Notes on the Vershik-Okounkov approach to the representation theory of the symmetric groups

1 Introduction

These notes[1] contain an expository account of the beautiful new approach to the complex finite dimensional irreducible representations of the symmetric group, developed by Anatoly Vershik and Andrei Okounkov [**4**].

The main task of any representation theory of the symmetric groups is to explain the appearence of Young diagrams and Young tableaux in the theory in a natural way. The traditional approach (see [**1, 2, 3**]) is indirect and rests upon nontrivial auxiliary constructions. The presence of Young tableaux in the theory is justified only in the end, after the proof of the branching rule. The main steps of the Vershik-Okounkov approach are as follows:

(i) A direct elementary argument shows that branching from S_n to S_{n-1} is simple, i.e., multiplicity free.

(ii) Consider an irreducible S_n-module V. Since the branching is simple the decomposition of V into irreducible S_{n-1}-modules is canonical. Each of these modules, in turn, decompose canonically into irreducible S_{n-2}-modules. Iterating this construction we get a canonical decomposition of V into irreducible S_1-modules, i.e., one dimensional subspaces. Thus there is a canonical basis of V, determined upto scalars, and called the the Gelfand-Tsetlin basis (GZ-basis).

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(iii) Let Z_n denote the center of $\mathbb{C}S_n$. The Gelfand-Tsetlin algebra (GZ-algebra), denoted GZ_n, is defined to be the (commutative) subalgebra of $\mathbb{C}S_n$ generated by $Z_1 \cup Z_2 \cup \cdots \cup Z_n$. It is shown that GZ_n consists of all elements in $\mathbb{C}S_n$ that act diagonally in the GZ-basis in every irreducible representation. Thus GZ_n is a maximal commutative subalgebra of $\mathbb{C}S_n$ and its dimension is equal to the sum of dimensions of the distinct inequivalent irreducible S_n-modules. It follows that any vector in the GZ-basis (in any irreducible representation) is uniquely determined by the eigenvalues of the elements of the GZ-algebra on this vector.

(iv) For $i = 1, 2, \ldots, n$ define $X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}S_n$. The X_i's are called the Young-Jucys-Murphy elements (YJM-elements) and it is shown that they generate the GZ-algebra. To a GZ-vector v (i.e., an element of the GZ-basis in some irreducible representation) we associate the tuple $\alpha(v) = (a_1, a_2, \ldots, a_n)$, where $a_i =$ eigenvalue of X_i on v and we let

$$\text{Spec }(n) = \{\alpha(v) \ : \ v \text{ is a GZ-vector}\}.$$

It follows from step (iii) that, for GZ-vectors u and v, $u = v$ iff $\alpha(u) = \alpha(v)$ and thus $\#\text{Spec }(n)$ is equal to the sum of the dimensions of the distinct irreducible inequivalent representations of S_n.

(v) In the final step we construct a bijection between Spec (n) and Tab (n) (= set of all standard Young tableaux on the letters $\{1, 2, \ldots, n\}$) such that tuples in Spec (n) whose GZ-vectors belong to the same irreducible representation go to standard Young tableaux of the same shape. This step is carried out inductively using an elementary analysis of the commutation relation

$$s_i^2 = 1, \ \ X_i X_{i+1} = X_{i+1} X_i, \ \ s_i X_i + 1 = X_{i+1} s_i.$$

where $s_i =$ the Coxeter generator $(i, i+1)$.

For more detailed explanations about the underlying philosophy of this method as well as many remarks on extensions and analogies with Lie theory, see the original papers [**4, 5**]. Our exposition closely follows [**4**] (we have added a few details here and there). In these notes we have restricted ourselves to only the most basic result, namely that the branching graph of the chain of symmetric groups is the Young graph. For the further development of the theory, see [**4, 5**].

Notation

A *partition* is a finite sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ of weakly decreasing positive integers. We say that μ has r parts and that μ is a partiton of $n = \mu_1 + \cdots + \mu_r$. Let $\mathcal{P}(n)$ denote the set of all partitions of n and let $\mathcal{P}_1(n)$ denote the set of all pairs (μ, i), where μ is a partition of n and i is a part in μ. A part is nontrivial if it is ≥ 2 and we let $\#\mu$ denote the sum of the nontrivial parts of μ. The number of all partitions of n is denoted $p(n)$.

Conjugacy classes in S_n are parametrized by partitions of n. Given a partition $\mu \in \mathcal{P}(n)$, let $c_\mu \in \mathbb{C}S_n$ denote the sum of all partitions in S_n with cycle type μ. It is well known that $\{c_\mu : \mu \in \mathcal{P}(n)\}$ is a basis of the center of $\mathbb{C}S_n$.

For a permutation $s \in S_n$ we denote by $\ell(s)$ the number of inversions in s. It is well known that the s can be written as a product of $\ell(s)$ Coxeter transpositions $s_i = (i, i{+}1),\ i = 1, 2, \ldots, n{-}1$ and that s cannot be written as a product of fewer Coxeter transpositions.

All our algebras are finite dimensional, over $\mathbb{C}$, and have units. Subalgebras contain the unit, and algebra homomorphisms preserve units. Given elements or subalgebras $A_1, A_2, \ldots, A_n$ of an algebra A we denote by $\langle A_1, A_2, \ldots, A_n \rangle$ the subalgebra of A generated by $A_1 \cup A_2 \cup \cdots \cup A_n$.

2 Gelfand-Tsetlin bases and Gelfand-Tsetlin algebras

In this section we shall introduce Gelfand-Tsetlin bases (GZ bases) and Gelfand-Tsetlin algebras (GZ algebras) for an inductive chain of finite groups with simple branching.

Let

$$\{1\} = G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \tag{1}$$

be an inductive chain of finite groups. Denote by $G_n^\wedge$ the set of equivalence classes of finite dimensional complex irreducible representations of G_n. Denote by V^λ the irreducible G_n module corresponding to $\lambda \in G_n^\wedge$. Define the following directed graph, called the *branching multigraph or Bratelli diagram* of this chain: its vertices are the elements of the set

$$\coprod_{n\geq 1} G_n^\wedge \quad \text{(disjoint union)}$$

and two vertices μ, λ are joined by k directed edges from μ to λ whenever $\mu \in G_{n-1}^\wedge$ and $\lambda \in G_n^\wedge$ for some n, and the multiplicity of μ in the restriction of λ to G_{n-1} is k. We call $G_n^\wedge$ the n^{th} *level* of the branching multigraph. We write $\mu \nearrow \lambda$ if there is an edge from μ to λ.

For the rest of this section assume that the branching multigraph defined above is actually a graph, i.e., the multiplicities of all restrictions are 0 or 1. We say that the *branching or multiplicities are simple*.

Consider the G_n-module V^λ, where $\lambda \in G_n^\wedge$. Since the branching is simple, the decomposition

$$V^\lambda = \bigoplus_\mu V^\mu,$$

where the sum is over all $\mu \in G^{\wedge}_{n-1}$ with $\mu \nearrow \lambda$, is canonical. Iterating this decomposition we obtain a canonical decomposition of V^λ into irreducible G_1-modules, i.e., one-dimensional subspaces,

$$V^\lambda = \bigoplus_T V_T, \tag{2}$$

where the sum is over all possible chains

$$T \;=\; \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n, \tag{3}$$

with $\lambda_i \in G^{\wedge}_i$ and $\lambda_n = \lambda$. By choosing a nonzero vector v_T in each one-dimensional space V_T above, we obtain a basis $\{v_T\}$ of V^λ, called the *Gelfand-Tsetlin basis (GZ-basis)*. From here till the end of Section 6 we consider the GZ-basis as fixed. In Section 7 we shall discuss an appropriate choice of these nonzero vectors (in the case of symmetric groups) so that all irreducibles are realized over $\mathbb{Q}$. By the definition of v_T, we have

$$\mathbb{C}[G_i] \cdot v_T \;=\; V^{\lambda_i}, \quad i = 1, 2, \ldots, n.$$

Also note that chains in (3) are in bijection with directed paths in the branching graph from the unique element λ_1 of $G^{\wedge}_1$ to λ.

We have identified a canonical basis (upto scalars), namely the GZ-basis, in each irreducible representation of G_n. A natural question at this point is to identify those elements of $\mathbb{C}[G_n]$ that act diagonally in this basis (in every irreducible representation). In other words, consider the algebra isomorphism

$$\mathbb{C}[G_n] \;\cong\; \bigoplus_{\lambda \in G^{\wedge}_n} \mathrm{End}(V^\lambda), \tag{4}$$

given by

$$g \mapsto (V^\lambda \xrightarrow{g} V^\lambda \;:\; \lambda \in G^{\wedge}_n), \quad g \in G_n.$$

Let $\mathrm{D}(V^\lambda)$ consists of all operators on V^λ diagonal in the GZ-basis of V^λ. Our question can now be stated as: what is the image under the isomorphism (4) of the subalgebra $\bigoplus_{\lambda \in G^{\wedge}_n} \mathrm{D}(V^\lambda)$ of $\bigoplus_{\lambda \in G^{\wedge}_n} \mathrm{End}(V^\lambda)$.

Let Z_n denote the center of the algebra $\mathbb{C}[G_n]$ and set $GZ_n = \langle Z_1, Z_2, \ldots, Z_n \rangle$. It is easy to see that GZ_n is commutative subalgebra of $\mathbb{C}S_n$. We call GZ_n the *Gelfand-Tsetlin algebra (GZ-algebra)* of the inductive family of group algebras.

Theorem 2.1. *GZ_n is the image of $\bigoplus_{\lambda \in G_n^\wedge} D(V^\lambda)$ under the isomorphism (4) above, i.e., GZ_n consists of all elements of $\mathbb{C}[G_n]$ that act diagonally in the GZ-basis in every irreducible representation of G_n. Thus GZ_n is a maximal commutative subalgebra of $\mathbb{C}[G_n]$ and its dimension is equal to $\sum_{\lambda \in G_n^\wedge}$ dim λ.*

Proof Consider the chain T from (3) above. For $i = 1, 2, \ldots, n$, let $p_{\lambda_i} \in Z_i$ denote the central idempotent corresponding to the representation $\lambda_i \in G_i^\wedge$. Define $p_T \in GZ_n$ by

$$p_T = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

A little reflection shows that the image of p_T under the isomorphism (4) is $(f_\mu \ : \ \mu \in G_n^\wedge)$, where $f_\mu = 0$, if $\mu \neq \lambda$ and f_λ is the projection on V_T (with respect to the decomposition (2) of V^λ.)

It follows that the image of GZ_n under (4) includes $\bigoplus_{\lambda \in G_n^\wedge} \mathrm{D}(V^\lambda)$, which is a maximal commutative subalgebra of $\bigoplus_{\lambda \in G_n^\wedge} \mathrm{End}(V^\lambda)$. Since GZ_n is commutative, the result follows. $\square$.

By a *GZ-vector* of G_n (determined upto scalars) we mean a vector in the GZ-basis of some irreducible representation of G_n. As an immediate consequence of the theorem above we get the following result.

Corollary 2.2. *(i) Let $v \in V^\lambda$, $\lambda \in G_n^\wedge$. If v is an eigenvector (for the action) of every element of GZ_n, then (a scalar multiple of) v belongs to the GZ-basis of V^λ.*

(ii) Let v, u be two GZ-vectors. If v and u have the same eigenvalues for every element of GZ_n, then $v = u$.

We shall find an explicit set of generators for the GZ-algebras of the symmetric groups in Section 4.

3 Simplicity of branching for symmetric groups

In this section we give a criterion for simple branching and use the theory of involutive algebras to verify this criterion in the case of symmetric groups.

Theorem 3.1. *Let M be a finite dimensional semisimple $\mathbb{C}$-algebra and let N be a semisimple subalgebra. Let $Z(M,N)$ denote the* centralizer *of this pair consisting of all elements of M that commute with N. Then $Z(M,N)$ is semisimple and the following conditions are equivalent:*

1. The restriction of any finite dimensional complex irreducible representation of M to N is multiplicity free.

2. The centralizer $Z(M,N)$ is commutative.

Proof By Wedderburn's theorem we may assume, without loss of generality, that $M = M_1 \oplus \cdots \oplus M_k$, where each M_i is a matrix algebra. We write elements of M as $(m_1, \ldots, m_k)$, where $m_i \in M_i$. For $i = 1, \ldots, k$, let N_i denote the image of N under the natural projection of M onto M_i. Being the homomorphic image of a semisimple algebra, N_i itself is semisimple.

We have $Z(M,N) = Z(M_1,N_1) \oplus \cdots \oplus Z(M_k,N_k)$. By the double centralizer theorem each $Z(M_i,N_i)$, and thus $Z(M,N)$, is semisimple.

For $i = 1, \ldots, k$, let V_i denote the set of all $(m_1, \ldots, m_k) \in M$ with $m_j = 0$ for $j \neq i$ and with all entries of m_i not in the first column equal to zero. Note that $V_1, \ldots, V_k$ are all the distinct inequivalent irreducible M-modules and that the decomposition of

V_i into irreducible N-modules is identical to the decomposition of V_i into irreducible N_i-modules.

It now follows from the double centralizer theorem that V_i is multiplicity free as a N_i-module, for all i iff all irreducile modules of $Z(M_i, N_i)$ have dimension 1, for all i iff $Z(M_i, N_i)$ is abelian, for all i iff $Z(M, N)$ is abelian.$\Box$

We shall now give an elegant proof that the restriction of an irreducible representation of S_n to S_{n-1} is multiplicity free. The proof exploits the additional structure of an involution that certain algebras possess (like $g \mapsto g^{-1}$ in groups and $A \mapsto A^*$ in matrices).

Recall the notion of an involutive algebra. Let F denote the real field or the complex field. If $F = \mathbb{C}$ and $\alpha \in \mathbb{C}$, then $\bar{\alpha}$ denotes the complex conjugate of α, and if $F = \mathbb{R}$ and $\alpha \in \mathbb{R}$, then $\bar{\alpha} = \alpha$.

An *involutive algebra* over F is an algebra A (over F) together with a conjugate linear anti-automorphism of order 2, i.e., a bijective mapping $x \mapsto x^*$ such that

$$(x+y)^* = x^* + y^*, (\alpha x)^* = \bar{\alpha} x^*, (xy)^* = y^* x^*, (x^*)^* = x,$$

for all $x, y \in A$, and $\alpha \in F$. We call x^* the *adjoint* of x.

An element $x \in A$ is said to be *normal* if $xx^* = x^*x$ and is said to be *hermitian* or *self-adjoint* if $x = x^*$.

Given an involutive algebra A over $\mathbb{R}$, we define an involutive algebra over $\mathbb{C}$, called the $*$-*complexification* of A as follows: The elements of the complexification are ordered pairs $(x, y) \in A \times A$. We write (x, y) as $x + iy$. The algebra operations are the natural ones. For $\alpha + i\beta \in \mathbb{C}$, $x, x_1, x_2, y, y_1, y_2 \in A$ we have

$$\begin{aligned}
(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2) \\
(x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\
(\alpha + i\beta)(x + iy) &= (\alpha x - \beta y) + i(\alpha y + \beta x) \\
(x + iy)^* &= x^* - iy^*
\end{aligned}$$

By a *real* element of the complexification we mean an element of the form $x = x + i0$, for some $x \in A$. Note that the adjoint of a real element in the complexification is the same as its adjoint in A.

Example Let F be $\mathbb{R}$ or $\mathbb{C}$ and let G be a finite group. Then the group algebra $F[G]$ is an involutive algebra under the involution $(\sum_i \alpha_i g_i)^* = \sum_i \bar{\alpha}_i g_i^{-1}$. It can also be easily checked that $\mathbb{C}[G]$ is the $*$-complexification of $\mathbb{R}[G]$.

Theorem 3.2. *Let A be a complex involutive algebra.*

(i) An element $x \in A$ is normal if and only if $x = y + iz$, for some self-adjoint $y, z \in A$ that commute.

(ii) A is commutative if and only if every element of A is normal.

(iii) If A is the $*$-complexification of a real involutive algebra, then A is commutative if every real element of A is self-adjoint.

Proof (i) (if) $xx^* = (y + iz)(y^* - iz^*) = yy^* + izy^* - iyz^* + zz^* = y^2 + z^2$, where in the last step we use the fact that y, z are commuting self-adjoints. Similarly we can show $x^*x = y^2 + z^2$.

(only if) Write $x = y + iz$, where $y = \frac{1}{2}(x + x^*)$ and $z = \frac{1}{2i}(x - x^*)$. It is easily checked that y and z are self-adjoint. Now

$$yz = \frac{1}{4i}(x^2 - xx^* + x^*x - (x^*)^2) = \frac{1}{4i}(x^2 - (x^*)^2),$$

where in the last step we use the normality of x. Similarly, $zy = \frac{1}{4i}(x^2 - (x^*)^2)$.

(ii) (only if) This is clear.

(if) Let $y, z \in A$ be two self-adjoint elements. Then

$$\begin{aligned} (y + iz)(y^* - iz^*) &= y^2 + z^2 + i(zy - yz) \\ (y^* - iz^*)(y + iz) &= y^2 + z^2 + i(yz - zy) \end{aligned}$$

Since $y + iz$ is normal (by hypothesis) we have $zy = yz$. So any two self-adjoint elements of A commute.

Let $x_1, x_2 \in A$. Since x_1, x_2 are normal we can write $x_1 = y_1 + iz_1$ and $x_2 = y_2 + iz_2$, where y_1, y_2, z_1, z_2 are self-adjoint (and all commute with each other, by the paragraph above). We now have

$$x_1x_2 = (y_1y_2 - z_1z_2) + i(z_1y_2 + y_1z_2) = x_2x_1.$$

(iii) Let y, z be real elements of A. Then yz is also real and $yz = (yz)^* = z^*y^* = zy$, so y and z commute. Let $x = y + iz \in A$, where y and z are real. By part (i) x is normal and by part (ii) A is commutative. □

Theorem 3.3. *The centralizer $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$ of the subalgebra $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$ is commutative.*

Proof The involutive subalgebra $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$ of $\mathbb{C}[S_n]$ is the $*$-complexification of $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$ (why?). It therefore suffices to show that every element of $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$ is self-adjoint.

Let

$$f = \sum_{\pi \in S_n} \alpha_\pi \pi, \quad \alpha_\pi \in \mathbb{R},$$

be an element of $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$.

Fix a permutation $\sigma \in S_n$. Now σ and σ^{-1} have the same cycle structure and are thus conjugate. There is a well known procedure to produce a permutation in S_n conjugating σ to σ^{-1}. Namely, first write down in a straight line the permutation σ in cycle form. Below this line write down σ^{-1} in cycle form, the only requirement being that the lengths of the corresponding cycles match. The permutation in S_n taking an element of the top row to the corresponding element of the bottom row conjugates σ to σ^{-1}. It is clear that we can so write down the two permutations that the letter n matches with itself. Thus we can choose $\tau \in S_{n-1}$ such that $\sigma^{-1} = \tau\sigma\tau^{-1}$.

Since $\tau \in S_{n-1}$ we have $\tau f = f\tau$ or

$$f = \tau f \tau^{-1} = \sum_{\pi \in S_n} \alpha_\pi (\tau\pi\tau^{-1}).$$

It follows that $\alpha_\sigma = \alpha_{\sigma^{-1}}$. Since σ was an arbitrary element in S_n it follows that $f^* = f$. $\square$

We denote the centralizer $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$ by $Z_{(n-1,1)}$. In the next section we shall provide another proof that $Z_{(n-1,1)}$ is commutative.

4 Young-Jucys-Murphy elements

From this section onwards we consider only the symmetric groups and we put $G_n = S_n$. Thus chains in the branching graph (as in Section 2) will refer to chains in the branching graph of the symmetric groups.

For $i = 2, \ldots, n$ define the following elements of $\mathbb{C}S_n$:

$$\begin{aligned} Y_i &= \text{Sum of all } i\text{-cycles in } S_{n-1}. \\ Y_i' &= \text{Sum of all } i\text{-cycles in } S_n \text{ containing } n. \end{aligned}$$

Note that $Y_n = 0$.

For $(\mu, i) \in \mathcal{P}_1(n)$ define $c_{(\mu,i)} \in \mathbb{C}S_n$ to be the sum of all permutations π in S_n satisfying

- type $(\pi) = \mu$.
- size of the cycle of π containing $n = i$.

Observe that each of $Y_2, \ldots, Y_{n-1}, Y_2', \ldots, Y_n'$ is equal to $c_{(\mu,i)}$, for suitable μ and i.

Lemma 4.1. *(i)* $\{c_{(\mu,i)} \mid (\mu, i) \in \mathcal{P}_1(n)\}$ *is a basis of* $Z_{(n-1,1)}$. *It follows that* $\langle Y_2, \ldots, Y_{n-1}, Y_2', \ldots, Y_n' \rangle \subseteq Z_{(n-1,1)}$.

(ii) $c_{(\mu,i)} \in \langle Y_2, \ldots, Y_k, Y_2', \ldots Y_k' \rangle$, *where* $k = \#\mu$.

(iii) $Z_{(n-1,1)} = \langle Y_2, \ldots, Y_{n-1}, Y'_2, \ldots, Y'_n \rangle$.

(iv) $Z_{n-1} = \langle Y_2, \ldots, Y_{n-1} \rangle$.

Proof (i) The first statement is left as an exercise for the reader (it is similar to the proof that $\{c_\mu \mid \mu \in \mathcal{P}(n)\}$ is a basis of Z_n). The second statement then follows from the observation made just before the statement of the lemma.

(ii) By induction on $\#\mu$. If $\#\mu = 0$, then $c_{(\mu,i)}$ is the identity permutation and the result is clearly true. Assume the result whenever $\#\mu \leq k$. Consider $(\mu, i) \in \mathcal{P}_1(n)$ with $\#\mu = k+1$. Let the nontrivial parts of μ be $\mu_1, \ldots, \mu_l$ (here we are not assuming that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l$). Consider the following two subcases:

(a) $i = 1$: Consider the product $Y_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l}$. Then we have, using part (i) (why?),

$$Y_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l} = \alpha_{(\mu,1)} c_{(\mu,1)} + \sum_{(\tau,1)} \alpha_{(\tau,1)} c_{(\tau,1)},$$

where $\alpha_{(\mu,1)} \neq 0$ and the sum is over all $(\tau, 1)$ with $\#\tau < \#\mu$. The result follows by induction.

(b) $i > 1$: Without loss of generality we may assume that $i = \mu_1$. Consider the product $Y'_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l}$. Then we have, using part (i) (why?),

$$Y'_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l} = \alpha_{(\mu,i)} c_{(\mu,i)} + \sum_{(\tau,j)} \alpha_{(\tau,j)} c_{(\tau,j)},$$

where $\alpha_{(\mu,i)} \neq 0$ and the sum is over all (τ, j) with $\#\tau < \#\mu$. The result follows by induction.

(iii) This follows from parts (i) and (ii).

(iv) This is similar to the proof of part (iii). □

For $i = 1, \ldots, n$ define the following elements of $\mathbb{C}S_n$:

$$\begin{aligned} X_i &= (1, i) + (2, i) + \cdots + (i-1, i) \\ &= \text{Sum of all 2-cycles in } S_i \; - \; \text{Sum of all 2-cycles in } S_{i-1}. \end{aligned}$$

Note that $X_1 = 0$. Also note that $X_i \notin Z_i$. In fact, X_i is the difference of an element in Z_i and an element in Z_{i-1}, so $X_i \in GZ_i \subseteq GZ_n$. These elements have remarkable properties and are called the *Young-Jucys-Murphy* (YJM) elements.

Theorem 4.2. *(i)* $Z_{(n-1,1)} = \langle Z_{n-1}, X_n \rangle$.

(ii) $GZ_n = \langle X_1, X_2, \ldots, X_n \rangle$.

Proof (i) Clearly $Z_{(n-1,1)} \supseteq \langle Z_{n-1}, X_n \rangle$ (note that $X_n = Y_2'$). To show the converse it is enough to show that $Y_2', \ldots, Y_n' \in \langle Z_{n-1}, X_n \rangle$. Since $Y_2' = X_n$ we have $Y_2' \in \langle Z_{n-1}, X_n \rangle$.

Assume that $Y_2', \ldots, Y_{k+1}' \in \langle Z_{n-1}, X_n \rangle$. We shall now show that $Y_{k+2}' \in \langle Z_{n-1}, X_n \rangle$. We write Y_{k+1}' as

$$\sum_{i_1, \ldots, i_k} (i_1, \ldots, i_k, n),$$

where the sum is over all distinct $i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$. In the following we use this summation convention implicitly.

Now consider the product $Y_{k+1}' X_n \in \langle Z_{n-1}, X_n \rangle$:

$$\left\{ \sum_{i_1, \ldots, i_k} (i_1, \ldots, i_k, n) \right\} \left\{ \sum_{i=1}^{n-1} (i, n) \right\}. \tag{5}$$

Take a typical element $(i_1, \ldots, i_k, n)(i, n)$ of this product. If $i \neq i_j$, for $j = 1, \ldots, k$ this product is $(i, i_1, \ldots, i_k, n)$. On the other hand if $i = i_j$, for some $1 \leq j \leq k$, this product becomes $(i_1, \ldots, i_j)(i_{j+1}, \ldots, n)$. It follows that the element (5) above is equal to

$$\sum_{i, i_1, \ldots, i_k} (i, i_1, \ldots, i_k, n) + \sum_{i_1, \ldots, i_k} \sum_{j=1}^{k} (i_1, \ldots, i_j)(i_{j+1}, \ldots, i_k, n), \tag{6}$$

where the first sum is over all distinct $i, i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$ and the second sum is over all distinct $i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$. We can rewrite (6) as

$$Y'_{k+2} + \sum_{(\mu,i)} \alpha_{(\mu,i)} c_{(\mu,i)},$$

where the sum is over all (μ, i) with $\#\mu \leq k+1$. By induction hypothesis and part (ii) of Lemma 4.1 it follows that $Y'_{k+2} \in \langle Z_{n-1}, X_n \rangle$.

(ii)The proof is by induction (the cases $n = 1, 2$ being clear). Assume we have proved that $GZ_{n-1} = \langle X_1, X_2, \ldots, X_{n-1} \rangle$. It remains to show that $GZ_n = \langle GZ_{n-1}, X_n \rangle$. The lhs clearly contains the rhs so it suffices to check that the lhs is contained in the rhs. For this it suffices to check that $Z_n \subseteq \langle GZ_{n-1}, X_n \rangle$. This follows from part (i) since $Z_n \subseteq Z_{(n-1,1)}$. $\square$

Note that Theorem 5(i) implies that $Z_{(n-1,1)}$ is commutative.

The GZ-basis in the case of symmetric groups is also called the *Young basis*, since it was first considered by A.Young (though not as a global basis as here). By Corollary 2.2(i) the Young (or GZ) vectors are common eigenvectors for GZ_n. Let v be a Young vector (for S_n). Define

$$\alpha(v) \;\; = \;\; (a_1, \ldots, a_n) \in \mathbb{C}^n,$$

where a_i = eigenvalue of X_i on v. We call $\alpha(v)$ the *weight* of v (note that $a_1 = 0$ since $X_1 = 0$). Set

$$\text{Spec } (n) = \{\alpha(v) \;:\; v \text{ is a Young vector}\}.$$

It follows from Corollary 2.2(ii) that

$$\dim GZ_n = \#\text{Spec } (n) = \sum_{\lambda \in S_n^\wedge} \dim \lambda.$$

By definition of Young vectors, the set Spec (n) is in natural bijection with chains T as in (3). Given $\alpha \in$ Spec (n) we denote by v_α the Young vector with weight α and by T_α the corresponding chain in the branching graph. Similarly, given a chain T as in (3) we denote the correponding weight vector $\alpha(v_T)$ by $\alpha(T)$. Thus we have 1-1 correspondences

$$T \mapsto \alpha(T), \quad \alpha \mapsto T_\alpha$$

between chains (3) and Spec (n).

There is a natural equivalence relation $\sim$ on Spec (n): for $\alpha, \beta \in$ Spec (n),

$$\begin{aligned} \alpha \sim \beta \quad &\Leftrightarrow \quad v_\alpha \text{ and } v_\beta \text{ belong to the same irreducible } S_n\text{-module} \\ &\Leftrightarrow \quad T_\alpha \text{ and } T_\beta \text{ end in the same vertex.} \end{aligned}$$

Clearly we have $\#(\text{Spec }(n)/\sim) = \#S_n^\wedge$.

5 Action of Coxeter generators on the Young basis and the algebra $H(2)$

The Young vectors are a simultaneous eigenbasis for the GZ-algebra. Let us now consider the action of the full algebra $\mathbb{C}S_n$ on the Young basis. The action of the Coxeter generators $s_i = (i, i+1)$, $i = 1, \dots, n-1$ on the Young basis is "local" in the sense of the following lemma.

Lemma 5.1. *Let T be a chain*

$$\lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n, \ \lambda_k \in S_k^\wedge \tag{7}$$

and let $1 \le i \le n-1$. Then $s_i \cdot v_T$ is a linear combination of vectors $v_{T'}$, where T' runs over chains of the form

$$\lambda'_1 \nearrow \lambda'_2 \nearrow \cdots \nearrow \lambda'_n, \ \lambda'_k \in S_k^\wedge \tag{8}$$

such that $\lambda'_k = \lambda_k$, for $k \neq i$. Moreover, the coefficients of this linear combination depend only on $\lambda_{i-1}, \lambda_i, \lambda_{i+1}$ and the choice of the scalar factors for the vectors in the Young basis. That is, the action of s_i affects only the ith level and depends only on levels $i-1, i$, and $i+1$ of the branching graph.

Proof Clearly, for $j \geq i+1$, we have $s_i \cdot v_T \in V^{\lambda_j}$. For $j \leq i-1$, the action of s_i on $V^{\lambda_{i+1}}$ is S_j-linear (since s_i commutes with all elements of S_j). Thus $s_i \cdot v_T$ belongs to the V^{λ_j}-isotypical component of $V^{\lambda_{i+1}}$. This proves the first part of the lemma. The moreover part is left as an exercise for the reader. □

We shall now give explicit formulas for the action of the Coxeter generators s_i on the Young basis vectors v_T in terms of the weights $\alpha(T)$. For this purpose, it will be useful to know the relations satisfied by the s_i's and the X_j's since the v_T's are eigenvectors for $X_1, \ldots, X_n$. The following relations are easily verified.

$$s_i X_j = X_j s_i, \ j \neq i, i+1, \tag{9}$$

$$s_i^2 = 1, \ X_i X_{i+1} = X_{i+1} X_i, \ s_i X_i + 1 = X_{i+1} s_i. \tag{10}$$

(The preceding lemma can also be proved using the commutation relation (9) above. We leave this as an exercise). Let T be as in (7) above and let $\alpha(T) = (a_1, a_2, \ldots, a_n)$. Let V be the subspace of $V^{\lambda_{i+1}}$ generated by v_T and $s_i \cdot v_T$ (note that V is atmost two dimensional). The relation (10) shows that V is invariant under the actions of s_i, X_i, and X_{i+1} and a study of this action will enable us to write down a formula for $s_i \cdot v_T$.

We are thus led to consider the algebra $H(2)$ generated by elements H_1, H_2, and s subject to the following relations:

$$s^2 = 1, \quad H_1 H_2 = H_2 H_1, \quad sH_1 + 1 = H_2 s. \tag{11}$$

Note that H_2 can be excluded because $H_2 = sH_1 s + s$, but it is most useful to include H_2 in the list of generators.

Lemma 5.2. *(i) All irreducible representations of $H(2)$ are atmost two dimensional.*

(ii) For $i = 1, 2, \ldots, n-1$ the image of $H(2)$ in $\mathbb{C}S_n$ obtained by setting $s = s_i = (i, i+1)$, $H_1 = X_1$, and $H_2 = X_{i+1}$ is semisimple, i.e., the subalgebra M of $\mathbb{C}S_n$ generated by s_i, X_i, and X_{i+1} is semisimple.

Proof (i) Let V be an irreducible $H(2)$ module. Since H_1 and H_2 commute they have a common eigenvector v. Let W be the space spanned by v and $s \cdot v$. Then dim $W \leq 2$ and (11) shows that W is a submodule of V. Since V is irreducible it follows that $W = V$.

(ii) Let $Mat(n)$ denote the algebra of complex $n! \times n!$ matrices, with rows and columns indexed by permutations in S_n. Consider the left regular representation of S_n. Writing this in matrix terms gives an embedding of $\mathbb{C}S_n$ into $Mat(n)$. Note that the matrix in $Mat(n)$ corresponding to a transposition in S_n is real and symmetric. Since X_i and X_{i+1} are sums of transpositions the matrices in $Mat(n)$ corresponding to them are also real and symmetric. It follows that the subalgebra M is closed under the matrix * operation ($A \mapsto (\bar{A})^t$). A standard result on finite dimensional $\mathbb{C}^*$-algebras shows that M is semisimple. $\square$

We can now explicitly describe the action of the Coxeter generators on the Young basis in terms of transformation of weights. In the following it is convenient to parametrize Young vectors by elements of Spec (n) rather than by chains T (recall that these sets are in natural bijection).

Theorem 5.3. *Let T be a chain as in (7) and let $\alpha(T) = (a_1, \ldots, a_n) \in$ Spec (n). Consider the Young vector $v_\alpha = v_T$. Then*

(i) $a_i \neq a_{i+1}$, for all i.

(ii) $a_{i+1} = a_i \pm 1$ if and only if $s_i \cdot v_\alpha = \pm v_\alpha$.

(iii) For $i = 1, \ldots, n-2$ the following statements are not true:

$$a_i = a_{i+1} + 1 = a_{i+2}, \; a_i = a_{i+1} - 1 = a_{i+2}.$$

(iv) If $a_{i+1} \neq a_i \pm 1$ then

$$\alpha' = s_i \cdot \alpha = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$$

belongs to Spec (n) and $\alpha' \sim \alpha$. Moreover the vector

$$v = \left(s_i - \frac{1}{a_{i+1} - a_i} \right) \cdot v_\alpha$$

is a scalar multiple of $v_{\alpha'}$. Thus, in the basis $\{v_\alpha, v\}$ the actions of X_i, X_{i+1}, and s_i are given by the matrices

$$\begin{bmatrix} a_i & 0 \\ 0 & a_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+1} & 0 \\ 0 & a_i \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{bmatrix}$$

respectively.

Proof (i) Suppose first that v_α and $s_i \cdot v_\alpha$ are linearly dependent. Then, since $s_i^2 = 1$, we have $s_i \cdot v_\alpha = \pm v_\alpha$. The relation (10) (equivalent to $s_i X_i s_i + s_i = X_{i+1}$) now shows $a_{i+1} = a_i \pm 1$.

Now assume that v_α and $s_i \cdot v_\alpha$ are linearly independent and let V be the subspace of $V^{\lambda_{i+1}}$ they span. Then, as checked before, V is invariant under the action of the algebra M and the matrices for the actions of X_i, X_{i+1} and s_i in the basis $\{v_\alpha, \; s_i \cdot v_\alpha\}$ of V are

$$\begin{bmatrix} a_i & -1 \\ 0 & a_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+1} & 1 \\ 0 & a_i \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively. The action of X_i on $V^{\lambda_{i+1}}$ is diagonalizable and thus, since V is X_i-invariant, the action of X_i on V is also diagonalizable. This implies $a_i \neq a_{i+1}$.

(ii) The if part has already been proved in part (i) above. Let us now prove the only if part. Suppose $a_{i+1} = a_i + 1$ (the case $a_{i+1} = a_i - 1$ is similar). Assume that v_α and $s_i \cdot v_\alpha$ are linearly independent and let V be the subspace of $V^{\lambda_{i+1}}$ they span. Now V is an M-module and M is semisimple. But it can be easily checked that there is only one 1-dimensional subspace of V, namely the space spanned by $s_i \cdot v_\alpha - v_\alpha$, that is invariant under the action of M, a contradiction. Thus v_α and $s_i \cdot v_\alpha$ are linearly dependent and this implies that $s_i \cdot v_\alpha = v_\alpha$.

(iii) Suppose $a_i = a_{i+1} - 1 = a_{i+2}$ (the proof in the other case is the same). By part (ii) we have $s_i \cdot v_\alpha = v_\alpha$ and $s_{i+1} \cdot v_\alpha = -v_\alpha$. Consider the Coxeter relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

and let both sides act on v_α. The lhs yields $-v_\alpha$ and the rhs yields v_α, a contradiction.

(iv) By part (ii) we have that v_α and $s_i \cdot v_\alpha$ are linearly independent. For $j \neq i, i+1$ we can easily check using commutativity of X_j and s_i that $X_j \cdot v = a_j v$. Similarly, using (10), we can easily check that $X_i \cdot v = a_{i+1} v$ and $X_{i+1} \cdot v = a_i v$. It follows from Corollary 2.2(i) that $\alpha' \in$ Spec (n) and from Corollary 2.2(ii) that v is a scalar multiple of $v_{\alpha'}$. Clearly $\alpha' \sim \alpha$ as $v \in V^{\lambda_n}$. The matrix representations of s_i, X_i and X_{i+1} are easily verified. □

If $\alpha = (a_1, \ldots, a_n) \in$ Spec (n) and $a_i \neq a_{i+1} \pm 1$, we say that the transposition s_i is *admissible* for α. Note that if $\alpha \in$ Spec (n) is obtained from $\beta \in$ Spec (n) by a sequence of admissible transpositions then $\alpha \sim \beta$.

In the next section we shall show that Spec (n) consists of integral vectors. This given, the matrix for the action of s_i from part (iv) of Theorem 5.3 suggests that if we choose the GZ-basis $\{v_T\}$ appropriately then all irreducible representations of S_n are defined over $\mathbb{Q}$. We shall show how to do this in Section 7.

6 Content vectors and Young tableaux

In the Vershik-Okounkov theory Young tableaux are related to irreducible S_n representations via their content vectors. Let us define these first.

Definition 6.1. *Let $\alpha = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. We say that α is a* content vector *if*

(i) $a_1 = 0$.

(ii) $\{a_i - 1, a_i + 1\} \cap \{a_1, a_2, \dots, a_{i-1}\} \neq \emptyset$, for all $i > 1$.

(iii) if $a_i = a_j = a$ for some $i < j$ then $\{a-1, a+1\} \subseteq \{a_{i+1}, \dots, a_{j-1}\}$ (i.e., between two occurrences of a there should also be occurrences of $a - 1$ and $a + 1$).

Condition (ii) in the definition above can be replaced (in the presence of conditions (i) and (iii)) by condition (ii') below.

(ii') For all $i > 1$, if $a_i > 0$ then $a_j = a_i - 1$ for some $j < i$ and if $a_i < 0$ then $a_j = a_i + 1$ for some $j < i$.

The set of all content vectors of length n is denoted Cont $(n) \subseteq \mathbb{Z}^n$.

Theorem 6.1. *Spec $(n) \subseteq$ Cont (n).*

Proof Let $\alpha = (a_1, \dots, a_n) \in$ Spec (n). Clearly $a_1 = 0$ as $X_1 = 0$. We verify conditions (ii) and (iii) by induction on n. Since $X_2 = (1, 2)$ we have $a_2 = \pm 1$ and thus condition (ii) is verified (and condition (iii) does not apply). Now assume $n \geq 3$.

If $a_{n-1} = a_n \pm 1$ there is nothing to prove, so assume this does not hold. Then the transposition $(n-1, n)$ is admissible for α and, by Theorem 5.3(iv), $(a_1, \dots, a_{n-2}, a_n, a_{n-1}) \in$ Spec (n). Now

$(a_1, \dots, a_{n-2}, a_n) \in \text{Spec}\,(n-1)$ and by the induction hyphothesis $\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-2}\} \neq \emptyset$. Thus condition (ii) is verified.

Now assume that $a_i = a_n = a$ for some $i < n$. We may assume that i is the largest possible index, i.e., a does not occur between a_i and a_n: $a \notin \{a_{i+1}, \dots, a_{n-1}\}$. Now assume that $a - 1 \notin \{a_{i+1}, \dots, a_{n-1}\}$. We shall derive a contradiction (the case where $a + 1 \notin \{a_{i+1}, \dots, a_{n-1}\}$ is similar).

By induction hypothesis the number $a+1$ occurs in $\{a_{i+1}, \dots, a_{n-1}\}$ at most once (for, if it occured twice, then by the induction hypothesis a would also occur contradicting our assumption). Thus there are two possibilities:

$$(a_i, \dots, a_n) = (a, *, \dots, *, a) \text{ or } (a_i, \dots, a_n) = (a, *, \dots, *, a{+}1, *, \dots, *, a),$$

where $*$ stands for a number different from $a - 1, a, a + 1$.

In the first case we can apply a sequence of admissible transpositions to infer that $(\dots, a, a, \dots) \in \text{Spec}\,(n)$, contradicting Theorem 5.3(i) and in the second case we can apply a sequence of admissible transpositions to infer that $(\dots, a, a + 1, a, \dots) \in \text{Spec}\,(n)$, contradicting Theorem 5.3(iii). □

If $\alpha = (a_1, \dots, a_n) \in \text{Cont}\,(n)$ and $a_i \neq a_{i+1} \pm 1$, we say that the transposition s_i is *admissible* for α. We define the following equivalence relation on Cont (n): $\alpha \approx \beta$ if β can be obtained from α by a sequence of (zero or more) admissible transpositions.

We are finally ready to introduce Young tableaux into the picture. Recall the definition of the *Young graph* $\mathbb{Y}$: its vertices are Young diagrams, and two vertices μ and τ are joined by a directed edge from μ to τ if and only if $\mu \subseteq \tau$ and $\tau - \mu$ is a single box. In this case we write $\mu \nearrow \tau$. The *content* $c(B)$ of a box B of a Young diagram is its x-coordinate $-$ its y-coordinate (our convention for drawing Young diagrams is akin to drawing matrices with x-axis running downwards and y axis running to the left). By Tab (τ) we denote the set of all paths in $\mathbb{Y}$ from the unique partition of 1 to

τ. Such paths are called *standard Young tableaux*. Given a path $T \in \text{Tab}\ (\tau)$, written as

$$\tau_1 \nearrow \tau_2 \nearrow \cdots \nearrow \tau_n = \tau,$$

a convenient way to represent it is to take the Young diagram of τ and write the numbers $1, 2, \ldots, n$ in the boxes $\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}$ respectively. Set

$$\text{Tab}\ (n) = \cup_\tau \text{Tab}\ (\tau),$$

where the union is over all partitions τ of n.

Let $T_1 \in \text{Tab}\ (n)$ and assume that i and $i+1$ do not appear in the same row or column of T_1. Then exchanging i and $i+1$ in T_1 produces another standard Young tableaux $T_2 \in \text{Tab}\ (n)$. We say that T_2 is obtained from T_1 by an *admissible transposition*. For $T_1, T_2 \in \text{Tab}\ (n)$, define $T_1 \approx T_2$ if T_2 can be obtained from T_1 by a sequence of (zero or more) admissible transpositions (it is easily seen that $\approx$ is an equivalence relation).

The proof of the following combinatorial lemma is left as an exercise.

Lemma 6.2. *Let $\Phi : Tab\ (n) \longrightarrow Cont\ (n)$ be defined as follows. Given*

$$T = \tau_1 \nearrow \tau_2 \nearrow \cdots \nearrow \tau_n \in Tab\ (n),$$

Define

$$\Phi(T) = (c(\tau_1), c(\tau_2 - \tau_1), \ldots, c(\tau_n - \tau_{n-1})).$$

Then Φ is a bijection which takes $\approx$-equivalent standard Young tableaux to $\approx$-equivalent content vectors.

Lemma 6.3. *Let $T_1, T_2 \in Tab\ (n)$. Then $T_1 \approx T_2$ if and only if the Young diagrams of T_1 and T_2 have the same shape.*

Proof The only if part is obvious. To prove the if part we proceed as follows. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of n. Define the following element R of Tab (μ): in the first row write down the numbers $1, 2, \ldots, \mu_1$ (in increasing order), in the second row write down the numbers $\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2$ (in increasing order) and so on till the last row. We show that any $T \in$ Tab (μ) satisfies $T \approx R$. This will prove the if part. Consider the last box of the last row of μ. Let i be written in this box of T. Exchange i and $i+1$ in T (which is clearly an admissible transposition). Now repeat this procedure with $i+1$ and $i+2$, then $i+2$ and $i+3$, and finally $n-1$ and n. At the end of this sequence of admissible transpositions we have the number n written in the last box of the last row of μ. Now repeat the same procedure for $n-1, n-2, \ldots, 2$. $\square$

Let us make a remark about the proof of Lemma 6.3. Let s denote the permutation that maps R to T. Then the proof shows that R can be obtained from T by a sequence of $\ell(s)$ admissible transpositions. Thus T can be obtained from R by a sequence of $\ell(s)$ admissible transpositions.

We can now present one of the central results in the representation theory of the symmetric groups.

Theorem 6.4. *(i) Spec (n) = Cont (n) and the equivalence relations $\sim$ and $\approx$ coincide.*

(ii) The map Φ^{-1} : Spec $(n) \longrightarrow$ Tab (n) is a bijection and, for $\alpha, \beta \in$ Spec (n) we have $\alpha \sim \beta$ if and only if $\Phi^{-1}(\alpha)$ and $\Phi^{-1}(\beta)$ have the same Young diagram.

(iii) The branching graph of the chain of symmetric groups is the Young graph $\mathbb{Y}$.

Proof We have

- Spec $(n) \subseteq$ Cont (n).
- If $\alpha \in$ Spec (n), $\beta \in$ Cont (n), and $\alpha \approx \beta$ then it is easily seen that $\beta \in$ Spec (n) and $\alpha \sim \beta$. It follows that given an

$\sim$-equivalence class $\mathcal{A}$ of Spec (n) and an $\approx$-equivalence class $\mathcal{B}$ of Cont (n), either $\mathcal{A} \cap \mathcal{B} = \emptyset$ or $\mathcal{B} \subseteq \mathcal{A}$.

- $\#(\text{Spec }(n)/\sim) = p(n)$, since the number of irreducible S_n-representations is equal to the number of conjugacy classes in S_n,
- $\#(\text{Cont }(n)/\approx) = p(n)$, by Lemmas 6.2 and 6.3.

The four statements above imply part (i). Parts (ii) and (iii) are now clear. □

7 Young's seminormal and orthogonal forms

We now discuss the choice of the scalar factors in the Young basis $\{v_T\}$, so that all irreducible representations of S_n are defined over $\mathbb{Q}$.

Given the results of Section 6, we can now parametrize irreducible S_n-modules by partitions of n and we can parametrize the Young basis vectors in an S_n-irreducible (parametrized by $\mu \in \mathcal{P}(n)$) by standard Young tableaux of shape μ.

Fix a partition μ of n and consider the irreducible S_n-module V^μ. Let R be the tableau defined in the proof of Lemma 6.3. Choose any nonzero vector $v_R \in V_R$. Now consider a tableau $T \in \text{Tab }(\mu)$. Let s be the permutation that maps R to T. Define

$$v_T = p_T(s \cdot v_R),$$

and define $\ell(T) = \ell(s)$. Recall, from Section 2, that p_T denotes the projection onto V_T. We will now show that $v_T \neq 0$. It will then follow that $\{v_T : T \in \text{Tab }(n)\}$ is a basis of V^μ.

Before proceeding further we observe the following: let $T \in$ Tab (μ) and choose a nonzero vector $v \in V_T$. Theorem 5.3(iv) shows that, if s_j is an admissible transposition for T, then $s_j \cdot v$ is the sum of a nonzero rational multiple of v and a nonzero vector in $V_{s_j \cdot T}$. If s_j is not admissible for T, then Theorem 5.3(ii) shows that $s_j \cdot v = \pm v$.

Fix $T \in$ Tab (μ) and let s be the permutation that maps R to T. Let $k = \ell(T)$. The proof of Lemma 6.3 shows that we can write

$$T = s \cdot R = c_1 c_2 \cdots c_k \cdot R,$$

where each c_j is an admissible transposition (w.r.t $c_{j+1} \cdots c_k \cdot R$). It now follows (why?) from the definition of v_T, the observation above, and the fact that s cannot be written as a product of fewer than k Coxeter transpositions that

$$s \cdot v_R = v_T + \sum_Q u_Q,$$

where $v_T \neq 0$, $u_Q \in V_Q$ and the sum is over all $Q \in$ Tab (μ) with $\ell(Q) < \ell(T)$.

Theorem 7.1. *Consider the basis $\{v_T : T \in \textit{Tab}\ (n)\}$ of V^μ defined above. Fix $T \in \textit{Tab}\ (n)$ and let $\alpha(T) = (a_1, \ldots, a_n)$. Let s_i be a Coxeter generator. The action of s_i on v_T is as follows.*

- *If i and $i+1$ are in the same column of T then s_i leaves v_T unchanged.*
- *If i and $i+1$ are in the same row of T then s_i multiplies v_T by -1.*
- *Suppose i and $i+1$ are not in the same row or column of T. Let $S = s_i \cdot T$ (i.e, swap i and $i+1$ in T).*

If $\ell(S) = \ell(T)+1$ then the action of s_i in the two dimensional subspace with ordered basis $\{v_T, v_S\}$ is given by the matrix

$$\begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1-\frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i-a_{i+1}} \end{bmatrix}$$

If $\ell(S) = \ell(T)-1$ then the action of s_i in the two dimensional subspace with ordered basis $\{v_T, v_S\}$ is given by the transpose of the matrix above.

Thus, all irreducible representations of S_n are defined over $\mathbb{Q}$.

Proof Items (i) and (ii) in the statement of the theorem follow from Theorem 5.3(ii).

Let us now consider item (iii). Assume that $\ell(S) = \ell(T) + 1$.

We have (why?), using the expression for $s \cdot v_R$ above,

$$\begin{aligned} s_i \cdot v_T &= s_i \cdot (s \cdot v_R - \sum_Q u_Q) \\ &= s_i \cdot (c_1 c_2 \cdots c_k \cdot v_R - \sum_Q u_Q) \\ &= v_S + \sum_{Q'} u_{Q'} \end{aligned}$$

where $u_{Q'} \in V_{Q'}$ and the sum is over all $Q' \in \text{Tab}\,(\mu)$ with $\ell(Q') \leq \ell(T)$. It now follows (why?) from Theorem 5.3(iv) that

$$s_i \cdot v_T = \frac{1}{a_{i+1} - a_i} v_T + v_S$$

The expression for $s_i \cdot v_S$ can be obtained by applying s_i to both sides of the equation above.

If $\ell(S) = \ell(T) - 1$ then we write $T = s_i \cdot S$ and switch T and S in the formulas for v_T, v_S above along with switching a_i and a_{i+1}. Doing this is equivalent to transposing the matrix. □

Another way to prove the theorem above is to directly verify the Coxeter relations (we leave this as an exercise).

The basis and the action described above is called Young's *semi-normal form* of V^μ. Now let us consider Young's *orthogonal form* for V^μ. This is defined over $\mathbb{R}$. Since V^μ is irreducible there is a unique (upto scalars) (why?) S_n-invariant inner product on V^μ. Pick one such inner product and normalize the vectors $\{v_T\}$ defined above (we use the same notation for the normalized vectors). Note that the vectors $\{v_T\}$ are orthonormal (why?). The following result now follows from the previous result.

Theorem 7.2. *Consider the orthonormal basis* $\{v_T : T \in Tab\ (n)\}$ *of* V^μ *defined above. Fix a chain* T *and let* $\alpha(T) = (a_1, \ldots, a_n)$*. Let* s_i *be a Coxeter generator and put* $r = a_{i+1} - a_i$*. The action of* s_i *on* v_T *is as follows.*

- *If* i *and* $i+1$ *are in the same column of* T *then* s_i *leaves* v_T *unchanged.*
- *If* i *and* $i+1$ *are in the same row of* T *then* s_i *multiplies* v_T *by* -1*.*
- *Suppose* i *and* $i+1$ *are not in the same row or column of* T*. Let* $S = s_i \cdot T$ *(i.e, swap* i *and* $i+1$ *in* T*).*

  The action of s_i *in the two dimensional subspace with ordered basis* $\{v_T, v_S\}$ *is given by the matrix*

$$\begin{bmatrix} r^{-1} & \sqrt{1-r^{-2}} \\ \sqrt{1-r^{-2}} & -r^{-1} \end{bmatrix}$$

The number r is called the *axial distance*. It is the difference of the contents of the corresponding boxes in the Young diagram.

References

[1] W. Fulton and J. Harris, *Representation Theory*, Springer, 1991.

[2] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.

[3] B. Sagan, *The Symmetric Group*, Springer, 2000.

[4] A. M. Vershik and A. Yu. Okounkov, *A New Approach to the Representation Theory of the Symmetric Groups. II*, arXiv:math.RT/0503040.

[5] A. M. Vershik, *A New Approach to the Representation Theory of the Symmetric Groups. III*, arXiv:math.RT/0609258.

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