unimodular element. $\square$

Now we prove (1.4) mentioned in the introduction.

Theorem 3.4. *Let R be a ring containing $\mathbb{Q}$ of* **even** *dimension n with ht$(\mathcal{J}(R)) \geq 2$. Let P be a projective $R[T, T^{-1}]$-module of rank n with trivial determinant. Assume there exists a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset R[T, T^{-1}]$ is an ideal of height n such that I is generated by n elements. Then P has a unimodular element.*

Proof. Let $\chi : R[T,T^{-1}] \overset{\sim}{\to} \wedge^n(P)$ be an isomorphism. Let $e(P,\chi) = (I,\omega_I) \in E(R[T,T^{-1}])$ be obtained from the pair (α,χ). By (2.8 (3)), it is enough to prove that $e(P,\chi) = (I,w_I) = 0$ in $E(R[T,T^{-1}])$.

Suppose ω_I is given by $I = (g_1,\cdots,g_n) + I^2$. Also I is generated by n elements, say, $I = (f_1,\cdots,f_n)$. By (2.9), there exists $\tau \in GL_n(R[T,T^{-1}]/I)$ such that $(\overline{f}_1,\cdots,\overline{f}_n) = (\overline{g}_1,\cdots,\overline{g}_n)\tau$, where "bar" denotes reduction modulo I. Let $U,V \in R[T,T^{-1}]$ be such that $det(\tau) = \overline{U}$ and $UV = 1$ modulo I.

Claim: The unimodular row $(U,f_1,\cdots,f_n)$ over $R[T,T^{-1}])$ is completable.

First we show that the theorem follows from the claim. Since n is even, by (2.4), there exists $\eta \in M_n(R[T,T^{-1}])$ with $det(\eta) = \overline{V}$ such that if $(f_1,\cdots,f_n)\eta = (h_1,\cdots,h_n)$, then $I = (h_1,\cdots,h_n)$. Further, $(\overline{g}_1,\cdots,\overline{g}_n)\tau \circ \overline{\eta} = (\overline{h}_1,\cdots,\overline{h}_n)$. Note $\tau \circ \overline{\eta} \in SL_n(R[T,T^{-1}]/I)$.

Let $\mathcal{A}$ be the ring obtained from $R[T,T^{-1}]$ by inverting all special monic polynomials. Since $\dim(\mathcal{A}/I\mathcal{A}) = 0$, we have $SL_n(\mathcal{A}/I\mathcal{A}) = E_n(\mathcal{A}/I\mathcal{A})$. Let $\Theta \in SL_n(\mathcal{A})$ be a lift of $(\tau\circ\overline{\eta})\otimes\mathcal{A}$. Then $I\mathcal{A} = (h_1,\cdots,h_n)\Theta^{-1}$ and $(\overline{h}_1,\cdots,\overline{h}_n)\overline{\Theta^{-1}} = (\overline{g}_1,\cdots,\overline{g}_n)$. In other words, $(I\mathcal{A},\omega_I\otimes\mathcal{A}) = 0$ in $E(\mathcal{A})$. Since ht$(\mathcal{J}(R)) \geq 2$, by (2.8 (4)), the canonical map $E(R[T,T^{-1}]) \longrightarrow E(\mathcal{A})$ is injective. Therefore, $(I,\omega_I) = 0$ in $E(R[T,T^{-1}])$. This completes the proof. So we just need to prove the claim.

Proof of the claim: Let Q be the stably free $R[T,T^{-1}]$-module associated to the unimodular row $(U,f_1,\cdots,f_n)$. If S is the set of all monic polynomials in $R[T]$, then $S^{-1}R[T,T^{-1}] = R(T)$. Applying (2.5), we have P_S is free and hence there exists a monic polynomial $f \in R[T]$ such that P_f is free. In particular, P_f is extended from R. Hence by (2.13), there exists a projective $R[T^{-1}]$-module Q_1 such that $Q \simeq Q_1 \otimes_{R[T^{-1}]} R[T,T^{-1}]$. Since $(Q_1 \oplus R[T^{-1}])_{T^{-1}} \simeq Q\oplus R[T,T^{-1}] \simeq R[T,T^{-1}]^{n+1}$, by Quillen and Suslin [Q, Su 1], we get $Q_1\oplus R[T^{-1}] \simeq R[T^{-1}]^{n+1}$. By (2.3), Q_1 is extended from R, say, $Q_1 \simeq Q_2 \otimes_R R[T^{-1}]$. Since ht$(\mathcal{J}(R)) \geq 1$, by Bass [B], Q_2 is free. Hence Q is free, i.e. $(V,f_1,\cdots,f_n)$ is completable. This proves the claim. $\square$

Next result is the converse of (3.4).

Theorem 3.5. *Let R be a ring containing $\mathbb{Q}$ of* **even** *dimension n with ht$(\mathcal{J}(R) \geq 2$. Let P be a projective $R[T,T^{-1}]$-module of rank n with trivial determinant. Let $I \subset R[T,T^{-1}]$ be an ideal of height n which is generated by n elements. Suppose that P has a unimodular element. Then there exists a surjection $\alpha : P \twoheadrightarrow I$.*

Proof. Let $\chi : R[T,T^{-1}] \overset{\sim}{\to} \wedge^n(P)$ be an isomorphism and $e(P,\chi) \in E(R[T,T^{-1}])$ be obtained from the pair (P,χ). Since P has a unimodular element, by (2.8 (3)), $e(P,\chi) = 0$ in $E(R[T,T^{-1}])$.

Let $I = (f_1,\cdots,f_n)$ and ω_I be the orientation of I induced by $f_1,\cdots,f_n$. Then $(I,\omega_I) = 0 = e(P,\chi)$ in $E(R[T,T^{-1}])$. By (2.8 (2)), there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I,ω_I) is induced from (α,χ). $\square$

Theorem 3.6. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$ and $D = R[X,Y]/(XY)$. Let P be a projective D-module of rank n with determinant L. Suppose there are surjections $\alpha : P \twoheadrightarrow I$*

and $\phi : L \oplus D^{n-1} \twoheadrightarrow I$, where I is an ideal of D of height n. Assume further that $P/(X,Y)P$ has a unimodular element. Then P has a unimodular element.

Proof. Without loss of generality, we may assume that R is reduced. We continue to denote the images of X and Y in D by X and Y. We give proof in two steps.

Step 1: Assume L is extended from R. Let "bar" and "tilde" denote reductions modulo (Y) and (X) respectively. We get surjections $\overline{\alpha} : \overline{P} \twoheadrightarrow \overline{I}$ and $\widetilde{\alpha} : \widetilde{P} \twoheadrightarrow \widetilde{I}$ induced from α. Note that $\text{ht}(\overline{I}) \geq n$ and $\text{ht}(\widetilde{I}) \geq n$. If $\text{ht}(\overline{I}) > n$, then $\overline{I} = R[X]$ and $\overline{P}$ has a unimodular element. Assume $\text{ht}(\overline{I}) = n$. Since $\overline{D} = R[X]$ and $\overline{P}/X\overline{P} = P/(X,Y)P$ has a unimodular element, by (3.2), $\overline{P}$ has a unimodular element. Similarly, $\widetilde{P}$ has a unimodular element.

Let $p_1 \in Um(\overline{P})$. Then $\widetilde{p}_1 \in Um(\widetilde{\overline{P}})$. Since $det(\widetilde{P}) = \widetilde{L}$ is extended from R, it follows from [B-L-Ra, Theorem 5.2, Remark 5.3] that the natural map $Um(\widetilde{P}) \twoheadrightarrow Um(\widetilde{\overline{P}})$ is surjective. Therefore there exists $p_2 \in Um(\widetilde{P})$ such that $\overline{p}_2 = \widetilde{p}_1$. Since the following square of rings

$$\begin{array}{ccc} D & \longrightarrow & R[X] \\ \downarrow & & \downarrow \\ R[Y] & \longrightarrow & R \end{array}$$

is Cartesian with vertical maps surjective, $p_1 \in Um(\overline{P})$ and $p_2 \in Um(\widetilde{P})$ will patch up to give a unimodular element of P.

Step 2: Assume L is not necessarily extended from R. Since R is reduced, by [K-Z 2, Lemma 3.7], there exists a ring S such that

(1) $R \hookrightarrow S \hookrightarrow Q(R)$,
(2) S is a finite R-module,
(3) $R \hookrightarrow S$ is subintegral and
(4) $L \otimes_R S$ is extended from S.

Note that $R \hookrightarrow S$ is actually a finite subintegral extension. Hence, by [K-Z 2, Lemma 2.11], $R[X,Y]/(XY) \hookrightarrow S[X,Y]/(XY)$ is also subintegral. Since $L \otimes_R S$ is extended from S, by Step 1, $P \otimes S[X,Y]/(XY)$ has a unimodular element. Since R contains $\mathbb{Q}$, by [D-Z 1, Corollary 3.14], P has a unimodular element. $\square$

Using [B-L-Ra, Theorem 5.2, Remark 5.3] and following the proof of (3.6), we can prove the following result.

Theorem 3.7. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$ and $D = R[Y_1, \ldots, Y_m]/Y_1(Y_2, \ldots, Y_m)$. Let P be a projective D-module of rank n with determinant L. Suppose there are surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus D^{n-1} \twoheadrightarrow I$, where I is an ideal of D of height n. Assume further that $P/(Y_1, \ldots, Y_m)P$ has a unimodular element. Then P has a unimodular element.*

4. Applications

In this section, we give some applications of results proved earlier. When $L = R$, the following result is proved in [B-RS 2, Theorem 4.5].

Theorem 4.1. *Let R be an affine algebra over an algebraically closed field of characteristic 0 with dim$(R) = n \geq 3$. Let P be a projective $R[T]$-module of rank n with determinant L. Suppose I is an ideal of $R[T]$ of height n such that there are two surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus R[T]^{n-1} \twoheadrightarrow I$. Then P has a unimodular element.*

Proof. Let "bar" denote reduction modulo (T). Then α induces a surjection $\overline{\alpha} : \overline{P} \twoheadrightarrow I(0)$. We can assume ht$(I(0)) \geq n$. If ht$(I(0)) > n$, then $I(0) = R$ and $\overline{P}$ has a unimodular element. Assume ht$(I(0)) = n$ and fix an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(P)$. Let $e(\overline{P}, \overline{\chi}) = (I(0), \omega_{I(0)})$ in $E(R, \overline{L})$ be induced from $(\overline{\alpha}, \overline{\chi})$. Since $\overline{\phi} : \overline{L} \oplus R^{n-1} \twoheadrightarrow I(0)$ is a surjection, by (2.11), $e(\overline{P}, \overline{\chi}) = 0$ in $E(R, \overline{L})$. By [B-RS 1, Corollary 4.4], P/TP has a unimodular element. Rest of the proof is exactly as in (3.2) using (2.10). □

Corollary 4.2. *Let R be an affine algebra over an algebraically closed field of characteristic 0 with dim$(R) = n \geq 3$ and $D = R[X, Y_1 \ldots, Y_m]/X(Y_1, \ldots, Y_m)$. Let P be a projective D-module of rank n with determinant L. Suppose I is an ideal of D of height n such that there are two surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus D^{n-1} \twoheadrightarrow I$. Then P has a unimodular element.*

Proof. Follow the proof of (3.6) and use (4.1). □

Theorem 4.3. *Let R be an affine algebra over an algebraically closed field of characteristic 0 such that dim$(R) = n \geq 4$ is* **even***. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose I is an ideal of A of height n such that there are two surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus A^{n-1} \twoheadrightarrow I$. Then P has a unimodular element.*

Proof. By (2.11) and [B-RS 1, Corollary 4.4], we get $P/(T_1, \cdots, T_k)P$ has a unimodular element. By (3.3), we are done. □

In the following special case, we can remove the restriction on dimension in (3.3). This result improves [B-RS 2, Theorem 4.4].

Proposition 4.4. *Let R be a ring of dimension $n \geq 2$ containing $\mathbb{Q}$ and P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose $\mathcal{M} \subset A$ is a maximal ideal of height n such that there are two surjections $\alpha : P \twoheadrightarrow \mathcal{M}$ and $\phi : L \oplus A^{n-1} \twoheadrightarrow \mathcal{M}$. Then P has a unimodular element.*

Proof. Assume $k = 1$. If $\mathcal{M}$ contains a monic polynomial $f \in R[T_1]$, then P_f has a unimodular element. Using [B-RS 2, Theorem 3.4], P has a unimodular element. Assume $\mathcal{M}$ does not contain a monic polynomial. Since $T_1 \notin \mathcal{M}$, ideal $\mathcal{M} + (T_1) = R[T_1]$. By surjection α, we get P/T_1P has a unimodular element. Follow the proof of (3.2) and use [B-RS 2, Theorem 4.4] to conclude that P has a unimodular element. We are done when $k = 1$.

Assume $k \geq 2$ and use induction on k. For rest of the proof, follow the proof of (3.3). □

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