Radial eigenvectors of the Laplacian of the nonbinary hypercube

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Abstract

Let q be a positive integer ≥ 2. Define a $(n+1)\times(n+1)$ real, symmetric, tridiagonal matrix M, with rows and columns indexed by $\{0,1,\ldots,n\}$, and with entries given by:

$$M(i,j) \;=\; \begin{cases} -\sqrt{(q-1)j(n-j+1)} & \text{if } i=j-1,\\ j+(q-1)(n-j) & \text{if } i=j,\\ -\sqrt{(q-1)(j+1)(n-j)} & \text{if } i=j+1,\\ 0 & \text{if } |i-j|\geq 2. \end{cases}$$

The $(n+1)$-dimensional space of radial vectors for the nonbinary hypercube $C_q(n)$ is invariant under the Laplacian and M arises as the matrix of the Laplacian with respect to a suitable orthonormal basis of radial vectors. We show that the eigenvalues of M are $0,q,2q,\ldots,nq$ by explicitly writing down the eigenvectors.

1 Introduction

For a finite set S we denote by $V(S)$ the complex vector space with S as basis. For a simple graph G on the vertex set S, recall that the *adjacency operator* $A: V(S)\rightarrow V(S)$ of G is defined, for $v\in S$, by $A(v)=\sum_u u$, where the sum is over all vertices u adjacent to v.

Let $B(n) = \{a = (a_1, \ldots, a_n) : a_i = 0 \text{ or } 1 \text{ for all } i\}$ denote the set of all sequences of 0's and 1's of length n. The *binary hypercube* $C(n)$ is the simple graph with $B(n)$ as the vertex set and with $a, b \in B(n)$ connected by an edge provided they differ in exactly one coordinate.

The spectrum of the adjacency operator A of $C(n)$ is easily determined using the graph operation of Cartesian products (see Section 1.4.6 in [**BH**] or Section 2.5 in [**CDS**] where this operation is called sum). We recall this briefly (without proof). First we find that the adjacency eigenvalues of $C(1)$ are $\alpha = 1$ and $\beta = -1$. Now $C(n)$ is the n-fold Cartesian product of $C(1)$ and thus the eigenvalues of A are $(n-i)\alpha + i\beta = n - 2i$ with multiplicity $\binom{n}{i}$, for $i = 0, 1, \ldots, n$. This method also gives an explicit set of eigenvectors (using tensor products of adjacency eigenvectors of $C(1)$) of A.

In the theory of lumped Markov chains the following subspace of $V(B(n))$, invariant under A, is considered. Define the following vectors in $V(B(n))$:

$$u_i \;=\; \sum_a a, \quad i = 0, 1, \ldots, n, \tag{1}$$

where the sum is over all $a = (a_1, \ldots, a_n) \in B(n)$ containing i 1's. Let $R(n)$ be the subspace of $V(B(n))$ spanned by $\{u_0, \ldots, u_n\}$. Clearly, $\dim(R(n)) = n + 1$. Elements of $R(n)$ are called *radial vectors* in $V(B(n))$. It is easy to see that $R(n)$ is invariant under A. We have

Theorem 1.1 *The eigenvalues of $A : R(n) \to R(n)$ are $n - 2i$, $i = 0, 1, \ldots, n$ and the (canonically defined) eigenvectors can be explicitly written down.*

Theorem 1.1 is proved, in an equivalent form, in [**CST**] (see Theorem 2.6.4) in the context of radial harmonic analysis on the binary hypercube. In this note we present a nonbinary generalization of Theorem 1.1. The main point of this generalization is that the correct formulation of the nonbinary analog, so as to obtain integral eigenvalues, involves the Laplacian operator of a nonregular graph. In the binary case the adjacency and Laplacian formulations are equivalent due to the regularity of $C(n)$.

2 Nonbinary hypercube

Let $q \geq 2$ be a positive integer and define

$$B_q(n) = \{(a_1, \ldots, a_n) : a_i \in \{0, 1, \ldots, q-1\} \text{ for all } i\}.$$

For $a \in B_q(n)$ define the *support* of a by $S(a) = \{i \in \{1, 2, \ldots, n\} : a_i \neq 0\}$. We define the *nonbinary hypercube* $C_q(n)$ to be the simple graph with $B_q(n)$ as the vertex set and with $a, b \in B_q(n)$ connected by an edge provided $S(a) \subseteq S(b)$ or $S(b) \subseteq S(a)$, $||S(a)| - |S(b)|| = 1$, and $a_i = b_i$, for $i \in S(a) \cap S(b)$. That is, a and b are connected by an edge provided they differ in exactly one coordinate and one of a or b is 0 in that coordinate. The degree of a vertex a of $C_q(n)$ is $|S(a)| + (q-1)(n-|S(a)|)$. Note that, while $C(n) = C_2(n)$ is a regular graph (of degree n), $C_q(n)$ is not regular for $q \geq 3$. Note also that $C_q(n)$ is different from what is usually called the nonbinary Hamming graph on q letters (which is regular).

Let A_q denote the adjacency operator of $C_q(n)$ and let $deg : V(B_q(n)) \to V(B_q(n))$ denote the *degree operator* given by: $deg(a) =$ (degree of a) $a,\ a \in B_q(n)$. Recall that the *Laplacian operator* L_q of $C_q(n)$ is defined by $L_q = deg - A_q$.

Define the following vectors in $V(B_q(n))$:

$$v_i \;=\; \sum_a a, \quad i = 0, 1, \ldots, n, \tag{2}$$

where the sum is over all $a \in B_q(n)$ with $|S(a)| = i$. Let $R_q(n)$ be the subspace of $V(B_q(n))$ spanned by $\{v_0, \ldots, v_n\}$. Clearly, $\dim(R_q(n)) = n+1$. Elements of $R_q(n)$ are called *radial vectors* in $V(B_q(n))$. It is easy to see that $R_q(n)$ is invariant under L_q. Before determining the eigenvalues of $L_q : R_q(n) \to R_q(n)$ we briefly discuss the entire Laplacian spectrum of $C_q(n)$, i.e., the eigenvalues of $L_q : V(B_q(n)) \to V(B_q(n))$. First we find that the Laplacian spectrum of $C_q(1)$ is $\{0, q, 1\}$, with 0 and q of multiplicity 1, and 1 of multiplicity $q-2$. Now $C_q(n)$ is the n-fold Cartesian product of $C_q(1)$ and thus its Laplacian spectrum is $\mathcal{S} = \{qt + s : s, t \geq 0,\ s + t \leq n\}$. Given $\sigma \in \mathcal{S}$, let $\mathcal{M}_\sigma = \{(s,t) : s, t \geq 0,\ s+t \leq n,\ qt + s = \sigma\}$. Then the multiplicity of $\sigma \in \mathcal{S}$ is $\sum_{(s,t)\in\mathcal{M}_\sigma} \binom{n}{t}\binom{n-t}{s}(q-2)^s$. An explicit set of Laplacian eigenvectors (using tensor products of Laplacian eigenvectors of $C_q(1)$) can also be written down.

Since $C_2(n)$ is regular of degree n it follows from Theorem 1.1 that the eigenvalues of $L_2 : R_2(n) \to R_2(n)$ are $0, 2, 4, \ldots, 2n$. The following result is the nonbinary generalization.

Theorem 2.1 *The eigenvalues of $L_q : R_q(n) \to R_q(n)$ are $0, q, 2q, \ldots, nq$ and the (canonically defined) eigenvectors can be explicitly written down.*

Proof Fix $0 \leq t \leq n$. Define a vector $w_t \in V(B_q(n))$ as follows:

$$w_t \;=\; \sum_{a \in B_q(n)} \left\{ \sum_b (-1)^{|S(a)\cap S(b)|} \right\} a, \tag{3}$$

where the inner sum is over all $b \in B_q(n)$ satisfying $|S(b)| = t$ and $i \in S(a) \cap S(b)$ implies $a_i = b_i$ (i.e., a and b agree on the intersection of their supports). It is easily seen that $w_t \in R_q(n)$. We claim that $L_q(w_t) = qtw_t$. To prove the claim we introduce a notational device.

The coefficient of x^k in a polynomial $f(x)$ is denoted $[x^k](f(x))$. The derivative of $f(x)$ is denoted $\mathcal{D}(f(x))$. We have

$$[x^{k-1}](\mathcal{D}(f(x))) = k([x^k](f(x))).$$

Fix $a \in B_q(n)$ with $|S(a)| = k$. The number of $b \in B_q(n)$ with $|S(b)| = t$, $|S(a) \cap S(b)| = j$, and with a, b agreeing on their support is easily seen to be $\binom{k}{j}\binom{n-k}{t-j}(q-1)^{t-j}$.

Thus the coefficient of a in w_t equals

$$\sum_{j=0}^{t} (-1)^j \binom{k}{j} \binom{n-k}{t-j} (q-1)^{t-j} = [x^t]((1-x)^k (1+(q-1)x)^{n-k}).$$

Now the coefficient of a in $L_q(w_t) = (deg - A_q)(w_t)$ is equal to

$$\begin{aligned}
& (k+(n-k)(q-1))\left\{[x^t]((1-x)^k(1+(q-1)x)^{n-k})\right\} \\
& -(n-k)(q-1)\left\{[x^t]((1-x)^{k+1}(1+(q-1)x)^{n-k-1})\right\} \\
& -k\left\{[x^t]((1-x)^{k-1}(1+(q-1)x)^{n-k+1})\right\} \\
= \;& (n-k)(q-1)\left\{[x^t]((1-x)^k(1+(q-1)x)^{n-k-1}(1+(q-1)x-(1-x)))\right\} \\
& -k\left\{[x^t]((1-x)^{k-1}(1+(q-1)x)^{n-k}(1+(q-1)x-(1-x)))\right\} \\
= \;& q(n-k)(q-1)\left\{[x^{t-1}]((1-x)^k(1+(q-1)x)^{n-k-1})\right\} \\
& -qk\left\{[x^{t-1}]((1-x)^{k-1}(1+(q-1)x)^{n-k})\right\} \\
= \;& q\left\{[x^{t-1}](\mathcal{D}((1-x)^k(1+(q-1)x)^{n-k}))\right\} \\
= \;& qt\left\{[x^t]((1-x)^k(1+(q-1)x)^{n-k})\right\},
\end{aligned}$$

completing the proof.□

Make $V(B_q(n))$ into an inner product space by defining the standard basis $B_q(n)$ to be orthonormal, i.e., $\langle a, b \rangle = \delta(a, b)$ (Kronecker delta) for $a, b \in B_q(n)$. The *length* $\sqrt{\langle v, v \rangle}$ of $v \in V(B_q(n))$ is denoted $\| v \|$.

Set $z_i = \frac{v_i}{\|v_i\|}, 0 \leq i \leq n$. We shall now write down the matrix of $L_q : R_q(n) \to R_q(n)$ with respect to the (ordered) orthonormal basis $\{z_0, z_1, \ldots, z_n\}$.

Define a $(n+1) \times (n+1)$ real, symmetric, tridiagonal matrix M, with rows and columns indexed by $\{0, 1, \ldots, n\}$, and with entries given by:

$$M(i,j) = \begin{cases} -\sqrt{(q-1)j(n-j+1)} & \text{if } i = j-1, \\ j + (q-1)(n-j) & \text{if } i = j, \\ -\sqrt{(q-1)(j+1)(n-j)} & \text{if } i = j+1, \\ 0 & \text{if } |i-j| \geq 2. \end{cases}$$

We have, for $j = 0, 1, \ldots, n$,

$$\begin{aligned} \| v_j \| &= \sqrt{\binom{n}{j}(q-1)^j}, \\ A_q(v_j) &= (j+1)v_{j+1} + (q-1)(n-j+1)v_{j-1}. \end{aligned}$$

Thus it follows that, for $j = 0, 1, \ldots, n$, we have

$$A_q(z_j) = \sqrt{(q-1)(j+1)(n-j)}z_{j+1} + \sqrt{(q-1)j(n-j+1)}z_{j-1},$$

and the matrix of $L_q : R_q(n) \to R_q(n)$ with respect to $\{z_0, \ldots, z_n\}$ is M. Thus the eigenvalues of M are $0, q, 2q, \ldots, nq$ and the eigenvectors w_t, $0 \leq t \leq n$, given in (3) above are orthogonal.

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