SOME PROBLEMS MOTIVATED BY THE NOTION OF THRESHOLD GRAPHS

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Abstract. We motivate and discuss four open problems in polyhedral combinatorics related to threshold graphs, degree sequences of graphs and hypergraphs, and balanced subgraphs of 2-colored graphs.

1. **Introduction**

A simple graph G is *threshold* if every induced subgraph of G contains a dominating or an isolated vertex. These graphs were introduced by Chvátal and Hammer [CH] in 1977. Their original motivation came from aggregation of inequalities in integer programming. Since then they have been studied from several different perspectives and a large literature has grown around them. A comprehensive monograph on the subject is the book by Mahadev and Peled [MP]. As explained in Chapter 3 of this book, the study of threshold sequences (i.e., degree sequences of threshold graphs) goes hand in hand with the study of degree sequences of (simple) graphs. Threshold sequences satisfy many of the criteria for degree sequences in an extremal way. In this expository paper (addressed to the beginning graduate student in graph theory) we motivate and discuss four open problems suggested by this connection between degree sequences and threshold graphs. Briefly, these four problems are concerened with the following topics:

- Degree sequences of ideal hypergraphs.
- Face numbers of the degree partition polytope.

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- Dimension of the alternating cone corresponding to a degree partition.
- Cone of balanced subgraphs of a 2-colored graph.

The last of these topics is not really about degree sequences but is a problem in non-bipartite matching theory (which we can think of as generalized degree sequence theory).

Our main references for this paper are [BSS, BPS1, BPS2]. Since these are freely available on the arXiv we have chosen not to include any proofs in this paper but have concentrated on motivation and clear statements of problems and results. We have also used, without proof, two characterizations of threshold graphs (in Sections 2 and 4). We leave their easy proofs as exercises. For more characterizations, generalizations, and applications of threshold graphs see the monograph [MP].

Notation

Given a simple graph $G = ([n], E)$ on the vertex set $[n] = \{1, 2, \ldots, n\}$, the degree d_j of a vertex j is the number of edges with j as an endpoint and $d_G = (d_1, d_2, \ldots, d_n)$ is the *degree sequence* of G. The *degree partition* of G is obtained by rearranging d_G in weakly decreasing order. Let $DS(n)$ denote the set of all degree sequences of simple graphs on the vertex set $[n]$ and let $DP(n)$ denote the set of all degree partitions of n-vertex simple graphs (note that some of the entries of a degree partition may be zero. It is usual to have only nonzero terms in a partition, but in this paper it is convenient to have this slight generality). The degree sequence (respectively, degree partition) of a threshold graph is called a *threshold sequence* (respectively, *threshold partition*). Let $TS(n)$ (respectively, $TP(n)$) denote the set of all threshold sequences (respectively, threshold partitions) of length n. If $(d_1, \ldots, d_n) \in TP(n)$ then either $d_1 = n - 1$ or $d_n = 0$. Applying this recursively we see that $\#TP(n) = 2^{n-1}$.

2. **Degree sequences of ideal hypergraphs**

Several different characterizations of threshold graphs can be generalized to the hypergraph setting to serve as definitions of threshold hypergraphs (see [G, RRST]). For example, one of the characterizations of threshold sequences is that they are precisely the extreme points of the polytope of degree sequences [K, PS]. In [BS] it was shown that much of this basic theory generalizes to the polytope of degree sequences of hypergraphs whose extreme points can then be defined to be the analog of threshold sequences for hypergraphs). Unlike the case of graphs these different definitions of thresholdness are not equivalent and yield diferent families of hypergraphs. A survey of the inclusions among these families is given in the paper of Klivans and Reiner [KR]. Here we consider another result on degree partitions of graphs, involving threshold partitions and majorization, that generalizes to hypergraphs and that has relevance to the long standing open question of characterizing degree partitions of simple uniform hypergraphs.

Let $S(n, r)$ denote the set of all r-subsets of $[n]$. Write each r-subset of $[n]$ in increasing order of its elements. Partially order $S(n, r)$ by componentwise $\leq$, i.e., $\{x_1 < x_2 < \cdots < x_r\} \leq \{y_1 < y_2 < \cdots < y_r\}$ iff $x_i \leq y_i$, for all i. An *r-graph* (or an *r-uniform hypergraph*) is a pair $([n], E)$, where $[n]$ is the set of *vertices* and $E \subseteq S(n, r)$ is the set of *edges*. Given an r-graph $H = ([n], E)$, the degree d_j of a vertex j is the number of edges in E containing j and $d_H = (d_1, \ldots, d_n)$ is the *degree sequence* of H. The *degree partition* of H is its degree sequence rearranged in weakly decreasing order.

An r-graph $H = ([n], E)$ is said to be an *r-ideal* if E is an order ideal of $S(n, r)$, i.e., $X \in E$ and $Y \leq X$ implies $Y \in E$. It is easy to see that the degree sequence of an r-ideal is a partition. For graphs, one of the standard characterizations of threshold partitions is that they are precisely the degree partitions of order ideals in $S(n, 2)$ (the proof of this statement is left as an exercise for the reader).

Let $a = (a(1), \ldots, a(n))$ and $b = (b(1), \ldots, b(n))$ be real sequences of length n. Denote the i-th largest component of a

(respectively, b) by $a[i]$ (respectively, $b[i]$). We say that *a majorizes* b, denoted $a \succeq b$, if

$$\sum_{i=1}^{k} a[i] \geq \sum_{i=1}^{k} b[i], \quad k = 1, \ldots, n,$$

with equality for $k = n$. It was proved in [PS, RG] that a weakly decreasing nonnegative integral sequence is a degree partition of a graph if and only if it is majorized by a threshold partition. This fact generalizes to hypergraphs, with essentially the same proof (see [BSS] (arXiv version) and [KR]): A weakly decreasing nonnegative integral sequence is a degree partition of an r-graph if and only if it is majorized by the degree partition of an r-ideal.

We do not know whether this characterization can be used in any way to give an efficient (i.e., polynomial time) algorithm for recognizing degree partitions of r-graphs. In any case, this characterization suggests the following problem.

Problem 1: Characterize or find an efficient algorithm for recognizing degree partitons of r-ideals.

3. Face numbers of the degree partition polytope

Three basic questions on degree sequences and degree partitions of simple graphs are

- Characterize degree sequences (or, equivalently, degree partitions) of simple graphs.
- Count the number of degree sequences of simple graphs.
- Count the number of degree partitions of simple graphs.

We remark that in the second and third items above we are concerned with exact enumeration and not with (the equally important) asymptotic enumeration. As is well known, degree sequences of simple graphs were characterized by Erdös and Gallai [EG]. We now recall this characterization from the polytope point of view. For later use we shall write down the characterizations for both degree sequences and degree partitions although they are trivially equivalent.

We first define two polytopes in $\mathbb{R}^n$. The polytope $\mathsf{K(n)}$ is defined to be the solution set of the following system of linear inequalities:

$$\sum_{i\in S} x_i - \sum_{i\in T} x_i \leq \#S(n-1-\#T), \tag{1}$$

where $S, T \subseteq [n],\ S\cup T \neq \emptyset,\ S\cap T = \emptyset$. Note that taking $S=\{i\},\ T=\emptyset$ gives $x_i \leq n-1$ and taking $S=\emptyset,\ T=\{i\}$ gives $x_i \geq 0$, showing that $\mathsf{K(n)}$ is indeed a polytope.

The polytope $\mathsf{F(n)}$ is defined to be the solution set of the following system of linear inequalities:

$$x_1 \geq x_2 \geq \cdots \geq x_n, \tag{2}$$

$$\sum_{i=1}^{k} x_i - \sum_{i=n-l+1}^{n} x_i \leq k(n-1-l), \quad 1 \leq k+l \leq n. \tag{3}$$

Note that (3) is obtained from (1) by taking $S=\{1,\ldots,k\}$ and $T=\{n-l+1,\ldots,n\}$. Intuitively, $\mathsf{K(n)}$ is obtained by symmetrizing $\mathsf{F(n)}$ and $\mathsf{F(n)}$ is the asymmetric part of $\mathsf{K(n)}$. Also note that $\mathsf{K(n)}$ has exponentially many defining inequalities while $\mathsf{F(n)}$ has only quadratically many defining inequalities.

We now recall the Erdös-Gallai criterion for degree partitions, as linearized by Fulkerson-Hoffman-McAndrew (see [FHM, MP]): given a nonnegative integral sequence $d=(d_1,\ldots,d_n)$, we have $d\in DP(n)$ (respectively, $d \in DS(n)$) if and only if $d_1+\cdots+d_n$ is even and $d \in \mathsf{F(n)}$ (respectively, $d\in\mathsf{K(n)}$). This characterization suggests the following definitions: given a simple graph G on the vertex set $[n]$ we define its *extended degree sequence* to be $(e, d_1, \ldots, d_n)$, where $(d_1,\ldots,d_n)$ is the degree sequence of G and e is the number of edges in G (note that $2e = d_1+\cdots+d_n$). Similarly, we define the *extended degree partition* of G. Define $\mathsf{DS(n)}$ (respectively, $\widehat{\mathsf{DS}}(n)$), the *polytope of degree sequences* (respectively, *polytope of extended degree sequences*), to be the convex hull (in $\mathbb{R}^n$) (respectively, in $\mathbb{R}^{n+1}$) of all degree sequences (respectively, extended degree sequences) of simple graphs on the vertex set $[n]$. Similarly, we define $\mathsf{DP(n)}$, the *polytope of degree partitions*, and $\widehat{\mathsf{DP}}(n)$, the *polytope of extended degree partitions*. Note that

- Since the number of edges depends linearly on the degrees the face numbers of $\widehat{\mathsf{DS}}(n)$ (respectively, $\widehat{\mathsf{DP}}(n)$) and $\mathsf{DS(n)}$ (respectively, $\mathsf{DP(n)}$) are equal.
- It follows from the Erdös-Gallai criterion (why?) that the number of degree sequences (respectively, degree partitions) of length n is equal to the number of lattice points in $\widehat{\mathsf{DS}}(n)$ (respectively, $\widehat{\mathsf{DP}}(n)$).

The study of $\mathsf{DS(n)}$ was begun by Koren [K] who showed that its extreme points are precisely the threshold sequences and that $\mathsf{DS(n)} = \mathsf{K(n)}$ (observe that the Erdös-Gallai criterion only shows $\mathsf{DS(n)} \subseteq \mathsf{K(n)}$). Beissinger and Peled [BP] determined the (exponential) generating function of the number of threshold sequences. Peled and Srinivasan [PS] determined the edges and facets of $\mathsf{DS(n)}$ and gave another proof of Koren's linear inequality description. Finally, Stanley [S2] obtained detailed information on $\mathsf{DS(n)}$ and $\widehat{\mathsf{DS}}(n)$ including generating functions for all face numbers, volume, and number of lattice points. In particular, he obtained a formula for the number of degree sequences of length n. It is natural to ask the same questions for the polytope $\mathsf{DP(n)}$. The first few steps were taken in the paper [BSS] whose main result we now describe.

As explained above, we can think of $\mathsf{DS(n)}$ as the symmetrization of $\mathsf{DP(n)}$ and $\mathsf{DP(n)}$ as the asymmetric part of $\mathsf{DS(n)}$. Let us consider an example of the operation of taking the asymmetric part. Let P be an integral polytope (i.e, a polytope with integral extreme points) in $\mathbb{R}^n$ that is closed under permutations of its points, i.e., $x \in P$ implies $\pi.x \in P$, for all permutations π of $[n]$. For example, $\mathsf{DS(n)}$ is such a polytope. Let E denote the set of extreme points of P and let $E_d \subseteq E$ denote the set of extreme points that have weakly decreasing coordinates. There are two natural ways to define the *asymmetric part* of P. In terms of lattice points we define the asymmetric part of P as the polytope

$$P_d = \text{ convex hull of } \{(x_1, x_2, \ldots, x_n) \in P \cap \mathbb{N}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

In terms of linear inequalities we define the asymmetric part of P as the polytope P_l obtained by adding the inequalities $x_1 \geq \cdots \geq x_n$ to the list of inequalities defining P. It is easily seen that $P_d \subseteq P_l$ and $E_d \subseteq$ set of extreme points of P_d. Equality need not hold in these two inclusions. For instance, consider the polytope P in $\mathbb{R}^2$ defined by: $x_1, x_2 \geq 0,\ x_1 + x_2 \leq 3$. Then it is easily checked that P_d is strictly contained in P_l. If we take P to be the polytope in $\mathbb{R}^2$ defined by $x_1, x_2 \geq 0,\ x_1+x_2 \leq 2$, then we can check that $P_d = P_l$ but P_d has an extreme point $(1,1)$ that is not contained in E_d. It is proved in [BSS] that, in the case $P = \mathsf{DS(n)}$, we do have $P_d = P_l$ and set of extreme points of $P_d = E_d$. More precisely, it is shown that $\mathsf{DP(n)} = \mathsf{F(n)}$ and that the set of extreme points of $\mathsf{DP(n)}$ are precisely the 2^{n-1} threshold partitions. Two other face numbers of $\mathsf{DP(n)}$ are also calculated. It is shown that, for $n \geq 4$, $\mathsf{DP(n)}$ has $2^{n-2}(2n-3)$ edges, and $(n^2 - 3n + 12)/2$ facets.

This result suggests the following problem.

Problem 2 Determine all the face numbers of $\mathsf{DP(n)}$. In particular (in analogy with the face numbers of the hypercube), is it true that the number of dimension k faces of $\mathsf{DP(n)}$, for $k = 0, 1, \ldots, n-1$, is of the form $P_k(n)2^{n-1-k}$, where $P_k(n)$ is a polynomial in n.

4. **Dimension of the alternating cone**

We start with another fundamental characterization of threshold graphs. Namely, a simple graph $G = (V, E)$ is threshold if and only if there are real vertex weights $c(v)$, $v \in V$ such that, for all distinct $u, v \in V$, $c(u) + c(v) > 0$ if $\{u, v\} \in E$ and $c(u) + c(v) < 0$ if $\{u, v\} \notin E$ (we leave this characterization also as an exercise).

Let $G = (V, E)$ be an undirected graph (we allow parallel edges but not loops). Assume that the edges of G are colored red or blue, the coloring being given by $\mathcal{C} : E \to \{R, B\}$. We say that $(G, \mathcal{C})$ is a *2-colored graph.* Consider the real vector space $\mathbb{R}^E$, with coordinates indexed by the set of edges of G. We write an element $x \in \mathbb{R}^E$ as $x = (x(e) : e \in E)$. For a subset

$F \subseteq E$ and $v \in V$, $F(v)$ denotes the set of all edges in F incident with v. For a subset $F \subseteq E$, F_R (respectively, F_B) denotes the set of red (respectively, blue) edges in F. For an edge $e \in E$, the characteristic vector $\chi(e) \in \mathbb{R}^E$ is defined by $\chi(e)(f) = \begin{cases} 1, & \text{if } f = e \\ 0, & \text{if } f \neq e \end{cases}$. The *red degree* $r(v)$ (respectively, *blue degree* $b(v)$) of a vertex $v \in V$ is the number of red (respectively, blue) edges incident with v.

The *cone of closed alternating walks*, or simply the *alternating cone*, $\mathcal{A}(G, \mathcal{C})$ of a 2-colored graph $(G, \mathcal{C})$ (denoted simply by $\mathcal{A}(G)$ when the coloring $\mathcal{C}$ is understood) is defined to be the set of all vectors $x = (x(e) : e \in E)$ in $\mathbb{R}^E$ satisfying the following system of homogeneous linear inequalities:

$$\sum_{e \in E_R(v)} x(e) - \sum_{e \in E_B(v)} x(e) = 0, \qquad v \in V, \tag{4}$$

$$x(e) \geq 0, \qquad e \in E. \tag{5}$$

We refer to (4) as the *balance condition* at vertex v. Our motivation for defining the alternating cone is as follows.

Let $G = (V, E)$ be a simple graph. We think of the elements of E as 2-element subsets of V. The 2-colored simple graph *associated* to G is the complete graph $\widehat{G} = \left(V, \binom{V}{2}\right)$, where $e = \{i, j\} \in \binom{V}{2}$ is colored red if $e \in E$ and colored blue if $e \notin E$. The basic observation is that G is threshold if and only if $\mathcal{A}(\widehat{G}) = \{0\}$. This can be seen as follows.

Given $e \in \binom{V}{2}$, let $\tau(e) = (\tau(e)(v) : v \in V) \in \mathbb{R}^V$ denote the incidence vector of e, where $\tau(e)(v)$ is 1 if v is an endpoint of e and 0 otherwise. Let $\mathcal{C}_R(G)$ denote the cone in $\mathbb{R}^V$ generated by the incidence vectors of the edges E, and let $\mathcal{C}_B(G)$ denote the cone generated by the incidence vectors of the nonedges $\binom{V}{2} - E$. If we write (4) in matrix notation, the columns correspond to the incidence vectors of edges and the negatives of the incidence vectors of nonedges. It follows that $\mathcal{A}(\widehat{G}) = \{0\}$ if and only if $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$.

Now assume that G is threshold with vertex weights $c(v),\ v \in V$ as in the characterization stated at the beginning of this section. This means that $\mathcal{C}_R(G)$ and $\mathcal{C}_B(G)$ are on opposite sides of the hyperplane $\sum_{v\in V} c(v)x(v) = 0$. Hence $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$.

Conversely, suppose that $\mathcal{C}_R(G) \cap \mathcal{C}_B(G) = \{0\}$. Then by the hyperplane separation theorem, there is a hyperplane $\sum_{v\in V} c(v)x(v) = 0$ such that all nonzero vectors $(p(v) : v \in V) \in \mathcal{C}_R(G)$ satisfy $\sum_{v\in V} c(v)p(v) > 0$, and all nonzero vectors $(q(v) : v \in V) \in \mathcal{C}_B(G)$ satisfy $\sum_{v\in V} c(v)q(v) < 0$. Thus $\{u, v\} \in E$ implies $c(u) + c(v) > 0$, and $\{u, v\} \notin E$ implies $c(u) + c(v) < 0$.

A closed walk on the edges of a 2-colored graph with succesive edges alternating in color is called a *closed alternating walk*. A simple alternating walk argument shows that integral vectors in the alternating cone are sums of characteristic vectors of closed alternating walks (thereby justifying our terminology). In the paper [BPS1] alternating walk arguments are used to study basic properties of the alternating cone relating to dimension, extreme rays, and existence of integral vectors satisfying given lower and upper bounds on the edges. Here we focus on a problem involving the dimension of the alternating cone.

It is well-known that for a simple graph G, the property of $\dim \mathcal{A}(\widehat{G})$ being 0 (i.e., G being threshold) depends only on the degree partition of G. This property is generalized in [BPS1] by showing that, for any 2-colored graph $(G, \mathcal{C})$, $\dim \mathcal{A}(G, \mathcal{C})$ depends only on the red degree sequence of $(G, \mathcal{C})$. As a consequence it follows that $\dim \mathcal{A}(\widehat{G})$ depends only on the degree partition of G. Following [AP], the dimension of $\mathcal{A}(\widehat{G})$ can be related to the concept of majorization as follows. Partially order the set $DP(n)$ of all degree partitions of length n by majorization, i.e., $d \geq e$ iff $d \succeq e$. The set of maximal elements of this poset is precisely the set of threshold partitions ([PS, RG]). Define a map $A : DP(n) \to \mathbb{N}$ by $A(d) = \dim \mathcal{A}(\widehat{G})$, where G is any simple graph with degree partiton d. It is shown in

[BPS1] that A is an order-reversing map (i.e., d majorizes e implies $A(d) \leq A(e)$). Thus, we can think of $A(d)$ as a kind of measure of how non-threshold the degree partition d is.

Given a degree partiton $d = (d_1, \ldots, d_n) \in DP(n)$, the procedure to determine $A(d)$ given in [BPS1] needs a realization of d as the degree partition of a simple graph. However, it is easy to test whether $A(d) = 0$ without such a realization. Namely, $A(d) = 0$ if and only if either $d_1 = n-1$ and $A((d_2-1, \ldots, d_n-1)) = 0$ or $d_n = 0$ and $A((d_1, \ldots, d_{n-1})) = 0$. We can ask a similar question for higher values of $A(d)$.

Problem 3: Given a degree partition $d \in DP(n)$, is there an algorithm to determine $A(d)$ that works with only the entries of d.

5. Cone of balanced subgraphs

Consider a directed graph. Assign a nonnegative real weight to every arc so that at every vertex, the total weight of the incoming arcs is equal to the total weight of the outgoing arcs. The set of all such assignments forms a convex polyhedral cone in the arc space, called the *cone of circulations*, and is a basic object of study in network flow theory. It is a fundamental fact that the cone of circulations, defined above via linear inequalities, can also be defined as the cone generated by the characteristic vectors of directed circuits. Moreover, an integral vector in the cone of circulations can be written as a nonnegative integral combination of characteristic vectors of directed circuits.

What about undirected analogs of the cone of circulations? Thinking in terms of linear inequalities we can consider the alternating cone of a 2-colored graph as a possible analog. However, there are integral vectors in the alternating cone that do not intuitively correspond to a circulation. For example, consider a graph G on six vertices that consists of two disjoint triangles together with an edge e with endpoints in the two triangles. Consider the unique matching M in G of size three. Color the edges in M red and the other four edges blue. The integral vector that assigns the value 2 to e and 1 to all other edges is in the alternating cone and does not correspond to a circulation in an intuitive sense.

Thinking combinatorially we can consider Seymour's cycle cone [S] as an undirected analog of the cone of circulations. Let $G = (V, E)$ be a graph and let $\mathcal{Z}(G)$ denote the convex polyhedral cone in $\mathbb{R}^E$ generated by the characteristic vectors of the cycles in G. We call $\mathcal{Z}(G)$ the *cycle cone* of G. Seymour [S] characterized the rational vectors in the cycle cone and made a conjecture concerning integral vectors in the cycle cone. Let us recall these results. By a *sum* (respectively, *fractional sum*) of cycles we mean a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of cycles.

Given a nonempty proper subset X of V, the subset $D \subseteq E$ of edges between X and $V - X$ will be called a *cut*. Let D be a cut, $e \in D$, and C a cycle in G. If C contains e, then C must also contain an edge in $D - \{e\}$. Thus the characteristic vector $\chi(C)$ of C satisfies the following inequality

$$x(e) \leq \sum_{f \in D-e} x(f), \tag{6}$$

where we write $D - e$ for $D - \{e\}$. We call (6) the *cut condition* for the pair (D, e). A nonnegative real vector on the edges that satisfies the cut condition for every pair (D, e) is said to be *cut admissible*. Seymour [S] proved that a vector $y \in \mathbb{N}^E$ is a fractional sum of cycles if and only if it is cut admissible. He also conjectured that if $y \in 2\mathbb{N}^E$ is a fractional sum of cycles then y is a sum of cycles.

We can combine the features of both the alternating cone and the cycle cone in a single model. Let $G = (V, E)$ be a graph. A (spanning) subgraph of G is said to be *even* if the degree of each vertex in the subgraph is even. By Euler's theorem the characteristic vector of an even subgraph is a sum of cycles and thus the cone generated by the (characteristic vectors of) even subgraphs is the same as the cycle cone. Now assume that E is colored with two colors, say red and blue. A vector $x = (x(e) : e \in E)$ in $\mathbb{R}^E$ is said to be *balanced* if it satifies the equation (4). A *balanced subgraph* is a subgraph whose characteristic vector is balanced (i.e., red degree equals blue degree at every vertex). By a *sum* (respectively, *fractional sum*) of balanced subgraphs we mean a nonnegative integral (respectively, nonnegative rational) combination of characteristic vectors of balanced subgraphs. The 2-color analog of the cycle cone is obtained by replacing even subgraphs with balanced subgraphs. For a 2-colored graph $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$, define $\mathcal{B}(G, \mathcal{C}) \subseteq \mathbb{R}^E$ to be the convex polyhedral cone generated by the characteristic vectors of balanced subgraphs in $(G, \mathcal{C})$. We call $\mathcal{B}(G, \mathcal{C})$ the *cone of balanced subgraphs*.

Consider a balanced subgraph in a 2-colored graph. By ignoring the colors and applying Euler's theorem we can write its characteristic vector as a sum of cycles. This shows that a sum of balanced subgraphs is a balanced sum of cycles. We conjecture that the converse of this observation is also true:

Conjecture 4: Let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a 2-colored graph and let $y \in \mathbb{N}^E$. If y is a balanced sum of cycles then y is a sum of balanced subgraphs.

In [BPS2] it is proved that a balanced sum of cycles is a fractional sum of balanced subgraphs. The proof is modelled on Seymour's proof characterizing rational vectors in the cycle cone and is based on induction and the following colored Kotzig's lemma: let $G = (V, E)$, $\mathcal{C} : E \to \{R, B\}$ be a bridgeless 2-colored graph and assume that every vertex has incident edges of both colors. Then there exists a nonempty balanced subgraph in G.

Conjecture 4 can be seen as strengthening both the hypothesis and conclusion of colored Kotzig's lemma. Given a bridgeless graph $G = (V, E)$, with edges colored red and blue, there is a positive vector $y \in \mathbb{N}^E$ that is a sum of cycles. Colored Kotzig's lemma assumes that E has edges of both colors at every vertex and concludes that G has a balanced subgraph, whereas Conjecture 4 assumes more, namely that y is balanced and concludes more, namely that y is a sum of balanced subgraphs.

Finally, we would like to make a few remarks concerning Conjecture 4. Seymour's conjecture states that a fractional sum of cycles that is an even integer on every edge is a sum of cycles. Analogously, in the 2-colored case, we can conjecture that a balanced fractional sum of cycles that is an even integer on every edge is a sum of balanced subgraphs. This latter conjecture follows from Seymour's conjecture and Conjecture 4. Is there any relation between these conjectures? Note that the hypothesis in Seymour's conjecture (that of a vector being a fractional sum of cycles) is well characterized whereas we do not know whether the hypothesis in Conjecture 4 is well characterized.

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