

# Eigenbasis for a weighted adjacency matrix associated with the projective geometry $B_q(n)$

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*To the memory of Reddy*

## Abstract

In a recent article

*Projective geometries,  $Q$ -polynomial structures, and quantum groups*

Terwilliger (arXiv:2407.14964) defined a certain weighted adjacency matrix, depending on a free (positive real) parameter, associated with the projective geometry, and showed (among many other results) that it is diagonalizable, with the eigenvalues and their multiplicities explicitly written down, and that it satisfies the  $Q$ -polynomial property (with respect to the zero subspace).

In this note we

- Write down an explicit eigenbasis for this matrix.
- Evaluate the adjacency matrix-eigenvector products, yielding a new proof for the eigenvalues and their multiplicities.
- Evaluate the dual adjacency matrix-eigenvector products and directly show that the action of the dual adjacency matrix on the eigenspaces of the adjacency matrix is block-tridiagonal, yielding a new proof of the  $Q$ -polynomial property.

**Key Words.** Projective geometry, weighted adjacency matrix,  $Q$ -polynomial property.

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## 1 Introduction

Let  $q$  be a prime power and let  $B_q(n)$  denote the *projective geometry*, i.e., the poset (under inclusion) of all subspaces of  $\mathbb{F}_q^n$ , the  $n$ -dimensional  $\mathbb{F}_q$ -vector space of all column vectors with  $n$  components.

The set of  $k$ -dimensional subspaces in  $B_q(n)$  is denoted  $B_q(n, k)$  and its cardinality is the  $q$ -binomial coefficient  $\binom{n}{k}_q$ . The *Galois number*

$$G_q(n) = \sum_{k=0}^n \binom{n}{k}_q$$

is the total number of subspaces in  $B_q(n)$ .

Let  $\varphi$  be a positive real number. In [T5], Terwilliger defined the following (complex)  $B_q(n) \times B_q(n)$  *weighted adjacency matrix*  $A_n$  with entries given by

$$A_n(X, Y) = \begin{cases} 1 & \text{if } Y \subseteq X, \dim(X) = \dim(Y) + 1, \\ \varphi q^{\dim(X)} & \text{if } X \subseteq Y, \dim(Y) = \dim(X) + 1, \\ \frac{\varphi-1}{q-1} q^{\dim(X)} & \text{if } X = Y, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Terwilliger [T5] develops a beautiful theory of these matrices (generalizing the  $\varphi = 1$  case studied in [T3]. Note that in this case the diagonal entries are 0). The topics studied include: diagonalizability, eigenvalues and their multiplicities, dual matrix and the (generalized)  $Q$ -polynomial property, interpretation of the  $Q$ -polynomial structure using the quantum group  $U_{q^{1/2}}(\mathfrak{sl}_2)$  in the equitable presentation, and split decompositions in the new  $Q$ -polynomial structure. Here we consider only the following two basic results.

**Theorem 1.1. (Terwilliger [T5])**  $A_n$  is diagonalizable. The eigenvalues of  $A_n$  are

$$\frac{\varphi q^{n-k} - q^k}{q-1} \text{ with multiplicity } \binom{n}{k}_q, \quad k = 0, 1, \dots, n.$$

Let us make a few informal remarks about the proof of Theorem 1.1 in [T5]. It proceeds by considering the irreducible modules for the Terwilliger algebra  $T_n$  generated by  $A_n$  and the dual adjacency matrix  $A_n^*$  (defined below). The decomposition of the standard module (the complex vector space of column vectors with components indexed by  $B_q(n)$ ) into an orthogonal (under the standard inner product) direct sum of irreducible  $T_n$ -submodules is known ([T1]). With respect to a suitable basis of an irreducible  $T_n$ -submodule  $W$ , the matrix representing the restriction of  $A_n$  to  $W$  is also known [T1] and (a diagonally similar matrix) appears in Terwilliger's Leonard system classification [T2] and as such its eigenvalues were known. Coupled with the known decomposition into irreducibles mentioned above this yields the proof of Theorem 1.1. As pointed out by Terwilliger (in a personal communication) the eigenvectors for the restriction of  $A_n$  to  $W$  can also be written down with components given by certain  ${}_3\phi_2$  hypergeometric series. However this procedure does not give us an attractive (global) eigenbasis for  $A_n$  itself.

Let  $E_n(k)$ ,  $k = 0, 1, \dots, n$  denote the eigenspace of  $A_n$  corresponding to the eigenvalue  $\frac{\varphi q^{n-k} - q^k}{q-1}$ . Set  $E_n(-1) = E_n(n+1) = \mathbf{0}$  (the zero subspace).

In [T5] Terwilliger defines the *dual adjacency matrix*  $A_n^*$ , a diagonal  $B_q(n) \times B_q(n)$  matrix, with diagonal entries given by

$$A_n^*(X, X) = q^{-\dim(X)}, \quad X \in B_q(n).$$

Note that the eigenvalues of  $A_n^*$  are  $q^{-k}$  with multiplicities  $\binom{n}{k}_q$ ,  $k = 0, 1, \dots, n$ . Also note that  $A_n$  acts on the eigenspaces of  $A_n^*$  in a block tridiagonal fashion.

Terwilliger [T4] (see Section 20) initiated the extension of the  $Q$ -polynomial property to graphs that are not necessarily distance-regular and gave the first attractive example [T3] of this concept. In the context of the present paper, and the matrix  $A_n$ , the (generalized)  $Q$ -polynomial property with respect to  $\mathbf{0}$  says the following.

**Theorem 1.2. (Terwilliger [T5])**

$$A_n^*(E_n(k)) \subseteq E_n(k-1) \oplus E_n(k) \oplus E_n(k+1), \quad k = 0, 1, \dots, n. \quad (2)$$

That is,  $A_n^*$  acts on the eigenspaces of  $A_n$  in a block tridiagonal way. The tridiagonal relations satisfied by  $A_n, A_n^*$  are first derived in [T5] and are then used to prove Theorem 1.2.

The purpose of this note is to write down an explicit eigenbasis for  $A_n$ . This yields new proofs of Theorems 1.1, 1.2. We show that the method given in [GS] to write down an eigenbasis of  $A_n$  in the  $\varphi = 1$  case easily extends to the general case with a few small changes to account for the parameter  $\varphi$ .

In a combinatorial study of vector spaces indexing sets for the objects studied (bases, eigenvectors etc.) play an important role. The multiplicities in Theorem 1.1 suggest that the eigenvectors may be indexed by subspaces (just like in the classical case of the  $n$ -cube the eigenvectors of the adjacency matrix may be indexed by subsets). However, there does not seem to be any natural way of indexing the eigenvectors by subspaces. We now describe the indexing set to be used.

The Goldman-Rota recurrence for the Galois numbers ([GoR, KC]) is the identity

$$G_q(n+1) = 2G_q(n) + (q^n - 1)G_q(n-1), \quad n \geq 1, \quad G_q(0) = 1, \quad G_q(1) = 2. \quad (3)$$

More generally, we have the following identity,

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n - 1)\binom{n-1}{k-1}_q, \quad n, k \geq 1, \quad (4)$$

with  $\binom{0}{k}_q = \delta(0, k)$  and  $\binom{n}{0}_q = 1$ . Note that (3) follows by summing (4) over  $k$ .

Let  $\mathcal{I}_q(n)$  denote the set of all distinct irreducible characters (all of degree 1) of the finite abelian group  $\mathbb{F}_q^n$  and let  $\mathcal{N}_q(n)$  denote the set of all distinct nontrivial irreducible characters of  $\mathbb{F}_q^n$ . So  $|\mathcal{N}_q(n)| = q^n - 1$ .

For  $n \geq 0$ , inductively define a set  $\mathcal{E}_q(n)$  consisting of sequences as follows (here  $()$  denotes the empty sequence):

$$\begin{aligned} \mathcal{E}_q(0) &= \{()\}, \\ \mathcal{E}_q(1) &= \{(0), (1)\}, \\ \mathcal{E}_q(n) &= \{(\alpha_1, \dots, \alpha_t) \mid (\alpha_1, \dots, \alpha_{t-1}) \in \mathcal{E}_q(n-1), \alpha_t \in \{0, 1\}\} \\ &\quad \cup \{(\alpha_1, \dots, \alpha_t) \mid (\alpha_1, \dots, \alpha_{t-1}) \in \mathcal{E}_q(n-2), \alpha_t \in \mathcal{N}_q(n-1)\}, \quad n \geq 2. \end{aligned}$$

Given  $\alpha \in \mathcal{E}_q(n)$ , let  $N(\alpha)$  denote the number of nonzero entries in the sequence  $\alpha$  (note that a nonzero entry is either 1 or an element of  $\mathcal{N}_q(m)$  for some  $m$ ). Set

$$\begin{aligned}\mathcal{E}_q(n, k) &= \{\alpha \in \mathcal{E}_q(n) \mid N(\alpha) = k\}, \\ e_q(n, k) &= |\mathcal{E}_q(n, k)|.\end{aligned}$$

It is easy to see that

$$e_q(n+1, k) = e_q(n, k) + e_q(n, k-1) + (q^n - 1)e_q(n-1, k-1), \quad n, k \geq 1, \quad (5)$$

with  $e_q(0, k) = \delta(0, k)$  and  $e_q(n, 0) = 1$ , the same recurrence (with the same initial conditions) as (4). Thus  $e_q(n, k) = \binom{n}{k}_q$  and  $|\mathcal{E}_q(n)| = |B_q(n)|$ .

Another relevant structure in the present situation is an inner product (different from the standard inner product) on the (complex) vector space of column vectors with components indexed by  $B_q(n)$ .

For  $X \in B_q(n)$  with  $\dim(X) = k$  define

$$\pi(X) = \frac{\varphi^k q^{\binom{k}{2}}}{P_q(n)},$$

where  $P_q(n) = \prod_{k=0}^{n-1} (1 + \varphi q^k)$ . We have

$$\sum_{X \in B_q(n)} \pi(X) = \frac{\sum_{k=0}^n \varphi^k q^{\binom{k}{2}} \binom{n}{k}_q}{P_q(n)} = 1$$

where the second equality follows by the  $q$ -binomial theorem (so  $\pi$  is a probability vector on  $B_q(n)$ ).

In the statistical literature the distribution  $(p_0, p_1, \dots, p_n)$ , where  $p_k = \frac{\varphi^k q^{\binom{k}{2}} \binom{n}{k}_q}{P_q(n)}$ , is called Kemp's distribution, a two parameter analog of the binomial distribution (which corresponds to  $\varphi = 1$ ,  $q = 1$ ) (see, for example, [KK]).

Given vectors  $u, v$  with components indexed by  $B_q(n)$  define

$$\langle u, v \rangle_\pi = \sum_{X \in B_q(n)} \overline{u(X)} v(X) \pi(X). \quad (6)$$

Since  $P_q(n)$  is independent of  $k$ , for orthogonality of vectors we do not need the denominator in the definition of  $\pi(X)$ . It is there only to make  $\pi$  a probability vector.

It follows from Lemma 10.1 in [T5] (also see Section 2 of the present paper) that  $A_n$  is self-adjoint with respect to the inner product (6).

We now state our two main results.

**Theorem 1.3.** *For each  $\alpha \in \mathcal{E}_q(n)$  we define a column vector  $v_\alpha$ , with components indexed by  $B_q(n)$ , such that*

- (i)  $A_n v_\alpha = \frac{\varphi q^{n-k} - q^k}{q-1} v_\alpha$ , where  $k = N(\alpha)$ .
- (ii)  $\{v_\alpha \mid \alpha \in \mathcal{E}_q(n)\}$  is an eigenbasis of  $A_n$ .
- (iii)  $\{v_\alpha \mid \alpha \in \mathcal{E}_q(n)\}$  is orthogonal with respect to the inner product (6).

Theorem 1.1 is an immediate consequence of parts (i) and (ii) of Theorem 1.3.

Our second main result, in Section 4, considers the action of  $A_n^*$  on the eigenspaces of  $A_n$ . We evaluate  $A_n^* v_\alpha$  and write it as a linear combination of the eigenbasis  $\{v_\beta\}$  and directly show (2), yielding an alternative proof of Theorem 1.2. While the construction of the eigenbasis of  $A_n$  is based on the recurrence (4) for subspaces, writing down the action of  $A_n^*$  on this eigenbasis is based on the recurrence (5) for the indexing set.

In Section 5 we work out a detailed example to illustrate our algorithms.

Our approach to proving Theorem 1.3 and Theorem 1.2 is inductive (which explains our notation  $A_n, A_n^*$ , instead of  $A, A^*$ , as in [T5]). It is based on a decomposition of the space of column vectors with components indexed by  $B_q(n)$  that was worked out in [S] (the indexing set for the eigenbasis is suggested by this decomposition).

## 2 A decomposition of $\mathbb{C}[B_q(n)]$

In this section we recall the decomposition from [S]. All the omitted proofs may be found in Section 2 of [S]. This section is virtually the same as Section 3 of [GS] and is included here primarily for the readers convenience.

Before recalling this decomposition we first observe that  $A_n$  is self-adjoint under the inner product (6). Let  $D_n$  be the  $B_q(n) \times B_q(n)$  diagonal matrix with diagonal entry in row  $X$ , column  $X$  given by  $\sqrt{\varphi^k q^{\binom{k}{2}}}$ , where  $k = \dim(X)$ .

**Lemma 2.1.** ([T5], Lemma 10.1)  $D_n A_n D_n^{-1}$  is symmetric. Thus  $A_n$  is diagonalizable with real eigenvalues.

**Proof.** For  $X \in B_q(n, k), Y \in B_q(n, r)$  the entry in row  $X$ , column  $Y$  of  $D_n A_n D_n^{-1}$  is given by

$$\begin{aligned} & \sqrt{\varphi^k q^{\binom{k}{2}}} A_n(X, Y) \sqrt{\varphi^{-r} q^{-\binom{r}{2}}} \\ &= \begin{cases} \sqrt{\varphi^k q^{\binom{k}{2}}} q^k \sqrt{\varphi^{-(k+1)} q^{-\binom{k+1}{2}}} = \sqrt{\varphi q^k} & \text{if } X \subseteq Y \text{ and } r = k + 1, \\ \sqrt{\varphi^k q^{\binom{k}{2}}} \sqrt{\varphi^{-(k-1)} q^{-\binom{k-1}{2}}} = \sqrt{\varphi q^r} & \text{if } Y \subseteq X \text{ and } r = k - 1, \\ \frac{\varphi-1}{q-1} q^k & \text{if } X = Y, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sqrt{\varphi q^{\min\{\dim(X), \dim(Y)\}}} & \text{if } X \subseteq Y \text{ or } Y \subseteq X, \text{ and } |\dim(X) - \dim(Y)| = 1, \\ \frac{\varphi-1}{q-1} q^{\dim(X)} & \text{if } X = Y, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

yielding a symmetric matrix.  $\square$

Since  $P_q(n)$  is independent of  $k$ , the argument showing that  $D_n A_n D_n^{-1}$  is symmetric shows that  $A_n$  is self-adjoint with respect to the inner product (6).

For a finite set  $S$ , we denote the complex vector space with  $S$  as basis by  $\mathbb{C}[S]$ . We can think of  $\mathbb{C}[S]$  as the space of all (complex) column vectors with components indexed by  $S$ . Thus, for  $s \in S$ , the vector  $s \in \mathbb{C}[S]$  corresponds to the column vector with 1 in component  $s$  and 0's elsewhere.

We have (vector space direct sum)

$$\mathbb{C}[B_q(n)] = \mathbb{C}[B_q(n, 0)] \oplus \mathbb{C}[B_q(n, 1)] \oplus \cdots \oplus \mathbb{C}[B_q(n, n)].$$

An element  $v \in \mathbb{C}[B_q(n)]$  can be uniquely written as  $v = v_0 + v_1 + \cdots + v_n$ , where  $v_k \in \mathbb{C}[B_q(n, k)]$  for all  $k$ . The  $v_k$  are the *homogeneous components* of  $v$ . We say  $v$  is *homogeneous* if  $v \in \mathbb{C}[B_q(n, k)]$  for some  $k$ . A subspace of  $\mathbb{C}[B_q(n)]$  is *homogeneous* if it contains the homogeneous components of all its elements.

Let  $0 \leq k \leq n$ . Define a linear operator  $I_{n,k} : \mathbb{C}[B_q(n)] \rightarrow \mathbb{C}[B_q(n)]$  by  $I_{n,k}(X) = 0$  if  $\dim(X) \neq k$  and  $I_{n,k}(X) = X$  if  $\dim(X) = k$ ,  $X \in B_q(n)$ . The  $k^{th}$  *up operator*  $U_{n,k} : \mathbb{C}[B_q(n)] \rightarrow \mathbb{C}[B_q(n)]$  is defined, for  $X \in B_q(n)$ , by  $U_{n,k}(X) = 0$  if  $\dim(X) \neq k$  and  $U_{n,k}(X) = \sum_Y Y$ , where the sum is over all  $Y \in B_q(n)$  covering  $X$ , if  $\dim(X) = k$ . Similarly we define the  $k^{th}$  *down operator*  $D_{n,k} : \mathbb{C}[B_q(n)] \rightarrow \mathbb{C}[B_q(n)]$  (we have  $U_{n,n} = D_{n,0} = 0$ ). Set  $U_n = U_{n,0} + U_{n,1} + \cdots + U_{n,n}$  and  $D_n = D_{n,0} + D_{n,1} + \cdots + D_{n,n}$ , called, respectively, the *up and down operators* on  $\mathbb{C}[B_q(n)]$ .

It is easily seen that  $A_n$  is the matrix of the operator

$$\mathcal{A}_n = U_n + \sum_{k=0}^n \varphi q^{k-1} D_{n,k} + \sum_{k=0}^n \frac{\varphi - 1}{q - 1} q^k I_{n,k}$$

with respect to the *standard basis*  $B_q(n)$ .

Define  $\mathcal{A}_n^* : \mathbb{C}[B_q(n)] \rightarrow \mathbb{C}[B_q(n)]$  by

$$\mathcal{A}_n^*(X) = q^{-\dim(X)} X, \quad X \in B_q(n).$$

In Section 4 we shall study the action of  $\mathcal{A}_n^*$  on the eigenspaces of  $\mathcal{A}_n$ .

The decomposition of  $\mathbb{C}[B_q(n)]$  to be studied gives a linear algebraic interpretation to (3) and (4). Denote the standard basis vectors of  $\mathbb{F}_q^n$  by the column vectors  $e_1, \dots, e_n$ . We identify  $\mathbb{F}_q^k$ , for  $k < n$ , with the subspace of  $\mathbb{F}_q^n$  consisting of all vectors with the last  $n - k$  components zero. So  $B_q(k)$  consists of all subspaces of  $\mathbb{F}_q^n$  contained in the subspace spanned by  $e_1, \dots, e_k$ .

Define  $B_q^\partial(n+1)$  to be the collection of all subspaces in  $B_q(n+1)$  not contained in the hyperplane  $\mathbb{F}_q^n$ , i.e.,

$$B_q^\partial(n+1) = B_q(n+1) - B_q(n) = \{X \in B_q(n+1) : X \not\subseteq \mathbb{F}_q^n\}, \quad n \geq 0.$$

For  $1 \leq k \leq n+1$ , let  $B_q^\partial(n+1, k)$  denote the set of all subspaces in  $B_q^\partial(n+1)$  with dimension  $k$ . We consider  $B_q^\partial(n+1)$  as an induced subposet of  $B_q(n+1)$ .

We have a direct sum decomposition

$$\mathbb{C}[B_q(n+1)] = \mathbb{C}[B_q(n)] \oplus \mathbb{C}[B_q^\partial(n+1)]. \quad (7)$$

We shall now give a further decomposition of  $\mathbb{C}[B_q^\partial(n+1)]$ .

Let  $H(n+1, \mathbb{F}_q)$  denote the subgroup of  $GL(n+1, \mathbb{F}_q)$  consisting of all matrices of the form

$$\begin{bmatrix} & a_1 \\ & \cdot \\ I & \cdot \\ & a_n \\ 0 \dots 0 & 1 \end{bmatrix},$$

where  $I$  is the  $n \times n$  identity matrix.

The additive abelian group  $\mathbb{F}_q^n$  is isomorphic to  $H(n+1, \mathbb{F}_q)$  via  $\phi : \mathbb{F}_q^n \rightarrow H(n+1, \mathbb{F}_q)$  given by

$$\phi \left( \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \right) \rightarrow \begin{bmatrix} & a_1 \\ & \cdot \\ I & \cdot \\ & a_n \\ 0 \dots 0 & 1 \end{bmatrix}.$$

Recall that  $\mathcal{I}_q(n)$  denotes the set of all distinct irreducible characters of the finite abelian group  $H(n+1, \mathbb{F}_q)$  and  $\mathcal{N}_q(n)$  denotes the set of all distinct nontrivial irreducible characters of  $H(n+1, \mathbb{F}_q)$ .

There is a natural (left) permutation action of  $H(n+1, \mathbb{F}_q)$  on  $B_q^\partial(n+1)$  and  $B_q^\partial(n+1, k)$ . This permutation action induces representations of  $H(n+1, \mathbb{F}_q)$  on  $\mathbb{C}[B_q^\partial(n+1)]$  and  $\mathbb{C}[B_q^\partial(n+1, k)]$ .

For  $\chi \in \mathcal{I}_q(n)$ , let  $W(\chi)$  (respectively,  $W(\chi, k)$ ) denote the isotypical component of  $\mathbb{C}[B_q^\partial(n+1)]$  (respectively,  $\mathbb{C}[B_q^\partial(n+1, k)]$ ) corresponding to the irreducible representation of  $H(n+1, \mathbb{F}_q)$  with character  $\chi$ . When  $\chi$  is the trivial character we denote  $W(\chi)$  (respectively,  $W(\chi, k)$ ) by  $W(0)$  (respectively,  $W(0, k)$ ). We have the following decompositions, (note that  $W(\chi, n+1)$ , for  $\chi \in \mathcal{N}_q(n)$ , is the zero module).

$$W(0) = W(0, 1) \oplus \dots \oplus W(0, n+1), \quad (8)$$

$$W(\chi) = W(\chi, 1) \oplus \dots \oplus W(\chi, n), \quad \chi \in \mathcal{N}_q(n), \quad (9)$$

$$\mathbb{C}[B_q^\partial(n+1)] = W(0) \oplus \left( \bigoplus_{\chi \in \mathcal{N}_q(n)} W(\chi) \right). \quad (10)$$

Note that  $\mathbb{C}[B_q(n)]$ ,  $\mathbb{C}[B_q^\partial(n+1)]$ ,  $W(0)$ ,  $W(\chi)$  are homogeneous subspaces of  $\mathbb{C}[B_q(n+1)]$ .

Now  $GL(n+1, \mathbb{F}_q)$  acts on  $B_q(n+1)$  and  $\mathbb{C}[B_q(n+1)]$  and the action of  $U_{n+1}$  commutes with the action of  $GL(n+1, \mathbb{F}_q)$  (and hence with the action of  $H(n+1, \mathbb{F}_q)$ ). Also,  $\mathbb{C}[B_q^\partial(n+1)]$  is clearly closed under  $U_{n+1}$ . Thus

$$W(0), W(\chi), \chi \in \mathcal{N}_q(n) \text{ are } U_{n+1}\text{-closed.} \quad (11)$$

Define an equivalence relation  $\sim$  on  $B_q^\partial(n+1)$  by  $X \sim Y$  iff  $X \cap \mathbb{F}_q^n = Y \cap \mathbb{F}_q^n$ . Denote the equivalence class of  $X \in B_q^\partial(n+1)$  by  $[X]$ . For a subspace  $X \in B_q(n)$ , define  $\widehat{X}$  to be the subspace in  $B_q^\partial(n+1)$  spanned by  $X$  and  $e_{n+1}$ .

**Lemma 2.2.** *Let  $X, Y \in B_q^\partial(n+1)$  and  $Z, T \in B_q(n)$ . Then*

- (i)  *$\dim(X \cap \mathbb{F}_q^n) = \dim X - 1$  and  $\widehat{X \cap \mathbb{F}_q^n} \in [X]$ .*
- (ii)  *$Z \leq T$  iff  $\widehat{Z} \leq \widehat{T}$ .*
- (iii)  *$Y$  covers  $X$  iff*
  - (a)  *$Y \cap \mathbb{F}_q^n$  covers  $X \cap \mathbb{F}_q^n$  and*
  - (b)  *$Y = \text{span}((Y \cap \mathbb{F}_q^n) \cup \{v\})$  for any  $v \in X - \mathbb{F}_q^n$ .*
- (iv) *The number of subspaces  $Z' \in B_q^\partial(n+1)$  with  $Z' \cap \mathbb{F}_q^n = Z$  is  $q^l$ , where  $l = n - \dim Z$ . Thus,  $|[X]| = q^{n+1-k}$ , where  $k = \dim X$ .  $\square$*

For  $X \in B_q^\partial(n+1)$ , let  $G_X \subseteq H(n+1, \mathbb{F}_q)$  denote the stabilizer of  $X$ .

**Lemma 2.3.** *Let  $X, Y \in B_q^\partial(n+1)$ . Then*

- (i) *The orbit of  $X$  under the action of  $H(n+1, \mathbb{F}_q)$  is  $[X]$ .*
- (ii) *Suppose  $Y$  covers  $X$ . Then the bipartite graph of the covering relations between  $[Y]$  and  $[X]$  is regular with degrees  $q$  (on the  $[Y]$  side) and  $1$  (on the  $[X]$  side).*
- (iii) *Suppose  $X \subseteq Y$ . Then  $G_X \subseteq G_Y$ .  $\square$*

Consider  $\mathbb{C}[B_q(n+1)]$ . For  $X \in B_q(n)$  define

$$\theta_n(X) = \sum_Y Y,$$

where the sum is over all  $Y \in B_q^\partial(n+1)$  covering  $X$ . Equivalently, the sum is over all  $Y \in B_q^\partial(n+1)$  with  $Y \cap \mathbb{F}_q^n = X$ , i.e.,  $Y \in [\widehat{X}]$ . It follows from Lemma 2.3(i) that

$$\theta_n : \mathbb{C}[B_q(n)] \rightarrow W(0) \tag{12}$$

is a linear isomorphism.

Combining (7) and (10) we have the decomposition

$$\mathbb{C}[B_q(n+1)] = (\mathbb{C}[B_q(n)] \oplus W(0)) \oplus \left( \bigoplus_{\chi \in \mathcal{N}_q(n)} W(\chi) \right), \tag{13}$$

where, by (11) and (12),

$$\mathbb{C}[B_q(n)] \oplus W(0) \text{ is } U_{n+1}\text{-closed.} \tag{14}$$

Let  $\psi_k$  (respectively,  $\psi$ ) denote the character of the permutation representation of  $H(n+1, \mathbb{F}_q)$  on  $\mathbb{C}[B_q^\partial(n+1, k)]$  (respectively,  $\mathbb{C}[B_q^\partial(n+1)]$ ) corresponding to the left action. Clearly  $\psi = \sum_{k=1}^{n+1} \psi_k$ . Below  $[\cdot, \cdot]$  denotes character inner product and the  $q$ -binomial coefficient  $\binom{n}{k}_q$  is taken to be zero when  $n$  or  $k$  is  $< 0$ .

**Theorem 2.4.** (i) Let  $\chi \in \mathcal{I}_q(n)$  be the trivial character. Then  $[\chi, \psi_k] = \binom{n}{k-1}_q$ ,  $1 \leq k \leq n+1$ .

(ii) Let  $\chi \in \mathcal{N}_q(n)$ . Then  $[\chi, \psi_k] = \binom{n-1}{k-1}_q$ ,  $1 \leq k \leq n+1$ .  $\square$

**Remark 2.5.** Using Theorem 2.4 we see that

$$\dim(W(\chi)) = \sum_{k=1}^{n+1} \binom{n-1}{k-1}_q = G_q(n-1), \quad \chi \in \mathcal{N}_q(n). \quad (15)$$

Now, by taking dimensions on both sides of (13) and using (12), (15) we get the Goldman-Rota identity (3). More generally, by restricting to dimension  $k$  on both sides of (13), we get the identity (4).

For  $\chi \in \mathcal{I}_q(n)$ , define the following element of the group algebra of  $H(n+1, \mathbb{F}_q)$ :

$$p(\chi) = \sum_g \overline{\chi(g)} g,$$

where the sum is over all  $g \in H(n+1, \mathbb{F}_q)$ . For  $1 \leq k \leq n+1$ , the map

$$p(\chi) : \mathbb{C}[B_q^\partial(n+1, k)] \rightarrow \mathbb{C}[B_q^\partial(n+1, k)], \quad (16)$$

given by  $v \mapsto \sum_{g \in H(n+1, \mathbb{F}_q)} \overline{\chi(g)} gv$ , is a nonzero multiple of the  $H(n+1, \mathbb{F}_q)$ -linear projection onto  $W(\chi, k)$ . Similarly for  $p(\chi) : \mathbb{C}[B_q^\partial(n+1)] \rightarrow \mathbb{C}[B_q^\partial(n+1)]$ .

For future reference we record the following observation:

$$p(\chi)(\widehat{Y}) \text{ and } p(\chi)(\widehat{Z}) \text{ have disjoint supports, for } Y \neq Z \in B_q(n). \quad (17)$$

**Lemma 2.6.** Let  $X \in B_q^\partial(n+1)$  and  $\chi \in \mathcal{I}_q(n)$ . Then  $p(\chi)(X) = 0$  iff  $\chi : G_X \rightarrow \mathbb{C}^*$  is a nontrivial character of  $G_X$ .  $\square$

**Theorem 2.7.** (i) Let  $\chi \in \mathcal{I}_q(n)$ ,  $X, Y \in B_q^\partial(n+1)$  with  $X = hY$  for some  $h \in H(n+1, \mathbb{F}_q)$ . Then

$$p(\chi)(X) = \overline{\chi(h^{-1})} p(\chi)(Y).$$

(ii) Let  $\chi \in \mathcal{I}_q(n)$ . Then  $\{p(\chi)(\widehat{X}) : X \in B_q(n, k-1) \text{ with } p(\chi)(\widehat{X}) \neq 0\}$  is a basis of  $W(\chi, k)$ ,  $1 \leq k \leq n+1$ .

(iii) Let  $\chi \in \mathcal{I}_q(n)$  and let  $X, Y \in B_q(n)$  with  $X$  covering  $Y$ .

$$p(\chi)(\widehat{X}) \neq 0 \text{ implies } p(\chi)(\widehat{Y}) \neq 0. \quad \square$$

Let  $\chi \in \mathcal{N}_q(n)$ . By Theorem 2.4(ii) we have  $\dim(W(\chi, n)) = 1$ . It thus follows by Theorem 2.7(ii) and (17) above that there is a unique element  $X(\chi) \in B_q(n, n-1)$  such that  $p(\chi)(\widehat{X(\chi)}) \neq 0$ . Moreover,

**Lemma 2.8.** Let  $Y \in B_q(n, n-1)$ . Then

$$|\{\chi \in \mathcal{N}_q(n) \mid X(\chi) = Y\}| = q-1. \quad \square$$

### 3 Eigenvectors of $\mathcal{A}_n$

Consider the decomposition

$$\mathbb{C}[B_q(n+1)] = (\mathbb{C}[B_q(n)] \oplus W(0)) \oplus \left( \bigoplus_{\chi \in \mathcal{N}_q(n)} W(\chi) \right). \quad (18)$$

We claim that

$$\mathbb{C}[B_q(n)] \oplus W(0) \text{ and } W(\chi), \chi \in \mathcal{N}_q(n) \text{ are } I_{n+1,k}, D_{n+1,k}\text{-closed.} \quad (19)$$

This can be seen as follows. Since  $\mathbb{C}[B_q(n)]$ ,  $W(0)$ ,  $W(\chi)$  are homogeneous subspaces of  $\mathbb{C}[B_q(n+1)]$  it is easily seen that they are closed under  $I_{n+1,k}$ . Consider the *standard inner product* on  $\mathbb{C}[B_q(n+1)]$  (i.e., declare  $B_q(n+1)$  to be an orthonormal basis), which is  $GL(n+1, \mathbb{F}_q)$ -invariant (and hence  $H(n+1, \mathbb{F}_q)$ -invariant). It follows that  $W(0)$ ,  $W(\chi)$ ,  $\chi \in \mathcal{N}_q(n)$  are orthogonal and hence the decomposition (18) is orthogonal. Since  $D_{n+1}$  is the adjoint of  $U_{n+1}$ , it now follows from (11) and (14) that  $\mathbb{C}[B_q(n)] \oplus W(0)$  and  $W(\chi)$  are closed under  $D_{n+1}$ . The claim now follows from homogeneity.

Thus  $\mathbb{C}[B_q(n)] \oplus W(0)$  and  $W(\chi)$ ,  $\chi \in \mathcal{N}_q(n)$  are closed under  $\mathcal{A}_n$ .

Let  $X \in B_q(n, k)$ . By Lemma 2.2(iv) we see that  $\theta_n(X)$  is a sum of  $q^{n-k}$  subspaces in  $B_q^\partial(n+1, k+1)$ . So we can express  $D_{n+1,k+1}(\theta_n(X))$  in the following form

$$D_{n+1,k+1}(\theta_n(X)) = q^{n-k}X + v, \quad (20)$$

for some  $v \in \mathbb{C}[B_q^\partial(n+1, k)]$ . A little reflection shows that  $v \in W(0)$ . Defining

$$D'_{n+1,k+1}(\theta_n(X)) = v$$

gives a linear map

$$D'_{n+1,k+1} : W(0) \rightarrow W(0), \quad 0 \leq k \leq n,$$

that takes the vector  $\theta_n(X)$  for  $X \in B_q(n, k)$  to the vector  $v$  above.

Define  $\mathcal{A}'_n : W(0) \rightarrow W(0)$  by

$$\mathcal{A}'_n = U_{n+1} + \sum_{k=0}^n \varphi q^k D'_{n+1,k+1} + \sum_{k=0}^n \frac{\varphi - 1}{q - 1} q^{k+1} I_{n+1,k+1}.$$

We have the following relations (the first of which follows from (20)):

$$\mathcal{A}_{n+1}(\theta_n(v)) = \varphi q^n v + \mathcal{A}'_n(\theta_n(v)), \quad v \in \mathbb{C}[B_q(n)], \quad (21)$$

$$\mathcal{A}_{n+1}(v) = \mathcal{A}_n(v) + \theta_n(v), \quad v \in \mathbb{C}[B_q(n)]. \quad (22)$$

We now write down the matrix of  $\mathcal{A}'_n$  with respect to the basis  $\{\theta_n(X) \mid X \in B_q(n)\}$  of  $W(0)$ .

It follows from Lemma 2.2(iii) and Lemma 2.3(ii) that

$$U_{n+1}(\theta_n(X)) = \sum_Y q \theta_n(Y) = q \theta_n(U_n(X)), \quad X \in B_q(n),$$

where the middle sum is over all  $Y \in B_q(n)$  covering  $X$ . Similarly, it follows that

$$q^k D'_{n+1,k+1}(\theta_n(Y)) = \sum_X q \{q^{k-1} \theta_n(X)\} = q \theta_n(q^{k-1} D_{n,k}(Y)), \quad Y \in B_q(n, k),$$

where the middle sum is over all  $X \in B_q(n)$  covered by  $Y$ . Clearly,

$$q^{k+1} I_{n+1,k+1}(\theta_n(X)) = q \theta_n(q^k I_{n,k}(X)), \quad X \in B_q(n, k).$$

Thus we see that

$$\text{Matrix of } \mathcal{A}'_n \text{ with respect to the basis } \{\theta_n(X) \mid X \in B_q(n)\} \text{ is } qA_n. \quad (23)$$

For a finite vector space  $X$  over  $\mathbb{F}_q$  we denote by  $B_q(X)$  the set of all subspaces of  $X$  and we denote by  $U_X$  (respectively,  $D_X$ ) the up operator (respectively, down operator) on  $\mathbb{C}[B_q(X)]$ .

Let  $(V_1, f_1)$  be a pair consisting of a finite dimensional vector space  $V_1$  (over  $\mathbb{C}$ ) and a linear operator  $f_1$  on  $V$ . Let  $(V_2, f_2)$  be another such pair. By an isomorphism of pairs  $(V_1, f_1)$  and  $(V_2, f_2)$  we mean a linear isomorphism  $\tau : V_1 \rightarrow V_2$  such that  $\tau(f_1(v)) = f_2(\tau(v))$ ,  $v \in V_1$ .

**Theorem 3.1.** *Let  $\chi \in \mathcal{N}_q(n)$  and  $X = X(\chi)$ . Define*

$$\lambda(\chi) : \mathbb{C}[B_q(X)] \rightarrow W(\chi)$$

by  $Y \mapsto q^{-\dim(Y)} p(\chi)(\widehat{Y})$ ,  $Y \in B_q(X)$ .

*Then*

- (i)  $\lambda(\chi)$  is an isomorphism of pairs  $(\mathbb{C}[B_q(X)], qU_X)$  and  $(W(\chi), U_{n+1})$ .
- (ii)  $\lambda(\chi)$  is an isomorphism of pairs  $(\mathbb{C}[B_q(X)], D_X)$  and  $(W(\chi), D_{n+1})$ .

**Proof.** This is Theorem 4.2 in [GS].  $\square$

Before proceeding further we introduce some notation. Let  $X \in B_q(n, n-1)$ . The pairs  $(\mathbb{C}[B_q(X)], U_X)$  and  $(\mathbb{C}[B_q(n-1)], U_{n-1})$  are clearly isomorphic with many possible isomorphisms. We now define a canonical isomorphism, based on the concept of a matrix in Schubert normal form.

A  $n \times k$  matrix  $M$  over  $\mathbb{F}_q$  is in *Schubert normal form* (or, *column reduced echelon form*) provided

- (i) Every column is nonzero.
- (ii) The last nonzero entry in every column is a 1. Let the last nonzero entry in column  $j$  occur in row  $r_j$ .
- (iii) We have  $r_1 < r_2 < \dots < r_k$  and the submatrix of  $M$  formed by the rows  $r_1, r_2, \dots, r_k$  is the  $k \times k$  identity matrix.

It is well known that every  $k$  dimensional subspace of  $\mathbb{F}_q^n$  is the column space of a unique  $n \times k$  matrix in Schubert normal form.

Let  $X \in B_q(n, n-1)$  and let  $M(X)$  be the  $n \times (n-1)$  matrix in Schubert normal form with column space  $X$ . The map  $\tau(X) : \mathbb{F}_q^{n-1} \rightarrow X$  given by  $e_j \mapsto$  column  $j$  of  $M(X)$  is clearly a linear isomorphism and this isomorphism gives rise to an isomorphism

$$\mu(X) : \mathbb{C}[B_q(n-1)] \rightarrow \mathbb{C}[B_q(X)]$$

of pairs  $(\mathbb{C}[B_q(n-1)], U_{n-1})$  and  $(\mathbb{C}[B_q(X)], U_X)$  (and also of pairs  $(\mathbb{C}[B_q(n-1)], D_{n-1})$  and  $(\mathbb{C}[B_q(X)], D_X)$ ) given by  $\mu(X)(Y) = \tau(X)(Y)$ ,  $Y \in B_q(n-1)$ .

Since  $\dim(\hat{Y}) = 1 + \dim(Y)$ , the following result now follows from Theorem 3.1.

**Theorem 3.2.** *Let  $\chi \in \mathcal{N}_q(n)$  and  $X = X(\chi)$ . Then the composition  $\lambda(\chi)\mu(X)$  is an isomorphism of pairs  $(\mathbb{C}[B_q(n-1)], q\mathcal{A}_{n-1})$  and  $(W(\chi), \mathcal{A}_{n+1})$ .  $\square$*

**Lemma 3.3.** *Let  $v \in \mathbb{C}[B_q(n)]$  satisfy*

$$\mathcal{A}_n(v) = \frac{\varphi q^{n-k} - q^k}{q-1} v.$$

*Then*

$$\begin{aligned} \mathcal{A}_{n+1}(q^k v + \theta_n(v)) &= \frac{\varphi q^{n+1-k} - q^k}{q-1} (q^k v + \theta_n(v)), \\ \mathcal{A}_{n+1}(\varphi q^{n-k} v - \theta_n(v)) &= \frac{\varphi q^{n+1-(k+1)} - q^{k+1}}{q-1} (\varphi q^{n-k} v - \theta_n(v)). \end{aligned}$$

**Proof.** By (23) we have

$$\mathcal{A}'_n(\theta_n(v)) = \frac{q(\varphi q^{n-k} - q^k)}{q-1} v.$$

We have, by (21) and (22),

$$\begin{aligned} \mathcal{A}_{n+1}(q^k v + \theta_n(v)) &= q^k \mathcal{A}_{n+1}(v) + \mathcal{A}_{n+1}(\theta_n(v)) \\ &= q^k (\mathcal{A}_n(v) + \theta_n(v)) + \varphi q^n v + \mathcal{A}'_n(\theta_n(v)) \\ &= \left( \varphi q^{n-k} + \frac{\varphi q^{n-k} - q^k}{q-1} \right) q^k v + \left( q^k + \frac{q(\varphi q^{n-k} - q^k)}{q-1} \right) \theta_n(v) \\ &= \frac{\varphi q^{n+1-k} - q^k}{q-1} (q^k v + \theta_n(v)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{n+1}(\varphi q^{n-k} v - \theta_n(v)) &= \varphi q^{n-k} \mathcal{A}_{n+1}(v) - \mathcal{A}_{n+1}(\theta_n(v)) \\ &= \varphi q^{n-k} (\mathcal{A}_n(v) + \theta_n(v)) - \varphi q^n v - \mathcal{A}'_n(\theta_n(v)) \\ &= \left( -q^k + \frac{\varphi q^{n-k} - q^k}{q-1} \right) \varphi q^{n-k} v - \left( -\varphi q^{n-k} + \frac{q(\varphi q^{n-k} - q^k)}{q-1} \right) \theta_n(v) \\ &= \frac{\varphi q^{n+1-(k+1)} - q^{k+1}}{q-1} (\varphi q^{n-k} v - \theta_n(v)). \quad \square \end{aligned}$$

**Proof of Theorem 1.3 (parts (i) and (ii))**

The proof is by induction on  $n$ , the cases  $n = 0, 1$  being clear by taking

$$v_{()} = \mathbf{0}, \quad v_{(0)} = \mathbf{0} + \mathbb{F}_q, \quad v_{(1)} = \varphi \mathbf{0} - \mathbb{F}_q.$$

Let  $n \geq 1$  and consider  $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathcal{E}_q(n+1)$ . Set  $\beta = (\alpha_1, \dots, \alpha_{t-1})$  and  $k = N(\beta)$ . We have three cases:

(a)  $\alpha_t = 0$ : We have  $v_\beta \in \mathbb{C}[B_q(n)]$ . Define

$$v_\alpha = q^k v_\beta + \theta_n(v_\beta) \in \mathbb{C}[B_q(n)] \oplus W(0). \quad (24)$$

(b)  $\alpha_t = 1$ : We have  $v_\beta \in \mathbb{C}[B_q(n)]$ . Define

$$v_\alpha = \varphi q^{n-k} v_\beta - \theta_n(v_\beta) \in \mathbb{C}[B_q(n)] \oplus W(0). \quad (25)$$

(c)  $\alpha_t = \chi$ ,  $\chi \in \mathcal{N}_q(n)$ : We have  $v_\beta \in \mathbb{C}[B_q(n-1)]$ . Set  $X = X(\chi)$  and define

$$v_\alpha = \lambda(\chi) \mu(X)(v_\beta) \in W(\chi). \quad (26)$$

Let us now check assertions (i) and (ii) in the statement of the theorem, beginning with (i). We have three cases.

(a)  $\alpha_t = 0$ : Follows from Lemma 3.3.

(b)  $\alpha_t = 1$ : Follows from Lemma 3.3.

(c)  $\alpha_t = \chi$ ,  $\chi \in \mathcal{N}_q(n)$ : Set  $X = X(\chi)$ . It follows from Theorem 3.2 that

$$\begin{aligned} \mathcal{A}_{n+1}(v_\alpha) &= q \frac{\varphi q^{n-1-k} - q^k}{q-1} v_\alpha \\ &= \frac{\varphi q^{n+1-(k+1)} - q^{k+1}}{q-1} v_\alpha. \end{aligned}$$

Assertion (ii) follows from the induction hypothesis using the decomposition (18), the isomorphism (12) and observing that the determinant of the  $2 \times 2$  matrix

$$\begin{bmatrix} q^k & 1 \\ \varphi q^{n-k} & -1 \end{bmatrix} \quad (27)$$

is nonzero.  $\square$

**Remark** It is interesting to see where exactly in the proof above the diagonal entries of  $A_n$  are used since the main ingredients of the proof, (23), (21), (22), and Theorem 3.2, all hold even when the diagonal entries are 0. However, the base case of the induction ( $n = 0, 1$ ) needs the diagonal entries. In effect, the presence of the diagonal entries results in the nice formula for the eigenvalues which makes possible the calculations in the proof of Lemma 3.3.

We denote the basis given in part (ii) of Theorem 1.3 by  $\mathcal{B}_n$ . Note that equations (24), (25), (26) give an inductive procedure to write down  $\mathcal{B}_{n+1}$  given  $\mathcal{B}_n$  and  $\mathcal{B}_{n-1}$ . Note also that (up to scalars) this basis is canonical in the sense that we have not made any choices anywhere.

**Proof of Theorem 1.3 (part (iii))**

**Proof.** The proof is by induction on  $n$ , the cases  $n = 0, 1$  being clear.

Let  $n \geq 1$ . We consider two cases:

(i) Let  $\beta = (\beta_1, \dots, \beta_{t-1}) \in \mathcal{E}_q(n)$ . Set  $k = N(\beta)$  and

$$\alpha = (\beta_1, \dots, \beta_{t-1}, 0), \quad \alpha' = (\beta_1, \dots, \beta_{t-1}, 1).$$

Given a vectors  $u, v \in \mathbb{C}[B_q(n)]$ , we shall write  $\langle u, v \rangle_n$  for the inner product (6) calculated in  $\mathbb{C}[B_q(n)]$  and  $\langle u, v \rangle_{n+1}$  for the inner product calculated in  $\mathbb{C}[B_q(n+1)]$ . We have, for  $X \in B_q(n, k)$ ,

$$\begin{aligned} \langle X, X \rangle_n &= \frac{\varphi^k q^{\binom{k}{2}}}{P_q(n)}, \quad \langle X, X \rangle_{n+1} = \frac{\varphi^k q^{\binom{k}{2}}}{P_q(n+1)} = \frac{1}{1+\varphi q^n} \langle X, X \rangle_n, \\ \langle \theta_n(X), \theta_n(X) \rangle_{n+1} &= \frac{\varphi^{k+1} q^{\binom{k+1}{2}}}{P_q(n+1)} q^{n-k} = \frac{\varphi q^n}{1+\varphi q^n} \langle X, X \rangle_n. \end{aligned}$$

It follows that

$$\langle v, v \rangle_{n+1} = \frac{1}{1+\varphi q^n} \langle v, v \rangle_n, \quad \langle \theta_n(v), \theta_n(v) \rangle_{n+1} = \frac{\varphi q^n}{1+\varphi q^n} \langle v, v \rangle_n, \quad v \in \mathbb{C}[B_q(n)]. \quad (28)$$

Note that the scalar factors on the right hand side are uniform across all vectors and do not depend on  $k$ . Thus, since  $\mathbb{C}[B_q(n)]$  and  $W(0)$  are orthogonal in  $\mathbb{C}[B_q(n+1)]$ , it follows by the induction hypothesis that  $\{v_\beta, \theta_n(v_\beta) \mid \beta \in \mathcal{E}_q(n)\}$  is an orthogonal basis of  $\mathbb{C}[B_q(n)] \oplus W(0)$ . We have

$$v_\alpha = q^k v_\beta + \theta_n(v_\beta), \quad v_{\alpha'} = \varphi q^{n-k} v_\beta - \theta_n(v_\beta). \quad (29)$$

Since  $v_\beta$  is orthogonal to  $\theta_n(v_\beta)$  we have, using (28),

$$\begin{aligned} \langle v_\alpha, v_{\alpha'} \rangle_{n+1} &= \varphi q^n \langle v_\beta, v_\beta \rangle_{n+1} - \langle \theta_n(v_\beta), \theta_n(v_\beta) \rangle_{n+1} \\ &= \frac{\varphi q^n}{1+\varphi q^n} \langle v_\beta, v_\beta \rangle_n - \frac{\varphi q^n}{1+\varphi q^n} \langle v_\beta, v_\beta \rangle_n \\ &= 0. \end{aligned}$$

From the isomorphism  $\theta_n$  we now see that

$$\{v_\alpha, v_{\alpha'} \mid \beta \in \mathcal{E}_n(q)\}$$

is an orthogonal basis of  $\mathbb{C}[B_q(n)] \oplus W(0)$ .

(ii) Let  $\beta = (\beta_1, \dots, \beta_{t-1}) \in \mathcal{E}_q(n-1)$  and let  $\chi \in \mathcal{N}_q(n)$ . Set  $\alpha = (\beta_1, \dots, \beta_{t-1}, \chi) \in \mathcal{E}_q(n+1)$  and  $X = X(\chi)$ , where  $X \in B_q(n, n-1)$ . We have  $v_\alpha = \lambda(\chi) \mu(X)(v_\beta)$ .

Let  $Y \in B_q(X)$  with  $\dim(Y) = k$ . We have

$$\langle Y, Y \rangle_{n-1} = \frac{\varphi^k q^{\binom{k}{2}}}{P_q(n-1)}.$$

Now observe the following:  $p(\chi)(\widehat{Y})$  is a linear combination of the elements of the orbit  $[\widehat{Y}]$ , whose cardinality is  $q^{n-k}$ . The number of elements  $g \in H(n+1, \mathbb{F}_q)$  with  $g \cdot \widehat{Y} = \widehat{Y}$  is  $q^k$  and by Lemma 2.6 each such  $g$  satisfies  $\chi(g) = 1$ . So, for  $Z \in [\widehat{Y}]$ , if  $g_1 \cdot \widehat{Y} = g_2 \cdot \widehat{Y} = Z$  then  $\chi(g_1) = \chi(g_2)$ . Thus we have

$$\langle \lambda(\chi)(Y), \lambda(\chi)(Y) \rangle_{n+1} = q^{-2k} \frac{\varphi^{k+1} q^{\binom{k+1}{2}}}{P_q(n+1)} q^{2k} q^{n-k} = \frac{\varphi q^n}{(1+\varphi q^{n-1})(1+\varphi q^n)} \langle Y, Y \rangle_{n-1}.$$

It follows from (17) that

$$\langle \lambda(\chi)\mu(X)(u), \lambda(\chi)\mu(X)(v) \rangle_{n+1} = \frac{\varphi q^n}{(1+\varphi q^{n-1})(1+\varphi q^n)} \langle u, v \rangle_{n-1}, \quad u, v \in \mathbb{C}[B_q(n-1)]. \quad (30)$$

From the isomorphism  $\lambda(\chi)\mu(X)$  we now see that

$$\{v_\alpha \mid \beta \in \mathcal{E}_q(n-1)\}$$

is an orthogonal basis of  $W(\chi)$ .

That completes the proof.  $\square$

## 4 Action of $\mathcal{A}_n^*$ on the eigenspaces of $\mathcal{A}_n$

For  $k = 0, 1, \dots, n$  define  $E_n(k) \subseteq \mathbb{C}[B_q(n)]$  to be the eigenspace of  $\mathcal{A}_n$  with eigenvalue  $\frac{\varphi q^{n-k} - q^k}{q-1}$ . In this section we consider the action of  $\mathcal{A}_n^*$  on the eigenspaces of  $\mathcal{A}_n$  and give a new proof of Theorem 1.2.

Any homogeneous subspace of  $\mathbb{C}[B_q(n)]$  is closed under the action of  $\mathcal{A}_n^*$  and so the subspaces  $\mathbb{C}[B_q(n)], W(0), W(\chi), \chi \in \mathcal{N}_q(n)$  of  $\mathbb{C}[B_q(n+1)]$  are all closed under  $\mathcal{A}_{n+1}^*$ .

Note the following relations in  $\mathbb{C}[B_q(n)] \oplus W(0)$ :

$$\begin{aligned} \mathcal{A}_{n+1}^*(X) &= \mathcal{A}_n^*(X), \quad X \in B_q(n), \\ \mathcal{A}_{n+1}^*(\theta_n(X)) &= q^{-1} \theta_n(\mathcal{A}_n^*(X)), \quad X \in B_q(n). \end{aligned}$$

It follows that

$$\mathcal{A}_{n+1}^*(v) = \mathcal{A}_n^*(v), \quad v \in \mathbb{C}[B_q(n)], \quad (31)$$

$$\mathcal{A}_{n+1}^*(\theta_n(v)) = q^{-1} \theta_n(\mathcal{A}_n^*(v)), \quad v \in \mathbb{C}[B_q(n)]. \quad (32)$$

Let  $\chi \in \mathcal{N}_q(n)$  and  $X = X(\chi)$ . Then we have

$$\mathcal{A}_{n+1}^*(\lambda(\chi)\mu(X)(Y)) = q^{-1} \lambda(\chi)\mu(X)(\mathcal{A}_{n-1}^*(Y)), \quad Y \in B_q(n-1).$$

It follows that

$$\mathcal{A}_{n+1}^*(\lambda(\chi)\mu(X)(v)) = q^{-1} \lambda(\chi)\mu(X)(\mathcal{A}_{n-1}^*(v)), \quad v \in \mathbb{C}[B_q(n-1)]. \quad (33)$$

Before giving the proof of Theorem 1.2 we introduce some notation. For a element  $\tau \in \mathcal{E}_q(n)$  we denote by  $\tau^0$  the element of  $\mathcal{E}_q(n+1)$  obtained by adding a zero after the last element of  $\tau$ . Similarly for  $\tau^1$ . For a element  $\tau \in \mathcal{E}_q(n-1)$  and  $\chi \in \mathcal{N}_q(n)$  we denote by  $\tau^\chi$  the element of  $\mathcal{E}_q(n+1)$  obtained by adding a  $\chi$  after the last element of  $\tau$ .

**Proof of Theorem 1.2** We shall show by induction on  $n$  how to write  $\mathcal{A}_n^*(v_\alpha)$  as a linear combination of the vectors  $v_\beta$ , where  $N(\beta) \in \{k-1, k, k+1\}$ ,  $k = N(\alpha)$ . The cases  $n = 0, 1$  are clear.

Consider  $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathcal{E}_q(n+1)$ . Set  $\beta = (\alpha_1, \dots, \alpha_{t-1})$  and  $k = N(\beta)$ . There are three cases to consider:

(i)  $\alpha_t = 0$ : We have  $v_\alpha = q^k v_\beta + \theta_n(v_\beta)$ , where  $v_\beta \in E_n(k)$  and  $v_\alpha \in E_{n+1}(k)$ .

By the induction hypothesis we have

$$\mathcal{A}_n^*(v_\beta) = \sum_{j=k-1}^{k+1} u_j, \quad u_j \in E_n(j), \quad j = k-1, k, k+1. \quad (34)$$

and we have linear combinations

$$u_{k+1} = \sum_{\tau \in \mathcal{E}_q(n, k+1)} a_\tau v_\tau, \quad u_k = \sum_{\tau \in \mathcal{E}_q(n, k)} b_\tau v_\tau, \quad u_{k-1} = \sum_{\tau \in \mathcal{E}_q(n, k-1)} c_\tau v_\tau. \quad (35)$$

We have, by (31) and (32),

$$\begin{aligned} \mathcal{A}_{n+1}^*(v_\alpha) &= \mathcal{A}_{n+1}^*(q^k v_\beta + \theta_n(v_\beta)) \\ &= q^k \mathcal{A}_n^*(v_\beta) + q^{-1} \theta_n(\mathcal{A}_n^*(v_\beta)) \\ &= \sum_{j=k-1}^{k+1} (q^k u_j + q^{-1} \theta_n(u_j)). \end{aligned}$$

We consider the three terms on the right hand side separately.

(a) We have

$$\begin{aligned} q^k u_{k+1} + q^{-1} \theta_n(u_{k+1}) &= q^{-1} (q^{k+1} u_{k+1} + \theta_n(u_{k+1})) \\ &= q^{-1} \left( \sum_{\tau \in \mathcal{E}_q(n, k+1)} a_\tau (q^{k+1} v_\tau + \theta_n(v_\tau)) \right) \\ &= \sum_{\tau \in \mathcal{E}_q(n, k+1)} q^{-1} a_\tau v_{\tau^0}, \end{aligned}$$

which belongs to  $E_{n+1}(k+1)$ .

(b) We have

$$q^k u_k + q^{-1} \theta_n(u_k) = q^{-1} (q^{k+1} u_k + \theta_n(u_k)).$$

By the nonsingularity of the matrix (27), the vector above is in the span of the vectors  $q^k u_k + \theta_n(u_k) \in E_{n+1}(k)$  and  $\varphi q^{n-k} u_k - \theta_n(u_k) \in E_{n+1}(k+1)$  (by Lemma 3.3). We make this calculation below.

$$\begin{aligned}
& q^{-1}(q^{k+1}u_k + \theta_n(u_k)) \\
&= q^{-1} \left( \frac{\varphi q^{n-k} + q^{k+1}}{\varphi q^{n-k} + q^k} (q^k u_k + \theta_n(u_k)) + \frac{q^{k+1} - q^k}{\varphi q^{n-k} + q^k} (\varphi q^{n-k} u_k - \theta_n(u_k)) \right) \\
&= \sum_{\tau \in \mathcal{E}_q(n,k)} \frac{q^{-1} b_\tau (\varphi q^{n-k} + q^{k+1})}{\varphi q^{n-k} + q^k} v_{\tau^0} + \sum_{\tau \in \mathcal{E}_q(n,k)} \frac{q^{-1} b_\tau (q^{k+1} - q^k)}{\varphi q^{n-k} + q^k} v_{\tau^1},
\end{aligned}$$

which belongs to  $E_{n+1}(k) \oplus E_{n+1}(k+1)$ .

(c) We have

$$q^k u_{k-1} + q^{-1} \theta_n(u_{k-1}) = q^{-1}(q^{k+1} u_{k-1} + \theta_n(u_{k-1})).$$

We compute

$$\begin{aligned}
& q^{-1}(q^{k+1} u_{k-1} + \theta_n(u_{k-1})) \\
&= q^{-1} \left( \frac{\varphi q^{n-k+1} + q^{k+1}}{\varphi q^{n-k+1} + q^{k-1}} (q^{k-1} u_{k-1} + \theta_n(u_{k-1})) + \frac{q^{k+1} - q^{k-1}}{\varphi q^{n-k+1} + q^{k-1}} (\varphi q^{n-k+1} u_{k-1} - \theta_n(u_{k-1})) \right) \\
&= \sum_{\tau \in \mathcal{E}_q(n,k-1)} \frac{q^{-1} c_\tau (\varphi q^{n-k+1} + q^{k+1})}{\varphi q^{n-k+1} + q^{k-1}} v_{\tau^0} + \sum_{\tau \in \mathcal{E}_q(n,k-1)} \frac{q^{-1} c_\tau (q^{k+1} - q^{k-1})}{\varphi q^{n-k+1} + q^{k-1}} v_{\tau^1},
\end{aligned}$$

which belongs to  $E_{n+1}(k-1) \oplus E_{n+1}(k)$ .

That completes the proof of the first case.

(ii)  $\alpha_t = 1$ : We have  $v_\alpha = \varphi q^{n-k} v_\beta - \theta_n(v_\beta)$ , where  $v_\beta \in E_n(k)$  and  $v_\alpha \in E_{n+1}(k+1)$ .

By the induction hypothesis we may assume that (34) and (35) hold.

We have, by (31) and (32),

$$\begin{aligned}
\mathcal{A}_{n+1}^*(v_\alpha) &= \mathcal{A}_{n+1}^*(\varphi q^{n-k} v_\beta - \theta_n(v_\beta)) \\
&= \varphi q^{n-k} \mathcal{A}_n^*(v_\beta) - q^{-1} \theta_n(\mathcal{A}_n^*(v_\beta)) \\
&= \sum_{j=k-1}^{k+1} (\varphi q^{n-k} u_j - q^{-1} \theta_n(u_j)).
\end{aligned}$$

We consider the three terms on the right hand side separately.

(a) We have

$$\begin{aligned}
\varphi q^{n-k} u_{k-1} - q^{-1} \theta_n(u_{k-1}) &= q^{-1} (\varphi q^{n-k+1} u_{k-1} - \theta_n(u_{k-1})) \\
&= q^{-1} \left( \sum_{\tau \in \mathcal{E}_q(n,k-1)} c_\tau (\varphi q^{n-k+1} v_\tau - \theta_n(v_\tau)) \right) \\
&= \sum_{\tau \in \mathcal{E}_q(n,k-1)} q^{-1} c_\tau v_{\tau^1},
\end{aligned}$$

which belongs to  $E_{n+1}(k)$ .

(b) We have

$$\varphi q^{n-k} u_k - q^{-1} \theta_n(u_k) = q^{-1} (\varphi q^{n-k+1} u_k - \theta_n(u_k)).$$

We compute

$$\begin{aligned} & q^{-1} (\varphi q^{n-k+1} u_k - \theta_n(u_k)) \\ &= q^{-1} \left( \frac{\varphi(q^{n-k+1} - q^{n-k})}{\varphi q^{n-k} + q^k} (q^k u_k + \theta_n(u_k)) + \frac{\varphi q^{n-k+1} + q^k}{\varphi q^{n-k} + q^k} (\varphi q^{n-k} u_k - \theta_n(u_k)) \right) \\ &= \sum_{\tau \in \mathcal{E}_q(n,k)} \frac{q^{-1} b_\tau (q^{n-k+1} - q^{n-k})}{\varphi q^{n-k} + q^k} v_{\tau^0} + \sum_{\tau \in \mathcal{E}_q(n,k)} \frac{q^{-1} b_\tau (\varphi q^{n-k+1} + q^k)}{\varphi q^{n-k} + q^k} v_{\tau^1}, \end{aligned}$$

which belongs to  $E_{n+1}(k) \oplus E_{n+1}(k+1)$ .

(c) We have

$$\varphi q^{n-k} u_{k+1} - q^{-1} \theta_n(u_{k+1}) = q^{-1} (\varphi q^{n-k+1} u_{k+1} - \theta_n(u_{k+1})).$$

We compute

$$\begin{aligned} & q^{-1} (\varphi q^{n-k+1} u_{k+1} - \theta_n(u_{k+1})) \\ &= q^{-1} \left( \frac{\varphi(q^{n-k+1} - q^{n-k-1})}{\varphi q^{n-k-1} + q^{k+1}} (q^{k+1} u_{k+1} + \theta_n(u_{k+1})) + \frac{\varphi q^{n-k+1} + q^{k+1}}{\varphi q^{n-k-1} + q^{k+1}} (\varphi q^{n-k-1} u_{k+1} - \theta_n(u_{k+1})) \right) \\ &= \sum_{\tau \in \mathcal{E}_q(n,k+1)} \frac{q^{-1} a_\tau (q^{n-k+1} - q^{n-k-1})}{\varphi q^{n-k-1} + q^{k+1}} v_{\tau^0} + \sum_{\tau \in \mathcal{E}_q(n,k+1)} \frac{q^{-1} a_\tau (\varphi q^{n-k+1} + q^{k+1})}{\varphi q^{n-k-1} + q^{k+1}} v_{\tau^1}, \end{aligned}$$

which belongs to  $E_{n+1}(k+1) \oplus E_{n+1}(k+2)$ .

That completes the proof of the second case.

(iii)  $\alpha_t = \chi$ , for some  $\chi \in \mathcal{N}_q(n)$ : Let  $X = X(\chi)$ . Then  $v_\alpha = \lambda(\chi) \mu(X)(v_\beta) \in W(\chi)$ , where  $v_\beta \in E_{n-1}(k)$  and  $v_\alpha \in E_{n+1}(k+1)$ .

By the induction hypothesis we have

$$\mathcal{A}_{n-1}^*(v_\beta) = \sum_{j=k-1}^{k+1} u_j, \quad u_j \in E_{n-1}(j), \quad j = k-1, k, k+1.$$

and we have linear combinations

$$u_{k+1} = \sum_{\tau \in \mathcal{E}_q(n-1,k+1)} a_\tau v_\tau, \quad u_k = \sum_{\tau \in \mathcal{E}_q(n-1,k)} b_\tau v_\tau, \quad u_{k-1} = \sum_{\tau \in \mathcal{E}_q(n-1,k-1)} c_\tau v_\tau.$$

From Theorem 3.2 and (33) we now have

$$\mathcal{A}_{n+1}^*(v_\alpha) = q^{-1}\lambda(\chi)\mu(X)(\mathcal{A}_{n-1}^*(v_\beta)),$$

and

$$\begin{aligned} \mathcal{A}_{n+1}^*(v_\alpha) = & \sum_{\tau \in \mathcal{E}_q(n-1, k+1)} q^{-1}a_\tau v_{\tau^\chi} + \sum_{\tau \in \mathcal{E}_q(n-1, k)} q^{-1}b_\tau v_{\tau^\chi} + \sum_{\tau \in \mathcal{E}_q(n-1, k-1)} q^{-1}c_\tau v_{\tau^\chi}, \end{aligned}$$

which belongs to  $E_{n+1}(k) \oplus E_{n+1}(k+1) \oplus E_{n+1}(k+2)$ .  $\square$

## 5 A detailed example

Given  $\alpha \in \mathcal{E}_q(n, k)$ , the algorithm of Section 3 writes  $v_\alpha$  as a linear combination of the standard basis vectors, say,  $v_\alpha = \sum_{X \in B_q(n)} c_X X$ . Then  $\mathcal{A}_n^*(v_\alpha) = \sum_{X \in B_q(n)} q^{-\dim(X)} c_X X$ .

On the other hand, the algorithm of Section 4 writes  $\mathcal{A}_n^*(v_\alpha)$  as a linear combination of the  $v_\beta$ , with  $N(\beta) \in \{k-1, k, k+1\}$ . We can tally this expression for  $\mathcal{A}_n^*(v_\alpha)$  with the expression above by substituting for  $v_\beta$  in the standard basis and expanding.

We now take  $q = 2$  and carry out this process for  $n = 0, 1, 2$ .

We do not distinguish between a subspace and the (unique) matrix in Schubert normal form whose column space is that subspace. The Schubert normal form for the zero subspace (in any dimension) is denoted  $[\ ]$ .

(i)  $n = 0$  : We have  $\mathcal{E}_2(0) = \{(\ )\}$ , and

$$v_{(\ )} = [\ ] \in E_0(0).$$

(ii)  $n = 1$  : We have  $\mathcal{E}_2(1) = \{(0), (1)\}$  and

$$\begin{aligned} v_{(0)} &= [\ ] + [1] \in E_1(0), \\ v_{(1)} &= \varphi[\ ] - [1] \in E_1(1). \end{aligned}$$

(iii)  $n = 2$  : We denote the (unique) nontrivial character of  $\mathbb{F}_2$  by  $\chi$ .

The number of subspaces in  $B_2(2)$  is 5 and the five elements of  $\mathcal{E}_2(2)$  are

$$\{(0, 0), (0, 1), (1, 0), (1, 1), (\chi)\}.$$

The element  $p(\chi)$  of the group algebra of  $H(2, \mathbb{F}_2)$  is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly  $X(\chi) = \mathbf{0}$ . We now write down the eigenbasis of  $\mathcal{A}_2$  using Theorem 1.3. Recall our convention that we treat  $\mathbb{F}_2^1$  as the hyperplane in  $\mathbb{F}_2^2$  consisting of all column vectors with last component 0.

We have

$$\begin{aligned} v_{(0,0)} &= v_{(0)} + \theta_1(v_{(0)}) \\ &= \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \theta_1 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in E_2(0). \end{aligned}$$

$$\begin{aligned} v_{(0,1)} &= 2\varphi v_{(0)} - \theta_1(v_{(0)}) \\ &= 2\varphi \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \theta_1 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= 2\varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in E_2(1), \end{aligned}$$

$$\begin{aligned} v_{(1,0)} &= 2v_{(1)} + \theta_1(v_{(1)}) \\ &= 2 \left( \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \theta_1 \left( \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= 2\varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \varphi \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in E_2(1), \end{aligned}$$

$$\begin{aligned} v_{(\chi)} &= \lambda(\chi)\mu(\mathbf{0})(v_{(0)}) \\ &= \lambda(\chi)\mu(\mathbf{0}) \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= \lambda(\chi) \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= p(\chi) \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in E_2(1). \end{aligned}$$

$$\begin{aligned} v_{(1,1)} &= \varphi v_{(1)} - \theta_1(v_{(1)}) \\ &= \varphi \left( \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \theta_1 \left( \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= \varphi^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \varphi \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in E_2(2). \end{aligned}$$

Now we consider the action of  $\mathcal{A}_n^*$  on our eigenbasis. The base cases for the induction are

(i)  $n = 0$  : We have

$$\mathcal{A}_0^*(v_{()} ) = [ ] = v_{()}.$$

(ii)  $n = 1$  : We have

$$\begin{aligned}\mathcal{A}_1^*(v_{(0)}) &= [ ] + 2^{-1}[1] \\ &= \frac{\varphi + 2}{2(\varphi + 1)} v_{(0)} + \frac{1}{2(\varphi + 1)} v_{(1)}, \\ \mathcal{A}_1^*(v_{(1)}) &= \varphi[ ] - 2^{-1}[1] \\ &= \frac{\varphi}{2(\varphi + 1)} v_{(0)} + \frac{2\varphi + 1}{2(\varphi + 1)} v_{(1)}.\end{aligned}$$

(iii)  $n = 2$  : We consider each vector  $v_\alpha$ ,  $\alpha \in \mathcal{E}_2(2)$ , and apply the algorithm of Section 4 to write  $\mathcal{A}_2^*(v_\alpha)$  as a linear combination of the  $v_\beta$ , and then substitute the standard basis expressions for  $v_\beta$  and expand. In each case we find that the expression for  $\mathcal{A}_2^*(v_\alpha)$  agrees with the expression for  $v_\alpha$  upto a multiplicative factor of  $2^{-\dim(X)}$  for each subspace  $X \in B_2(2)$ .

(a) Using the expression for  $\mathcal{A}_1^*(v_{(0)})$  above and subcases (i)(b) and (i)(a) in the proof of Theorem 1.2 we get

$$\begin{aligned}\mathcal{A}_2^*(v_{(0,0)}) &= \frac{2(\varphi + 1)(\varphi + 2)}{4(\varphi + 1)(2\varphi + 1)} v_{(0,0)} + \frac{\varphi + 2}{4(\varphi + 1)(2\varphi + 1)} v_{(0,1)} + \frac{1}{4(\varphi + 1)} v_{(1,0)} \\ &= [ ] + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

(b) Using the expression for  $\mathcal{A}_1^*(v_{(0)})$  above and subcases (ii)(b) and (ii)(c) in the proof of Theorem 1.2 we get

$$\begin{aligned}\mathcal{A}_2^*(v_{(0,1)}) &= \frac{2\varphi(\varphi + 2)}{4(\varphi + 1)(2\varphi + 1)} v_{(0,0)} + \frac{(\varphi + 2)(4\varphi + 1)}{4(\varphi + 1)(2\varphi + 1)} v_{(0,1)} \\ &\quad + \frac{3\varphi}{4(\varphi + 1)(\varphi + 2)} v_{(1,0)} + \frac{2(2\varphi + 1)}{4(\varphi + 1)(\varphi + 2)} v_{(1,1)} \\ &= 2\varphi[ ] + \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

(c) Using the expression for  $\mathcal{A}_1^*(v_{(1)})$  above and subcases (i)(b) and (i)(c) in the proof of Theorem 1.2 we get

$$\begin{aligned}\mathcal{A}_2^*(v_{(1,0)}) &= \frac{2\varphi(\varphi + 2)}{4(\varphi + 1)(2\varphi + 1)} v_{(0,0)} + \frac{3\varphi}{4(\varphi + 1)(2\varphi + 1)} v_{(0,1)} \\ &\quad + \frac{(2\varphi + 1)(\varphi + 4)}{4(\varphi + 1)(\varphi + 2)} v_{(1,0)} + \frac{2(2\varphi + 1)}{4(\varphi + 1)(\varphi + 2)} v_{(1,1)} \\ &= 2\varphi[ ] - \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\varphi}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{\varphi}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

(d) Using the expression for  $\mathcal{A}_0^*(v_{()})$  above and case (iii) of the proof of Theorem 1.2 we get

$$\begin{aligned}\mathcal{A}_2^*(v_{(x)}) &= \frac{1}{2}v_{(x)} \\ &= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned}$$

(e) Using the expression for  $\mathcal{A}_1^*(v_{(1)})$  above and subcases (ii)(b) and (ii)(a) in the proof of Theorem 1.2 we get

$$\begin{aligned}\mathcal{A}_2^*(v_{(1,1)}) &= \frac{\varphi(2\varphi+1)}{4(\varphi+1)(\varphi+2)} v_{(1,0)} + \frac{2(2\varphi+1)(\varphi+1)}{4(\varphi+1)(\varphi+2)} v_{(1,1)} + \frac{\varphi}{4(\varphi+1)} v_{(0,1)} \\ &= \varphi^2[ ] - \frac{\varphi}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{\varphi}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\varphi}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

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