The Eulerian generating function of q-derangements

Murali K. Srinivasan
Department of Mathematics
Indian Institute of Technology, Bombay
Powai, Mumbai, INDIA 400076
email: mks@math.iitb.ac.in

Abstract

Let $\mathbb{F}_q$ denote the finite field with q elements. For nonnegative integers n, k, let $d_q(n,k)$ denote the number of $n \times n$ $\mathbb{F}_q$-matrices having k as the sum of the dimensions of the eigenspaces (of the eigenvalues lying in $\mathbb{F}_q$). Let $d_q(n) = d_q(n,0)$, i.e., $d_q(n)$ denotes the number of $n \times n$ $\mathbb{F}_q$-matrices having no eigenvalues in $\mathbb{F}_q$. The Eulerian generating function of $d_q(n)$ has been well studied in the last twenty years (Kung [**12**], Neumann and Praeger [**15**], and Stong [**19**]). The main tools have been the rational canonical form, nilpotent matrices, and a q-series identity of Euler. In this paper we take an elementary approach to this problem, based on Möbius inversion, and find the following bivariate generating function:

$$\sum_{n,k\geq 0} d_q(n,k)\, y^k \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!} =$$
$$\left(\sum_{n\geq 0}(y-1)(y-q)\cdots(y-q^{n-1})\,\frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right)^q \left(\sum_{n\geq 0} q^{n^2}\frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right).$$

Key Words: Eulerian generating functions, q-analog of derangements, eigenvalue-free matrices over a finite field.
AMS Subject Classification (2000): 05A15, 05A30.

1 Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements and let $r_q(n)$ denote the number of monic polynomials of degree n in $\mathbb{F}_q[x]$ that have no root in $\mathbb{F}_q$. Zsigmondy [**21**] proved in 1894 that $r_q(n) = \sum_{k=0}^{t}(-1)^k\binom{q}{k}q^{n-k}$, where $t = \min\,\{q,n\}$.

Let $d(n)$ denote the number of derangements on n letters, i.e., the number of permutations of n letters that have no fixed points. The problem of counting derangements has a long history, going back to Montmort in 1713, and it is well known (see, for instance, Example 2.2.1 in Stanley **[17]** or Example 4 of Section 2.3 in Wilf [**20**]) that $d(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!$.

Let $d_q(n)$ denote the number of $n \times n$ $\mathbb{F}_q$-matrices whose characteristic polynomials have no root in $\mathbb{F}_q$, i.e., $d_q(n)$ denotes the number of $n \times n$ $\mathbb{F}_q$-matrices that have no eigenvalue in $\mathbb{F}_q$. Clearly, this is a natural counting coefficient to consider in view of Zsigmondy's result. On the other hand, an eigenvalue-free $n \times n$ matrix does not fix any one dimensional subspace of $(\mathbb{F}_q)^n$ and thus $d_q(n)$ can be considered as a q-analog of $d(n)$. We call an eigenvalue free square $\mathbb{F}_q$-matrix a *q-derangement*.

The analogies between the three counting problems considered above are best brought out in the language of generating functions. We use the symbol $\mathbb{N}$ for the set of nonnegative integers and $\mathbb{P}$ for the set of positive integers.

Let K denote a field of characteristic 0. Given a function $f : \mathbb{N} \to K$ define its *exponential generating function* by

$$E_f(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!},$$

and define its *Eulerian generating function* by

$$Q_f(x) = \sum_{n \geq 0} f(n) \frac{x^n}{q^{\binom{n}{2}} (\mathbf{n})!},$$

where $(\mathbf{n})! = (\mathbf{1})(\mathbf{2}) \cdots (\mathbf{n})$ and $(\mathbf{k}) = 1 + q + q^2 + \cdots + q^{k-1}$. Our definition of the Eulerian generating function of a sequence is different from that used in **[9]** and **[17]** but is convenient for our purposes. In general, if $M = \{M(n)\}_{n \geq 0}$ is a sequence consisting only of nonzero terms we define the *generating function of f with denominator sequence M* to be

$$\sum_{n \geq 0} f(n) \frac{x^n}{M(n)}.$$

Define the following power series

$$\begin{aligned}
\exp(x) &= \sum_{n\geq 0} \frac{x^n}{n!}, \\
\exp_q(x) &= \sum_{n\geq 0} \frac{x^n}{(\mathbf{n})!}, \\
\mathrm{Mon}_q(x) &= \sum_{n\geq 0} q^n x^n, \\
\mathrm{Perm}(x) &= \sum_{n\geq 0} n! \frac{x^n}{n!}, \\
\mathrm{Mat}_q(x) &= \sum_{n\geq 0} q^{n^2} \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}.
\end{aligned}$$

We can think of $\mathrm{Mon}_q(x)$, $\mathrm{Perm}(x)$, and $\mathrm{Mat}_q(x)$ as, respectively, the ordinary, exponential, and Eulerian generating functions for the number of monic polynomials of degree n, the number of permutations on n letters, and the number of $n \times n$ $\mathbb{F}_q$-matrices.

The formulas for $r_q(n)$ and $d(n)$ given above can be stated in terms of generating functions as follows:

$$\sum_{n\geq 0} r_q(n)x^n = (1-x)^q \, \mathrm{Mon}_q(x) \tag{1}$$

$$\sum_{n\geq 0} d(n)\frac{x^n}{n!} = \exp(-x) \, \mathrm{Perm}(x) \tag{2}$$

The following result is a corollary of our main theorem (Theorem 2 below).

Theorem 1

$$\sum_{n\geq 0} d_q(n) \, \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!} = \left(\exp_q(-x)\right)^q \ \mathrm{Mat}_q(x).$$

We now discuss proof techniques, along with history, for the problem of counting q-derangements. Let us first consider formulas (1) and (2) above. There are two common ways of proving (2): either by specializing Pólya's cycle index or by an inclusion-exclusion argument. Introducing a variable for each monic irreducible in $\mathbb{F}_q[x]$ and using unique factorization in $\mathbb{F}_q[x]$ leads to a "cycle index" type generating function for all monic polynomials in $\mathbb{F}_q[x]$ and an appropriate specialization then easily yields (1). A simple inclusion-exclusion argument can also be given for (1).

Motivated by Pólya's cycle index for the symmetric group, Kung [**12**] defined a cycle index for the conjugation action of $\mathrm{GL}(n,\mathbb{F}_q)$ on itself. The cycle decomposition of a permutation is here replaced by the rational canonical form of a matrix. This definition is easily extended to the conjugation action of $\mathrm{GL}(n,\mathbb{F}_q)$ on all $n \times n$ $\mathbb{F}_q$-matrices. For the many applications of the vector space cycle index see the survey paper of Fulman [**7**]. In principle, Theorem 1 should follow by an appropriate specialization of the cycle index. Let us see what is involved in this. We do not recall here the definition of the vector space cycle index but instead note the following consequence of the main theorem on the factorization of the cycle index (see Fulman [**7**], Kung [**12**], Neumann and Praeger [**15**], and Stong [**19**]): let $c_q(n)$ denote the number of nilpotent $n \times n$ $\mathbb{F}_q$-matrices. Then

$$\left(\sum_{n\geq 0} d_q(n)\ \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right)\left(\sum_{n\geq 0} c_q(n)\ \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right)^q \quad = \quad \mathrm{Mat}_q(x). \tag{3}$$

Now recall the following fundamental identity (see Corollary 2.2 in [**1**] or Section 3.15 in [**17**])

$$(\exp_q(x))^{-1} = \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{x^n}{(\mathbf{n})!} = \sum_{n\geq 0}(-1)^n q^{n(n-1)}\frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}.$$

Given identity (3) above, it now follows that proving Theorem 1 is equivalent to showing $c_q(n) = q^{n(n-1)}$. This is precisely the Fine-Herstein theorem [**5**] , proved in 1958. We remark that the references cited here work in the space of all nonsingular matrices (rather than all matrices) and use $\{\#\mathrm{GL}(n,\mathbb{F}_q)\}_{n\geq 0}$ as the denominator sequence for their generating functions. This makes no significant difference. For instance, Theorems 4.1 and 4.2 (taken together) in Neumann and Praeger [**15**] and identity (5) in Stong [**19**], though stated differently, are in essence equivalent to our Theorem 1.

In this paper we count q-derangements directly, without recourse to nilpotents and the rational canonical form. Our method generalizes the inclusion-exclusion proof of formulas (1) and (2) above. Our main result (Theorem 2 below) is a natural bivariate generalization of Theorem 1. As is clear from the discussion in the preceding paragraph, Theorem 1 (with the present proof) together with (3) above, yields the Fine-Herstein theorem as a corollary.

For nonnegative integers n, k, define $d_q(n,k)$ to be the number of $n \times n$ $\mathbb{F}_q$-matrices having k as the sum of the dimensions of the eigenspaces (of the eigenvalues lying in $\mathbb{F}_q$). Note that $d_q(n) = d_q(n,0)$. In Section 3 we show that

Theorem 2

$$\sum_{n,k\geq 0} d_q(n,k)\; y^k \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!} =$$
$$\left(\sum_{n\geq 0}(y-1)(y-q)\cdots(y-q^{n-1})\;\frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right)^q \left(\sum_{n\geq 0} q^{n^2}\frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!}\right).$$

Substituting $y = 0$ we get Theorem 1.

Our proof of Theorem 2 proceeds in three steps: First we give a combinatorial interpretation to multiplication of Eulerian generating functions. Then, using Möbius inversion, we write down a formula for $d_q(n,k)$. Finally, using the combinatorial interpretation, we extract the coefficient of $y^k x^n / q^{\binom{n}{2}}(\mathbf{n})!$ on the right hand side and show that it is equal to $d_q(n,k)$.

2 Eulerian Generating Functions

In this section we give a combinatorial interpretation to multiplication of Eulerian generating functions. Multiplication and composition of generating functions can be studied from several different perspectives (see [**2, 3, 4, 6, 8, 9, 10, 13, 14, 17, 18, 20**]). We follow the approach (for exponential generating functions) given in Chapter 5, Section 1 of [**18**].

Eulerian generating functions can be thought of as q-analogs of exponential generating functions. There are two natural choices for the denominator sequence of Eulerian generating functions. If we consider multiplication of exponential generating functions from the point of view of binomial posets then the natural choice of the denominator sequence for the q-analog is $\{(\mathbf{n})!\}_{n\geq 0}$ (see Section 3.15 of [**17**]). On the other hand, if we consider multiplication of exponential generating functions from the point of view of species theory (Section 1.3 in [**3**] and Section 5.1 in [**18**]) then, for the q-analog, we get the denominator sequence $\{q^{\binom{n}{2}}(\mathbf{n})!\}_{n\geq 0}$.

Let V be a finite dimensional vector space over $\mathbb{F}_q$. A *weak ordered partition* of V is a sequence $\pi = (V_1, V_2, \ldots, V_k)$ of subspaces of V such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ (direct sum). Note that this allows some of the V_i's to be the zero subspace. The tuple $(c_1, c_2, \ldots, c_k)$, where $c_i = \dim V_i$ is said to be the *type* of π and we say that π has k *blocks*. We write $O_k(V)$ for the set of all weak ordered partitions of V into k blocks. Let $W_1 \subseteq V$ be a subspace. By a *complement* of W_1 in V we mean a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Lemma 3 *Let V be an n-dimensional vector space over $\mathbb{F}_q$ and let $W \subseteq V$ be a subspace of dimension k. Then the number of complements of W is $q^{k(n-k)}$.*

Proof The number of linearly independent sequences $(v_1, v_2, \ldots, v_{n-k})$ such that $V = W \oplus \text{Span}(\{v_1, \ldots, v_{n-k}\})$ is $(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{n-1})$. It follows that the number of complements of W is given by

$$\frac{(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{n-1})}{(q^{n-k} - 1)(q^{n-k} - q) \cdots (q^{n-k} - q^{n-k-1})} = q^{k(n-k)},$$

completing the proof.□

Lemma 4 *Let V be an n-dimensional vector space over $\mathbb{F}_q$ and let $c = (c_1, \ldots, c_k) \in \mathbb{N}^k$ with $\sum_i c_i = n$. Then the number of $\pi \in O_k(V)$ with type $(\pi) = c$ is given by*

$$\frac{q^{\binom{n}{2}} (\mathbf{n})!}{q^{\binom{c_1}{2}} (\mathbf{c_1})! \cdots q^{\binom{c_k}{2}} (\mathbf{c_k})!}$$

Proof We use the well known result that the number of k-dimensional subspaces of V is given by $\binom{\mathbf{n}}{\mathbf{k}} = \frac{(\mathbf{n})!}{(\mathbf{k})!(\mathbf{n-k})!}$ (Proposition 1.3.18 in [**17**]). To choose $\pi = (V_1, V_2, \ldots, V_k) \in O_k(V)$ with type$(\pi) = c$, we proceed as follows: First choose V_1 of dimension c_1. Then choose a complement of V_1. In this complement, choose V_2 of dimension c_2. And so on. It follows that the number of elements in $O_k(V)$ of type c is

$$\binom{\mathbf{n}}{\mathbf{c_1}} \times q^{c_1(n-c_1)} \times \binom{\mathbf{n - c_1}}{\mathbf{c_2}} \times q^{c_2(n-c_1-c_2)} \times \cdots \times \binom{\mathbf{n - c_1 - \cdots - c_{k-1}}}{\mathbf{c_k}} \times q^{c_k(n-c_1-\cdots-c_k)}.$$

This expression is easily seen to be equal to the expression given in the statement of the lemma.□

Theorem 5 *Fix $k \in \mathbb{P}$ and functions $f_1, f_2, \ldots, f_k : \mathbb{N} \to K$. Define a new function $h : \mathbb{N} \to K$ by*

$$h(\dim V) = \sum f_1(\dim V_1) f_2(\dim V_2) \cdots f_k(\dim V_k),$$

where the sum is over all $(V_1, \ldots, V_k) \in O_k(V)$. Then the Eulerian generating function of h is given by

$$Q_h(x) = Q_{f_1}(x) Q_{f_2}(x) \cdots Q_{f_k}(x).$$

Proof This is immediate from Lemma 4.□

Though not required for the problem considered in the next section, for completeness we now give the composition formula. By a *partition* of a finite dimensional vector space V over $\mathbb{F}_q$ we mean a set $\{V_1, V_2, \ldots, V_k\}$ of nonzero subspaces such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$. Let $\Pi(V)$ denote the set of all partitions of V.

Theorem 6 *Given functions $f : \mathbb{P} \to K$ and $g : \mathbb{N} \to K$ with $g(0) = 1$, define a new function $h : \mathbb{N} \to K$ by $h(0) = 1$ and*

$$h(\dim V) = \sum_{\sigma = \{V_1, \ldots, V_k\}} f(\dim V_1) f(\dim V_2) \cdots f(\dim V_k) g(k), \quad \dim V > 0,$$

where the sum ranges over all $\sigma \in \Pi(V)$. Then

$$Q_h(x) = E_g(Q_f(x)).$$

We omit the proof as it is virtually identical to the proof of Theorem 5.1.4 in [**18**]. It is also a special case of the composition formula for exponential structures (see [**16**]). For another perspective on this formula and some applications see [**2**], [**13, 14**].

3 Square Matrices with no Eigenvalue in $\mathbb{F}_q$

Let V be a finite dimensional vector space over $\mathbb{F}_q$ and let $\phi : V \to V$ be a linear map. For $\alpha \in \mathbb{F}_q$, the subspace $S(\phi, \alpha) = \{v \in V : \phi(v) = \alpha v\}$ is called the *α-invariant subspace* of ϕ. Note that this could be the zero subspace.

For notational convenience, for the rest of this section we fix a bijection $\Gamma : [q] \to \mathbb{F}_q$, where $[q] = \{1, 2, \ldots, q\}$.

Before discussing eigenvalue free matrices, we first consider the case of diagonalizable matrices. For a nonnegative integer n, define

$$D_q(n) = \text{Number of diagonalizable } n \times n\ \mathbb{F}_q\text{-matrices.}$$

We have the following Eulerian generating function for the sequence $\{D_q(n)\}_{n\geq 0}$:

Theorem 7
$$\sum_{n\geq 0} D_q(n) \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!} = \left(\sum_{n\geq 0} \frac{x^n}{q^{\binom{n}{2}}(\mathbf{n})!} \right)^q .$$

Proof Let V be a finite dimensional vector space over $\mathbb{F}_q$. There is a bijection between diagonalizable operators on V and $O_q(V)$: with a diagonalizable linear map $\phi : V \to V$ associate the sequence $(S(\phi, \Gamma(1)), \ldots, S(\phi, \Gamma(q))) \in O_q(V)$. The result now follows from Theorem 5, applied to the constant functions
$$f_1, f_2, \ldots, f_q : \mathbb{N} \to K,$$
where $f_i(n) = 1$, for all i and n. □

Diagonalizable $n \times n$ matrices are one end of a spectrum, the other end being $n \times n$ matrices having no eigenvalue. We now relate these two families of matrices through Möbius inversion on an appropriate poset. Specifically, we define a rank-n graded poset and two functions F and G related by Möbius inversion on this poset, such that the following holds: The sum of the values of F at the rank-n elements equals $D_q(n)$ and the value of G at the minimum element is $d_q(n)$. More generally, $d_q(n, k)$ is the sum of the values of G at the rank-k elements.

Let $B_q(n)$ denote the poset of subspaces (under inclusion) of an n-dimensional vector space over $\mathbb{F}_q$. Let V be a finite dimensional vector space over $\mathbb{F}_q$. Define a poset $P(V)$, whose underlying set is

$$P(V) = \coprod_W O_q(W),$$

where the (disjoint) union is over all subspaces W of V and where $(V_1, V_2, \ldots, V_q) \leq (W_1, W_2, \ldots, W_q)$ if and only if $V_i \subseteq W_i$, for all i. It is easily seen that $P(V)$ is a rank-n graded poset with rank function given by rank $(V_1, \ldots, V_q) = \sum_{i=1}^q r(V_i)$ (we denote the dimension of a vector space V by $r(V)$). We denote by $P_k(V)$ the set of elements of $P(V)$ of rank k. The Möbius function of $P(V)$ is readily calculated. Given $(V_1, \ldots, V_q) \le (W_1, \ldots, W_q)$ it is easy to see that the interval $[(V_1, \ldots, V_q), (W_1, \ldots, W_q)]$ is order isomorphic to

$$B_q(r(W_1) - r(V_1)) \times B_q(r(W_2) - r(V_2)) \times \cdots \times B_q(r(W_q) - r(V_q)).$$

It follows by the Product theorem (Proposition 3.8.2 in [**17**]) and Hall's theorem ([**11**]) on the Möbius function of $B_q(n)$ (Example 3.10.2 in [**17**]) that the Möbius function of $P(V)$ is given by: for $(V_1, \ldots, V_q) \le (W_1, \ldots, W_q)$

$$\mu((V_1, \ldots, V_q), (W_1, \ldots, W_q)) = \prod_{i=1}^{q} (-1)^{r(W_i) - r(V_i)} \; q^{\binom{r(W_i) - r(V_i)}{2}}.$$

Let $\dim V = n$. Define functions $F : P(V) \to K$ and $G : P(V) \to K$ as follows

$$F((V_1, \ldots, V_q)) = \# \left\{ \phi : V \to V : \phi \text{ is } \mathbb{F}_q\text{-linear and } V_i \subseteq S(\phi, \Gamma(i)), \text{ for all } i \in [q] \right\},$$

$$G((V_1, \ldots, V_q)) = \# \left\{ \phi : V \to V : \phi \text{ is } \mathbb{F}_q\text{-linear and } V_i = S(\phi, \Gamma(i)), \text{ for all } i \in [q] \right\}.$$

A little reflection shows that

$$F((V_1, \ldots, V_q)) = q^{n(n - \sum_{i=1}^{q} r(V_i))},$$

$$F((V_1, \ldots, V_q)) = \sum G((W_1, \ldots, W_q)),$$

where the sum is over all $(W_1, \ldots, W_q) \in P(V)$ satisfying $(V_1, \ldots, V_q) \le (W_1, \ldots, W_q)$.

It follows by the Möbius inversion formula that

$$\begin{aligned}
& d_q(n,k) \\
& = \sum_{(V_1,\ldots,V_q) \in P_k(V)} G((V_1, \ldots, V_q)) \\
& = \sum_{(V_1,\ldots,V_q) \in P_k(V)} \; \sum_{(W_1,\ldots,W_q) \ge (V_1,\ldots,V_q)} \left\{ \prod_{i=1}^{q} (-1)^{r(W_i) - r(V_i)} \; q^{\binom{r(W_i) - r(V_i)}{2}} \right\} \; q^{n(n - \sum_{j=1}^{q} r(W_j))}.
\end{aligned}$$

Interchanging the order of summation we see that $d_q(n,k)$ is equal to

$$\sum_{(W_1,\ldots,W_q) \in P(V)} \left\{ \sum_{(W_1,\ldots,W_q) \ge (V_1,\ldots,V_q) \in P_k(V)} \; \prod_{i=1}^{q} (-1)^{r(W_i) - r(V_i)} \; q^{\binom{r(W_i) - r(V_i)}{2}} \right\} \; q^{n(n - \sum_{j=1}^{q} r(W_j))}.$$

Before giving the proof of Theorem 2, we recall the following formula, which is a version of the q-binomial theorem (for a proof, see the solution to Exercise 45a of Chapter 3 of [**17**]).

Lemma 8 *Let W be a finite dimensional vector space over $\mathbb{F}_q$. Then*

$$\sum_U (-1)^{r(W)-r(U)} q^{\binom{r(W)-r(U)}{2}} y^{r(U)} = (y-1)(y-q)\cdots(y-q^{r(W)-1}),$$

where the sum is over all subspaces U of W.

Proof of Theorem 2 Let V be an n-dimensional vector space over $\mathbb{F}_q$. We can think of the right-hand side as a product of $q+1$ Eulerian generating functions (in x). Thus, from Theorem 5, we see that the coefficient of $x^n/q^{\binom{n}{2}}(\mathbf{n})!$ in the rhs is given by

$$\sum_{(W_1,\ldots,W_{q+1})\in O_{q+1}(V)} \left(\prod_{i=1}^{q} \left\{ (y-1)(y-q)\cdots(y-q^{r(W_i)-1}) \right\} \right) q^{r(W_{q+1})^2}.$$

Now $r(W_{q+1}) = n - \sum_{j=1}^{q} r(W_j)$. For fixed $W_1,\ldots,W_q$, the number of choices for W_{q+1}, by Lemma 3, is q^t, where $t = (\sum_{j=1}^{q} r(W_j))(n - \sum_{j=1}^{q} r(W_j))$. Since

$$q^{r(W_{q+1})^2}\, q^t = q^{n(n-\sum_{j=1}^{q} r(W_j))},$$

we see that the expression above can be rewritten as

$$\sum_{(W_1,\ldots,W_q)\in P(V)} \left(\prod_{i=1}^{q} \left\{ (y-1)(y-q)\cdots(y-q^{r(W_i)-1}) \right\} \right) q^{n(n-\sum_{j=1}^{q} r(W_j))}.$$

Applying Lemma 8, the expression above can be further rewritten as

$$\sum_{(W_1,\ldots,W_q)\in P(V)} \left(\prod_{i=1}^{q} \left\{ \sum_{V_i\subseteq W_i} (-1)^{r(W_i)-r(V_i)} q^{\binom{r(W_i)-r(V_i)}{2}} y^{r(V_i)} \right\} \right) q^{n(n-\sum_{j=1}^{q} r(W_j))}.$$

The coefficient of y^k in the expression above is now easily seen to equal the formula for $d_q(n,k)$ displayed before Lemma 8. That completes the proof.$\Box$

References

[1] G. Andrews, *The theory of partitions*, Addison-Wesley, 1976.

[2] E. A. Bender, and J. R. Goldman, Enumerative uses of generating functions, *Indiana Univ. Math. J.* **20** 1970/71, 753-765.

[3] F. Bergeron, G.Labelle, and P.Leroux, *Combinatorial species and tree-like structures*, Cambridge University Press, 1998.

[4] P. Doubilet, R. Stanley, and G. -C. Rota, *On the foundations of combinatorial theory VI: the idea of generating functions*, Sixth Berkeley Symposium on Mathematics, Statistics, and Probability, volume 2: Probability Theory, University of California 1972, 267-318.

[5] N. J. Fine, and I. N. Herstein, The probability that a matrix be nilpotent, *Illinois J. Math.* **2** 1958, 499-504.

[6] D. Foata, and M. P. Schützenberger, *Theorie geométrique des polynômes Eulériens*, Lecture Notes in Mathematics, no. 138, Springer-Verlag, 1970.

[7] J. Fulman, Random matrix theory over finite fields, *Bull. Amer. Math. Soc. (N.S.)* **39**(1) 2002, 51-85.

[8] I. Gessel, A q-analog of the exponential formula, *Discrete Math.* **40** 1982, 69-80.

[9] J. R. Goldman, and G. -C. Rota, On the foundations of combinatorial Theory IV: finite vector spaces and Eulerian generating functions, *Stud. Appl. Math.* **49**(3) 1970, 239-258.

[10] I. P. Goulden, and D. M. Jackson, *Combinatorial enumeration*, John Wiley and Sons, 1983.

[11] P. Hall, The Eulerian functions of a group, *Quaterly Journal of Mathematics* **7** 1936, 134-151.

[12] J. P. S. Kung, The cycle structure of a linear transformation over a finite field, *Linear Algebra Appl.* **36** 1981, 141-155.

[13] K. E. Morrison, q-Exponential families, *Electronic Journal of Combinatorics* **11** 2004, #R36.

[14] K. E. Morrison, An introduction to q-Species, *Electronic Journal of Combinatorics* **12** 2005, #R62.

[15] P. M. Neumann, and C. E. Praeger, Derangements and eigenvalue-free elements in finite classical groups, *J. London Math. Soc. (2)* **58** 1998, 564-586.

[16] R. P. Stanley, Exponential structures, *Stud. Appl. Math.* **59**(1) 1978, 73-82.

[17] R. P. Stanley, *Enumerative Combinatorics - Volume 1*, Cambridge University Press, 1997.

[18] R. P. Stanley, *Enumerative Combinatorics - Volume 2*, Cambridge University Press, 1999.

[19] R. Stong, Some asymptotic results on finite vector spaces, *Adv. in Appl. Math.* **9**(2) 1988, 167-199.

[20] H. S. Wilf, *generatingfunctionology* Second Edition, Academic Press, 1994.

[21] K. Zsigmondy, Ueber die Anzahl derjenigen ganzen ganzzahligen Functionen nten Grades von x, welche in Bezug auf einen gegebenen Primzahlmodul eine vorgeschriebene Anzahl von Wurzeln besitzen, *Sitzungsber. Wien Abt. II* **103**, 1894, 135-144.