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Eigenvalues and Eigenvectors (continued)

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Theorem (Schur) V - fdvs over $\mathbb{C}$.
$T: V \to V$, linear operator.

There is an ordered basis B of V such that $M_B(T)$ is upper triangular.

Proof Let $f(x) = \det(T - xI)$ = characteristic polynomial of T.
Let $f(\lambda)=0$, i.e., $\lambda \in \mathbb{C}$ is an eigenvalue of T.
Let $0 \neq v$ be an eigenvector for λ: $T(v) = \lambda v$.
Extend v to a basis of V: $(v = v_1, v_2, \ldots, v_n)$.

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Write $V = V_1 \oplus V_2$
$V_1 = \text{span}\{v_1\}$
$V_2 = \text{span}\{v_2, \ldots, v_n\}$

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Define $T': V_2 \to V_2$ as follows: For $v \in V_2$ write $T(v) = v' + v''$, where $v' \in V_1$, $v'' \in V_2$.
Define $T'(v) = v''$ (Check this is linear)?
By induction, there is a basis B' of V_2 such that $M_{B'}(T')$ is upper triangular. Take the basis $B = (v_1, B')$ of V. Then

$$M_B(T) = \left[\begin{array}{c|ccc} \lambda & x & \cdots & x \\ \hline 0 & & & \\ \vdots & & M_{B'}(T') & \\ 0 & & & \end{array}\right]$$

is upper triangular. □

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Cayley-Hamilton Theorem

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A - $n \times n$ matrix over $\mathbb{C}$

$$f(x) = \det(xI - A) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n.$$

Then $f(A) \stackrel{\text{def}}{=} A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0$.

Proof Consider $T_A: \mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Ax$. By Schur's theorem there is an ordered basis $B = (v_1, v_2, \ldots, v_n)$ of $\mathbb{C}^n$ such that

$$M_B(T_A) = M = P^{-1}AP \text{ is upper triangular.}$$

$$M = \begin{bmatrix} m_{11} & & * \\ & m_{22} & \\ 0 & & \ddots \; m_{nn} \end{bmatrix}$$

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Now

$$f(x) = \det(xI - A) = \det(xI - M) = (x - m_{11})(x - m_{22})\cdots(x - m_{nn})$$

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Thus we can write $f(A) = (A - m_{nn}I)\cdots(A - m_{11}I)$ (WHY?)

Now,
$f(A)(v_1) = (A - m_{nn}I)\cdots(A - m_{11}I)(v_1) = 0.$
$f(A)(v_2) = (A - m_{nn}I)\cdots(A - m_{33}I)(A - m_{11}I)(A - m_{22}I)(v_2) = 0$ (WHY?)
⋮ and so on.

Thus $f(A)(v_1) = f(A)(v_2) = \cdots = f(A)(v_n) = 0$. Since $\{v_1, \ldots, v_n\}$ is a basis, $f(A) = 0$. □

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Minimal polynomials

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$F[x]$ = Set of all polynomials with coefficients in F.

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$.

If $a_n \neq 0$ then $n = \deg(p)$.

If $a_n = 1$ then p is a monic polynomial.

$\deg(\text{nonzero Constant}) = 0$, $\deg(0) = -1$.

Division algorithm in $F[x]$ $\quad p, d \in F[x], d \neq 0$

There exist unique $q, r \in F[x]$ with

$p = qd + r$, with $\deg r < \deg d$.

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Uniqueness

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$p = q_1 d + r_1 = q_2 d + r_2$,

$\deg r_1, \deg r_2 < \deg d$

$(q_1 - q_2) d = r_2 - r_1$

If both sides are non-zero then $\deg(\text{LHS}) \geq \deg d > \deg(\text{RHS})$, a contradiction.

So both sides are zero $\Rightarrow q_1 = q_2,\ r_1 = r_2$.

Existence $\quad p = a_0 + \cdots + a_n x^n$

$d = b_0 + \cdots + b_m x^m$, $b_m \neq 0$

Proof by induction on $\deg(p)$. clear for $\deg(p) < \deg(d)$.

$p = 0 \cdot d + p$.

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Now assume $n \geq m$:

$p - \frac{a_n}{b_m} x^{n-m} d$ has degree $< n$.

So, by induction,

$$p - \frac{a_n}{b_m} x^{n-m} d = qd + r, \quad \deg r < \deg d.$$

or $p = \left(q + \frac{a_n}{b_m} x^{n-m}\right) d + r$.