

Circuits and connectivity in ordered matroids

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ABSTRACT

An ordered matroid is a partially ordered set equipped with a geometric closure operator. This structure abstracts the combinatorial properties of semimodular lattices in much the same way that matroids abstract the combinatorial properties of geometric lattices. Circuits in an ordered matroid are defined and cryptomorphisms relating the circuit family and the closure operator are given. An ordered sum operation for ordered matroids is introduced and a unique ordered decomposition theorem is proved.

1. INTRODUCTION

An ordered matroid is a partially ordered set equipped with a geometric closure operator. This structure, introduced by Faigle [4], abstracts the combinatorial properties of semi-modular lattices in much the same way that matroids abstract the combinatorial properties of geometric lattices. In his paper Faigle gave cryptomorphic descriptions of an ordered matroid in terms of its rank function, family of flats, family of hyperplanes, and family of independent sets and associated a dual ordered matroid with every ordered matroid. The object of this paper is to study circuits and 2-connectivity in ordered matroids. Alternative names that have been used for ordered matroids and related structures are pregeometries on posets [4] and F-geometries [6] (F-geometries are more general structures than ordered matroids. The B-independent sets of an ordered matroid satisfy the axioms of a F-geometry (see [4])).

In section 2 we present the basic theory of ordered matroids, as developed in Faigle [4], and give a few examples. Our treatment is slightly different from the one given in [4], but these can easily be shown to be equivalent.

In section 3 we define circuits and relate them to hyperplanes via duality. Unlike the case in matroid theory, the circuits of an ordered matroid are not the complementary subsets of the hyperplanes of the dual ordered matroid. A circuit turns out to be the set of maximal elements in the complementary subset of a hyperplane in the dual ordered matroid (this relation between circuits and hyperplanes motivated our definition of a circuit). Since in the case of matroids the underlying set is trivially ordered (i.e., an antichain) this relation reduces to the usual one between circuits and hyperplanes in a pair of dual matroids.

Next we relate the circuit family to the closure operator. The action of the closure operator of an ordered matroid is reducible to an ordered analog of the familiar process of "Completion of Circuits" in matroids [2]. This leads to a circuit set characterization of

ordered matroids.

Section 4 develops a theory of 2-connection in ordered matroids. An ordered sum operation for ordered matroids is introduced and a unique ordered decomposition theorem is proved.

2. ORDERED MATROIDS

$P = (P, \leq)$ denotes a finite poset. A subset $C \subseteq P$ is a *chain* if any two elements of C are comparable, and is an *antichain* if any two elements of C are incomparable. A subset $A \subseteq P$ is an *ideal* of P if

$$a \leq b \Rightarrow a \in A, \quad \forall b \in A, a \in P.$$

Dually, a subset $A \subseteq P$ is a *filter* of P if

$$a \leq b \Rightarrow b \in A, \quad \forall b \in P, a \in A.$$

$\mathcal{F}(P)$ denotes the lattice of ideals of P . For a subset $A \subseteq P$, $A^-(A^+)$ denotes the set of minimal (maximal) elements of A under the induced order and $[A]([A]^d)$ denotes the ideal (filter) generated by A , i.e.,

$$[A] = \{a \in P : \exists b \in A \text{ with } a \leq b\},$$

$$[A]^d = \{a \in P : \exists b \in A \text{ with } b \leq a\}.$$

The height of an element $a \in P$ will be denoted by $h(a)$, i.e., $h(a) = \max\{|C|-1 : C \subseteq P \text{ a chain, } a \in C^+\}$. Every poset P has an *order dual* $P^* = (P, \leq^*)$ which is defined as the set P together with the order relation

$$a \leq^* b \text{ in } P^* \Leftrightarrow b \leq a \text{ in } P.$$

So $(P^*)^* = P$ and the ideal generated by $A \subseteq P$ in P^* is $[A]^d$.

Let $(L, \wedge, \vee)$ denote a finite lattice. An element $a \neq 0 \in L$ is *join irreducible* if for all $A \subseteq L$, $a = \sup A$ implies $a = b$ for some $b \in A$. L is *semimodular* if for all $a, b \in L$, a covers $a \wedge b$ implies $a \vee b$ covers b .

A *closure operator* on P is a function $cl : \mathcal{F}(P) \rightarrow \mathcal{F}(P)$ satisfying the following:

$$A \subseteq cl(A), \quad \forall A \in \mathcal{F}(P), \quad (1)$$

$$A \subseteq B \Rightarrow cl(A) \subseteq cl(B), \quad \forall A, B \in \mathcal{F}(P), \quad (2)$$

$$cl(cl(A)) = cl(A), \quad \forall A \in \mathcal{F}(P). \quad (3)$$

An ideal $A \in \mathcal{F}(P)$ is a *flat* if $A = cl(A)$. It is easy to see that the intersection of flats is a flat. A closure operator on P is *geometric* if

$$a \in cl(A \cup b) - cl(A) \Rightarrow b \in cl(A \cup a) - cl(A), \quad \forall A \in \mathcal{F}(P), a, b \in (P - A)^-. \quad (4)$$

An *ordered matroid* G is a pair (P, cl) , where P is a finite poset and cl is a geometric closure operator on P . To every ordered matroid G on P we associate the lattice $L(P)$ of flats of G , ordered by inclusion. For any pair A, B of flats, $A \wedge B = A \cap B$ and $A \vee B = cl(A \cup B)$. For $A \in \mathcal{F}(P)$, define $r(A) = h(cl(A))$, where $h(cl(A))$ is the height of the flat $cl(A)$ in $L(P)$. $r(A)$ is the *rank* of A . Faigle [4] proved

Theorem 1.

- i) The lattice of flats of an ordered matroid is semimodular.
- ii) If L is a semimodular lattice with poset of join irreducibles P , then P together with $cl : \mathcal{F}(P) \rightarrow \mathcal{F}(P)$ given by $cl(A) = \{p \in P : p \leq \sup A\}$ is an ordered matroid and $\psi : L \rightarrow L(P)$ given by $\psi(a) = \{p \in P : p \leq a\}$ is a lattice isomorphism. ■

Theorem 2. A function $r : \mathcal{F}(P) \rightarrow \mathbb{N}$ is the rank function of some ordered matroid G on P if and only if r satisfies

$$r(\emptyset) = 0, \quad (5)$$

$$A \subseteq B \Rightarrow 0 \leq r(B) - r(A) \leq |B - A|, \quad \forall A, B \in \mathcal{F}(P), \quad (6)$$

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B), \quad \forall A, B \in \mathcal{F}(P). \quad (7)$$

Moreover, r determines G uniquely. ■

For future reference we note the following. Let G be an ordered matroid on P with closure operator cl and rank function r . Then

$$cl(A) = \{a \in P : r(A \cup [a]) = r(A)\}, \quad \forall A \in \mathcal{F}(P). \quad (8)$$

If the elements of P are pairwise incomparable, an ordered matroid on P is just a matroid. Below we mention a few other examples:

a) For any partial order P , $\mathcal{F}(P)$ is a distributive lattice and is the lattice of flats of the *free ordered matroid* on P . The following Proposition is from [5].

Proposition 3. *A finite semimodular lattice L is distributive if and only if the number of join irreducibles of L is equal to the height of L .*

Proof: If L is distributive with $P \subseteq L$ the poset of join irreducibles, then by Birkhoff's theorem [1], $L \cong \mathcal{F}(P)$. Clearly the height of L is equal to $|P|$.

Conversely, let L be a semimodular lattice, with poset of join irreducibles P , satisfying the hypotheses. By Theorem 1 there exists an ordered matroid G on P such that $L \cong L(P)$. By the hypotheses $r(P) = |P|$. From Theorem 2 it now follows that $r(A) = |A|$, for all $A \in \mathcal{F}(P)$. By (8), every $A \in \mathcal{F}(P)$ is a flat of G and hence $L \cong \mathcal{F}(P)$. So L is distributive. ■

b) Let I be a collection of ideals of P satisfying

- i) $\emptyset \in I$,
- ii) $A \subseteq B \in I, A \in \mathcal{F}(P) \Rightarrow A \in I$,
- iii) $A, B \in I, |B| > |A| \Rightarrow \exists a \in B - A \text{ s.t. } A \cup a \in I$.

For $A \in \mathcal{F}(P)$, define $r(A) = \max\{|B| : B \subseteq A, B \in I\}$. It is shown in [3] that r satisfies the rank axioms of an ordered matroid. This structure is called a *distributive supermatroid*.

c) A special case of the above example is the class of transversal matroids on posets : Let $\{A_1, \dots, A_n\}$ be a family of ideals of P and define $\mathcal{I}$ to be the collection of those ideals of P that are partial transversals of $\{A_1, A_2, \dots, A_n\}$.

d) Let P consist of disjoint chains whose elements are incomparable, and consider an integral polymatroid on the set of chains of P . For any $A \in \mathcal{F}(P)$, define $r(A)$ to be the vector rank in the polymatroid of the integral vector corresponding to A (indicating the number of elements of each chain that are contained in A).

e) Let P be the collection of indecomposable subgroups of a finite abelian group, partially ordered by inclusion. For any ideal $A \in \mathcal{F}(P)$, define $cl(A) = \{p \in P : p \subseteq \langle A \rangle\}$, where $\langle A \rangle$ is the subgroup generated by A .

f) Let P be a partially ordered set and F be a field. Let $\psi : P \rightarrow PG(n, F)$ be an order preserving map such that, for all $a \in P$,

$$\dim(\psi(a)) \leq \dim \left(\left\langle \bigcup_{b \in [a] - a} \psi(b) \right\rangle \right) + 1,$$

where $\langle A \rangle$ denotes the subspace generated by A . For any ideal $A \in \mathcal{F}(P)$, define $r(A) = \dim \left(\left\langle \bigcup_{b \in A} \psi(b) \right\rangle \right)$.

We now discuss the operations of restriction, contraction, and duality (see [4]). Let G be an ordered matroid on P (with rank function r and closure operator cl) and let $A \in \mathcal{F}(P)$. The restriction of G to A , denoted by $G|A$ or $G \setminus (P - A)$, is the ordered matroid on A whose closure operator cl_A is given by

$$cl_A(B) = cl(B) \cap A, \quad \forall B \in \mathcal{F}(A).$$

The contraction of G to $P - A$, denoted by $G \cdot (P - A)$ or G/A , is the ordered matroid on $P - A$ whose closure operator $cl_{P/A}$ is given by

$$cl_{P/A}(B) = cl(A \cup B) - A, \quad \forall B \in \mathcal{F}(P - A).$$

The rank functions of $G|A$ and $G \cdot (P - A)$ are given by $r_A(B) = r(B)$ and $r_{P/A}(B) = r(A \cup B) - r(A)$ respectively.

A minor of G is an ordered matroid of the form $(G \setminus (P - A)) / B$ for some $A \in \mathcal{F}(P)$, $B \in \mathcal{F}(A)$. Equivalently, a minor can be defined as an ordered matroid of the form $(G/B) \setminus (P - A)$ for some $A \in \mathcal{F}(P)$, $B \in \mathcal{F}(A)$. The dual of G is the ordered matroid G^* on P^* whose rank function r^* is given by

$$r^*(A) = |A| + r(P - A) - r(P), \quad \forall A \in \mathcal{F}(P^*). \quad (9)$$

$$(G^*)^* = G, (G/A)^* = G^* \setminus A \text{ for } A \in \mathcal{F}(P), \text{ and } (G \setminus A)^* = G^*/A \text{ for } A \in \mathcal{F}(P^*).$$

3. CIRCUITS AND HYPERPLANES

We first establish a relation between the closure operators of an ordered matroid and its dual. Let cl_1 and cl_2 be closure operators on P and P^* respectively. We say that cl_1 and cl_2 are *dual* if

$$a \in cl_1(A - a) \Leftrightarrow a \notin cl_2(P - A), \quad \forall A \in \mathcal{F}(P), a \in A^+.$$

Theorem 4. cl_1 and cl_2 are dual if and only if cl_1 and cl_2 are geometric and they define dual ordered matroids.

Proof: (If) From (8) and (9) we have

$$\begin{aligned} a \in cl_1(A - a) &\Leftrightarrow r(A) = r(A - a) \\ &\Leftrightarrow r^*(P - A) < r^*((P - A) \cup a) \\ &\Leftrightarrow a \notin cl_2(P - A). \end{aligned}$$

(only if) For $A \in \mathcal{F}(P)$ and $a, b \in (P - A)^-$ assume that $a \in cl_1(A \cup b) - cl_1(A)$. So $a \notin cl_2((P - A) - a - b)$ and $a \in cl_2((P - A) - a)$. Hence $cl_2((P - A) - a - b) \subsetneq cl_2((P - A) - a)$ and so $b \notin cl_2((P - A) - a - b)$. Thus $b \in cl_1(A \cup a)$ and cl_1 is geometric. Similarly

for cl_2 . Denote the rank functions of the matroids defined by cl_1 and cl_2 by r and r^* respectively. Let $a_1, \dots, a_n$ be a listing of the elements of A such that $A_i = \{a_1, \dots, a_i\} \in \mathcal{F}(P)$, for all $1 \leq i \leq n$. Set $A_0 = \emptyset$. We shall show that $|A| - r(A) = r^*(P) - r^*(P - A)$. This will prove that cl_1 and cl_2 define dual ordered matroids.

For $1 \leq i \leq n$, we have from (8)

$$\begin{aligned} r(A_i) = r(A_{i-1}) &\Leftrightarrow a_i \in cl_1(A_{i-1}) \\ &\Leftrightarrow a_i \notin cl_2((P - A_{i-1}) - a_i) \\ &\Leftrightarrow r^*(P - A_{i-1}) > r^*(P - A_i). \end{aligned}$$

So,

$$\begin{aligned} |A| - r(A) &= \left| \left\{ i : r(A_i) = r(A_{i-1}) \right\} \right| \\ &= \left| \left\{ i : r^*(P - A_{i-1}) > r^*(P - A_i) \right\} \right| \\ &= r^*(P) - r^*(P - A). \end{aligned}$$

■

Let G be an ordered matroid on P . A subset $C \subseteq P$ is called a *circuit* if

$$\emptyset \neq C, \tag{10}$$

$$C \text{ is an antichain of } P, \tag{11}$$

$$a \in cl([C] - a), \quad \forall a \in C, \tag{12}$$

$$a \notin cl([C] - a - b), \quad \forall a, b \in C, a \neq b. \tag{13}$$

Another way of defining a circuit is as follows: Let $C \subseteq P$ be a non-empty antichain. Consider the minor $(G[[C]]) \cdot C$. Since C is an antichain, this minor is a matroid. C is a circuit of G if and only if $(G[[C]]) \cdot C$ is a circuit in the matroid sense of the term.

A flat $A \in L(P)$ is called a *hyperplane* if

$$A \neq P, \quad (14)$$

$$(P - A)^- \subseteq cl(A \cup a), \quad \forall a \in (P - A)^-. \quad (15)$$

An equivalent definition of a hyperplane is as follows: Let $A \in \mathcal{F}(P)$ and consider the minor $(G \cdot (P - A))|(P - A)^-$, which is a matroid. A is a hyperplane if and only if $(G \cdot (P - A))|(P - A)^-$ is a rank 1 matroid consisting of a single parallel class. Faigle [4] has defined a hyperplane to be a meet irreducible element in the lattice of flats, i.e., $A \neq P$ and $A = A_1 \cap A_2$ for $A_1, A_2 \in L(P)$ implies $A = A_1$ or $A = A_2$. The definition we have used can be shown to be equivalent to that used in [4].

We denote the family of circuits of G by $\mathcal{C}$ and the family of hyperplanes by $\mathcal{H}$.

Proposition 5. $C \subseteq P$ is a circuit of G if and only if $P - [C]$ is a hyperplane of G^* .

Proof: We may assume that C is a non-empty antichain of P . From Theorem 4 we have

$$a \in cl([C] - a) \Leftrightarrow a \notin cl^*(P - [C]), \quad \forall a \in C,$$

$$a \notin cl([C] - a - b) \Leftrightarrow a \in cl^*((P - [C]) \cup b), \quad \forall a, b \in C, a \neq b,$$

(Here cl^* denotes the closure operator of G^*). The Proposition follows from the above equivalences. ■

Proposition 6. Let $A \in \mathcal{F}(P)$, $a \in P - A$, and $a \in cl(A)$. Then there exists a $C \in \mathcal{C}$ such that $a \in C \subseteq A \cup a$.

Proof: We first assume that $a \in (P - A)^-$. Let B be a minimal ideal with respect to the following property: $[a] - a \subseteq B \subseteq A$ and $a \in cl(B)$. We assert that $C = (B \cup a)^+$ is a circuit. Clearly C is an antichain of P and $a \in C \subseteq A \cup a$. We shall now verify (12) and (13) in the definition of a circuit.

First note that $[C] = B \cup a$ and therefore, by the defining property of B , $a \in cl([C] - a)$. By the minimality of B , $a \notin cl(B - b)$ for all $b \in C$, $b \neq a$. Therefore $a \in cl(B) - cl(B - b)$ and by the exchange axiom $b \in cl((B - b) \cup a) - cl(B - b)$. So $b \in cl((B - b) \cup a) = cl([C] - b)$ and (12) is verified.

By the minimality of B , $a \notin cl(B - b) = cl([C] - a - b)$ and $b \notin cl(B - b) = cl([C] - a - b)$ for all $b \in C$, $a \neq b$. Now consider $b_1, b_2 \in C$ such that $b_1 \neq b_2$ and $b_1, b_2 \neq a$. Assume that $b_1 \in cl([C] - b_1 - b_2) = cl((B \cup a) - b_1 - b_2)$. By the minimality of B , $b_1 \in cl((B \cup a) - b_1 - b_2) - cl(B - b_1 - b_2)$ and therefore by the exchange axiom $a \in cl(B - b_2) - cl(B - b_1 - b_2)$, which contradicts the minimality of B . Therefore $b_1 \notin cl([C] - b_1 - b_2)$ and (13) is verified.

In the remaining case $a \notin (P - A)^-$. Consider the ideal $A \cup ([a] - a)$. Since $a \in cl(A \cup ([a] - a))$ we get, from the previous case, a circuit $C \in \mathcal{C}$ such that $a \in C \subseteq A \cup [a]$. As C is an antichain we have $C \subseteq A \cup a$. ■

We note that the converse of the above proposition is not true: $A \in \mathcal{F}(P)$, $a \in P - A$ and $C \in \mathcal{C}$ with $a \in C \subseteq A \cup a$ do not necessarily imply $a \in cl(A)$ (See Theorem 8).

We now observe two properties of circuits in ordered matroids with a view to characterizing $\mathcal{C}$ (note that in case P is an antichain, (i) says that $\mathcal{C}$ is a clutter and (ii) reduces to the strong circuit elimination axiom in matroids).

Proposition 7.

- i) $C_1 \neq C_2, C_1 \subseteq [C_2] \Rightarrow C_1 \subseteq [C_2] - C_2, \quad \forall C_1, C_2 \in \mathcal{C}.$
- ii) $C_1 \neq C_2, a \in C_1 \cap C_2, b \in C_1 - [C_2]$
 $\Rightarrow \exists C_3 \in \mathcal{C} \quad s.t. \quad b \in C_3 \subseteq ([C_1 \cup C_2] - a), \quad \forall C_1, C_2 \in \mathcal{C}.$

Proof:

- i) Assume that $C_1 \cap C_2 \neq \emptyset$ and let $b \in C_1 \cap C_2$. Since $C_1 \neq C_2$, we have $b \in cl([C_1] - b) \subseteq cl([C_2] - b - b_1)$ for some $b_1 \in C_2 - C_1$. But this contradicts the fact that

C_2 is a circuit.

ii) Consider the ideal $[C_1 \cup C_2] - a - b$. Since C_2 is a circuit and $b \notin C_2$, $a \in cl([C_1 \cup C_2] - a - b)$. From the above and the fact that C_1 is a circuit it follows that $b \in cl([C_1 \cup C_2] - a - b)$. Therefore from Proposition 6 we get the existence of a $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq ([C_1 \cup C_2] - a)$. $\blacksquare$

Theorem 8. A family $\mathcal{C}$ of antichains of a poset P is the family of circuits of an ordered matroid G on P if and only if

$$\emptyset \notin \mathcal{C}, \quad (16)$$

$$C_1 \neq C_2, C_1 \subseteq [C_2] \Rightarrow C_1 \subseteq [C_2] - C_2, \quad \forall C_1, C_2 \in \mathcal{C}, \quad (17)$$

$$\begin{aligned} C_1 \neq C_2, a \in C_1 \cap C_2, b \in C_1 - [C_2] \\ \Rightarrow \exists C_3 \in \mathcal{C} \text{ s.t. } b \in C_3 \subseteq ([C_1 \cup C_2] - a), \quad \forall C_1, C_2 \in \mathcal{C}. \end{aligned} \quad (18)$$

Moreover $\mathcal{C}$ determines G uniquely.

Proof: Assume $\mathcal{C}$ satisfies (16), (17), and (18) and define $cl : \mathcal{F}(P) \rightarrow \mathcal{F}(P)$ as follows

$$cl(A) = A \cup B, \quad \text{where}$$

$$B = \left\{ a \in P - A : b \in [a] \cap (P - A) \Rightarrow \exists C \in \mathcal{C} \text{ with } b \in C \subseteq A \cup b \right\}, \quad \forall A \in \mathcal{F}(P).$$

We shall show that cl satisfies (1), (2), (3), and (4). From the definition of cl it is clear that $cl(A) \in \mathcal{F}(P)$ and that (1) and (2) are satisfied.

Let $a, b \in (P - A)^-$ and $a \in cl(A \cup b) - cl(A)$. So there exists a $C \in \mathcal{C}$ such that $a, b \in C \subseteq A \cup a \cup b$. Thus $b \in cl(A \cup a)$. To complete the proof that (4) holds it remains to show that $b \notin cl(A)$. This will follow from the next paragraph where we show that (3) holds.

Let $d \in (P - cl(A))^-$. Assume that $d \in cl(cl(A)) - cl(A)$, i.e., there exists a $C_1 \in \mathcal{C}$ such that $d \in C_1 \subseteq cl(A) \cup d$ and there exists a $e \in C_1$ such that $e \in cl(A) - A$. Since

$e \in cl(A)$, there exists a $C_2 \in \mathcal{C}$ such that $e \in C_2 \subseteq A \cup e$. Clearly $d \in C_1 - [C_2]$. By applying (18) we get a $C_3 \in \mathcal{C}$ such that $d \in C_3 \subseteq ([C_1 \cup C_2] - e)$. We note that $|[C_3] \cap (cl(A) - A)| < |[C_1] \cap (cl(A) - A)|$, and therefore by iterating the above process we will arrive at a $C \in \mathcal{C}$ such that $d \in C \subseteq A \cup d$. So $d \in cl(A)$, which is a contradiction and therefore $cl(cl(A)) = cl(A)$. Thus (P, cl) is an ordered matroid.

We have to show now that $\mathcal{C}$ equals the set of circuits of (P, cl) . Let $C \in \mathcal{C}$. Clearly C satisfies (10) and (11). By the definition of cl , $a \in cl([C] - a)$ for all $a \in C$. If $a \in cl([C] - a - b)$ for some $a, b \in C$, $a \neq b$ it will follow that there is a C_1 in $\mathcal{C}$ such that $a \in C_1 \subseteq [C] - b$. Then $C_1 \neq C$, $C_1 \subseteq [C]$ and $C_1 \not\subseteq [C] - C$. This violates (17) and therefore (12) and (13) are satisfied. So C is a circuit of (P, cl) .

Now let C be a circuit of (P, cl) and $a \in C$. So $a \in cl([C] - a)$ and thus there exists a $C_1 \in \mathcal{C}$ such that $a \in C_1 \subseteq [C]$. If $C_1 \neq C$, then there exists a $b \in C - C_1$ such that $C_1 \subseteq [C] - b$. Now, since C_1 is also a circuit of (P, cl) , $a \in cl([C_1] - a) \subseteq cl([C] - a - b)$ which contradicts the fact that C is a circuit of (P, cl) . So $C = C_1$. ■

By dualizing the above theorem we get the following:

Theorem 9. A family $\mathcal{H}$ of ideals of a poset P is the family of hyperplanes of an ordered matroid G on P if and only if

$$P \notin \mathcal{H}, \quad (19)$$

$$H_1 \neq H_2, H_1 \subseteq H_2 \Rightarrow (P - H_1)^- \subseteq H_2 \quad \forall H_1, H_2 \in \mathcal{H}, \quad (20)$$

$$\begin{aligned} H_1 \neq H_2, a \notin H_1 \cup H_2, a \in (P - (H_1 \cap H_2))^- , b \in (H_2 - H_1)^- \\ \Rightarrow \exists H_3 \in \mathcal{H} \text{ s.t. } b \notin H_3 \supseteq ((H_1 \cap H_2) \cup a), \quad \forall H_1, H_2 \in \mathcal{H}. \end{aligned} \quad (21)$$

Moreover $\mathcal{H}$ determines G uniquely. ■

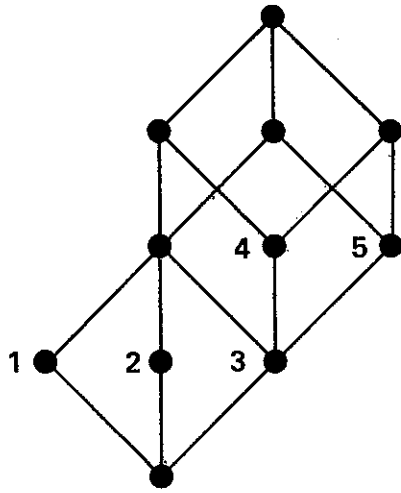


Figure 1

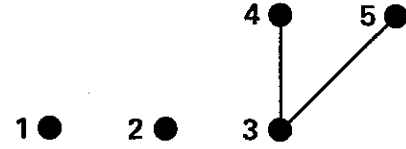


Figure 2

Example 1. Let L denote the semimodular lattice of Figure 1 and let P be the poset of join irreducibles of L (Figure 2).

Let G denote the ordered matroid (P, cl) , where cl is defined as in Theorem 1 (ii). From Figure 1 and Theorem 1 (ii) we see that the lattice $L(P)$ of flats of G is the following family of flats, ordered by inclusion.

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{3, 4\}, \{3, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Figures 3 and 4 depict $L(P^*)$ and P^* respectively.

Hyperplanes of $G = \{\{1\}, \{2\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{3, 4, 5\}\}.$

Circuits of $G^* = \{\{2, 3\}, \{1, 3\}, \{5\}, \{4\}, \{1, 2\}\}.$

Hyperplanes of $G^* = \{\{4, 5\}\}.$

Circuits of $G = \{\{1, 2, 3\}\}.$

Note that 4 and 5 are *loops* in G^* (single element circuits) and *isthmuses* in G (belonging to no circuits).

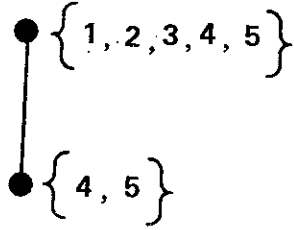


Figure 3

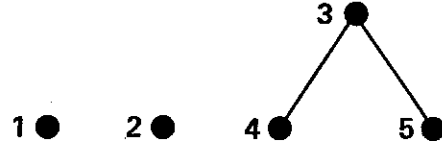


Figure 4

We now look at $\mathcal{C}_A$ and $\mathcal{C}_{P/A}$, the family of circuits of $G|A$ and $G \cdot (P - A)$ respectively.

Proposition 10. *Let $\mathcal{C}$ be the family of circuits of an ordered matroid G on P and let $A \in \mathcal{F}(P)$.*

- i) $\mathcal{C}_A = \{C \in \mathcal{C} : C \subseteq A\}$.
- ii) Let $\mathcal{C}' = \{C - A : C \in \mathcal{C} \text{ and } \emptyset \neq (C - A)\}$. Then

$$\mathcal{C}_{P/A} = \{B \in \mathcal{C}' : D \subseteq [B] \Rightarrow D \subseteq [B] - B, \quad \forall D \in \mathcal{C}', D \neq B\}. \quad (22)$$

Proof:

i) Clear.

ii) Let $B \in \mathcal{C}_{P/A}$ and $b \in B$. So $b \in cl((A \cup [B]) - b)$. By Proposition 6 there exists a $C \in \mathcal{C}$ such that $b \in C \subseteq A \cup [B]$. We assert that $B \subseteq C$. If not, then $C \subseteq (A \cup [B]) - b_1$ for some $b_1 \in B, b_1 \neq b$ and therefore $b \in cl((A \cup [B]) - b - b_1)$. But this contradicts the fact that $B \in \mathcal{C}_{P/A}$. Hence $B = C - A$ and therefore $B \in \mathcal{C}'$. We now show that $D \subseteq [B]$ implies $D \subseteq [B] - B$ for all $D \in \mathcal{C}', D \neq B$. Assume not and let $D \in \mathcal{C}'$ be such that $D \neq B, D \subseteq [B]$, and $d \in D \cap B$. So there exists a $d_1 \in B - D$ such that $d \in cl((A \cup [D]) - d) \subseteq cl((A \cup [B]) - d - d_1)$. But this contradicts the fact that $B \in \mathcal{C}_{P/A}$. We have now shown that $\mathcal{C}_{P/A}$ is contained in the R.H.S of (22).

Now let $B \in \mathcal{C}'$ be such that $D \subseteq [B]$ implies $D \subseteq [B] - B$ for all $D \in \mathcal{C}', D \neq B$. We have to show that $B \in \mathcal{C}_{P/A}$. (10) and (11) in the definition of a circuit are clearly satisfied. Also, it is easy to see that $b \in cl(A \cup [B]) - b$ for all $b \in B$ and therefore (12) is satisfied. Assume (13) fails and let $a, b \in B$ be such that $a \in cl((A \cup [B]) - a - b)$. By Proposition 6 there exists a $C \in \mathcal{C}$ such that $a \in C \subseteq (A \cup [B]) - b$. Put $D = C - A$. Then $D \in \mathcal{C}', D \subseteq [B]$ and $D \not\subseteq [B] - B$. This contradicts the choice of B . ■

We now introduce a weak circuit elimination axiom (23) and show that it is equivalent to (18). A little reflection shows that, in the case of matroids, (23) reduces to the usual circuit elimination axiom.

Proposition 11. *Let $\mathcal{C}$ be a family of antichains of a poset P satisfying (16), and (17). Then $\mathcal{C}$ satisfies (18) if and only if $\mathcal{C}$ satisfies the following*

$$C_1 \neq C_2, a \in C_1 \cap C_2 \Rightarrow \exists C_3, C_4 \in \mathcal{C} \text{ s.t.}$$

- i) $C_3, C_4 \subseteq ([C_1 \cup C_2] - a)$,
- ii) $C_3 \cap (C_1 - [C_2]) \neq \emptyset$,
- iii) $C_4 \cap (C_2 - [C_1]) \neq \emptyset$. (23)

Proof: First note that if a, C_1, C_2 are as in (23), then $C_1 - [C_2]$ and $C_2 - [C_1]$ are both non-empty since (17) is satisfied. So (17), and (18) imply (23). The proof of the other direction is by induction on $||C_1 \cup C_2||$ (the Proposition is clearly true if $||C_1 \cup C_2|| \leq 3$). Let $b \in C_1 - [C_2]$. By (23) there exists a $C_4 \in \mathcal{C}$ such that $C_4 \subseteq ([C_1 \cup C_2] - a)$ and $d \in C_4 \cap (C_2 - [C_1])$. We may assume that $b \notin C_4$, as otherwise we are done. Since $b \notin C_4$, we have $||C_4 \cup C_2|| < ||C_1 \cup C_2||$. Now $a \in C_2 - [C_4]$, $d \in C_2 \cap C_4$ and therefore by the induction hypothesis we get a $C_3 \subseteq ([C_2 \cup C_4] - d)$ such that $a \in C_3$. Since $d \notin C_3$, we have $||C_1 \cup C_3|| < ||C_1 \cup C_2||$. Now $a \in C_1 \cap C_3$, $b \in C_1 - [C_3]$ and therefore by again applying the induction hypothesis we get a $C_5 \subseteq ([C_1 \cup C_3] - a)$ such that $b \in C_5$. Clearly $C_5 \subseteq ([C_1 \cup C_2] - a)$. ■

Let G be an ordered matroid on P with circuit family $\mathcal{C}$. A subset $I \subseteq P$ is *independent* if

- i) I is an antichain of P ,
- ii) $a \notin cl([I] - a), \quad \forall a \in I$.

Independent sets as defined here are the maximal elements of the “independent sets” of [4], which are ideals of P .

Proposition 12. $I \subseteq P$ is independent if and only if I is an antichain and

$$C \subseteq [I] \Rightarrow C \subseteq [I] - I, \quad \forall C \in \mathcal{C}.$$

Proof: (if) Let $a \in I$. If $a \in cl([I] - a)$ then from Proposition 6 we have a circuit $C \in \mathcal{C}$ such that $a \in C \subseteq [I]$, which is a contradiction.

(only if) Let $C \subseteq [I]$ and assume that $a \in C \cap I$. Since C is a circuit $a \in cl([C] - a) \subseteq cl([I] - a)$, which is a contradiction. So $C \subseteq [I] - I$. ■

4. CONNECTIVITY

Let $(P_1, \dots, P_n)$ be a sequence of posets on pairwise disjoint sets. A poset P , whose underlying set is $\bigcup_{i=1}^n P_i$, is called an *ordered sum* of $(P_1, \dots, P_n)$ if

$$P_1 \cup \dots \cup P_i \in \mathcal{F}(P), \quad \forall i \in \{1, 2, \dots, n\}.$$

We note that P is not uniquely determined by $P_i, 1 \leq i \leq n$. To indicate this we write

$$P = P_1 \otimes_P \dots \otimes_P P_n.$$

For any ideal $A \in \mathcal{F}(P)$, $A \cap P_i$ is an ideal of P_i . Also, we note that

$$P^* = P_n^* \otimes_{P^*} \dots \otimes_{P^*} P_1^*.$$

Let $(G_1, \dots, G_n)$ be a sequence of ordered matroids on pairwise disjoint posets $P_1, P_2, \dots, P_n$ respectively. Denote the rank function, closure operator, circuit family, and family of independent sets of G_i by $r_i, cl_i, \mathcal{C}_i$, and $\mathcal{I}_i$ respectively. An ordered matroid G on P is called an *ordered sum* of $(G_1, \dots, G_n)$ if

$$P = P_1 \otimes_P \dots \otimes_P P_n, \quad (24)$$

$$r(A) = \sum_{i=1}^n r_i(A \cap P_i), \quad \forall A \in \mathcal{F}(P). \quad (25)$$

It is easily verified that r satisfies the rank axioms of Theorem 2. We write

$$G = G_1 \otimes_P \dots \otimes_P G_n.$$

It follows from (9) that

$$G^* = G_n^* \otimes_{P^*} \dots \otimes_{P^*} G_1^*. \quad (26)$$

Denote by $r, cl, \mathcal{C}$, and $\mathcal{I}$ the rank function, closure operator, circuit family, and family of independent sets of G . In the following Proposition $[X]^d([X])$ refers to the filter (ideal) generated by X in P .

Proposition 13.

- i) For $A \in \mathcal{F}(P)$, let $A_i = P_i \cap A$, $1 \leq i \leq n$ and let $B_1 = cl_1(A_1)$. Define B_i , $2 \leq i \leq n$ as follows

$$B_i = B_{i-1} \cup (cl_i(A_i) - [(P_i \cup \dots \cup P_{i-1}) - B_{i-1}]^d).$$

Then $cl(A) = B_n$.

- ii) $C \in \mathcal{C}$ if and only if $C \subseteq P_i$ and $C \in \mathcal{C}_i$ for some i .
iii) $I \in \mathcal{I}$ if and only if I is an antichain of P and $I \cap P_i \in \mathcal{I}_i$, for all $1 \leq i \leq n$.

Proof: i) From (8) and (25) it follows that $B_1 = P_1 \cap cl(A)$. Assume inductively that $B_{i-1} = (P_1 \cup \dots \cup P_{i-1}) \cap cl(A)$. As $cl(A)$ is an ideal of P we have $cl(A) \cap$

$([(P_1 \cup \dots \cup P_{i-1}) - B_{i-1}]^d) = \emptyset$. By (8) and (25) it now follows that $P_i \cap cl(A) = cl_i(A_i) - [(P_1 \cup \dots \cup P_{i-1}) - B_{i-1}]^d$. Therefore $B_i = (P_i \cup \dots \cup P_i) \cap cl(A)$. So $B_n = cl(A)$.

ii) (if) It can be easily verified from part i) that C satisfies (10), (11), (12), and (13).

(only if) Let $C \in \mathcal{C}$ and assume that there exist $a \in C \cap P_i$ and $b \in C \cap P_j$ for some $i \neq j$. From part i) it follows that $a \in cl([C] - a - b)$, which is a contradiction. So $C \subseteq P_i$ for some i . Again appealing to part i), it follows that $C \in \mathcal{C}_i$.

iii) Follows from part i). ■

A *separator* of an ordered matroid G on P is an ideal $A \in \mathcal{F}(P)$ such that

$$G = G|A \otimes_P G \cdot (P - A). \quad (27)$$

In the following Proposition $[X]$ refers to the ideal generated by X in P .

Proposition 14. *The following are equivalent for $A \in \mathcal{F}(P)$:*

i) A is a separator.

ii) The rank function $r_{P/A}$ of $G \cdot (P - A)$ is given by

$$r_{P/A}(B) = r([B]) - r([B] - B), \quad \forall B \in \mathcal{F}(P - A).$$

iii) A is a modular ideal, i.e.,

$$r(A) + r(B) = r(A \cup B) + r(A \cap B), \quad \forall B \in \mathcal{F}(P).$$

iv) $C \in \mathcal{C}$ implies $C \subseteq A$ or $C \subseteq P - A$.

Proof: i) $\Rightarrow$ ii) Let $B \in \mathcal{F}(P - A)$. From (25) and (27) we have

$$r([B]) = r([B] - B) + r_{P/A}(B).$$

ii) $\Rightarrow$ iii) Let $B \in \mathcal{F}(P)$. Then

$$\begin{aligned} r(A \cup B) - r(A) &= r_{P/A}(B - A) \\ &= r([B - A]) - r([B - A] - (B - A)) \\ &\geq r(B) - r(A \cap B). \end{aligned}$$

The last inequality follows from the submodularity of r . So $r(A \cup B) + r(A \cap B) \geq r(A) + r(B)$. Again by the submodularity of r , $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$. Hence, A is modular.

iii) $\Rightarrow$ iv) Let $B \in \mathcal{F}(P)$. Then

$$r(B) = r(A \cap B) + (r(A \cup B) - r(A)) = r(A \cap B) + r_{P/A}(B - A).$$

Now assume that $C \in \mathcal{C}$ is a circuit such that $a \in A \cap C$ and $b \in C - A$. Then

$$\begin{aligned} r([C]) &= r(A \cap [C]) + r_{P/A}([C] - A), \\ r([C] - a) &= r((A \cap [C]) - a) + r_{P/A}([C] - A), \\ r([C] - b) &= r(A \cap [C]) + r_{P/A}([C] - A - b). \end{aligned}$$

Since C is a circuit we have, from (8) and (12), $r([C]) = r([C] - a) = r([C] - b)$. From the above we get

$$r([C] - a - b) = r((A \cap [C]) - a) + r_{P/A}([C] - A - b) = r([C]).$$

This implies that $a \in cl([C] - a - b)$, which contradicts the fact that C is a circuit.

iv) $\Rightarrow$ i) From Proposition 10 we have $\mathcal{C}_{P/A} = \{C \in \mathcal{C} : C \subseteq P - A\}$. Since the circuit family uniquely determines the ordered matroid, it follows from Proposition 13 ii) that $G = G|A \otimes_P G \cdot (P - A)$. ■

Proposition 15. Let G be an ordered matroid on P and let $A \in \mathcal{F}(P)$ be a separator of G .

i) $\emptyset$ and P are separators. The union and intersection of separators is again a separator. Hence the separators form a distributive sublattice of $\mathcal{F}(P)$.

ii) $B \in \mathcal{F}(A)$ is a separator of $G|A$ if and only if B is a separator of G .

iii) $B \in \mathcal{F}(P - A)$ is a separator of $G \cdot (P - A)$ if and only if $A \cup B$ is a separator of G .

Proof: i) follows easily from Proposition 14 iv). ii) and iii) follow from Proposition 14 iv) and Proposition 10. ■

An ordered matroid which only possesses the trivial separators $\emptyset$ and P is called *connected*.

Theorem 16. Let G be an ordered matroid on P . Then G admits an ordered decomposition $G = G_1 \otimes_P \dots \otimes_P G_n$ where G_i is a connected ordered matroid (on P_i), for all i . Moreover, if $G = G'_1 \otimes_P \dots \otimes_P G'_m$ is another ordered decomposition of G with G'_i a connected ordered matroid on Q_i , for all i , then $m = n$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that $P_i = Q_{\sigma(i)}$ and $G_i = G'_{\sigma(i)}$, for all i .

Proof: Let $\emptyset = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = P$ be a unrefinable chain in the lattice of separators of G . Denote this chain by S . Put $P_i = A_i - A_{i-1}$ and $G_i = (G|A_i) \cdot (A_i - A_{i-1})$, for $1 \leq i \leq n$. Since A_i is a separator of G , for all i , it follows from Proposition 14 iv) that for every circuit C of G , $C \subseteq A_i - A_{i-1}$ for some $i \in \{1, \dots, n\}$. Therefore, by Proposition 10, the circuits of G_i are the circuits of G contained in $A_i - A_{i-1}$. Since the circuit family uniquely determines the ordered matroid, it follows from Proposition 13 ii) that $G = G_1 \otimes_P \dots \otimes_P G_n$. From Proposition 14 ii) and iii) and the unrefinability of S we get that G_i is connected, for all i .

Let $G = G'_1 \otimes_P \dots \otimes_P G'_m$ be another ordered decomposition of G with G'_i a connected ordered matroid on Q_i , for all i . Put $B_i = Q_1 \cup \dots \cup Q_i$, for all $1 \leq i \leq m$. Then, by using arguments similar to those in the preceding paragraph, it can be easily shown that B_i is a separator, for all i , and that $\emptyset = B_0 \subseteq B_1 \subseteq \dots \subseteq B_m = P$ is a unrefinable chain of separators of G . Denote this chain by S' . From the unrefinability of S and S' and from Proposition 15 i) it follows by a standard argument that $m = n$ and $P_i = Q_{\sigma(i)}$, for all i , for some permutation σ of $\{1, \dots, n\}$. $G_i = G'_{\sigma(i)}$ because they have the same circuits. ■

The ordered matroids $G_1, \dots, G_n$ in the ordered decomposition above are called the *components* of G .

Example 2.

a) Let G be the free ordered matroid on P . Then G has no circuits and the lattice of separators of G is $\mathcal{F}(P)$ itself. The components of G are therefore the rank 1 matroids on the elements of P and any linear extension of P gives rise to an ordered decomposition of G into connected components.

b) Let P be the poset of Figure 2 and let G be the ordered matroid on P of Example 1. The lattice of separators of G is shown in Figure 5.

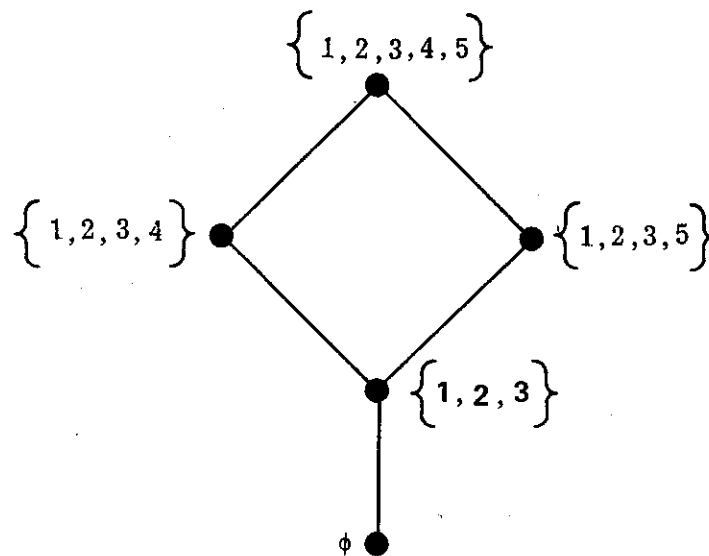


Figure 5

The components of G are $G_1 = G|_{\{1, 2, 3\}}$, $G_2 = (G|_{\{1, 2, 3, 4\}}) \cdot \{4\}$ and $G_3 = G \cdot \{5\}$. $G = G_1 \otimes_P G_2 \otimes_P G_3$ and $G = G_1 \otimes_P G_3 \otimes_P G_2$ are all the ordered decompositions of G into connected components.

Circuits in ordered matroids do not enjoy the transitivity property as the following example shows.

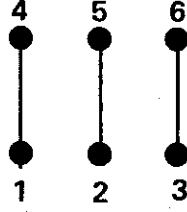


Figure 6

Example 3. Let P be the poset of Figure 6.

Let G be the ordered matroid on P whose circuit family is given by

$$\mathcal{C} = \{\{2, 4\}, \{3, 4\}, \{1, 5\}, \{3, 5\}, \{1, 6\}, \{2, 6\}, \{1, 2, 3\}\}.$$

2, 4, and 6 are mutually incomparable in P and there are circuits through 4 and 2 and through 2 and 6. However, G contains no circuit through 4 and 6.

Proposition 17. Let G be an ordered matroid on P and let $a, b \in P$, $a \neq b$. Then there is a circuit containing a and b if and only if there is a cocircuit containing a and b . Hence A is a separator of G if and only if $P - A$ is a separator of G^* .

Proof: We may assume that a and b are incomparable. Let C be a circuit of G containing a and b . We shall produce a hyperplane A of G such that $a, b \in (P - A)^-$. By Proposition 5, $(P - A)^-$ will then be a circuit of G^* .

Let $A \supseteq cl([C] - a - b)$ be a flat of G such that $a, b \in (P - A)^-$ (as C is a circuit of G we can always set $A = cl([C] - a - b)$). If A is a hyperplane of G we are done. Otherwise, consider $cl(A \cup a)$. Since $A \supseteq cl([C] - a - b)$, $b \in cl(A \cup a)$. Since A is not a hyperplane of G , $d \notin cl(A \cup a)$ for some $d \in (P - A)^-$. Now $cl(A \cup d) \supseteq cl([C] - a - b)$

and $a, b \in (P - cl(A \cup d))^-$. Continuing the above process we get a hyperplane of G satisfying the required conditions. ■

The following Theorem follows from Proposition 17 and (26).

Theorem 18. *Let G be an ordered matroid on P . If $G = G_1 \otimes_P \dots \otimes_P G_n$ is an ordered decomposition of G into connected components, then $G^* = G_n^* \otimes_{P^*} \dots \otimes_{P^*} G_1^*$ is an ordered decomposition of G^* into connected components. Hence the components of G^* are the duals of the components of G .* ■

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