

Eigenvalues and Eigenvectors (Continued)

Theorem (Schur) V -tdvs over \mathbb{C} .

$T: V \rightarrow V$, linear operator.

There is an ordered basis B of V such that $M_B(T)$ is upper triangular.

Proof Let $f(x) = \det(T - xI)$ = characteristic polynomial of T .

Let $f(\lambda) = 0$, i.e., $\lambda \in \mathbb{C}$ is an eigenvalue of T .

Let $0 \neq v$ be an eigenvector for λ : $T(v) = \lambda v$.

Extend v to a basis of V : $(v = v_1, v_2, \dots, v_n)$.



CDEEP
IIT Bombay

Write $V = V_1 \oplus V_2$

$V_1 = \text{span}\{v_1\}$

$V_2 = \text{span}\{v_2, \dots, v_n\}$

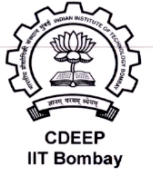
Define $T': V_2 \rightarrow V_2$ as follows: For $v \in V_2$

write $T(v) = v' + v''$, where $v' \in V_1$, $v'' \in V_2$.

Define $T'(v) = v''$ (check this is linear?).

By induction, there is a basis B' of V_2 such that $M_{B'}(T')$ is upper triangular. Take the basis $B = (v_1, B')$ of V . Then

$$M_B(T) = \begin{bmatrix} \lambda & x & \dots & \dots & x \\ 0 & \boxed{M_{B'}(T')} & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \text{ is upper triangular. } \square$$



CDEEP
IIT Bombay

Cayley-Hamilton Theorem

A - $n \times n$ matrix over \mathbb{C}

$$f(x) = \det(xI - A) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$$

$$\text{Then } f(A) \stackrel{\text{def}}{=} A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = 0.$$

Proof Consider $T_A: \mathbb{C}^n \rightarrow \mathbb{C}^n$. By Schur's theorem there

is an ordered basis $B = (v_1, v_2, \dots, v_n)$ such that

$M_B(T_A) = M = P^{-1}AP$ is upper triangular.

$$M = \begin{bmatrix} m_{11} & & & * \\ & m_{22} & & \\ & 0 & \dots & \\ & & & m_{nn} \end{bmatrix}$$



CDEEP
IIT Bombay

Now

$$f(x) = \det(xI - A) = \det(xI - M) \\ = (x - m_{11})(x - m_{22}) \dots (x - m_{nn})$$

Thus we can write $f(A) = (A - m_{nn}I) \dots (A - m_{11}I)$ (WHY?)

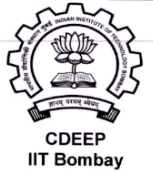
Now,

$$f(A)(v_1) = (A - m_{nn}I) \dots (A - m_{11}I)(v_1) = 0.$$

$$f(A)(v_2) = (A - m_{nn}I) \dots (A - m_{33}I)(A - m_{11}I)(A - m_{22}I)(v_2) \\ = 0 \text{ (WHY?)}$$

and so on.

Thus $f(A)(v_1) = f(A)(v_2) = \dots = f(A)(v_n) = 0$. Since $\{v_1, \dots, v_n\}$ is a basis, $f(A) = 0$. \square



CDEEP
IIT Bombay

Slide No: 165

Minimal polynomials

$F[x]$ = Set of all polynomials with coefficients in F .

$$p(x) = a_0 + a_1x + \dots + a_nx^n.$$

If $a_n \neq 0$ then $n = \deg(p)$.

If $a_n = 1$ then p is a monic polynomial.

$\deg(\text{nonzero constant}) = 0$, $\deg(0) = -1$.

Division algorithm in $F[x]$ $f, d \in F[x], d \neq 0$

There exist unique $q, r \in F[x]$ with
 $f = qd + r$, with $\deg r < \deg d$.

CDEEP
IIT Bombay

Slide No: 166

Uniqueness

$$p = q_1d + r_1 = q_2d + r_2,$$

$$\deg r_1, \deg r_2 < \deg d$$

$$(q_1 - q_2)d = r_2 - r_1$$

If both sides are non-zero then $\deg(\text{LHS}) \geq \deg d > \deg(\text{RHS})$,
 a contradiction.

So both sides are zero $\Rightarrow q_1 = q_2, r_1 = r_2$.

Existence

$$p = a_0 + \dots + a_nx^n$$

$$d = b_0 + \dots + b_mx^m, b_m \neq 0$$

Proof by induction on $\deg(p)$. Clear for $\deg(p) < \deg(d)$.
 $p = 0 \cdot d + p$.

CDEEP
IIT BombayCourse Name
MA 401 Linear AlgebraLecture No.
21Instructor's Name
Prof. Murali. K. SrinivasanCourse Name
MA 401 Linear AlgebraLecture No.
21Instructor's Name
Prof. Murali. K. Srinivasan

Slide No: 167

Now assume $n \geq m$:

$$p - \frac{a_n}{b_m} x^{n-m} d \text{ has degree } < n.$$

So, by induction,

$$p - \frac{a_n}{b_m} x^{n-m} d = qd + r, \quad \deg r < \deg d.$$

$$\text{or } p = \left(q + \frac{a_n}{b_m} x^{n-m} \right) d + r.$$

CDEEP
IIT BombayCourse Name
MA 401 Linear AlgebraLecture No.
21Instructor's Name
Prof. Murali. K. Srinivasan