Notes on explicit block diagonalization

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Abstract

In these expository notes we present a unified approach to explicit block diagonalization of the commutant of the symmetric group action on the Boolean algebra and of the nonbinary and q-analogs of this commutant.

1 Introduction

We present a unified approach to explicit block diagonalization in three classical cases: the commutant of the symmetric group action on the Boolean algebra and the nonbinary and q-analogs of this commutant.

Let $B(n)$ denote the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ and, for a prime power q, let $B(q, n)$ denote the set of all subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$. Let $p \geq 2$ and let $A(p)$ denote the alphabet $\{L_0, L_1, \ldots, L_{p-1}\}$ with p letters. Define $B_p(n) = \{(a_1, \ldots, a_n) : a_i \in A(p) \text{ for all } i\}$, the set of all n-tuples of elements of $A(p)$ (we use $\{L_0, \ldots, L_{p-1}\}$ rather than $\{0, \ldots, p-1\}$ as the alphabet for later convenience. We do not want to confuse the letter 0 with the vector 0).

Let S_n denote the symmetric group on n letters and let $S_p(n)$ denote the wreath product $S_{p-1} \sim S_n$. The natural actions of S_n on $B(n)$, $S_p(n)$ on $B_p(n)$ (permute the n coordinates followed by independently permuting the nonzero letters $\{L_1, \ldots, L_{p-1}\}$ at each of the n coordinates), and $GL(n, \mathbb{F}_q)$ on $B(q, n)$ have been classical objects of study. Recently the problem of explicitly block diagonalizing the commutants of these actions has been extensively studied. In these expository notes we revisit these three results. Our main sources are the papers by Schrijver [**17**], Gijswijt, Schrijver, and Tanaka [**9**], and Terwilliger [**22**].

We emphasize that there are several other classical and recent references offering an alternative approach and different perspective on the topic of this paper. We mention Bachoc [**2**], Ceccherini-Silberstein, Scarabotti, and Tolli [**3**], Delsarte [**4, 5**], Dunkl [**6, 7**], Go [**10**], Eisfeld [**8**], Marco, and Parcet [**12, 13, 14**], Tarnanen, Aaltonen, and Goethals [**21**], and Vallentin [**23**].

In Section 2 we recall (without proof) a result of Terwilliger [**22**] on the singular values of the up operator on subspaces. In [**18**], the $q = 1$ case of this result, together with binomial inversion, was used to derive Schrijver's [**17**] explicit block diagonalization of the commutant of the S_n action on $B(n)$. In Section 3 we show that the general case of Terwilliger's result, together with q-binomial inversion, yields the explicit block diagonalization of the commutant of the $GL(n, \mathbb{F}_q)$ action on $B(q, n)$. In Section 4 we define the concept of upper Boolean decomposition and use it to reduce the explicit block diagonalization of the commutant of the $S_p(n)$ action on $B_p(n)$ to the binary case, i.e., the commutant of the S_n action on $B(n)$. The overall pattern of our proof is the same as in Gijswijt, Schrijver, and Tanaka [**9**] but the concept of upper Boolean decomposition adds useful additional insight to the reduction from the nonbinary to the binary case.

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2 Singular values

All undefined poset terminology is from [**20**]. Let P be a finite *graded poset* with *rank function* $r : P \to \mathbb{N} = \{0, 1, 2, \ldots\}$. The *rank* of P is $r(P) = \max\{r(x) : x \in P\}$ and, for $i = 0, 1, \ldots, r(P)$, P_i denotes the set of elements of P of rank i. For a subset $S \subseteq P$, we set $\text{rankset}(S) = \{r(x) : x \in S\}$.

For a finite set S, let $V(S)$ denote the complex vector space with S as basis. Let P be a graded poset with $n = r(P)$. Then we have $V(P) = V(P_0) \oplus V(P_1) \oplus \cdots \oplus V(P_n)$ (vector space direct sum). An element $v \in V(P)$ is *homogeneous* if $v \in V(P_i)$ for some i, and if $v \neq 0$, we extend the notion of rank to nonzero homogeneous elements by writing $r(v) = i$. Given an element $v \in V(P)$, write $v = v_0 + \cdots + v_n,\ v_i \in V(P_i),\ 0 \leq i \leq n$. We refer to the v_i as the *homogeneous components* of v. A subspace $W \subseteq V(P)$ is *homogeneous* if it contains the homogeneous components of each of its elements. For a homogeneous subspace $W \subseteq V(P)$ we set $\text{rankset}(W) = \{r(v) : v \text{ is a nonzero homogeneous element of } W\}$.

The *up operator* $U : V(P) \to V(P)$ is defined, for $x \in P$, by $U(x) = \sum_y y$, where the sum is over all y covering x. Similarly, the *down operator* $D : V(P) \to V(P)$ is defined, for $x \in P$, by $D(x) = \sum_y y$, where the sum is over all y covered by x.

Let $\langle, \rangle$ denote the standard inner product on $V(P)$, i.e., $\langle x, y \rangle = \delta(x, y)$ (Kronecker delta), for $x, y \in P$. The *length* $\sqrt{\langle v, v \rangle}$ of $v \in V(P)$ is denoted $\parallel v \parallel$.

In this paper we study three graded posets. The *Boolean algebra* is the rank-n graded poset obtained by partially ordering $B(n)$ by inclusion. The rank of a subset is given by cardinality. The *q-analog* of the Boolean algebra is the rank-n graded poset obtained by partially ordering $B(q,n)$ by inclusion. The rank of a subspace is given by dimension. We recall that, for $0 \le k \le n$, the *q-binomial coefficient*

$$\binom{n}{k}_q = \frac{(1)_q(2)_q \cdots (n)_q}{(1)_q \cdots (k)_q (1)_q \cdots (n-k)_q},$$

where $(i)_q = 1 + q + q^2 + \cdots + q^{i-1}$, denotes the cardinality of $B(q,n)_k$.

Given $a = (a_1, \ldots, a_n) \in B_p(n)$, define the *support* of a by $S(a) = \{i \in \{1, \ldots, n\} : a_i \neq L_0\}$. For $b = (b_1, \ldots, b_n) \in B_p(n)$, define $a \le b$ provided $S(a) \subseteq S(b)$ and $a_i = b_i$ for all $i \in S(a)$. It is easy to see that this makes $B_p(n)$ into a rank-n graded poset with rank of a given by $|S(a)|$. We call $B_p(n)$ the *nonbinary analog* of the Boolean algebra $B(n)$. Clearly, when $p = 2$, $B_p(n)$ is order isomorphic to $B(n)$.

We give $V(B(n))$, $V(B_p(n))$, and $V(B(q,n))$ the standard inner products. We use U to denote the up operator on all three of the posets $V(B(n))$, $V(B_p(n))$, and $V(B(q,n))$ and do not indicate the rank n (as in U_n, say) in the notation for U. The meaning of the symbol U is always clear from the context.

Let P be a graded poset. A *graded Jordan chain* in $V(P)$ is a sequence

$$s = (v_1, \ldots, v_h) \tag{1}$$

of nonzero homogeneous elements of $V(P)$ such that $U(v_{i-1}) = v_i$, for $i = 2, \ldots h$, and $U(v_h) = 0$ (note that the elements of this sequence are linearly independent, being nonzero and of different ranks). We say that s *starts* at rank $r(v_1)$ and *ends* at rank $r(v_h)$. A *graded Jordan basis* of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of graded Jordan chains in $V(P)$.

The graded Jordan chain (1) is said to be a *symmetric Jordan chain* (SJC) if the sum of the starting and ending ranks of s equals $r(P)$, i.e., $r(v_1) + r(v_h) = r(P)$ if $h \ge 2$, or $2r(v_1) = r(P)$ if $h = 1$. A *symmetric Jordan basis* (SJB) of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of symmetric Jordan chains in $V(P)$.

The graded Jordan chain (1) is said to be a *semisymmetric Jordan chain* (SSJC) if the sum of the starting and ending ranks of s is $\ge r(P)$. A *semisymmetric Jordan basis* (SSJB) of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of semisymmetric Jordan chains in $V(P)$. An SSJB is said to be *rank complete* if it contains graded Jordan chains starting at rank i and ending at rank j, for all $0 \le i \le j \le r(P)$, $i + j \ge r(P)$.

Suppose we have an orthogonal graded Jordan basis O of $V(P)$. Normalize the vectors in O to get an orthonormal basis O'. Let $(v_1, \ldots, v_h)$ be a graded Jordan chain in O. Put $v'_u = \frac{v_u}{\|v_u\|}$ and $\alpha_u = \frac{\|v_{u+1}\|}{\|v_u\|}$, $1 \le u \le h$ (we take $v'_0 = v'_{h+1} = 0$). We have, for $1 \le u \le h$,

$$U(v'_u) = \frac{U(v_u)}{\| v_u \|} = \frac{v_{u+1}}{\| v_u \|} = \alpha_u v'_{u+1}. \tag{2}$$

Thus the matrix of U with respect to O' is in block diagonal form, with a block corresponding to each (normalized) graded Jordan chain in O, and with the block corresponding to $(v'_1, \ldots, v'_h)$ above being a lower triangular matrix with subdiagonal $(\alpha_1, \ldots, \alpha_{h-1})$ and 0's elsewhere.

Now note that the matrices, in the standard basis P, of U and D are real and transposes of each other. Since O' is orthonormal with respect to the standard inner product, it follows that the matrices of U and D, in the basis O', must be adjoints of each other. Thus the matrix of D with respect to O' is in block diagonal form, with a block corresponding to each (normalized) graded Jordan chain in O, and with the block corresponding to $(v'_1, \ldots, v'_h)$ above being an upper triangular matrix with superdiagonal $(\alpha_1, \ldots, \alpha_{h-1})$ and 0's elsewhere. Thus, for $0 \leq u \leq h-1$, we have

$$D(v'_{u+1}) = \alpha_u v'_u. \tag{3}$$

It follows that the subspace spanned by each graded Jordan chain in O is closed under U and D. We use (2) and (3) without explicit mention in a few places.

The following result is due to Terwilliger [**22**] whose proof is based on the results of Dunkl [**7**]. For a proof based on Proctor's [**15**] $\mathfrak{sl}(2,\mathbb{C})$ method see [**19**].

Theorem 2.1 *There exists a SJB $J(q,n)$ of $V(B(q,n))$ such that*

(i) The elements of $J(q,n)$ are orthogonal with respect to $\langle , \rangle$ (the standard inner product).

(ii) (Singular Values) *Let $0 \leq k \leq n/2$ and let $(x_k, \ldots, x_{n-k})$ be any SJC in $J(q,n)$ starting at rank k and ending at rank $n-k$. Then we have, for $k \leq u < n-k$,*

$$\frac{\| x_{u+1} \|}{\| x_u \|} = \sqrt{q^k (u+1-k)_q (n-k-u)_q} \; . \tag{4}$$

Let $J'(q,n)$ denote the orthonormal basis of $V(B(q,n))$ obtained by normalizing $J(q,n)$.

Substituting $q = 1$ in Theorem 2.1 we get the following result.

Theorem 2.2 *There exists a SJB $J(n)$ of $V(B(n))$ such that*

(i) The elements of $J(n)$ are orthogonal with respect to $\langle , \rangle$ (the standard inner product).

(ii) (Singular Values) *Let $0 \leq k \leq n/2$ and let $(x_k, \ldots, x_{n-k})$ be any SJC in $J(n)$ starting at rank k and ending at rank $n-k$. Then we have, for $k \leq u < n-k$,*

$$\frac{\| x_{u+1} \|}{\| x_u \|} = \sqrt{(u+1-k)(n-k-u)} \; . \tag{5}$$

Let $J'(n)$ denote the orthonormal basis of $V(B(n))$ obtained by normalizing $J(n)$.

Theorem 2.2 was proved by Go [**10**] using the $\mathfrak{sl}(2,\mathbb{C})$ method. For an explicit construction of an orthogonal SJB $J(n)$, together with a representation theoretic interpretation, see [**18**]. It would be interesting to give an explicit construction of an orthogonal SJB $J(q,n)$ of $V(B(q,n))$.

3 q-Analog of $\mathbf{End}_{S_n}(V(B(n)))$

We represent elements of $\mathrm{End}(V(B(q,n)))$ (in the standard basis) as $B(q,n) \times B(q,n)$ matrices (we think of elements of $V(B(q,n))$ as column vectors with coordinates indexed by $B(q,n)$). For $X, Y \in B(q,n)$, the entry in row X, column Y of a matrix M will be denoted $M(X,Y)$. The matrix corresponding to $f \in \mathrm{End}(V(B(q,n)))$ is denoted M_f. We use similar notations for $B(q,n)_i \times B(q,n)_i$ matrices corresponding to elements of $\mathrm{End}(V(B(q,n)_i))$. The finite group $G(q,n) = GL(n, \mathbb{F}_q)$ has a rank and order preserving action on $B(q,n)$. Set

$$\begin{aligned} \mathcal{A}(q,n) &= \{M_f : f \in \mathrm{End}_{G(q,n)}(V(B(q,n)))\}, \\ \mathcal{B}(q,n,i) &= \{M_f : f \in \mathrm{End}_{G(q,n)}(V(B(q,n)_i))\}. \end{aligned}$$

Thus $\mathcal{A}(q,n)$ and $\mathcal{B}(q,n,i)$ are $*$-algebras of matrices.

Let $f : V(B(q,n)) \to V(B(q,n))$ be linear and $g \in G(q,n)$. Then

$$f(g(Y)) = \sum_X M_f(X, g(Y))X \text{ and } g(f(Y)) = \sum_X M_f(X,Y)g(X).$$

It follows that f is $G(q,n)$-linear if and only if

$$M_f(X,Y) = M_f(g(X), g(Y)), \text{ for all } X, Y \in B(q,n),\ g \in G(q,n), \tag{6}$$

i.e., M_f is constant on the orbits of the action of $G(q,n)$ on $B(q,n) \times B(q,n)$.

Now it is easily seen that $(X,Y), (X',Y') \in B(q,n) \times B(q,n)$ are in the same $G(q,n)$-orbit if and only if

$$\dim(X) = \dim(X'),\ \ \dim(Y) = \dim(Y'),\ \text{and } \dim(X \cap Y) = \dim(X' \cap Y'). \tag{7}$$

For $0 \leq i, j, t \leq n$ let $M_{i,j}^t$ be the $B(q,n) \times B(q,n)$ matrix given by

$$M_{i,j}^t(X,Y) = \begin{cases} 1 & \text{if } \dim(X) = i,\ \dim(Y) = j,\ \dim(X \cap Y) = t, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\{M_{i,j}^t \mid i + j - t \leq n,\ 0 \leq t \leq i, j\}$$

is a basis of $\mathcal{A}(q,n)$ and its cardinality is $\binom{n+3}{3}$.

Let $0 \leq i \leq n$. Consider the $G(q,n)$-action on $V(B(q,n)_i)$. Given $X, Y \in B(q,n)_i$, it follows from (7) that the pairs (X,Y) and (Y,X) are in the same orbit of the $G(q,n)$-action on $B(q,n)_i \times B(q,n)_i$. It thus follows from (6) that the algebra $\mathcal{B}(q,n,i)$ has a basis consisting of symmetric matrices and is hence commutative. Thus $V(B(q,n)_i)$ is a multiplicity free $G(q,n)$-module and the $*$-algebra $\mathcal{B}(q,n,i)$ can be diagonalized. We now consider the more general problem of block diagonalizing the $*$-algebra $\mathcal{A}(q,n)$.

Fix $i, j \in \{0, \ldots, n\}$. Then we have

$$M^t_{i,t} M^t_{t,j} = \sum_{u=0}^{n} \binom{u}{t}_q M^u_{i,j}, \qquad t = 0, \ldots, n,$$

since the entry of the left hand side in row X, column Y with $\dim(X) = i,\ \dim(Y) = j$ is equal to the number of common subspaces of X and Y of size t. Apply q-binomial inversion (see Exercise 2.47 in [**1**]) to get

$$M^t_{i,j} = \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q M^u_{i,u} M^u_{u,j}, \qquad t = 0, \ldots, n. \tag{8}$$

Before proceeding further we observe that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (n - 2k + 1)^2 = \binom{n+3}{3} = \dim \mathcal{A}(q, n), \tag{9}$$

since both sides (of the first equality) are polynomials in r (treating the cases $n = 2r$ and $n = 2r + 1$ separately) of degree 3 and agree for $r = 0, 1, 2, 3$.

For $i, j, k, t \in \{0, \ldots, n\}$ define

$$\beta^{n,t}_{i,j,k}(q) \;=\; \sum_{u=0}^{n} (-1)^{u-t}\, q^{\binom{u-t}{2} - ku} \binom{u}{t}_q \binom{n-2k}{u-k}_q \binom{n-k-u}{i-u}_q \binom{n-k-u}{j-u}_q .$$

For $0 \le k \le \lfloor n/2 \rfloor$ and $k \le i, j \le n - k$, define $E_{i,j,k}$ to be the $(n-2k+1) \times (n-2k+1)$ matrix, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row i and column j equal to 1 and all other entries 0. Let $\mathrm{Mat}(n \times n)$ denote the algebra of complex $n \times n$ matrices.

In the proof of the next result we will need the following alternate expression for the singular values:

$$\sqrt{q^k (u+1-k)_q (n-k-u)_q} \;=\; q^{\frac{k}{2}}\, (n-k-u)_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{u+1-k}_q^{-\frac{1}{2}} \tag{10}$$

We now present a q-analog of the explicit block diagonalization of $\mathrm{End}_{S_n}(V(B(n)))$ given by Schrijver [**17**].

Theorem 3.1 *Let $J(q, n)$ be an orthogonal SJB of $V(B(q, n))$ satisfying the conditions of Theorem 2.1. Define a $B(q, n) \times J'(q, n)$ unitary matrix $N(n)$ as follows: for $v \in J'(q, n)$, the column of $N(n)$ indexed by v is the coordinate vector of v in the standard basis $B(q, n)$. Then*

(i) $N(n)^* \mathcal{A}(q, n) N(n)$ consists of all $J'(q, n) \times J'(q, n)$ block diagonal matrices with a block corresponding to each (normalized) SJC in $J(q, n)$ and any two SJC's starting and

ending at the same rank give rise to identical blocks. Thus there are $\binom{n}{k}_q - \binom{n}{k-1}_q$ *identical blocks of size* $(n-2k+1)\times(n-2k+1)$*, for* $k = 0, \ldots, \lfloor n/2 \rfloor$*.*

(ii) Conjugating by $N(n)$ *and dropping duplicate blocks thus gives a positive semidefiniteness preserving* C^**-algebra isomorphism*

$$\Phi : \mathcal{A}(q,n) \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} Mat((n-2k+1)\times(n-2k+1)).$$

It will be convenient to re-index the rows and columns of a block corresponding to a SJC starting at rank k *and ending at rank* $n-k$ *by the set* $\{k, k+1, \ldots, n-k\}$*. Let* $i, j, t \in \{0, \ldots, n\}$*. Write*

$$\Phi(M^t_{i,j}) = (N_0, \ldots, N_{\lfloor n/2 \rfloor}).$$

Then, for $0 \le k \le \lfloor n/2 \rfloor$*,*

$$N_k = \begin{cases} q^{\frac{k(i+j)}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} \beta^{n,t}_{i,j,k}(q)\ E_{i,j,k} & \text{if } k \le i,j \le n-k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof (i) Let $i \ge u$ and let $Y \subseteq X$ with $X \in B(q,n)_i$ and $Y \in B(q,n)_u$. The number of chains of subspaces $X_u \subseteq X_{u+1} \subseteq \cdots \subseteq X_i$ with $X_u = Y$, $X_i = X$, and $\dim(X_l) = l$, for $u \le l \le i$ is clearly $(i-u)_q(i-u-1)_q \cdots (1)_q$. Thus the action of $M^u_{i,u}$ on $V(B(q,n)_u)$ is $\frac{1}{(i-u)_q(i-u-1)_q\cdots(1)_q}$ times the action of U^{i-u} on $V(B(q,n)_u)$.

Now the subspace spanned by each SJC in $J(q,n)$ is closed under U and D. It thus follows by (8) that the subspace spanned by each SJC in $J(q,n)$ is closed under $\mathcal{A}(q,n)$. The result now follows from Theorem 2.1(ii) and the dimension count (9).

(ii) Fix $0 \le k \le \lfloor n/2 \rfloor$. If both i, j are not elements of $\{k, \ldots, n-k\}$ then clearly $N_k = 0$. So we may assume $k \le i, j \le n-k$. Clearly, $N_k = \lambda E_{i,j,k}$ for some λ. We now find $\lambda = N_k(i,j)$.

Let $u \in \{0, \ldots, n\}$. Write $\Phi(M^u_{i,u}) = (A^u_0, \ldots, A^u_{\lfloor n/2 \rfloor})$. We claim that

$$A^u_k = \begin{cases} q^{\frac{k(i-u)}{2}} \binom{n-k-u}{i-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} E_{i,u,k} & \text{if } k \le u \le n-k, \\ 0 & \text{otherwise.} \end{cases}$$

The otherwise part of the claim is clear. If $k \le u \le n-k$ and $i < u$ then we have $A^u_k = 0$. This also follows from the right hand side since the q-binomial coefficient $\binom{a}{b}_q$ is 0 for $b < 0$. So we may assume that $k \le u \le n-k$ and $i \ge u$. Clearly, in this case we have $A^u_k = \alpha E_{i,u,k}$, for some α. We now determine $\alpha = A^u_k(i,u)$. We have using Theorem 2.1(ii) and the expression (10)

$$\begin{aligned} A^u_k(i,u) &= \frac{\prod_{w=u}^{i-1} \left\{ q^{\frac{k}{2}}\ (n-k-w)_q \binom{n-2k}{w-k}_q^{\frac{1}{2}} \binom{n-2k}{w+1-k}_q^{-\frac{1}{2}} \right\}}{(i-u)_q(i-u-1)_q \cdots (1)_q} \\ &= q^{\frac{k(i-u)}{2}} \binom{n-k-u}{i-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}}. \end{aligned}$$

Similarly, if we write $\Phi(M^u_{u,j}) = (B^u_0, \ldots, B^u_{\lfloor n/2 \rfloor})$, then we have

$$B^u_k = \begin{cases} q^{\frac{k(j-u)}{2}} \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} E_{u,j,k} & \text{if } k \le u \le n-k, \\ 0 & \text{otherwise.} \end{cases}$$

So from (8) we have

$$N_k = \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k B^u_k = \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k B^u_k.$$

Thus

$$\begin{aligned} &N_k(i,j) \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q \left\{ \sum_{l=k}^{n-k} A^u_k(i,l) B^u_k(l,j) \right\} \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k(i,u) B^u_k(u,j) \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q q^{\frac{k(i-u)}{2}} \binom{n-k-u}{i-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} \\ &\quad \times q^{\frac{k(j-u)}{2}} \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} \\ &= q^{\frac{k(i+j)}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} \left\{ \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}-ku} \binom{u}{t}_q \binom{n-k-u}{i-u}_q \right. \\ &\quad \left. \times \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q \right\}, \end{aligned}$$

completing the proof.□

We now explicitly diagonalize $\mathcal{B}(q,n,i)$. Let $0 \le i \le n$. We set $i^- = \max\{0, 2i-n\}$ and $m(i) = \min\{i, n-i\}$ (note that i^- and $m(i)$ depend on both i and n. The n will always be clear from the context). It follows from (6) that $\mathcal{B}(q,n,i)$ has a basis consisting of $M^t_{i,i}$, for $i^- \le t \le i$ (here we think of $M^t_{i,i}$ as $B(q,n)_i \times B(q,n)_i$ matrices). The cardinality of this basis is $1 + m(i)$. Since $\mathcal{B}(q,n,i)$ is commutative it follows that $V(B(q,n)_i)$ is a canonical orthogonal direct sum of $1 + m(i)$ common eigenspaces of the $M^t_{i,i}$, $i^- \le t \le i$ (these eigenspaces are the irreducible $G(q,n)$-submodules of $V(B(q,n)_i)$).

Let $J(q,n)$ be an orthogonal SJB of $V(B(q,n))$ satisfying the conditions of Theorem 2.1. For $k = 0, 1, \ldots, m(i)$ define

$$J(q,n,i,k) = \{v \in J(q,n) : r(v) = i \text{ and the Jordan chain containing } v \text{ starts at rank } k\}.$$

Let $W(q,n,i,k)$ be the subspace spanned by $J(q,n,i,k)$ (note that this subspace is nonzero). We have an orthogonal direct sum decomposition

$$V(B(q,n)_i) = \bigoplus_{k=0}^{m(i)} W(q,n,i,k).$$

It now follows from Theorem 3.1 that the $W(q,n,i,k)$ are the common eigenspaces of the $M^t_{i,i}$. The following result is due to Delsarte [**4**].

Theorem 3.2 *Let $0 \le i \le n$. For $i^- \le t \le i$ and $0 \le k \le m(i)$ the eigenvalue of $M^t_{i,i}$ on $W(q,n,i,k)$ is*

$$\sum_{u=0}^{n} (-1)^{u-t}\, q^{\binom{u-t}{2}+k(i-u)} \binom{u}{t}_q \binom{n-k-u}{i-u}_q \binom{i-k}{i-u}_q.$$

Proof Follows from substituting $j = i$ in Theorem 3.1 and noting that

$$\binom{n-2k}{i-k}_q^{-1} \binom{n-2k}{u-k}_q \binom{n-k-u}{i-u}_q = \binom{i-k}{i-u}_q. \quad \square$$

Set

$$\mathcal{A}(n) = \{M_f : f \in \mathrm{End}_{S_n}(V(B(n)))\},$$

and for $i,j,k,t \in \{0,\ldots,n\}$ define

$$\beta^{n,t}_{i,j,k} \;=\; \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t}\binom{n-2k}{u-k}\binom{n-k-u}{i-u}\binom{n-k-u}{j-u}.$$

Substituting $q = 1$ in Theorem 3.1 we get the following result of Schrijver [**17**]. We shall use this result in the next section.

Theorem 3.3 *Let $J(n)$ be an orthogonal SJB of $V(B(n))$ satisfying the conditions of Theorem 2.2. Define a $B(n) \times J'(n)$ unitary matrix $N(n)$ as follows: for $v \in J'(n)$, the column of $N(n)$ indexed by v is the coordinate vector of v in the standard basis $B(n)$. Then*

(i) $N(n)^*\mathcal{A}(n)N(n)$ consists of all $J'(n) \times J'(n)$ block diagonal matrices with a block corresponding to each (normalized) SJC in $J(n)$ and any two SJC's starting and ending at the same rank give rise to identical blocks. Thus there are $\binom{n}{k} - \binom{n}{k-1}$ identical blocks of size $(n-2k+1)\times(n-2k+1)$, for $k = 0,\ldots,\lfloor n/2 \rfloor$.

(ii) Conjugating by $N(n)$ and dropping duplicate blocks thus gives a positive semidefiniteness preserving C^*-algebra isomorphism

$$\Phi : \mathcal{A}(n) \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} Mat((n-2k+1)\times(n-2k+1)).$$

It will be convenient to re-index the rows and columns of a block corresponding to a SJC starting at rank k and ending at rank $n-k$ by the set $\{k, k+1, \ldots, n-k\}$. Let $i, j, t \in \{0, \ldots, n\}$. Write

$$\Phi(M_{i,j}^t) = (N_0, \ldots, N_{\lfloor n/2 \rfloor}).$$

Then, for $0 \leq k \leq \lfloor n/2 \rfloor$,

$$N_k = \begin{cases} \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta_{i,j,k}^{n,t} \, E_{i,j,k} & \text{if } k \leq i, j \leq n-k, \\ 0 & \text{otherwise.} \end{cases}$$

4 Nonbinary analog of $\mathbf{End}_{S_n}(V(B(n)))$

Let (V, f) be a pair consisting of a finite dimensional inner product space V (over $\mathbb{C}$) and a linear operator f on V. Let (W, g) be another such pair. By an isomorphism of pairs (V, f) and (W, g) we mean a linear isometry (i.e, an inner product preserving isomorphism) $\theta : V \to W$ such that $\theta(f(v)) = g(\theta(v)),\ v \in V$.

Consider the inner product space $V(B_p(n))$. An *upper Boolean subspace* of rank t is a homogeneous subspace $W \subseteq V(B_p(n))$ such that $\text{rankset}(W) = \{n-t, n-t+1, \ldots, n\}$, W is closed under the up operator U, and there is an isomorphism of pairs $(V(B(t)), \sqrt{p-1}\,U) \cong (W, U)$ that sends homogeneous elements to homogeneous elements and increases rank by $n-t$.

Consider the following identity

$$p^n \;=\; (p-2+2)^n \;=\; \sum_{l=0}^{n} \binom{n}{l} (p-2)^l 2^{n-l}. \tag{11}$$

We shall now give a linear algebraic interpretation to the identity above. For simplicity we denote the inner product space $V(A(p))$, with $A(p)$ as an orthonormal basis, by $V(p)$. Make the tensor product

$$\otimes_{i=1}^n V(p) = V(p) \otimes \cdots \otimes V(p) \;\; (n \text{ factors})$$

into an inner product space by defining

$$\langle v_1 \otimes \cdots \otimes v_n, u_1 \otimes \cdots \otimes u_n \rangle \;=\; \langle v_1, u_1 \rangle \cdots \langle v_n, u_n \rangle. \tag{12}$$

There is an isometry

$$V(B_p(n)) \;\cong\; \otimes_{i=1}^n V(p) \tag{13}$$

given by $a = (a_1, \ldots, a_n) \mapsto \overline{a} = a_1 \otimes \cdots \otimes a_n,\ a \in B_p(n)$. The rank function (on nonzero homogeneous elements) and the up and down operators, U and D, on $V(B_p(n))$ are transferred to $\otimes_{i=1}^n V(p)$ via the isomorphism above.

Fix a $(p-1)\times(p-1)$ unitary matrix $P=(m_{ij})$, with rows and columns indexed by $\{1,2,\ldots,p-1\}$, and with first row $\frac{1}{\sqrt{p-1}}(1,1,\ldots,1)$. For $i=1,\ldots,p-1$, define the vector $w_i \in V(p)$ by

$$w_i \;=\; \sum_{j=1}^{p-1} m_{ij}\, L_j. \tag{14}$$

Note that $w_1=\frac{1}{\sqrt{p-1}}(L_1+\cdots+L_{p-1})$ and that, for $i=2,\ldots,p-1$, the sum $\sum_{j=1}^{p-1} m_{ij}$ of the elements of row i of P is 0. Thus we have, in $V(p)$,

$$U(w_i)=D(w_i)=0,\ i=2,\ldots,p-1, \tag{15}$$
$$U(w_1)=D(L_0)=0, \tag{16}$$
$$U(L_0)=\sqrt{p-1}\,w_1,\ D(w_1)=\sqrt{p-1}\,L_0. \tag{17}$$

Set

$$\mathcal{S}_p(n) \;=\; \{(A,f): A\subseteq [n],\ f:A\rightarrow\{2,\ldots,p-1\}\}, \tag{18}$$
$$\mathcal{K}_p(n) \;=\; \{(A,f,B): (A,f)\in\mathcal{S}_p(n),\ B\subseteq [n]-A\}. \tag{19}$$

Note that

$$|\mathcal{S}_p(n)|=\sum_{l=0}^{n}\binom{n}{l}(p-2)^l,\quad |\mathcal{K}_p(n)|=\sum_{l=0}^{n}\binom{n}{l}(p-2)^l 2^{n-l}.$$

For $(A,f,B)\in\mathcal{K}_p(n)$ define a vector $v(A,f,B)=v_1\otimes\cdots\otimes v_n\in\otimes_{i=1}^n V(p)$ by

$$v_i \;=\; \begin{cases} w_{f(i)} & \text{if } i\in A,\\ w_1 & \text{if } i\in B,\\ L_0 & \text{if } i\in[n]-(A\cup B).\end{cases}$$

Note that $v(A,f,B)$ is a homogeneous vector in $\otimes_{i=1}^n V(p)$ of rank $|A|+|B|$. For $(A,f)\in\mathcal{S}_p(n)$, define $V_{(A,f)}$ to be the subspace of $\otimes_{i=1}^n V(p)$ spanned by the set $\{v(A,f,B): B\subseteq [n]-A\}$. Set $K_p(n)=\{v(A,f,B):(A,f,B)\in\mathcal{K}_p(n)\}$.

We have, using (15), (16), and (17), the following formula in $\otimes_{i=1}^n V(p)$:

$$U(v(A,f,B)) \;=\; \sqrt{p-1}\left\{\sum_{B'} v(A,f,B')\right\}, \tag{20}$$

where the sum is over all $B'\subseteq([n]-A)$ covering B.

It follows from the unitariness of P and the inner product formula (12) that

$$\langle v(A,f,B),v(A',f',B')\rangle \;=\; \delta((A,f,B),(A',f',B')), \tag{21}$$

where $(A,f,B),\ (A',f',B')\in\mathcal{K}_p(n)$.

We can summarize the discussion above in the following result.

Theorem 4.1 *(i) $K_p(n)$ is an orthonormal basis of $\otimes_{i=1}^n V(p)$.*

(ii) For $(A,f) \in \mathcal{S}_p(n)$, $V_{(A,f)}$ is an upper Boolean subspace of $\otimes_{i=1}^n V(p)$ of rank $n-|A|$ and with orthonormal basis $\{v(A,f,B) : B \subseteq [n]-A\}$.

(iii) We have the following orthogonal decomposition into upper Boolean subspaces:

$$\bigotimes_{i=1}^{n} V(p) \quad = \bigoplus_{(A,f)\in\mathcal{S}_p(n)} V_{(A,f)}, \tag{22}$$

with the right hand side having $(p-2)^l \binom{n}{l}$ upper Boolean subspaces of rank $n-l$, for each $l = 0, 1, \ldots, n$. $\square$

Certain nonbinary problems can be reduced to the corresponding binary problems via the basis $K_p(n)$. We now consider two examples of this (Theorems 4.2 and 4.5 below).

For $0 \le k \le n$ note that $0 \le k^- \le k$ and $k \le n + k^- - k$. For a SSJC c in $V(B_p(n))$, starting at rank i and ending at rank j, we define the *offset* of c to be $i + j - n$. It is easy to see that if an SSJC starts at rank k then its offset l satisfies $k^- \le l \le k$ and the chain ends at rank $n + l - k$. For $0 \le k \le n$ and $k^- \le l \le k$ set

$$\mu(n,k,l) = (p-2)^l \binom{n}{l} \left\{ \binom{n-l}{k-l} - \binom{n-l}{k-l-1} \right\}.$$

Note that when $p = 2$, $\mu(n,k,l)$ is nonzero only when $l = 0$. The following result is due to Terwilliger [**22**].

Theorem 4.2 *There exists a SSJB $J_p(n)$ of $V(B_p(n))$ such that*

(i) The elements of $J_p(n)$ are orthogonal with respect to $\langle , \rangle$ (the standard inner product).

(ii) (Singular Values) *Let $0 \le k \le n$, $k^- \le l \le k$ and let $(x_k, \ldots, x_{n+l-k})$ be any SSJC in $J_p(n)$ starting at rank k and having offset l. Then we have, for $k \le u < n + l - k$,*

$$\frac{\| x_{u+1} \|}{\| x_u \|} \quad = \quad \sqrt{(p-1)(u+1-k)(n+l-k-u)}\ . \tag{23}$$

(iii) Let $0 \le k \le n$ and $k^- \le l \le k$. Then $J_p(n)$ contains $\mu(n,k,l)$ SSJC's starting at rank k and having offset l. Thus, $J_p(n)$ is rank complete when $p \ge 3$ and is an SJB when $p = 2$.

Proof Let $V_{(A,f)}$, with $|A| = l$, be an upper Boolean subspace of rank $n - l$ in the decomposition (22). Let $\gamma : \{1, 2, \ldots, n-l\} \to [n] - A$ be the unique order preserving bijection, i.e., $\gamma(i) = i^{th}$ smallest element of $[n] - A$. Denote by $\Gamma : V(B(n-l)) \to V_{(A,f)}$ the linear isometry given by $\Gamma(X) = v(A, f, \gamma(X))$, $X \in B(n-l)$.

Use Theorem 2.2 to get an orthogonal SJB $J(n-l)$ of $V(B(n-l))$ with respect to $\sqrt{p-1}\, U$ (rather than just U) and transfer it to $V_{(A,f)}$ via Γ. Each SJC in $J(n-l)$ will get transferred to a SSJC in $\otimes_{i=1}^n V(p)$ of offset l and, using (5), we see that this SSJC will satisfy (23). The number of these SSJC's (in $V_{(A,f)}$) starting at rank k is $\binom{n-l}{k-l} - \binom{n-l}{k-l-1}$.

Doing this for every upper Boolean subspace in the decomposition (22) we get an orthogonal SSJB of $\otimes_{i=1}^n V(p)$. Transferring via the isometry (13) we get an orthogonal SSJB $J_p(n)$ of $V(B_p(n))$ satisfying (23). Since the number of upper Boolean subspaces in the decomposition (22) of rank $n-l$ is $(p-2)^l \binom{n}{l}$, Theorem 4.2 now follows.□

Denote by $J'_p(n)$ the orthonormal basis of $V(B_p(n))$ obtained by normalizing $J_p(n)$.

We represent elements of End$(V(B_p(n)))$ (in the standard basis) as $B_p(n) \times B_p(n)$ matrices. Our notation for these matrices is similar to that used in the previous section. The group $S_p(n)$ has a rank and order preserving action on $B_p(n)$. Set

$$\begin{aligned} \mathcal{A}_p(n) &= \{M_f : f \in \mathrm{End}_{S_p(n)}(V(B_p(n)))\}, \\ \mathcal{B}_p(n,i) &= \{M_f : f \in \mathrm{End}_{S_p(n)}(V(B_p(n)_i))\}. \end{aligned}$$

Thus $\mathcal{A}_p(n)$ and $\mathcal{B}_p(n,i)$ are $*$-algebras of matrices.

Let $f : V(B_p(n)) \to V(B_p(n))$ be linear and $\pi \in S_p(n)$. Then f is $S_p(n)$-linear if and only if

$$M_f(a,b) = M_f(\pi(a),\pi(b)), \text{ for all } a,b \in B_p(n),\ \pi \in S_p(n), \tag{24}$$

i.e., M_f is constant on the orbits of the action of $S_p(n)$ on $B_p(n) \times B_p(n)$. Now it is easily seen that $(a,b),(c,d) \in B_p(n) \times B_p(n)$ are in the same $S_p(n)$-orbit if and only if

$$\begin{aligned} &|S(a)| = |S(c)|,\ |S(b)| = |S(d)|,\ |S(a) \cap S(b)| = |S(c) \cap S(d)|, \\ &\text{and } |\{i \in S(a) \cap S(b) : a_i = b_i\}| = |\{i \in S(c) \cap S(d) : c_i = d_i\}|. \end{aligned} \tag{25}$$

For $0 \le i,j,t,s \le n$ let $M_{i,j}^{t,s}$ be the $B_p(n) \times B_p(n)$ matrix given by

$$M_{i,j}^{t,s}(a,b) = \begin{cases} 1 & \text{if } |S(a)| = i,\ |S(b)| = j,\ |S(a) \cap S(b)| = t,\ |\{i : a_i = b_i \neq L_0\}| = s, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{I}_p(n) = \{(i,j,t,s) : 0 \le s \le t \le i,j,\ \ i+j-t \le n\}.$$

It follows from (24) and (25) that $\{M_{i,j}^{t,s} : (i,j,t,s) \in \mathcal{I}_p(n)\}$ is a basis of $\mathcal{A}_p(n)$. Note that

$$p \ge 3 \text{ implies } |\mathcal{I}_p(n)| = \dim \mathcal{A}_p(n) = \binom{n+4}{4}, \tag{26}$$

since $(i,j,t,s) \in \mathcal{I}_p(n)$ if and only if $(i-t)+(j-t)+(t-s)+s \le n$ and all four terms are nonnegative. When $p=2$ this basis becomes $\{M_{i,j}^{t,t} : (i,j,t,t) \in \mathcal{I}_2(n)\}$ and its cardinality is $\binom{n+3}{3}$.

Let $0 \le i \le n$. Consider the $S_p(n)$-action on $V(B_p(n)_i)$, $0 \le i \le n$. Given $a,b \in B_p(n)_i$, it follows from (25) that the pairs (a,b) and (b,a) are in the same orbit of the

$S_p(n)$-action on $B_p(n)_i \times B_p(n)_i$. It thus follows from (24) that the algebra $\mathcal{B}_p(n,i)$ has a basis consisting of symmetric matrices and is hence commutative. Thus $V(B_p(n)_i)$ is multiplicity free as a $S_p(n)$-module and the $*$-algebra $\mathcal{B}_p(n,i)$ can be diagonalized. We now consider the more general problem of block diagonalizing the $*$-algebra $\mathcal{A}_p(n)$.

Before proceeding we observe that

$$\sum_{k=0}^{n} \sum_{l=k^-}^{k} (n+l-2k+1)^2 = \binom{n+4}{4}, \tag{27}$$

since both sides are polynomials in r of degree 4 (treating the cases $n = 2r$ and $n = 2r+1$ separately) and agree for $r = 0, 1, 2, 3, 4$.

Consider the linear operator on $V(B_p(n))$ whose matrix with respect to the standard basis $B_p(n)$ is $M^{t,s}_{i,j}$. Transfer this operator to $\otimes_{i=1}^n V(p)$ via the isomorphism (13) above and denote the resulting linear operator by $\mathcal{M}^{t,s}_{i,j}$. In Theorem 4.4 below we show that the action of $\mathcal{M}^{t,s}_{i,j}$ on the basis $K_p(n)$ mirrors the binary case.

Define linear operators $\mathcal{N}, \mathcal{Z}, \mathcal{R} : V(p) \to V(p)$ as follows

- $\mathcal{Z}(L_0) = L_0$ and $\mathcal{Z}(L_i) = 0$ for $i = 1, \ldots, p-1$,
- $\mathcal{N}(L_0) = 0$ and $\mathcal{N}(L_i) = L_i$ for $i = 1, \ldots, p-1$,
- $\mathcal{R}(L_0) = 0$ and $\mathcal{R}(L_i) = (L_1 + \cdots + L_{p-1}) - L_i$ for $i = 1, \ldots, p-1$.

Note that

$$\mathcal{R}(w_1) = (p-2)w_1, \tag{28}$$
$$\mathcal{R}(w_i) = -w_i, \; i = 2, \ldots, p-1, \tag{29}$$

where the second identity follows from the fact that the sum of the elements of row $i, \; i \geq 2$, of P is zero.

Let there be given a 5-tuple $\mathcal{X} = (S_U, S_D, S_{\mathcal{N}}, S_{\mathcal{Z}}, S_{\mathcal{R}})$ of pairwise disjoint subsets of $[n]$ with union $[n]$ (it is convenient to index the components of S in this fashion). Define a linear operator

$$F(\mathcal{X}) : \bigotimes_{i=1}^{n} V(p) \to \bigotimes_{i=1}^{n} V(p)$$

by $F(\mathcal{X}) = F_1 \otimes \cdots \otimes F_n$, where each F_i is U or D or $\mathcal{N}$ or $\mathcal{Z}$ or $\mathcal{R}$ according as $i \in S_U$ or S_D or $S_{\mathcal{N}}$ or $S_{\mathcal{Z}}$ or $S_{\mathcal{R}}$, respectively.

Let $b \in B_p(n)$. It follows from the definitions that

$$F(\mathcal{X})(\bar{b}) \neq 0 \quad \text{iff} \quad S_D \cup S_{\mathcal{N}} \cup S_{\mathcal{R}} = S(b), \; S_U \cup S_{\mathcal{Z}} = [n] - S(b). \tag{30}$$

Given a 5-tuple $r = (r_1, r_2, r_3, r_4, r_5)$ of nonnegative integers with sum n define $\Pi(r)$ to be the set of all 5-tuples $\mathcal{X} = (S_U, S_D, S_{\mathcal{N}}, S_{\mathcal{Z}}, S_{\mathcal{R}})$ of pairwise disjoint subsets of $[n]$ with union $[n]$ and with $|S_U| = r_1, |S_D| = r_2, |S_{\mathcal{N}}| = r_3, |S_{\mathcal{Z}}| = r_4$, and $|S_{\mathcal{R}}| = r_5$.

Lemma 4.3 *Let $(i,j,t,s) \in \mathcal{I}_p(n)$ and $r = (i-t, j-t, s, n+t-i-j, t-s)$. Then*

$$\mathcal{M}_{i,j}^{t,s} \;=\; \sum_{\mathcal{X}\in\Pi(r)} F(\mathcal{X}). \tag{31}$$

Proof Let $b = (b_1,\ldots,b_n) \in B_p(n)$ and $\mathcal{X} = (S_U, S_D, S_{\mathcal{N}}, S_{\mathcal{Z}}, S_{\mathcal{R}}) \in \Pi(r)$. We consider two cases:

(i) $|S(b)| \neq j$: In this case we have $\mathcal{M}_{i,j}^{t,s}(\bar{b}) = 0$. Now $|S_D| + |S_{\mathcal{N}}| + |S_{\mathcal{R}}| = j - t + s + t - s = j$. Thus, from (30), we also have $F(\mathcal{X})(\bar{b}) = 0$.

(ii) $|S(b)| = j$: Assume $F(\mathcal{X})(\bar{b}) \neq 0$. Then, from (30), we have that $F(\mathcal{X})(\bar{b}) = \sum_a \bar{a}$, where the sum is over all $a = (a_1,\ldots,a_n) \in B_p(n)_i$ with $S(a) = S_U \cup S_{\mathcal{N}} \cup S_{\mathcal{R}}$, $a_k \neq b_k,\ k \in S_{\mathcal{R}}$, and $a_k = b_k,\ k \in S_{\mathcal{N}}$.

Going over all elements of $\Pi(r)$ and summing we see that both sides of (31) evaluate to the same element on $\bar{b}$. □

Theorem 4.4 *Let $(A,f,B) \in \mathcal{K}_p(n)$ with $|A| = l$ and $(i,j,t,s) \in \mathcal{I}_p(n)$.*

(i) $\mathcal{M}_{i,j}^{t,s}(v(A,f,B)) = 0$ if $|B| \neq j - l$.

(ii) If $|B| = j - l$ then

$$\mathcal{M}_{i,j}^{t,s}(v(A,f,B)) =$$
$$(p-1)^{\frac{i+j}{2}-t} \left\{ \sum_{g=0}^{l} (-1)^{l-g} \binom{l}{g} \binom{t-l}{s-g} (p-2)^{t-l-s+g} \right\} \left(\sum_{B'} v(A,f,B') \right),$$

where the sum is over all $B' \subseteq ([n]-A)$ with $|B'| = i - l$ and $|B \cap B'| = t - l$.

Proof Let $r = (i-t, j-t, s, n+t-i-j, t-s)$ and let $\mathcal{X} = (S_U, S_D, S_{\mathcal{N}}, S_{\mathcal{Z}}, S_{\mathcal{R}}) \in \Pi(r)$. Assume that $F(\mathcal{X})(v(A,f,B)) \neq 0$. Then we must have (using (15) and the definitions of $\mathcal{N}, \mathcal{Z}$, and $\mathcal{R}$)

$$S_U \cup S_{\mathcal{Z}} = [n] - A - B,\ A \subseteq S_{\mathcal{N}} \cup S_{\mathcal{R}},\ S_D \subseteq B,\ S_{\mathcal{N}} \cup S_D \cup S_{\mathcal{R}} = A \cup B.$$

Thus $|B| = n - l - |S_U \cup S_{\mathcal{Z}}| = n - l - (i - t + n + t - i - j) = j - l$ (so part (i) follows).

Put $|A \cap S_{\mathcal{R}}| = l - g$. Then $|A \cap S_{\mathcal{N}}| = g$ and thus $|B \cap S_{\mathcal{N}}| = s - g$. We have $|B \cap S_{\mathcal{R}}| = |B - S_D - (B \cap S_{\mathcal{N}})| = j - l - j + t - s + g = t - l - s + g$.

We now have (using (17), (28), and (29))

$$F(\mathcal{X})(v(A,f,B)) \;=\; (-1)^{l-g}\,(p-2)^{t-s-l+g}\,(p-1)^{\frac{i+j}{2}-t}\,v(A,f,B'), \tag{32}$$

where $B' = S_U \cup (B - S_D)$ and $|B'| = i - t + j - l - j + t = i - l$, $|B \cap B'| = |B - S_D| = j - l - j + t = t - l$.

Formula (32) depends only on S_U, S_D and $|A \cap S_{\mathcal{R}}|$. Once S_U, S_D are fixed the number of choices for $S_{\mathcal{R}}$ with $|A \cap S_{\mathcal{R}}| = l - g$ is clearly $\binom{l}{g}\binom{t-l}{s-g}$.

Going over all elements of $\Pi(r)$ and summing we get the result. $\square$

For $i, j, k, t, s, l \in \{0, \ldots, n\}$ define

$$\alpha_{i,j,k,l}^{n,t,s} = (p-1)^{\frac{1}{2}(i+j)-t} \left\{ \sum_{g=0}^{l} (-1)^{l-g} \binom{l}{g} \binom{t-l}{s-g} (p-2)^{t-l-s+g} \right\} \beta_{i-l,j-l,k-l}^{n-l,t-l}.$$

For $0 \le k \le n$ and $k^- \le l \le k$, define $E_{i,j,k,l}$ to be the $(n+l-2k+1) \times (n+l-2k+1)$ matrix, with rows and columns indexed by $\{k, k+1, \ldots, n+l-k\}$, and with entry in row i and column j equal to 1 and all other entries 0.

The following result is due to Gijswijt, Schrijver, and Tanaka [**9**].

Theorem 4.5 *Let $p \ge 3$ and let $J_p(n)$ be an orthogonal SSJB of $V(B_p(n))$ satisfying the conditions of Theorem 4.2. Define a $B_p(n) \times J_p'(n)$ unitary matrix $M(n)$ as follows: for $v \in J_p'(n)$, the column of $M(n)$ indexed by v is the coordinate vector of v in the standard basis $B_p(n)$. Then*

(i) $M(n)^* \mathcal{A}_p(n) M(n)$ consists of all $J_p'(n) \times J_p'(n)$ block diagonal matrices with a block corresponding to each (normalized) SSJC in $J_p(n)$ and any two SSJC's starting and ending at the same rank give rise to identical blocks. Thus, for each $0 \le k \le n$, $k^- \le l \le k$, there are $\mu(n,k,l)$ identical blocks of size $(n+l-2k+1) \times (n+l-2k+1)$.

(ii) Conjugating by $M(n)$ and dropping duplicate blocks thus gives a positive semidefiniteness preserving C^*-algebra isomorphism

$$\Phi : \mathcal{A}_p(n) \cong \bigoplus_{k=0}^{n} \bigoplus_{l=k^-}^{k} Mat((n+l-2k+1) \times (n+l-2k+1)).$$

It will be convenient to re-index the rows and columns of a block corresponding to a SSJC starting at rank k and having offset l by the set $\{k, k+1, \ldots, n+l-k\}$. Let $i, j, t, s \in \{0, \ldots, n\}$. Write

$$\Phi(M_{i,j}^{t,s}) = (N_{k,l}),\ 0 \le k \le n,\ k^- \le l \le k.$$

Then

$$N_{k,l} = \begin{cases} \binom{n+l-2k}{i-k}^{-\frac{1}{2}} \binom{n+l-2k}{j-k}^{-\frac{1}{2}} \alpha_{i,j,k,l}^{n,t,s}\, E_{i,j,k,l} & \text{if } k \le i, j \le n+l-k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Follows from Theorem 3.3 using Theorems 4.1 and 4.4 and the dimension counts (26), (27). $\square$

Remark In [**18**] an explicit orthogonal SJB $J(n)$ of $V(B(n))$ was constructed and given a S_n-representation theoretic interpretation as the canonically defined Gelfand-Tsetlin basis of $V(B(n))$. This explicit basis from the binary case, together with a choice of the $(p-1)\times(p-1)$ unitary matrix P, leads to an explicit orthogonal SSJB $J_p(n)$ in the nonbinary case. Different choices of P lead to different SSJB's $J_p(n)$. One natural choice, used in [**9**], is the Fourier matrix. Another natural choice is the following. Consider the action of the symmetric group S_{p-1} on $V = V(\{L_1,\ldots,L_{p-1}\})$. Under this action V splits into two irreducibles, the one dimensional trivial representation and the $p-2$ dimensional standard representation consisting of all linear combinations of $L_1,\ldots,L_{p-1}$ with coefficients summing to 0. The first row of P is a basis of the trivial representation and rows $2,\ldots,p-1$ of P are a basis of the standard representation. Choose rows $2,\ldots,p-1$ to be the canonical Gelfand-Tsetlin basis of this representation. (Upto order) we can write them down explicitly as follows (see [**18**]): for $i=2,\ldots,p-1$ define $v_i=(i-1)L_i-(L_1+\cdots+L_{i-1})$ and $u_i=\frac{v_i}{\|v_i\|}$. So rows $2,\ldots,p-1$ of P are $u_2,\ldots,u_{p-1}$. The resulting matrix P is called the Helmert matrix (see Section 7.6 in [**16**]). It is interesting to study the resulting orthogonal SSJB $J_p(n)$ form the point of view of representation theory of the wreath product $S_p(n)$ (for which, see Appendix B of Chapter 1 in [**11**]).

We now explicitly diagonalize $\mathcal{B}_p(n,i)$.

Lemma 4.6 *Let $0\le i\le n$. Set*

$$\begin{aligned} L(i) &= \{(k,l) : i^- \le k \le i,\ 0\le l\le k\},\\ R(i) &= \{(k,l) : 0\le k\le n,\ k^-\le l\le k,\ k\le i\le n+l-k\}. \end{aligned}$$

Then $|L(i)| = |R(i)|$.

Proof The identity is clearly true when $i\le n/2$. Now assume $i>n/2$. Then the set $L(i)$ has cardinality $\sum_{k=2i-n}^{i}(k+1)$. The defining conditions on pairs (k,l) for membership in $R(i)$ can be rewritten as $0\le l\le k\le i,\ 0\le k-l\le n-i$. For $0\le j\le n-i$, the pairs (k,l) with $0\le l\le k\le i$ and $k-l=j$ are $(j,0),(j+1,1),\ldots,(i,i-j)$ and their number is $i-j+1$. Thus, for $i>n/2$, $|R(i)|=\sum_{j=0}^{n-i}(i-j+1)=\sum_{t=2i-n}^{i}(t+1)$. The result follows. □

Let $0\le i\le n$. It follows from (25) that $\mathcal{B}_p(n,i)$ has a basis consisting of $M_{i,i}^{t,s}$, for $(t,s)\in L(i)$ (here we think of $M_{i,i}^{t,s}$ as $B_p(n)_i\times B_p(n)_i$ matrices). The cardinality of this basis, by Lemma 4.6, is $\tau(i)$ (where $\tau(i)=|R(i)|$). Since $\mathcal{B}_p(n,i)$ is commutative it follows that $V(B_p(n)_i)$ is a canonical orthogonal direct sum of $\tau(i)$ common eigenspaces of the $M_{i,i}^{t,s}$, $(t,s)\in L(i)$ (these eigenspaces are the irreducible $S_p(n)$-submodules of $V(B_p(n)_i)$).

Let $0\le i\le n$. For $(k,l)\in R(i)$ define

$$J_p(n,i,k,l) = \{v\in J_p(n) : r(v)=i \text{ and the Jordan chain containing } v \text{ starts at rank } k \text{ and has offset } l\}. \quad (33)$$

Let $W_p(n,i,k,l)$ be the subspace spanned by $J_p(n,i,k,l)$ (note that this subspace is nonzero). We have an orthogonal direct sum decomposition

$$V(B_p(n)_i) = \bigoplus_{(k,l)\in R(i)} W_p(n,i,k,l).$$

It now follows from Theorem 4.5 that the $W_p(n,i,k,l)$ are the common eigenspaces of the $M_{i,i}^{t,s}$. The following result is due to Tarnanen, Aaltonen, and Goethals [**21**].

Theorem 4.7 *Let $0 \le i \le n$. For $(t,s) \in L(i)$ and $(k,l) \in R(i)$ the eigenvalue of $M_{i,i}^{t,s}$ on $W_p(n,i,k,l)$ is*

$$(p-1)^{i-t}\left\{\sum_{g=0}^{l}(-1)^{l-g}\binom{l}{g}\binom{t-l}{s-g}(p-2)^{t-l-s+g}\right\}$$
$$\times\left\{\sum_{u=0}^{n-l}(-1)^{u-t+l}\binom{u}{t-l}\binom{n-k-u}{i-l-u}\binom{i-k}{i-l-u}\right\}$$

Proof Follows from substituting $j = i$ in Theorem 4.5 and noting that

$$\binom{n+l-2k}{i-k}^{-1}\binom{n+l-2k}{u+l-k}\binom{n-k-u}{i-l-u} = \binom{i-k}{i-l-u}. \quad \square$$

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