CONES OF CLOSED ALTERNATING WALKS AND TRAILS

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Dedicated to the memory of Malka Peled

Abstract. Consider a graph whose edges have been colored red and blue. Assign a nonnegative real weight to every edge so that at every vertex, the sum of the weights of the incident red edges equals the sum of the weights of the incident blue edges. The set of all such assignments forms a convex polyhedral cone in the edge space, called the *alternating cone*. The integral (respectively, $\{0, 1\}$) vectors in the alternating cone are sums of characteristic vectors of closed alternating walks (respectively, trails). We study the basic properties of the alternating cone, determine its dimension and extreme rays, and relate its dimension to the majorization order on degree sequences. We consider whether the alternating cone has integral vectors in a given box, and use residual graph techniques to reduce this problem to the one of searching for an alternating trail connecting two given vertices. The latter problem, called *alternating reachability*, is solved in a companion paper along with related results.

1. Introduction and Summary

Consider a directed graph. Assign a nonnegative real weight to every arc so that at every vertex, the total weight of the incoming arcs is equal to the total weight of the outgoing arcs. The set of all such assignments forms a convex polyhedral cone in the arc space, called the cone of circulations, and is a basic object of study in network flow theory. For instance, placing integral upper and lower bounds on every arc and asking whether there is an integral vector in the cone of circulations meeting these bounds leads to Hoffman's circulation theorem (see the book [FF]). Now consider an undirected analog of the situation above. Take an undirected graph whose edges have been colored red and blue. Assign a nonnegative real weight to every edge so that at every vertex, the total weight of the incident red edges equals the total weight of the incident blue edges. The set of all such assignments forms a convex polyhedral cone in the edge space, called the alternating cone. In this paper and the companion paper [BPS2], we study the basic theory of the alternating cone. Here we consider its extreme rays, integral vectors, and dimension. We also relate it to threshold graphs and majorization order on degree sequences. We reduce the problem of finding an integral vector in the alternating cone whose components satisfy given upper and lower bounds to the problem of searching for an alternating trail connecting two given vertices in a 2-colored graph (recall that in the directed case, the corresponding problem is reduced to the problem of searching for a directed path from one given vertex to another in a suitable residual directed graph). This latter problem, called alternating reachability, generalizes the problem of searching for an augmenting path with respect to a matching in a non-bipartite graph and is solved in [BPS2] by generalizing the blossom forest algorithm of Edmonds. We now give precise definitions and an outline of our results.

Let $G = (V, E)$ be an undirected graph (we allow parallel edges but not loops). Assume that the edges of G are colored red or blue, the coloring being given by $\mathcal{C} : E \to \{R, B\}$. We say that $(G, \mathcal{C})$

Date: October 22, 2005, Revised October 12, 2006.

2000 *Mathematics Subject Classification.* 05C70, 90C27, 90C57.

Key words and phrases. colored graphs, alternating walks and trails.

UNP and MKS would like to thank Professor Martin Golumbic for his kind invitation to visit the Caesarea Edmond Benjamin de Rothschild Foundation Institute for Interdisciplinary Applications of Computer Science at the University of Haifa, Israel during May–June 2003, where part of this work was carried out. The warm hospitality and partial support of this visit from CRI is gratefully acknowledged.

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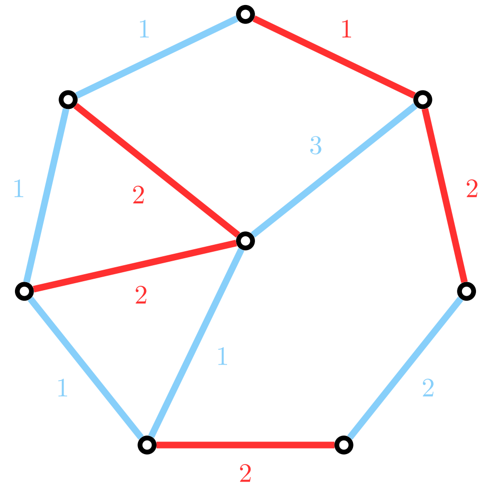

FIGURE 1. An integral vector in the alternating cone

is a *2-colored graph*. Consider the real vector space $\mathbb{R}^E$, with coordinates indexed by the set of edges of G. We write an element $x \in \mathbb{R}^E$ as $x = (x(e) : e \in E)$. For a subset $F \subseteq E$ and $v \in V$, $F(v)$ denotes the set of all edges in F incident with v. For a subset $F \subseteq E$, F_R (respectively, F_B) denotes the set of red (respectively, blue) edges in F. For an edge $e \in E$, the characteristic vector $\chi(e) \in \mathbb{R}^E$ is defined by $\chi(e)(f) = \begin{cases} 1, & \text{if } f = e \\ 0, & \text{if } f \neq e \end{cases}$. The *red degree* $r(v)$ (respectively, *blue degree* $b(v)$) of a vertex $v \in V$ is the number of red (respectively, blue) edges incident with v.

The *cone of closed alternating walks*, or simply the *alternating cone*, $\mathcal{A}(G, \mathcal{C})$ of a 2-colored graph $(G, \mathcal{C})$ (denoted simply by $\mathcal{A}(G)$ when the coloring $\mathcal{C}$ is understood) is defined to be the set of all vectors $x = (x(e) : e \in E)$ in $\mathbb{R}^E$ satisfying the following system of homogeneous linear inequalities:

$$\sum_{e \in E_R(v)} x(e) - \sum_{e \in E_B(v)} x(e) = 0, \qquad v \in V, \tag{1}$$

$$x(e) \geq 0, \qquad e \in E. \tag{2}$$

We refer to (1) as the *balance condition* at vertex v. Figure 1 illustrates a 2-colored graph together with an integral vector in its alternating cone.

If $G = (V, E)$ is a simple graph, we think of the elements of E as 2-element subsets of V. In this case the 2-colored simple graph *associated* to G is the complete graph $\widehat{G} = \left(V, \binom{V}{2}\right)$, where $e = \{i, j\} \in \binom{V}{2}$ is colored red if $e \in E$ and colored blue if $e \notin E$.

Let $G = (V, E)$ be a graph. A *walk* in G is a sequence

$$W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m), \qquad m \geq 0, \tag{3}$$

where $v_i \in V$ for all i, $e_j \in E$ for all j, and e_j has endpoints v_{j-1} and v_j for all j. We say that W is a v_0-v_m walk of length m. We call e_1 the *first* edge of W and e_m the *last* edge of W. We say that $v_1, v_2, \ldots, v_{m-1}$ are the *internal vertices* of the walk W. Note that since we are allowing repetitions, the vertices v_0, v_m could also be internal vertices. The walk W^R is the v_m-v_0 walk obtained by reversing the sequence (3). The characteristic vector of the walk W is defined to be $\chi(W) = \sum_{i=1}^{m} \chi(e_i)$.

The walk W is said to be

closed	when $v_0 = v_m$;
a trail	when the edges $e_1, \ldots, e_m$ are distinct;
a path	when the edges $e_1, \ldots, e_m$ are distinct and the vertices $v_0, \ldots, v_m$ are distinct;

a cycle when W is closed, the edges $e_1, \dots, e_m$ are distinct, and the vertices $v_0, \dots, v_{m-1}$ are distinct.

We have defined paths and cycles as special classes of walks. However, sometimes it is more convenient to think of paths and cycles as subgraphs, as is done usually. This will be clear from the context. If W_1 is a u-v walk and W_2 is a v-w walk, then the *concatenation* of W_1 and W_2, denoted $W_1 * W_2$, is the u-w walk obtained by walking from u to v along W_1 and continuing by walking from v to w along W_2. Note that if W_1 and W_2 are trails, then $W_1 * W_2$ is a trail whenever W_1 and W_2 have no edges in common.

Now let $(G, \mathcal{C})$ be a 2-colored graph. The walk W in (3) is said to be

internally alternating
when $\mathcal{C}(e_j) \neq \mathcal{C}(e_{j+1})$ for each $j = 1, \dots, m-1$;

alternating
when W is internally alternating and if W is closed we also have $\mathcal{C}(e_m) \neq \mathcal{C}(e_1)$ (note that a walk can be closed and internally alternating without being alternating, but if $v_0 \neq v_m$, there is no distinction between internally alternating and alternating walks and we use the word alternating in this case); a closed alternating walk (respectively, trail) is abbreviated as *CAW* (respectively, *CAT*);

an even alternating cycle
when W is a cycle of even length and W is alternating (Figure 1 depicts even alternating cycles and their characteristic vectors); an even alternating cycle will also be called simply an *alternating cycle*;

an odd internally alternating cycle with base v_0
when W is a v_0-v_0 cycle of odd length and W is internally alternating (Figure 3 depicts odd internally alternating cycles);

an alternating bicycle
when W is alternating and is of the form $W = W_1 * P * W_2 * P^R$, where W_1, W_2 are odd internally alternating cycles, P is a path between the bases of W_1 and W_2, and the internal vertices of W_1, P, and W_2 are disjoint (note that W_1 and W_2 may have the same base, in which case P is empty; Figure 4 depicts alternating bicycles and their characteristic vectors); clearly $\chi(W) = \chi(W_1) + 2\chi(P) + \chi(W_2)$.

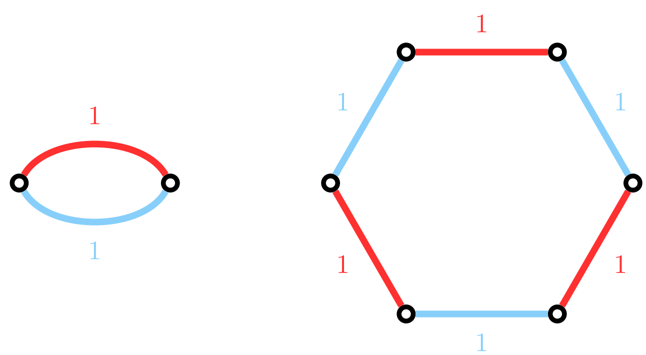

FIGURE 2. Even alternating cycles

A CAW W is said to be *irreducible* if $\chi(W)$ cannot be written as $\chi(W_1) + \chi(W_2)$ for any CAW's W_1 and W_2. For instance, alternating cycles and bicycles are easily seen to be irreducible. Similarly, a CAT T is said to be *irreducible* if $\chi(T)$ cannot be written as $\chi(T_1) + \chi(T_2)$ for any CAT's T_1 and T_2. Figure 5 depicts an irreducible CAW (with the direction of walk indicated by an arrow) and Figure 6 depicts an irreducible CAT. Irreducibility is easily seen.

Section 2 considers the integral vectors, extreme rays, and dimension of the alternating cone. We use a simple alternating walk argument to show that the extreme rays of the alternating cone are the characteristic vectors of alternating cycles and bicycles, the integral vectors in the alternating cone are sums of characteristic vectors of irreducible CAW's, and the $\{0, 1\}$-vectors in the alternating cone

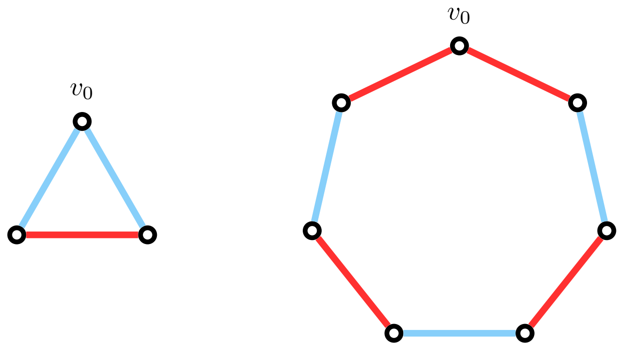

FIGURE 3. Odd alternating cycles with base v_0

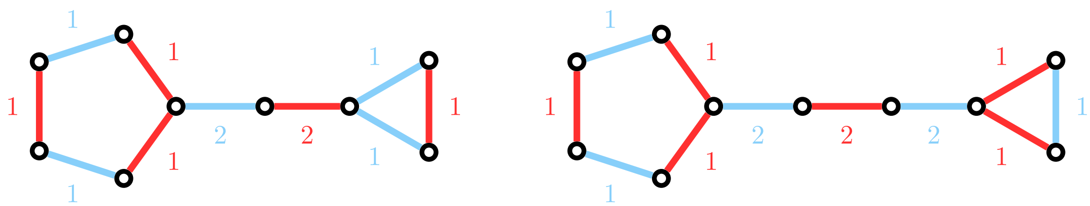

FIGURE 4. Alternating bicycles

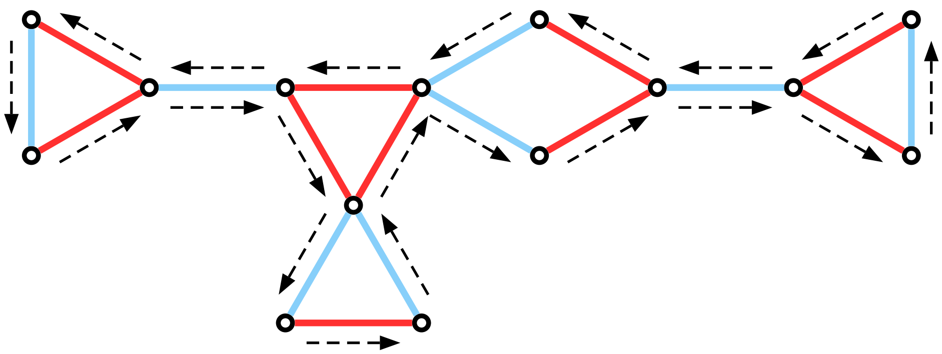

FIGURE 5. An irreducible CAW

are sums of characteristic vectors of irreducible CAT's. Using the characterization of the extreme rays, we obtain that a simple graph G is a threshold graph if and only if $\dim \mathcal{A}(\widehat{G}) = 0$ (this fact was our original motivation for defining the alternating cone). It is well-known that for a simple graph G, the property $\dim \mathcal{A}(\widehat{G}) = 0$ (i.e., G being threshold) depends only on the degree sequence of G. More generally, for any 2-colored graph $(G, \mathcal{C})$, we determine $\dim \mathcal{A}(G, \mathcal{C})$ in terms of the red degree sequence of $(G, \mathcal{C})$. We then relate this dimension to the concept of majorization (following [AP]). Consider the set $D(n)$ of all ordered degree sequences $d = (d_1, d_2, \ldots, d_n)$ of simple graphs on n vertices, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Partially order $D(n)$ by majorization (the definition is recalled in Section 2 before Lemma 2.9). It is well-known (see [MP] and [RG]) that the set of maximal elements of this poset is precisely the set of ordered degree sequences of threshold graphs. Define a map $A : D(n) \to \mathbb{N}$ by $A(d) = \dim \mathcal{A}(\widehat{G})$, where G is any simple graph with ordered degree sequence d. We show that A is an order-reversing map ($d_1 \succeq d_2$ implies $A(d_1) \leq A(d_2)$). Thus, we can think of $A(d)$ as a kind of measure of how non-threshold the degree sequence d is.

Section 3 is motivated by the following undirected analog of Hoffman's circulation problem for directed graphs: let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph. Assume that we are given nonnegative integral lower and upper bounds $l, u : E \to \mathbb{N}$ satisfying $l(e) \leq u(e)$ for all $e \in E$. We are interested in knowing whether there is a vector $y \in \mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$ satisfying $l(e) \leq y(e) \leq u(e)$

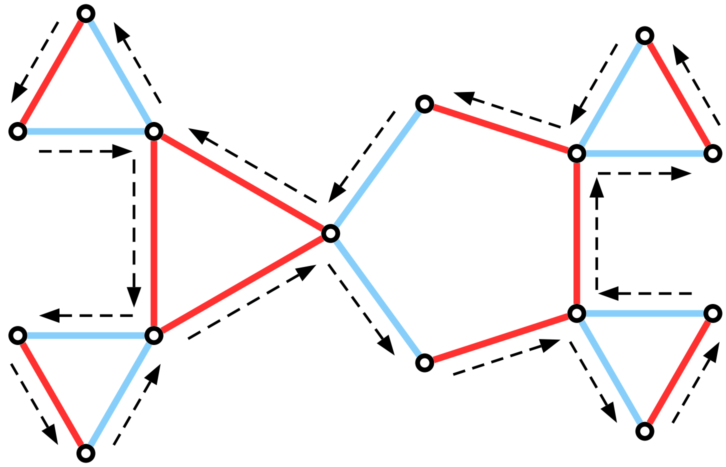

FIGURE 6. An irreducible CAT

for all $e \in E$. We use residual graph techniques to reduce this problem to the *alternating reachability problem*: given distinct vertices s, t in a 2-colored graph, is there an alternating s-t trail? Recall that in the directed case, the circulation problem is reduced to the *directed reachability problem*: given distinct vertices s, t in a directed graph, is there a directed s-t path? This is solved by a breadth-first search algorithm, which either finds a directed s-t path or produces an s-t cut set. In [BPS2] we give a polynomial-time algorithm to the alternating reachability problem generalizing the blossom forest algorithm of Edmonds for searching for an augmenting path with respect to a matching in a non-bipartite graph. The algorithm either finds an alternating s-t trail or produces an s-t Tutte set (which is an obstruction to the existence of an alternating s-t trail. For the definition of a Tutte set, see [BPS2]).

Circulations in directed graphs can be thought of in terms of flows. For example, the characteristic vector of a directed circuit corresponds to a unit of flow along the circuit. Such an interpretation is not available in the case of vectors in the alternating cone; the irreducible CAW of Figure 5 does not correspond to a flow in an intuitive sense. On the other hand, the characteristic vector of an irreducible CAT *can* be thought of as a unit of flow around the trail. For a 2-colored graph $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$, it is thus natural to consider the convex polyhedral cone $\mathcal{T}(G, \mathcal{C}) \subseteq \mathbb{R}^E$ generated by the characteristic vectors of the CAT's in $(G, \mathcal{C})$. We call $\mathcal{T}(G, \mathcal{C})$ the *cone of closed alternating trails*, or simply the *trail cone*, of $(G, \mathcal{C})$.

Consider a CAT in a 2-colored graph. Its characteristic vector satisfies the balance condition at every vertex. If we ignore the colors, the edge-set of the CAT is a disjoint union of the edge-sets of some cycles in the underlying graph. This shows that a nonnegative integral combination (that is to say, a linear combination with nonnegative integral *coefficients*) of characteristic vectors of CAT's satisfies the balance condition at every vertex and can be written as a nonnegative integral combination of characteristic vectors of cycles in the underlying graph G. Let $\mathcal{Z}(G)$ denote the cone in $\mathbb{R}^E$ generated by the characteristic vectors of the cycles in G. The linear inequalities defining $\mathcal{Z}(G)$ were determined by Seymour [S]. The observation above shows that $\mathcal{T}(G, \mathcal{C}) \subseteq \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$. In [BPS2] we prove that $\mathcal{T}(G, \mathcal{C}) = \mathcal{A}(G, \mathcal{C}) \cap \mathcal{Z}(G)$. The proof uses our solution to the alternating reachability problem.

We remark that in this paper we focus on graph-theoretical aspects of the alternating cone and not on algorithmic efficiency. We do consider algorithms, but always with a view to obtaining graph-theoretical results.

2. Extreme Rays and Dimension of the Alternating Cone

A simple graph $G = (V, E)$ is said to be *threshold* if there are real vertex weights $c(v)$, $v \in V$ such that every pair $e = \{u, v\} \in \binom{V}{2}$ satisfies $c(u) + c(v) > 0$ if $e \in E$ and $c(u) + c(v) < 0$ if $e \notin E$. Our initial motivation for defining the alternating cone was the following observation.

Theorem 2.1. *A simple graph $G = (V, E)$ is threshold if and only if* $\dim \mathcal{A}(\widehat{G}) = 0$.

Proof. Given $e \in \binom{V}{2}$, let $\tau(e) = (\tau(e)(v) : v \in V) \in \mathbb{R}^V$ denote the incidence vector of e, where $\tau(e)(v)$ is 1 if v is an endpoint of e and 0 otherwise. Let $\mathcal{C}_R(G)$ denote the cone in $\mathbb{R}^V$ generated by the incidence vectors of the edges E, and let $\mathcal{C}_B(G)$ denote the cone generated by the incidence vectors of the nonedges $\binom{V}{2} - E$. If we write (1) in matrix notation, the columns correspond to the incidence vectors of edges and the negatives of the incidence vectors of nonedges. It follows that $\mathcal{A}(\widehat{G}) = \{0\}$ if and only if $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$.

Only if: Assume that the weights $c(v)$, $v \in V$ satisfy the defining property of a threshold graph. This means that $\mathcal{C}_R(G)$ and $\mathcal{C}_B(G)$ are on opposite sides of the hyperplane $\sum_{v \in V} c(v)x(v) = 0$. Hence $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$.

If: Suppose $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$. Then by the separation theorem of convex polyhedral cones, there is a hyperplane $\sum_{v \in V} c(v)x(v) = 0$ such that all nonzero vectors $(p(v) : v \in V) \in \mathcal{C}_R(G)$ satisfy $\sum_{v \in V} c(v)p(v) > 0$, and all nonzero vectors $(q(v) : v \in V) \in \mathcal{C}_B(G)$ satisfy $\sum_{v \in V} c(v)q(v) < 0$. Thus $\{u, v\} \in E$ implies $c(u) + c(v) > 0$, and $\{u, v\} \notin E$ implies $c(u) + c(v) < 0$. $\square$

We now determine the extreme rays of the alternating cone.

Theorem 2.2. *Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph. Then*

(i) the extreme rays of the alternating cone $\mathcal{A}(G, \mathcal{C})$ are the characteristic vectors of the alternating cycles and bicycles in $(G, \mathcal{C})$;
(ii) every integral vector in the alternating cone is a nonnegative integral combination of the characteristic vectors of irreducible CAW's;
(iii) every $\{0, 1\}$-vector in the alternating cone is a nonnegative integral combination of the characteristic vectors of irreducible CAT's
(iv) the characteristic vector of an irreducible CAW is $\{0, 1, 2\}$-valued.

Proof. (i) Clearly, the characteristic vectors of alternating cycles and bicycles are extreme. To show the converse, we will express any rational vector in the alternating cone as a nonnegative rational combination of the characteristic vectors of alternating cycles and bicycles.

Let $a = (a(e) : e \in E)$ be a nonzero rational vector in $\mathcal{A}(G, \mathcal{C})$. Pick $e_1 \in E$ with $a(e_1) \neq 0$. Without loss of generality we may assume that e_1 is colored red. Let v_0 and v_1 be the endpoints of e_1. Build an alternating trail as follows: choose a blue edge e_2 incident at v_1 with $a(e_2) \neq 0$ (this is possible by the balance condition). Let the other endpoint of e_2 be v_2. Now choose a red edge e_3 incident at v_2 with $a(e_3) \neq 0$, and so on. At some stage we will revisit an already visited vertex. Suppose this happens for the first time when we choose edge e_{k+1}, i.e., we have built an alternating trail

$$(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k), \qquad k \geq 1, \tag{4}$$

where $v_0, v_1, \ldots, v_k$ are distinct, $\mathcal{C}(e_k) \neq \mathcal{C}(e_{k+1})$, and e_{k+1} has endpoints v_k and a vertex $u_0 \in \{v_0, v_1, \ldots, v_{k-1}\}$. Then we have found either an alternating cycle D, or an odd internally alternating cycle C with base u_0. In the first case, subtracting an appropriate multiple of $\chi(D)$ from a, we obtain another vector in the alternating cone whose support is strictly contained in the support of a. Thus, by induction on the size of the support, we are done.

In the second case, extend C to an alternating trail $C * T$ as follows: T starts with $T = (u_0, f_1, \ldots)$, where f_1 is an edge incident with u_0 and satisfies $a(f_1) \neq 0$ and $\mathcal{C}(f_1) \neq \mathcal{C}(e_{k+1})$. Let the other endpoint of f_1 be u_1. Now add to T an edge f_2 incident with u_1 and satisfying $a(f_2) \neq 0$ and

$\mathcal{C}(f_2) \neq \mathcal{C}(f_1)$, and so on. At some stage we will revisit an already visited vertex of the trail $C * T$. Suppose this happens for the first time when we choose edge f_{m+1}, i.e., we have

$$T \quad = \quad (u_0, f_1, u_1, f_2, u_2, \ldots, f_m, u_m), \qquad m \geq 1, \tag{5}$$

where $u_0, u_1, \ldots, u_m$ are distinct, none of $\{u_1, \ldots, u_m\}$ is on C (see Figure 7), $\mathcal{C}(f_{m+1}) \neq \mathcal{C}(f_m)$, one endpoint of f_{m+1} is u_m, and the other endpoint v of f_{m+1} is either in T or is a vertex of C different from u_0. Two cases arise:
Case (a): v is a vertex of T (see Figure 8). We have found either an alternating cycle or an alternating bicycle, and we are done by induction on the size of the support, as in the previous paragraph.
Case (b): v is not a vertex of T (see Figure 9). In this case, the edges $f_1, \ldots f_{m+1}$ together with an appropriate portion of C determine an alternating cycle, and we are done.

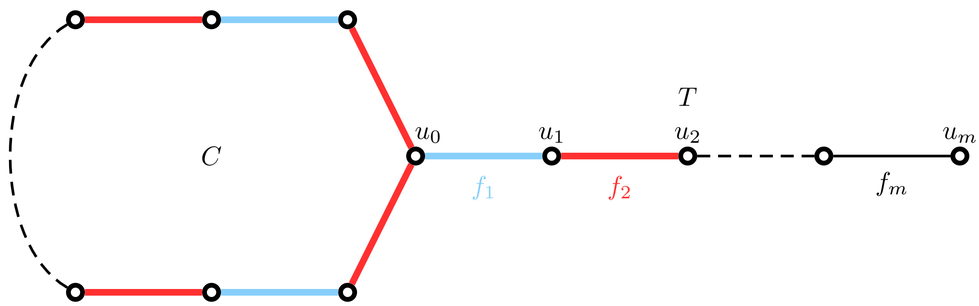

FIGURE 7. Illustrating the proof of Theorem 2.2 (i)

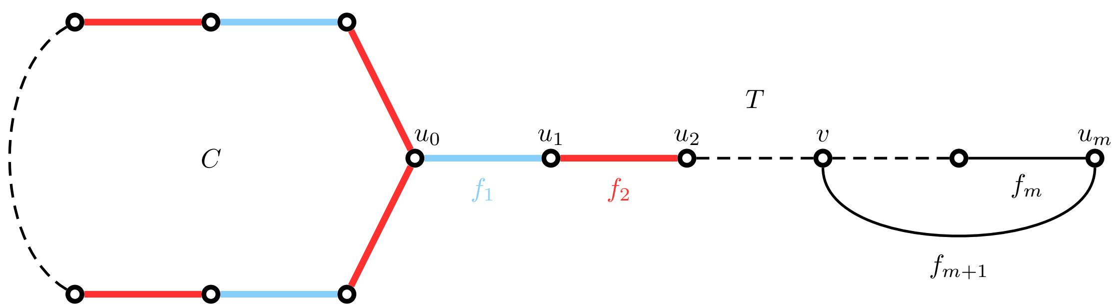

FIGURE 8. Illustrating the proof of Theorem 2.2 (i) Case (a)

Essentially the same argument as given above appears in [HIP] (in the context of edges and non-edges).

(ii) Let $a = (a(e) : e \in E)$ be a nonzero integral vector in the alternating cone. Pick an edge e_1 with $a(e_1) \neq 0$ and with end points v_0 and v_1. Assume that we have an internally alternating walk

$$W \quad = \quad (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m), \qquad m \geq 1, \tag{6}$$

with $\chi(W) \leq a$ (we can always start with the walk (v_0, e_1, v_1)). We show below that either we can extend W, or else there is a CAW (and hence an irreducible CAW) through e_1. Since we cannot extend indefinitely because of the condition $\chi(W) \leq a$, we are done. The following cases arise.
Case (a): $v_m \neq v_0$. Then $\chi(W)$ does not satisfy the balance condition at v_m, but a does, and since $\chi(W) \leq a$ and a is integral, we can find an edge e_{m+1} incident at v_m with $\mathcal{C}(e_{m+1}) \neq \mathcal{C}(e_m)$ such that $\chi(W)(e_{m+1}) < a(e_{m+1})$. Extend W by adding e_{m+1} and the other end point of e_{m+1}.

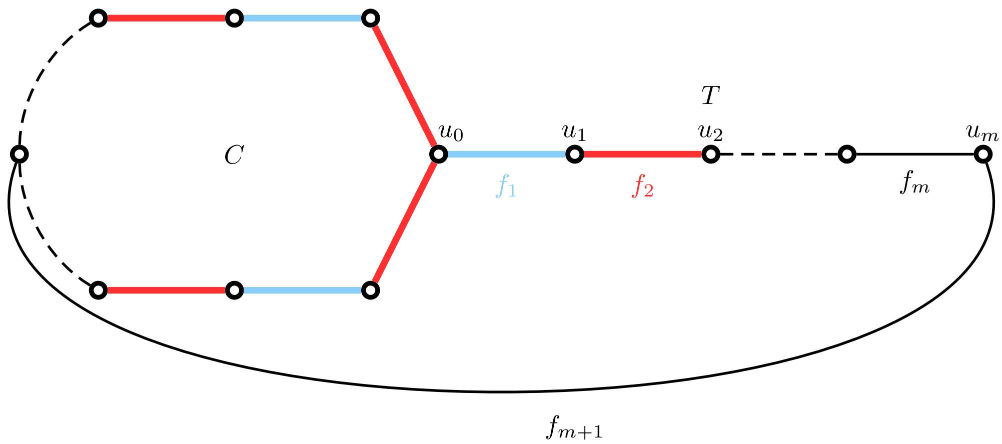

FIGURE 9. Illustrating the proof of Theorem 2.2 (i) Case (b)

Case (b): $v_m = v_0$ and $\mathcal{C}(e_1) = \mathcal{C}(e_m)$. We can extend W just as in case (a).
Case (c): $v_m = v_0$ and $\mathcal{C}(e_1) \neq \mathcal{C}(e_m)$. In this case W is a CAW.

(iii) This is a special case of (ii): if a is a $\{0,1\}$-vector, then the CAW's in (ii) must be CAT's.

(iv) Consider a walk W as in (3). This assigns a direction of traversal to each edge; for instance, the edge e_2 is traversed from v_1 to v_2. The direction of traversal may be different for two occurrences of the same edge. However, if a CAW W traverses an edge three or more times, then two of these directions must be the same, and this can be used to write $W = W_1 * W_2$ for two positive length CAW's W_1 and W_2, so W is not irreducible. □

As a corollary of Theorem 2.2, we derive the following well-known characterization of threshold graphs.

Corollary 2.3. *A simple graph G is not threshold if and only if $\widehat{G}$ contains an alternating cycle of length 4.*

Proof. **If:** Suppose $\{\{i,j\},\{j,k\},\{k,l\},\{l,i\}\}$ is an alternating 4-cycle in $\widehat{G}$ with the pairs $\{i,j\},\{k,l\}$ red and the other two pairs blue. Assume that G is threshold with vertex weights $c(v)$, $v \in V$ satisfying the defining property. Since $\{i,j\},\{k,l\}$ are red, we have $c(i)+c(j) > 0$, $c(k)+c(l) > 0$ and therefore $c(i)+c(j)+c(k)+c(l) > 0$. Similarly, since $\{j,k\},\{l,i\}$ are blue, we have $c(i)+c(j)+c(k)+c(l) < 0$, a contradiction.

Only if: Since G is not threshold, by Theorem 2.1 $\mathcal{A}(\widehat{G})$ has an extreme ray, which is an alternating cycle or an alternating bicycle by Theorem 2.2. Suppose that this extreme ray is an alternating cycle of length greater than 4. There is a chord of $\widehat{G}$ that splits this cycle into two even cycles (since $\widehat{G}$ is complete). Regardless of the color of the chord, one of these two cycles is alternating. Repeating this argument, we obtain an alternating cycle of length 4.

Now consider an extreme ray that is an alternating bicycle $W = W_1 * P * W_2 * P^R$. Let u and v be the bases of W_1 and W_2. Let u' (respectively, v') be any vertex of W_1 (respectively, W_2) different from u (respectively, v). Consider the edge $\{u',v'\}$ of $\widehat{G}$. W_1 determines two alternating u'-u paths, and one of them starts with an edge having color different from that of $\{u',v'\}$. Call this alternating path P_1. Similarly, using W_2, choose an alternating v'-v path P_2 that starts with an edge having color different from that of $\{u',v'\}$. We now have the alternating cycle $P_1 * P * P_2^R * (v', \{v',u'\}, u')$, and we can use the argument of the preceding paragraph. □

We now give a formula for the dimension of the alternating cone of a 2-colored graph.

A connected graph is said to be *odd unicyclic* if it contains precisely one cycle, and that cycle has odd length. In other words, an odd unicyclic graph is obtained from a tree by adding a new edge between two nonadjacent vertices of the tree so that the cycle created has odd length. A graph is a *pseudo forest* if each component of the graph is either acyclic or odd unicyclic. Pseudo forests are to be distinguished from 1-forests, which are graphs whose connected components have at most one cycle, even or odd. The motivation for studying 1-forests is combinatorial while pseudo forests have a linear algebraic origin (see Theorem 2.4 below). Recall from the proof of Theorem 2.1 that for an edge e, the vector $\tau(e) \in \mathbb{R}^V$ is the incidence vector of e, which is 1 in the two coordinates indexed by the endpoints of e, and is 0 elsewhere. The *incidence matrix* of a graph is the matrix whose columns are the incidence vectors of the edges. For a proof of the following result see [GKS].

Theorem 2.4. *For a graph $G = (V, E)$ and a set $X \subseteq E$, the set $\{\tau(e) : e \in X\}$ is linearly independent in $\mathbb{R}^V$ if and only if the graph (V, X) is a pseudo forest. In particular, the rank of the incidence matrix of G is equal to $\#V -$ number of bipartite components of G.*

For a graph $G = (V, E)$ and an integer sequence $d = (d(v) : v \in V)$, we use the notation

$$\mathcal{K}(d) = \{\mathcal{C} : E \to \{R, B\} : \text{the red degree of } v \text{ is } d(v) \text{ for all } v \in V\},$$

i.e., $\mathcal{K}(d)$ denotes the set of all 2-colorings of G having red degree sequence d.

Lemma 2.5. *Consider the 2-colored graph $G = (V, E)$ with a coloring $\mathcal{C} \in \mathcal{K}(d)$, and let $e \in E$. If $(G, \mathcal{C})$ has a CAW through e, then for each $\mathcal{C}' \in \mathcal{K}(d)$, $(G, \mathcal{C}')$ has a CAW through e.*

Proof. Let $\mathcal{C}' \in \mathcal{K}(d)$ and consider the spanning subgraph $G' = (V, E')$ of G, where E' consists of all the edges where $\mathcal{C}$ and $\mathcal{C}'$ disagree.

Since the red degrees (and thus also the blue degrees) in G agree under $\mathcal{C}$ and $\mathcal{C}'$, it follows that for each $v \in V$, the red degree of v in $(G', \mathcal{C}')$ is equal to the blue degree of v in $(G', \mathcal{C}')$. Thus the all-1 vector in $\mathbb{R}^{E'}$ is balanced in $(G', \mathcal{C}')$, i.e., is in $\mathcal{A}(G', \mathcal{C}')$. It follows from Theorem 2.2(iii) that for each $e \in E'$, $(G', \mathcal{C}')$ has a CAT through e, and therefore so does $(G, \mathcal{C}')$.

Now let W be a CAW through e in $(G, \mathcal{C})$. If $\mathcal{C}(e) \neq \mathcal{C}'(e)$, then we already know that $(G, \mathcal{C}')$ has a CAT through e, and we are done. So we may assume that $\mathcal{C}(e) = \mathcal{C}'(e)$. We will transform W into a CAW W' through e in $(G, \mathcal{C}')$. Let f be an edge in W with endpoints u and v. If $\mathcal{C}(f) = \mathcal{C}'(f)$, we do nothing. If $\mathcal{C}(f) \neq \mathcal{C}'(f)$, then $f \in E'$ and $(G, \mathcal{C}')$ has a CAT through f. Dropping f from this CAT, we obtain u-v alternating trail P in $(G, \mathcal{C}')$ whose first and last edges have the color $\mathcal{C}(f)$. We drop f from W and substitute the trail P in its place. Doing this for every edge f in W with $\mathcal{C}(f) \neq \mathcal{C}'(f)$, we obtain a CAW W' through e in $(G, \mathcal{C}')$. □

For a graph G and an integer sequence d, let E_d be the set of all edges e of G such that some 2-coloring in $\mathcal{K}(d)$ has a CAW through e (equivalently by Lemma 2.5, all 2-colorings in $\mathcal{K}(d)$ have a CAW through e).

Theorem 2.6. *Let $G = (V, E)$ be a graph and $\mathcal{C}$ a 2-coloring of G with red degree sequence d. Then*

$$\dim \mathcal{A}(G, \mathcal{C}) = \#E_d - \#V + b(V, E_d),$$

where $b(V, E_d)$ denotes the number of bipartite components of the graph (V, E_d).

Proof. An edge $e \in E$ is said to be *inessential* if $x(e) = 0$ for all $x \in \mathcal{A}(G, \mathcal{C})$. From Theorem 2.2(ii) it follows that e is inessential if and only if $(G, \mathcal{C})$ has no CAW through e. From basic polyhedral theory it now follows that $\dim \mathcal{A}(G, \mathcal{C})$ is equal to the nullity (i.e., number of columns minus rank) of the $\#V \times \#E_d$ vertex-edge incidence matrix of the graph (V, E_d). The expression for the dimension now follows from Theorem 2.4. □

From Theorem 2.6, $\dim \mathcal{A}(G, \mathcal{C})$ depends only on G and the red degree sequence of $\mathcal{C}$. In the case of the associated 2-colored graphs of simple graphs we can say more.

Lemma 2.7. *Let G_1 and G_2 be simple graphs with degree sequences d_1 and d_2. If d_1 is a rearrangement of d_2 (so in particular G_1 and G_2 have the same number of vertices), then* $\dim \mathcal{A}(\widehat{G}_1) = \dim \mathcal{A}(\widehat{G}_2)$.

Proof. Suppose that the permutation $\pi : V \to V$ rearranges d_1 into d_2. The result follows from the fact that π is an automorphism of the complete graph $\left(V, \binom{V}{2}\right)$. □

Lemma 2.7 fails for 2-colored graphs that are not complete:

Example 2.8. Figures 10 depicts two 2-colorings of a graph on the vertex set $\{1, 2, \dots, 7\}$ whose red degree sequences are permutations of each other (via the permutation π that fixes $2, 3, 7$ and exchanges 1 with 5 and 4 with 6). However, it is easily seen that the dimensions of the alternating cones of the 2-colored graphs are 1 and 0, respectively. The permutation π is not an automorphism of the underlying graph.

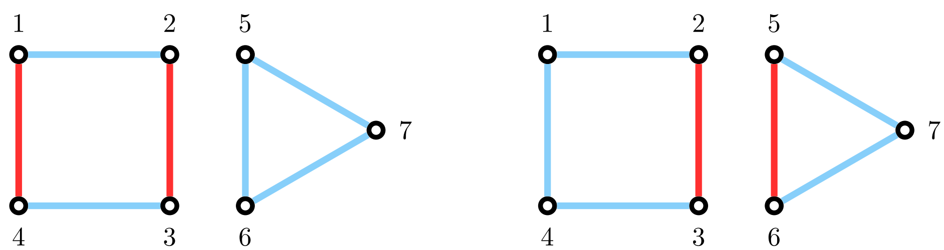

FIGURE 10. 2-colored graphs with the same red degree sequence: 1-dimensional alternating cone (left) and 0-dimensional alternating cone (right)

We now relate the dimension of the alternating cone to the concept of majorization. We begin with a few definitions.

Let $a = (a(1), \dots, a(n))$ and $b = (b(1), \dots, b(n))$ be real sequences of length n. Denote the i-th largest component of a (respectively, b) by $a[i]$ (respectively, $b[i]$). We say that a *majorizes* b, denoted by $a \succeq b$, if

$$\sum_{i=1}^{k} a[i] \geq \sum_{i=1}^{k} b[i], \qquad k = 1, \dots, n,$$

with equality for $k = n$. The majorization is *strict*, denoted by $a \succ b$, if at least one of the inequalities is strict, namely if a is not a permutation of b. We recall a fundamental lemma about majorization in integer sequences, called Muirhead's lemma. If $a = (a(1), \dots, a(n))$ is a sequence and there exist i and j such that $a(i) \geq a(j) + 2$, then the following operation is called a *unit transformation from i to j on a*: subtract 1 from $a(i)$ and add 1 to $a(j)$. Clearly, if b is obtained from a by a sequence of unit transformations, then $a \succ b$. The converse is also true for integer sequences.

Theorem 2.9 (Muirhead Lemma)**.** *If a and b are integer sequences and $a \succ b$, then some permutation of b can be obtained from a by a sequence of unit transformations.*

For a proof see [MP, MO].

Theorem 2.10. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be simple graphs with degree sequences d_1 and d_2. If $d_1 \succeq d_2$, then* $\dim \mathcal{A}(\widehat{G}_1) \leq \dim \mathcal{A}(\widehat{G}_2)$.

Proof. If d_2 is a rearrangement of d_1, the result follows from Lemma 2.7, so we may assume that $d_1 \succ d_2$. By Muirhead's lemma some permutation d_2' of d_2 can be obtained from d_1 by a finite sequence $d_1 \succ d \succ \cdots \succ d_2'$ of unit transformations. We will show that d is the degree sequence of a simple graph G satisfying $\dim \mathcal{A}(\widehat{G}_1) \leq \dim \mathcal{A}(\widehat{G})$. By Lemma 2.7 and induction on the number of unit transformations, the result will follow.

For notational convenience, let $V = \{1, 2, \ldots, n\}$, and suppose d is obtained from d_1 by a unit transformation from i to j, so that $d_1(i) \geq d_1(j) + 2$. This implies that there exist distinct vertices $k, l \neq i, j$ such that $\{i, k\}, \{i, l\}$ are edges of G_1 and $\{j, k\}, \{j, l\}$ are not. Let G be the graph with degree sequence d obtained from G_1 by dropping the edge $\{i, k\}$ and adding the edge $\{j, k\}$ (see Figure 11).

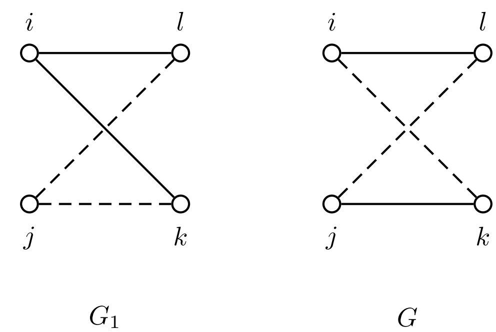

Figure 11. Illustrating the proof of Theorem 2.10

Consider the 2-colored graphs $\widehat{G}_1$ and $\widehat{G}$ with red degree sequences d_1 and d, respectively. We now show that $E_{d_1} \subseteq E_d$. By Theorem 2.6, it will then follow that $\dim \mathcal{A}(\widehat{G}_1) \leq \dim \mathcal{A}(\widehat{G})$, since $\#(E_d - E_{d_1}) \geq b(V, E_{d_1}) - b(V, E_d)$. This last inequality can be seen as follows: start with the graph (V, E_{d_1}) and add the edges $e \in E_d - E_{d_1}$ one at a time. If e connects two components C_1 and C_2, the number of bipartite components decreases by one or stays the same, according as C_1 and C_2 are both bipartite or not; if e connects two vertices in the same component C, the number of bipartite components stays the same if C is nonbipartite, and it stays the same or decreases by one if C is bipartite according to the parity of (any of the) cycles created by e.

Suppose $\{u, v\} \in E_{d_1}$. If $\{u, v\}$ is one of the pairs $\{i, k\}, \{j, k\}$ that changed status by going from G_1 to G, then Figure 11 depicts an alternating 4-cycle through $\{u, v\}$ in $\widehat{G}$, and thus $\{u, v\} \in E_d$ and we are done. So we may assume that $\{u, v\}$ is not one of these two pairs.

Since $\{u, v\} \in E_{d_1}$, $\widehat{G}_1$ has a CAW W through $\{u, v\}$. Replace every occurrence of

$$\ldots, i, \{i, k\}, k, \ldots,$$
$$\ldots, k, \{i, k\}, i, \ldots,$$
$$\ldots, k, \{k, j\}, j, \ldots,$$
$$\ldots, j, \{j, k\}, k, \ldots,$$

in W by (respectively)

$$\ldots, i, \{i, l\}, l, \{l, j\}, j, \{j, k\}, k, \ldots,$$
$$\ldots, k, \{k, j\}, j, \{j, l\}, l, \{l, i\}, i, \ldots,$$
$$\ldots, k, \{k, i\}, i, \{i, l\}, l, \{l, j\}, j, \ldots,$$
$$\ldots, j, \{j, l\}, l, \{l, i\}, i, \{i, k\}, k, \ldots,$$

keeping all other edges in W fixed. This yields a CAW through $\{u, v\}$ in G, and thus $\{u, v\} \in E_d$. $\square$

As stated in the introduction, Theorem 2.10 defines an order-reversing map $A : D(n) \to \mathbb{N}$, which maps the degree sequence of a simple graph G to the dimension of the alternating cone of the associated 2-colored graph $\widehat{G}$. Given $d = (d(1), \ldots, d(n)) \in D(n)$, there is a well-known algorithm working only with the numbers $d(1), \ldots, d(n)$ to determine whether $A(d) = 0$ (see [MP]). Motivated by this, we ask whether there is an algorithm working only with the numbers $d(1), \ldots, d(n)$ for computing $A(d)$.

3. Intersection of the Alternating Cone with a Box

Assume that we are given a 2-colored graph $G = (V, E)$, and for each $e \in E$ nonnegative integers $l(e), u(e)$ with $l(e) \le u(e)$. We ask if there is a rational vector $x \in \mathcal{A}(G, \mathcal{C})$ with $l(e) \le x(e) \le u(e)$ for all $e \in E$. The next theorem restricts the search to half-integral x.

Theorem 3.1. *Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph, and $l, u : E \to \mathbb{N}$ maps with $l(e) \le u(e)$ for all $e \in E$. If there exists a rational vector $x \in \mathcal{A}(G, \mathcal{C})$ with $l(e) \le x(e) \le u(e)$ for all $e \in E$, then there exists an integral $y \in \mathcal{A}(G, \mathcal{C})$ with $2l(e) \le y(e) \le 2u(e)$ for all $e \in E$.*

Proof. We use elementary polyhedral theory. Since by assumption a feasible solution exists, there exists a basic feasible solution $\overline{x} \in \mathcal{A}(G, \mathcal{C})$, with $l \le \overline{x} \le u$. In our case a basic feasible solution is obtained as follows. First choose a pseudo forest (V, X) such that the columns corresponding to X form a basis of the column space of the vertex-edge incidence matrix of G. For each $e \in E - X$ we have $\overline{x}(e) = l(e)$ or $\overline{x}(e) = u(e)$. Now solve for the remaining $\overline{x}(e)$, $e \in X$ using the balance condition at every node. Since l, u are integral and the determinant of the incidence matrix of an odd cycle is ± 2, the half-integrality of $\overline{x}$ easily follows, and $y = 2\overline{x}$ is as required. $\square$

Motivated by Theorem 3.1, we want to improve half-integrality to integrality, so we are led to the following problem. Let a 2-colored graph $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ and bounds $l, u : E \to \mathbb{N}$ be given. For $f \in \mathcal{A}(G, \mathcal{C})$, an edge e is called *feasible w.r.t.* f if $l(e) \le f(e) \le u(e)$, and f itself is called *feasible* if every edge is feasible w.r.t. f, *infeasible* otherwise. We ask if there is a feasible vector $f \in \mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$. We now reduce this problem to the problem of finding a CAT through a given edge in a 2-colored graph. This latter problem is easily reduced to the alternating reachability problem.

Let $f \in \mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$, f not necessarily feasible. The *residual 2-colored graph* $G(f) = (V, E(f))$ *of* f *w.r.t.* l, u is defined as follows. We take four disjoint copies E_1, E_2, E_3, E_4 of E, and denote the copy of $e \in E$ in E_i by e_i, $i = 1, \ldots, 4$. For each $e \in E$, we place e_1 in $E(f)$ with the color $\mathcal{C}(e)$ when $f(e) \le u(e) - 1$, place e_2 in $E(f)$ with the color $\mathcal{C}(e)$ when $f(e) \le u(e) - 2$, place e_3 in $E(f)$ with the color opposite $\mathcal{C}(e)$ when $f(e) \ge l(e) + 1$, and place e_4 in $E(f)$ with the color opposite $\mathcal{C}(e)$ when $f(e) \ge l(e) + 2$.

Suppose that $G(f)$ has a CAT T. We extend the characteristic vector $\chi(T)$ by adding zero components at all elements of $E_1 \cup E_2 \cup E_3 \cup E_4 - E(f)$. By *augmenting* f *along* T we mean replacing f with f_T given by

$$f_T(e) = f(e) + \chi(T)(e_1) + \chi(T)(e_2) - \chi(T)(e_3) - \chi(T)(e_4), \qquad e \in E.$$

Note that $f_T \in \mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$, and that in replacing f with f_T, feasible edges remains feasible, the infeasible edges of T move "in the right direction", i.e., become feasible or move closer to feasibility, and of course the edges out of T remain unchanged.

Theorem 3.2. *Suppose $f \in \mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$ is infeasible, but $\mathcal{A}(G, \mathcal{C}) \cap \mathbb{N}^E$ has a feasible vector. Then for each $e \in E$,*

- *(i) if $f(e) < l(e)$, then $G(f)$ has a CAT through e_1;*
- *(ii) if $f(e) > u(e)$, then $G(f)$ has a CAT through e_3.*

Proof. We define a 2-colored subgraph $G'(f) = (V, E'(f))$ of $G(f)$ by letting $E'(f) = E(f) \cap (E_1 \cup E_3)$ and restricting the 2-coloring of $G(f)$ to $G'(f)$.

Let $g \in \mathcal{A}(G,\mathcal{C}) \cap \mathbb{N}^E$ be a feasible vector. We define $h : E'(f) \to \mathbb{N}$ as follows: for $e_1 \in E'(f)$,

$$h(e_1) = \begin{cases} 0 & \text{if } g(e) - f(e) < 0, \\ g(e) - f(e) & \text{if } g(e) - f(e) \geq 0, \end{cases}$$

and for $e_3 \in E'(f)$,

$$h(e_3) = \begin{cases} 0 & \text{if } g(e) - f(e) > 0, \\ -(g(e) - f(e)) & \text{if } g(e) - f(e) \leq 0. \end{cases}$$

It is easy to check that h is an integral vector in the alternating cone of $G'(f)$.

(i) Assume that $e \in E$ with $f(e) < l(e)$. Then $e_1 \in E'(f)$ and $h(e_1) > 0$ (since g is feasible). By Theorem 2.2(ii), h can be written as a sum of characteristic vectors of irreducible CAW's in $G'(f)$, and thus $G'(f)$ has an irreducible CAW W through e_1. By Theorem 2.2(iv), $\chi(W)$ is $\{0, 1, 2\}$-valued. Suppose $\chi(W)(a_1) = 2$ for some a_1 (respectively, $\chi(W)(a_3) = 2$ for some a_3). Then $g(a) - f(a) \geq 2$ (respectively, $f(a) - g(a) \geq 2$). Since g is feasible, $a_2 \in E(f)$ (respectively, $a_4 \in E(f)$), and consequently $a_1 \in E(f)$ (respectively, $a_3 \in E(f)$) by the definition of $E(f)$. We then consider W as a subset of $E(f)$ and replace the double occurrence of a_1 (respectively, a_3) in W by a single occurrence of a_1 and of a_2 (respectively, of a_3 and of a_4). Doing this for all repeated edges in W transforms it into a CAT in $G(f)$ through e_1.

(ii) Similar to (i). □

The problem of finding a CAT through a given edge e in an edge-colored graph can be reduced to the alternating reachability problem as follows: let e have endpoints s and t. Remove e from the graph, add two new vertices s' and t', add two new edges with color $\mathcal{C}(e)$, one between s' and s and one between t' and t. Clearly the new graph has an alternating s'-t' trail if and only if the original graph has a CAT through e.

We can now use the following familiar scheme to look for a feasible integral vector. Start with an integral balanced $f : E \to \mathbb{N}$, for example $f = 0$. If f is infeasible, construct $G(f)$. Pick an edge $e \in E$ with $f(e) < l(e)$ (respectively, $f(e) > u(e)$), and find a CAT T through e_1 (respectively, e_3) in $G(f)$ if one exists (using the alternating reachability algorithm in [BPS2]), then augment f along T. As noted in the proof of Theorem 3.2, f_T is integral and balanced, feasible edges remain feasible, and in addition each infeasible edge in T, in particular e, has either become feasible or has moved closer to feasibility. Replace f with f_T and repeat. Since we are working with integral vectors, either we terminate with a feasible integral vector in time bounded by the total infeasibility, or else at some stage G_f has no CAT through e_1 (respectively, e_3), in which case no feasible integral vector exists by Theorem 3.2.

As stated in the introduction, in this paper we are not dealing with efficiency issues but only with the graph-theoretic aspects of the alternating cone. Our discussion motivates the alternating reachability problem and the problem of determining the linear inequalities defining the trail cone. These two problems are considered in [BPS2].

Acknowledgement: We thank the referees for their constructive suggestions that led to an improvement in the exposition.

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