THE CONE OF BALANCED SUBGRAPHS

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Dedicated to the memory of Malka Peled

Abstract. In this paper we study a 2-color analog of the cycle cone of a graph. Suppose the edges of a graph are colored red and blue. A nonnegative real vector on the edges is said to be *balanced* if the red sum equals the blue sum at every vertex. A *balanced subgraph* is a subgraph whose characteristic vector is balanced (i.e., red degree equals blue degree at every vertex). By a *sum* (respectively, *fractional sum*) of cycles we mean a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of cycles. Similarly, we define sum and fractional sum of balanced subgraphs. We show that a balanced sum of cycles is a fractional sum of balanced subgraphs.

1. Introduction

Let $G = (V, E)$ be a graph (we allow parallel edges but not loops) and let $\mathcal{Z}(G)$ denote the convex polyhedral cone in $\mathbb{R}^E$ generated by the characteristic vectors of the cycles in G. We call $\mathcal{Z}(G)$ the *cycle cone* of G. Seymour [S] found the linear inequalities determining this cone. Let us recall this result.

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Given a vector $q \in \mathbb{R}^E$ we denote its components using the notation $q = (q(e) : e \in E)$. To write linear inequalities satisfied by the components of vectors in $\mathbb{R}^E$ we use the variables $x(e), e \in E$. By a *sum* (respectively, *fractional sum*) of cycles we mean a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of cycles. Given a nonempty proper subset X of V, the subset $D \subseteq E$ of edges between X and $V - X$ will be called a *cut*. Let D be a cut, $e \in D$, and C a cycle in G. If C contains e, then C must also contain an edge in $D - \{e\}$. Thus the characteristic vector $\chi(C)$ of C satisfies the following linear inequality

$$x(e) \leq \sum_{f \in D-e} x(f), \tag{1}$$

where we write $D - e$ for $D - \{e\}$. We abbreviate the right-hand side of (1) by $x(D - e)$. We call (1) the *cut condition* for the pair (D, e). A vector in $\mathbb{R}^E$ is said to be *cut admissible* if it satisfies the following linear inequalities:

$$x(e) \leq x(D-e), \qquad \text{for all cuts } D \text{ and all } e \in D, \tag{2}$$

$$x(e) \geq 0, \qquad \text{for all } e \in E. \tag{3}$$

It follows that every vector in $\mathcal{Z}(G)$ is cut admissible. Being the solution set of a finite system of homogeneous linear inequalities, the set of all cut admissible vectors also forms a convex polyhedral cone in $\mathbb{R}^E$ and is thus equal to the set of all nonnegative real linear combinations of finitely many vectors. Moreover, these vectors may be taken to be rational (i.e., all components rational) since the coefficients in (2) and (3) are rational (see Chapter 7 in [Sc]). That every cut admissible vector is in $\mathcal{Z}(G)$ now follows from the following combinatorial result proved in [S].

Theorem 1.1. *Let $G = (V, E)$ be a graph. A vector $p \in \mathbb{Q}^E$ is a fractional sum of cycles if and only if p is cut admissible.*

The proof of Theorem 1.1 in [S] is based on induction and the Seymour-Giles lemma on cycles in bridgeless graphs which states the following: let $G = (V, E)$ be a bridgeless graph and let $\phi : V \to E$ map each vertex v to an edge incident with v. Then G has a nonempty cycle K such that for each vertex w of K, $\phi(w)$ is an edge of K.

The main object of study of this paper is a 2-color analog of the cycle cone. Let $G = (V, E)$ be a graph and assume that E is colored with two colors, say red and blue. A real vector in $\mathbb{R}^E$ is said to be *balanced* if the red sum equals the blue sum at every vertex. In order to motivate our main definition we first rephrase the definition of $\mathcal{Z}(G)$ in terms of the important concept of even subgraphs (see Section 2.6 in [BM]). A spanning subgraph of G (i.e., a subgraph of G whose vertex set is V) is said to be *even* if the degree of each vertex in the subgraph is even. By Euler's theorem the characteristic vector of an even subgraph is a sum of cycles and thus the cone generated by the (characteristic vectors of) even subgraphs is the same as the cycle cone. A *balanced subgraph* is a spanning subgraph whose characteristic vector is balanced (i.e., red degree equals blue degree at every vertex). By a *sum* (respectively, *fractional sum*) of balanced subgraphs we mean a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of balanced subgraphs.

The 2-color analog of the cycle cone is obtained by replacing even subgraphs with balanced subgraphs. Given a 2-colored graph $G = (V, E)$, with coloring given by $\mathcal{C} : E \to \{R, B\}$, define $\mathcal{B}(G, \mathcal{C}) \subseteq \mathbb{R}^E$ to be the convex polyhedral cone generated by the characteristic vectors of balanced subgraphs in $(G, \mathcal{C})$. We call $\mathcal{B}(G, \mathcal{C})$ the *cone of balanced subgraphs.*

Consider a balanced subgraph in a 2-colored graph. By ignoring the colors and applying Euler's theorem we can write its characteristic vector as a sum of cycles. This shows that a fractional sum of balanced subgraphs is a balanced fractional sum of cycles and thus is a balanced cut admissible vector. The main result of this paper is the converse of this observation.

Theorem 1.2. *Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph. A vector $p \in \mathbb{Q}^E$ is a fractional sum of balanced subgraphs if and only if p is balanced and cut admissible.*

Theorem 1.2 shows that $\mathcal{B}(G, \mathcal{C})$ equals the set of all balanced cut admissible vectors.

Let us say a few words here about the relation between Theorems 1.1 and 1.2.

We can easily derive Theorem 1.1 from Theorem 1.2. What we mean here is the use of Theorem 1.2 as a black box without going into any details of its proof. Let $G = (V, E)$ be a graph and let $p \in \mathbb{Q}^E$ be a cut admissible vector. We want to show that p is a fractional sum of cycles. For each $v \in V$, let $p(v) = \sum p(e)$, where the sum is over all edges e incident with v. Attach vertex disjoint triangles to each vertex of G to get a new graph G'. Two color the edges of G' as follows. The edges of G get the color blue. In each of the attached triangles, the two edges that touch a vertex of G get the color red and the other edge gets the color blue. Now define a (nonnegative rational) weight function p' on the edges of G' as follows. All the edges e in G have $p'(e) = p(e)$ and all the edges e of an attached triangle at vertex $v \in V$ get weight $p'(e) = p(v)/2$. Note the following:

- p' is balanced.
- Consider a triangle and a cut in a graph. Then the number of edges of the triangle contained in the cut is either 0 or 2. This fact together with the facts that p is cut admissible and that all the edges of the attached triangles get the same weight under p' shows that p' is cut admissible.
- Every balanced subgraph of G' consists of a vertex disjoint union of cycles of G together with the attached triangles at the vertices of these cycles.

It follows, by Theorem 1.2 and the first two items above, that p' is a fractional sum of balanced subgraphs from which it easily follows, by the last item above, that p is a fractional sum of cycles.

It would be very interesting if there were a similar proof of Theorem 1.2 using Theorem 1.1. Such a proof would proceed as follows. Given a balanced cut admissible vector p on the edges of a 2-colored graph G we produce a related graph G' and a cut admissible vector p' on the edges of G'. Theorem 1.1 will then show that p' is a fractional sum of cycles and somehow, using this information, we need to show that p is a fractional sum of balanced subgraphs. We do not know such a proof.

Our proof of Theorem 1.2 is closely modelled on the proof of Theorem 1.1 in [S] and runs parallel to it. Basically, we have to replace the use of Seymour-Giles lemma by an appropriate colored analog. In the original version of this paper (see [BPS2]) we used the Grossmann-Haggvist [GH] lemma for this purpose. This lemma states the following: a bridgeless graph whose edges have been colored red and blue so that both colors are present at every vertex contains a nonempty balanced subgraph (Yeo [Y] generalized this result to an arbitrary number of colors (see Theorem 2.1 in Section 2)). Subsequently, following a suggestion of Jácint Szabó, we formulated a technical result, the colored Seymour-Giles lemma (Theorem 2.2 in Section 2) that interpolates between the Seymour-Giles and Grossmann-Haggvist-Yeo lemmas. Using the colored Seymour-Giles lemma we prove another technical result, Theorem 3.1 in Section 3, that includes both Theorems 1.1 and 1.2 as special cases and whose proof is considerably shorter than the proof of Theorem 1.2 in [BPS2].

2. Colored Seymour-Giles lemma

In this section we formulate a colored Seymour-Giles lemma and show that it follows from the Grossmann-Haggvist-Yeo lemma. We begin with a few definitions and the statement of the Grossmann-Haggvist-Yeo lemma.

Let $G = (V, E)$ be a graph. A *trail* in G is a sequence

$$T \quad = \quad (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m), \qquad m \geq 0,$$

where $v_i \in V$ for all i, $e_j \in E$ for all j, e_j has endpoints v_{j-1} and v_j for all j, and all the e_j are distinct. We say that T is a $v_0 - v_m$ trail of *length* m. The trail T is said to be *closed* when $v_0 = v_m$.

Now assume that the edges of G are colored with a set S of colors, the coloring being given by $\mathcal{C} : E \to S$. The trail T above is *alternating* when $\mathcal{C}(e_j) \neq \mathcal{C}(e_{j+1})$ for each $j = 1, \ldots, m-1$ and if T is closed then $\mathcal{C}(e_m) \neq \mathcal{C}(e_1)$. A closed alternating trail is abbreviated as CAT. There is a close connection between CAT's and balanced subgraphs when the number of colors is two. Indeed, in the case of two colors, the characteristic vector of a CAT is balanced and, conversely, a simple alternating walk argument (see Theorem 2.2 (iii) in [BPS1]) shows that the characteristic vector of a balanced subgraph of a 2-colored graph can be written as a sum of characteristic vectors of edge-disjoint CAT's.

Theorem 2.1. *(Grossmann-Haggvist-Yeo Lemma* [Y]*) Let $G = (V, E)$ be a bridgeless graph with an edge-coloring $\mathcal{C} : E \to S$. Assume that for each $v \in V$, there are two edges with different colors incident at v. Then $(G, \mathcal{C})$ has a CAT of positive length.*

Actually, a slightly more general result is proved in [Y] but the statement given above follows immediately from it.

Let $G = (V, E)$ be a graph and let $\mathcal{C} : E \to S$ be an edge coloring. Suppose we are given a subset $A \subseteq V$ of vertices, called *alternating vertices*. Vertices in $V - A$ are called *special*. A *closed A-alternating trail (A-CAT)* in G is a closed trail

$$T \;=\; (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_0), \qquad m \geq 0,$$

that alternates at vertices in A, i.e., $v_0 \in A$ implies $\mathcal{C}(e_1) \neq \mathcal{C}(e_m)$ and $v_i \in A$, $1 \leq i \leq m-1$ implies $\mathcal{C}(e_i) \neq \mathcal{C}(e_{i+1})$. Now suppose we are given a function $\phi : V - A \to E$ that maps every special vertex $v \in V - A$ to an edge $\phi(v)$ incident with v. An *(A,ϕ)-closed trail (Aϕ-CT)* is an A-CAT T satisfying the following property: for every special vertex v, the number of edges in T incident with v is 0 or 2 and, in the case this number is 2, one of these edges is $\phi(v)$. Note that when every vertex is alternating an Aϕ-CT is a CAT and when every vertex is special an Aϕ-CT is a cycle C containing the edge $\phi(u)$ for every vertex u on C.

Theorem 2.2. *(Colored Seymour-Giles Lemma) Let $G = (V, E)$ be a bridgeless graph with an edge-coloring $\mathcal{C} : E \to S$. Assume that we are a given a subset $A \subseteq V$ of alternating vertices and a function $\phi : V - A \to E$ mapping every special vertex $v \in V - A$ to an edge $\phi(v)$ incident with v. Assume that for every alternating vertex $v \in A$, there are two edges with different colors incident with v. Then $(G, \mathcal{C})$ has a Aϕ-CT of positive length.*

Proof. Choose a color not in the set S and call this color black. Consider a special vertex $v \in V - A$. Subdivide each edge $e \in E_G(v) - \{\phi(v)\}$ ($E_G(v)$ = set of all edges in G with v as an endpoint) by introducing a new vertex (in the middle of) e. The two edges created get the following colors: the edge incident with v gets colored black and the other edge retains the color of e. Now do this procedure in turn for every special vertex. The resulting edge colored graph H is easily seen to satisfy the hypothesis of Theorem 2.1. A positive length CAT in H gives rise to the required Aϕ-CT in G. □

Szeider [Se] has made a detailed study of theorems equivalent (in the sense of being easily derivable from each other) to the classical Kotzig lemma of matching theory. In particular, he shows that the Seymour-Giles and Grossmann-Haggvist-Yeo lemmas are equivalent to Kotzig's lemma. Since the colored Seymour-Giles lemma trivially implies the Seymour-Giles lemma it follows that Theorem 2.2 is also equivalent to Kotzig's lemma.

3. The Cone of Balanced Subgraphs

In this section we prove Theorem 1.2.

In proving Theorem 1.2 by induction it is technically convenient to incorporate alternating and special vertices and prove the following result.

Theorem 3.1. *Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph and let $A \subseteq V$ be a subset of alternating vertices. A vector $p \in \mathbb{Q}^E$ is a fractional sum of A-CAT's if and only if p is cut admissible and satisfies the balance condition at every alternating vertex.*

When there are no alternating vertices an A-CAT is just a closed trail and, since a closed trail is a disjoint union of cycles, it follows that Theorem 3.1 implies Theorem 1.1. When every vertex is alternating an A-CAT is a CAT and hence Theorem 3.1 also implies Theorem 1.2.

Given a graph $G = (V, E)$, we denote by $\mathcal{K}(G)$ the set of all closed trails in G. If D denotes the cut of all edges between X and $V - X$, we say that X and $V - X$ are the two *sides* of the cut, and their *sizes* are $|X|$ and $|V - X|$. If, for some $e \in D$, the cut condition (1) holds with equality for $q \in \mathbb{R}^E$, the pair (D, e) is said to be *tight for q*.

Lemma 3.2. *Let $G = (V, E)$ be a graph and let $p : E \to \mathbb{Q}$. Let D be a cut in G, and let $e \in D$ be such that (D, e) is tight for p. Suppose p is a fractional sum of closed trails, i.e., p can be expressed as*

$$p = \sum_{C \in \mathcal{K}(G)} \alpha(C)\chi(C), \qquad \alpha(C) \in \mathbb{Q}^+.$$

Let $C \in \mathcal{K}(G)$ with $\alpha(C) > 0$. Then $C \cap D$ is either empty or equal to $\{e, h\}$ for some $h \in D - e$ (we think of C as a set of edges). In other words, (D, e) is tight for every $\chi(C)$ with $\alpha(C) > 0$.

Proof. We have

$$\begin{aligned}
\sum_{C \in \mathcal{K}(G)} |C \cap \{e\}| \, \alpha(C) &= \sum_{C \in \mathcal{K}(G),\, e \in C} \alpha(C) \\
&= p(e) \\
&= p(D - e) \\
&= \sum_{h \in D - e} \sum_{\substack{C \in \mathcal{K}(G) \\ h \in C}} \alpha(C) \\
&= \sum_{C \in \mathcal{K}(G)} |C \cap (D - e)| \, \alpha(C).
\end{aligned}$$

Since each $C \in \mathcal{K}(G)$ satisfies $|C \cap \{e\}| \leq |C \cap (D - e)|$, it follows that each $C \in \mathcal{K}(G)$ with $\alpha(C) > 0$ satisfies $|C \cap \{e\}| = |C \cap (D - e)|$. Since $|C \cap \{e\}| \in \{0, 1\}$, the result follows. □

We now fix some notation to be used in the statement and proof of the next result.

(4) Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph and $A \subseteq V$ a subset of alternating vertices.

(5) Let D be a cut in G with sides X and $V - X$. Set $A_X = A \cap X$ and $A_{V-X} = A \cap (V - X)$. For an edge $h \in D$ we let h_X and h_{V-X} denote the endpoints of h in X and $V - X$, respectively.

(6) Denote by G_X (respectively, G_{V-X}) the graph obtained from G by shrinking X (respectively, $V - X$) to a single vertex (and deleting the resulting loops). The vertex set of G_X is $V - X$ plus the shrunken vertex, which we denote by X. The edge set of G_X consists of all edges h of G that have at least one endpoint in $V - X$. If both endpoints of an edge $h \in E$ are in $V - X$, then it has the same endpoints in G_X; otherwise, if h has only one endpoint in $V - X$, then its endponts in G_X are h_{V-X} and X. We use a similar notation for the graph G_{V-X}.

(7) Let T be a closed trail in the graph G_X which contains exactly two edges of D, say e and h. Then by cyclically shifting T (and reversing direction, if necessary) we may arrange that T has the form (below $*$

denotes concatenation of trails)

$$(e_{V-X}, e, X) * (X, h, h_{V-X}) * T^{V-X},$$

where T^{V-X} is the h_{V-X}-e_{V-X} trail (that is the portion of T) whose vertices are in $V - X$.

Analogously, let T be a closed trail in the graph G_{V-X} which contains exactly two edges of D, say e and h. Then by cyclically shifting T (and reversing direction, if necessary) we may arrange that T has the form

$$(V - X, e, e_X) * T^X * (h_X, h, V - X),$$

where T^X is the e_X-h_X trail (that is the portion of T) whose vertices are in X.

We have the following important observation about this notation. Let $e, h \in D$. Let T_1 be a closed A_{V-X}-alternating trail in G_X that contains exactly two edges of D, e and h. Let T_2 be a closed A_X-alternating trail in G_{V-X} that contains exactly two edges of D, e and h. Then

$$(e_{V-X}, e, e_X) * T_2^X * (h_X, h, h_{V-X}) * T_1^{V-X},$$

is a closed A-alternating trail in G.

(8) $p : E \to \mathbb{N} - \{0\}$. Denote by p_X (respectively, p_{V-X}) the restriction of p to the edges of G_X (respectively, G_{V-X}).

Lemma 3.3. *Let the notation be as in items (4)-(8) above.*

- *(i) If p is cut admissible for G, then p_X (respectively, p_{V-X}) is cut admissible for G_X (respectively, G_{V-X}).*
- *(ii) Suppose that, for some $e \in D$, the pair (D, e) is tight for p and that p_X (respectively, p_{V-X}) is a sum of closed A_{V-X}-alternating trails (respectively, closed A_X-alternating trails) in G_X (respectively, G_{V-X}). Then p is a sum of closed A-alternating trails in G.*

Proof. (i) This follows since each cut in G_X, G_{V-X} is also a cut in G.

(ii) The hypothesis on p_X implies that there is a multiset L_X of closed A_{V-X}-alternating trails in G_X such that every edge h in G_X appears $p_X(h)$ times in the various trails contained in L_X. Similarly, there is a multiset L_{V-X} of closed A_X-alternating trails in G_{V-X} such that every edge h in G_{V-X} appears $p_{V-X}(h)$ times in the various trails contained in L_{V-X}.

Consider a trail in L_X or L_{V-X} that intersects D. By Lemma 3.2, the intersection of each such trail with D must be $\{e,h\}$, for some $h \in D-e$. For $h \in D-e$, let $L_X(h)$ (respectively, $L_{V-X}(h)$) consist of the multiset of trails in L_X (respectively, L_{V-X}) whose intersection with D is $\{e,h\}$. By the definition of L_X and L_{V-X}, p_X and p_{V-X}, we have $|L_X(h)| = p_X(h) = p(h) = p_{V-X}(h) = |L_{V-X}(h)|$ for each $h \in D-e$. For each $h \in D-e$, fix a bijection $\phi_h : L_X(h) \to L_{V-X}(h)$.

We now build a multiset L of closed A-alternating trails in G such that every edge h in G appears $p(h)$ times in the trails contained in L. This will prove the result.

We first take L to be empty and add trails to it as follows:

- Add to L all trails in L_X whose vertices are contained in $V-X$ (each such trail is added the same number of times as it appears in L_X).
- Add to L all trails in L_{V-X} whose vertices are contained in X.
- For every $h \in D-e$ and every $T \in L_X(h)$ add the trail

$$(e_{V-X}, e, e_X) * (\phi_h(T))^X * (h_X, h, h_{V-X}) * T^{V-X},$$

  to L.

It is easily checked that each edge h in G appears $p(h)$ times in the trails contained in L and that each trail in L is A-alternating. □

We now give the proof of the main result of this section.

Proof. (*of Theorem 3.1*) (only if): This is clear.

(if): Consider a vector $p : E \to \mathbb{Q}$ that satisfies the balance condition at every alternating vertex in A and that is cut admissible for G. Without loss of generality we may assume that $p(e) > 0$ for all $e \in E$ (we may drop edges e with $p(e) = 0$ and maintain the balance condition at alternating vertices and cut admissibility). The proof is by induction on the pairs $(|V|, |E|)$ ordered lexicographically.

The following two cases arise.

Case (i): there exists a cut D in G with sides X and $V-X$ of sizes at least 2, and an edge $e \in D$ such that (D,e) is tight for p.

Clearly p_X satisfies the balance condition at every vertex in A_{V-X} and, by Lemma 3.3(i), is cut admissible for G_X. Since G_X has fewer vertices than G, we see by induction that p_X is a fractional sum of closed A_{V-X}-alternating trails in G_X. Similarly, p_{V-X} is a fractional sum of closed A_X-alternating trails in G_{V-X}. Thus, for a suitably large positive integer M, Mp_X is a sum of closed A_{V-X}-alternating trails in G_X and Mp_{V-X} is a sum of closed A_X-alternating trails in G_{V-X}. Thus, by Lemma 3.3(ii), Mp is a sum of closed A-alternating trails in G and thus p is a fractional sum of closed A-alternating trails in G.

Case (ii): for each cut D in G with sides X and $V-X$ of sizes at least 2 and each $e \in D$, we have $p(e) < p(D-e)$.

It follows from the hypothesis for this case that if the pair (D, e) is tight for p then $D = E_G(v)$ for some $v \in V$. Define a map $\phi : V - A \to E$ as follows: let $v \in V - A$. If the pair $(E_G(v), e)$ is tight for p for some (unique) edge $e \in E_G(v)$ then put $\phi(v) = e$, otherwise let $\phi(v)$ be an arbitrary edge in $E_G(v)$. Since p is positive on every edge and cut admissible for G, it follows that G is bridgeless. Since p is positive and balanced at every alternating vertex, it follows that both colors are present at every alternating vertex. Thus p satisfies the hypothesis of Theorem 2.2 and it follows that G has a Aϕ-CT T.

Consider the vector $p_t = p - t\chi(T),\ t \geq 0$. We claim that

- For all $t \geq 0$, p_t is balanced at every alternating vertex in A.
- For all sufficiently small $t > 0$, p_t is positive, i.e, $p_t(e) > 0$, for all $e \in E$.
- For all sufficiently small $t > 0$, p_t is cut admissible.

The first claim follows from the fact that both p and $\chi(T)$ are balanced at the alternating vertices. The second claim follows from the fact that p is positive. We now show the third claim. Let D be a cut in G with sides X and $V-X$ and let $e \in D$. We have the following two subcases.

Case (a): $p(e) < p(D-e)$. Clearly $p_t(e) < p_t(D-e)$ for all sufficiently small $t > 0$.

Case (b): $p(e) = p(D-e)$. By assumption, one of X and $V-X$, say X, has size 1. Let v be the unique vertex in X. We claim that either T contains no edge of D or it contains precisely two edges of D, one of which is e . If v is special this follows from the definition of a Aϕ-CT. If v is alternating then, since p is positive and balanced at v and $p(e) = p(D-e)$, we have that all the edges in $D-e$ have the same color and this color is different than the color of e. The claim about T follows since T alternates at v. Thus $p_t(e) = p_t(D - e)$ for all $t \geq 0$.

From these considerations we see that the maximum value of t such that $p_t(e) \geq 0$ for all $e \in E$, p_t is balanced at every alternating vertex, and p_t is cut admissible for G is a positive finite rational t_0. Set $q = p_{t_0}$. The following two subcases arise:

Subcase (ii.1): $q(f) = 0$ for some $f \in E$. By dropping f we obtain a graph with the same number of vertices as G but with fewer edges. By induction, q is a fractional sum of A-CAT's and thus so is $p = q + t_0\chi(T)$.

Subcase (ii.2): $q(f) > 0$ for all $f \in E$. From case (b) above we see that $p(e) = p(D - e)$ implies $q(e) = q(D - e)$. Since q is positive on every edge, it must be that the cutoff determining t_0 occurs by case (a) above and not by case (b) or by the requirement that $p_t \geq 0$. Therefore there is a cut D^* and an edge $e^* \in D^*$ such that $p(e^*) < p(D^* - e^*)$ and $q(e^*) = q(D^* - e^*)$. Thus q is a positive rational vector, cut admissible for G, balanced at alternating vertices, and more pairs (D, e) are tight for q than for p. We may now repeat the whole argument with q in place of p. Since the total number of pairs (D, e) where D is a cut in G and $e \in D$ is finite, eventually we will reach case (i) or subcase (ii.1). □

Finally, we would like to propose the following conjecture. Theorem 1.2 has a reformulation that reads: a nonnegative rational vector p on the edges of a 2-colored graph is a fractional sum of balanced subgraphs if and only if p is a balanced fractional sum of cycles. We conjecture a stronger integral version of this statement.

Conjecture 3.1. *Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph. A vector $p \in \mathbb{N}^E$ is sum of balanced subgraphs if and only if p is a balanced sum of cycles.*

Note that, by Theorem 1.2, a balanced sum of cycles is a fractional sum of balanced subgraphs.

There is a well-known conjecture due to Seymour [S] asserting that a fractional sum of cycles that is an even integer on every edge is a sum of cycles. Analogously, in the 2-colored case, we can conjecture that a balanced fractional sum of cycles that is an even integer on every edge is a sum of balanced subgraphs. This latter conjecture follows from Seymour's conjecture and Conjecture 3.1. There is a significant difference between Seymour's conjecture and Conjecture 3.1. Namely, the hypothesis in Seymour's conjecture (that of a vector being a fractional sum of cycles) is well characterized (by means of cut admissibility) whereas we do not know whether the property of an edge weighted vector being a sum of cycles is well characterized.

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