Finite Element Galerkin Methods for Equations of Motion Arising in the Kelvin-Voigt Model

Thesis
Submitted in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

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2012

Acknowledgment

With the grace of Almighty and blessings of my parents and pedagogues, I have reached to the culmination of this thesis. I wish to express my deep and sincere gratitude to my supervisor, Professor Amiya K. Pani, whose wide knowledge and unflinching attitude encouraged me at all the stages of my research. His paternal advices buttressed me to conduct my work comfortably.

Words are short to express my deep sense of gratitude towards Neela Mam, my second supervisor, for her invaluable guidance, support and generous care. She created an amiable atmosphere for me in all respects.

I take this opportunity to sincerely acknowledge my RPC members, Professor S. Baskar and Professor Sivaji Ganesh Sista, for their valuable comments and criticism about my work during my progress seminar presentations.

I am grateful to UGC (University Grants Commission) for providing me the financial assistance without which this work would not have been possible. I am indebted to the Head, faculty members and non academic staff members of the department for their kind assistance.

Words fail me to express my appreciation to my friends Kalpesh, Anuradha and Bhawana for always being with me during ups and downs of the entire tenure at Indian Institute of Technology (IITB). I expand my thanks to Asha, Sajid, Deepjyoti, Arun, Pramod, Soham, Akshay, Ramesh, Rupali, Sangita, Shreedevi, Chandan for their endless love. All my friends played an important role in the successful completion of this research work.

Last but not the least, I would like to pay high regards to my mother, father, brother Satyam and sister Anchal for their constant inspiration throughout my research endeavor and lifting me high through this phase of life. I owe everything to them.

This thesis is dedicated to my parents for their unconditional love and eternal blessings.

Saumya

Declaration

I declare that this written submission represents my ideas in my own words and where others ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Abstract

The main goal of this dissertation is to study finite element Galerkin methods for the equations of motion described by Kelvin-Voigt viscoelastic fluids. We first prove the global existence of a unique weak solution by applying Faedo-Galerkin method and standard compactness arguments. Using the energy arguments, we establish new a *priori* bounds for the solution which exhibit exponential decay property. We next present semidiscrete approximation by discretizing spatial variable with the help of finite element method while keeping time variable continuous. For the semidiscrete scheme, we discuss optimal error estimate for the velocity in $L^\infty(\mathbf{H}^1)$-norm. For optimal $L^\infty(\mathbf{L^2})$ estimate, we appeal to the finite element solution of a linearized Kelvin-Voigt model and split the error in velocity into two parts: one due to linearization and the other to take care of nonlinear effect. For the error due to linearized problem, we introduce a new auxiliary projection called 'Sobolev-Stokes' projection and analyse its approximation properties. For the error due to nonlinearity, we use standard energy arguments and weighted estimates with exponential weight. Then, making use of these results, we derive optimal error bounds for the velocity in $L^\infty(\mathbf{L^2})$ as well as in $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm which again preserve the exponential decay property. In order to achieve complete discretization, we apply a backward Euler method and a second order backward difference scheme to discretize the semidiscrete problem in temporal direction. After discussing the solvability of the discrete problem, we have derived optimal rates of convergence for the error, which again preserve exponential decay property in time. Special care has been taken to introduce exponential weight in the discrete time level which plays a crucial role in exponential decay property. We also conduct several numerical experiments to support theoretical results. Finally, we implement a two-grid method based on Newton's type iteration. We derive a *priori* bounds for two-grid semidiscrete solutions. These bounds along with the estimates obtained for the Sobolev-Stokes projection help us in achieving optimal estimates of velocity in $L^\infty(\mathbf{L^2})$ as well as in $L^\infty(\mathbf{H}^1)$-norms. Then, we discuss two fully discrete schemes, based

on backward Euler and second order backward difference methods and obtain optimal error estimates. The theoretical results are verified by a few numerical examples. The dissertation concludes with a summary and a note on future plans.

Contents

Chapter 1

Introduction

1.1 Incompressible Viscous Fluids

The general equation describing the motion of an incompressible fluid in a bounded domain Ω in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is given by the following system of partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \sigma + \nabla p = \mathbf{F}(x,t), \;\; x \in \Omega, \;\; t > 0, \tag{1.1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \;\; x \in \Omega, \, t > 0, \tag{1.1.2}$$

with appropriate initial and boundary conditions. Here, $\sigma = (\sigma_{ik})_{1 \le i,k \le 2}$ (or $\sigma = (\sigma_{ik})_{1 \le i,k \le 3}$) denotes the stress tensor with $tr\ \sigma = 0$, $\mathbf{u} = (u_1, u_2)$ (or $\mathbf{u} = (u_1, u_2, u_3)$) represents the velocity vector, p is the pressure of the fluid and $\mathbf{F}$ is the external force. The defining relation between the stress tensor σ and the tensor of deformation velocities

$$\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{i,x_k} + \mathbf{u}_{k,x_i})$$

is called the equation of state or sometimes the rheological equation and it establishes the type of fluids under consideration. Depending on the relation between stress tensor σ and tensor of deformation velocities $\mathbf{D}$, broadly speaking, there are two types of fluids:

Newtonian fluids and non-Newtonian fluids. In Newtonian fluids, stress depends on

(i) the present state of deformation,

(ii) the local kinematic state of the immediate neighbourhood,

(iii) the rate of deformation linearly.

For example, when

$$\sigma = 2\nu \mathbf{D}, \tag{1.1.3}$$

the system (1.1.1)-(1.1.2) with the defining relation (1.1.3) represents the Navier-Stokes system of equations, given by:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{F}, \ \ \nabla \cdot \mathbf{u} = 0. \tag{1.1.4}$$

where ν is the kinematic coefficient of viscosity. This has been a basic model for describing flow at moderate velocities of the majority of the incompressible viscous fluids that are encountered in practice.

Clearly, non-Newtonian fluids do not satisfy at least one of the conditions (i)-(iii). The class of viscoelastic fluids constitutes an important group among the non-Newtonian fluids. The key property of viscoelastic fluids is the occurrence of intermolecular interactions between the fluid particles. In other words, these fluids exhibit the presence of long molecular chains. During the flow, surrounding fluid particles deform the molecular chains by stretching it out with the flow directions and then the tendency of the fluid particles to retract from its stretched position results in stress tensor. As name clearly suggests, these fluids are governed by viscous and elastic forces.

Polymeric fluids are important examples of this category as they exhibit viscoelastic behavior in a distinctive manner. A few more examples are molten plastics, engine oils, paints, ointments, gels, and many biological fluids like egg white and blood. Moreover, chemical,

pharmaceutical, food and oil industries involve processes with viscoelastic fluids.
Apart from the several applications of this model in the study of organic polymers and food industry, it also appears in the mechanisms of diffuse axonal injury that are unexplained by traumatic brain injury models. For more detailed description, we refer to [13], [14], [19].

1.2 Kelvin-Voigt Fluids

In the mid-twentieth century, models of viscoelastic fluids which take into account the prehistory of the flow and are not subject to the Newtonian flow have been proposed. One such model is called Kelvin-Voigt model ([52], [79], [96], [97]) and its rheological relation or equation of state has the form:

$$\sigma = 2\nu(1+\kappa\nu^{-1}\frac{\partial}{\partial t})\mathbf{D}, \;\; \kappa,\nu > 0, \tag{1.2.1}$$

where ν is the kinematic coefficient of viscosity and κ is the retardation time. This fluid is characterized by the fact that after instantaneous removal of the stress, the velocity of the fluid does not vanish instantaneously but dies out like $\exp(\kappa^{-1}t)$ [79]. The relation (1.2.1) differs from the Newtonian model in the sense that it has an additional term $\kappa\frac{\partial}{\partial t}\mathbf{D}$, that takes into account the relaxation property of the fluid.
Further, theory has been introduced for linear viscoelastic fluids representing finite number of discretely distributed relaxation and retardation times with the defining relation:

$$\left(1+\sum_{l=1}^{L}\frac{\partial^l}{\partial t^l}\right)\sigma = 2\nu\left(1+\sum_{m=1}^{M}\kappa_m\nu^{-1}\frac{\partial^m}{\partial t^m}\right)\mathbf{D}, \tag{1.2.2}$$

where $M = L+1$, $L = 0,1,\cdots$. In this thesis, we deal only with the case $L = 0$.
Using the rheological relation (1.2.1), the equations of motion arising from the Kelvin-Voigt's model give rise to the following system of partial differential equations :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t), \;\; x\in\Omega, \;\; t>0, \tag{1.2.3}$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \;\; x \in \Omega, \, t > 0, \tag{1.2.4}$$

with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \;\; \text{in } \Omega, \qquad \mathbf{u} = 0, \quad \text{on } \partial\Omega, \; t \geq 0. \tag{1.2.5}$$

Here, $\partial\Omega$ denotes the boundary of Ω. For analysis purposes, we assume that the right hand side function $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term.

The aim of this thesis is to analyse the Kelvin-Voigt viscoelastic fluid flow model. We study semidiscrete, fully discrete, two-grid semidiscrete and fully discrete error analysis for the equations of motion described by Kelvin-Voigt fluids.

1.3 Preliminaries

In this section, we provide some notations and preliminaries to be used in the subsequent chapters. Throughout the thesis, only real valued functions are considered. Let Ω be a bounded domain in $\mathbb{R}^d$, $(d = 2,3)$ with boundary $\partial\Omega$ and let $\mathbf{x} = (x_1, x_2) \in \Omega$ (or $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$), with $d\mathbf{x} = dx_1 dx_2$ (or $d\mathbf{x} = dx_1 dx_2 dx_3$). Let $\alpha = (\alpha_1, \alpha_2)$ (or $\alpha = (\alpha_1, \alpha_2, \alpha_3)$) be a multi index with non-negative integers α_i, $i = 1,2$ (or α_i, $i = 1,2,3$) such that length of α, that is, $|\alpha| = \alpha_1 + \alpha_2$ (or $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$). Further, let us denote $D^\alpha \phi$ to be the α^{th} order partial derivative of ϕ, defined as

$$D^\alpha \phi = \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \quad \text{or} \quad D^\alpha \phi = \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

Next we introduce Lebesgue spaces $L^p(\Omega), 1 \leq p \leq \infty$, which will be used very often in the analysis. Define $L^p(\Omega), 1 \leq p < \infty$, the space of real functions on Ω with the pth power absolutely integrable for the Lebesgue measure $d\mathbf{x} = dx_1 dx_2$ (or $d\mathbf{x} = dx_1 dx_2 dx_3$). That is,

$$L^p(\Omega) := \{\phi : \int_\Omega |\phi(\mathbf{x})|^p d\mathbf{x} < \infty\}.$$

The norm on these spaces is defined as

$$\|\phi\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}, \ 1 \leq p < \infty$$

and for $p = \infty$, $L^\infty(\Omega)$ denotes the space of essentially bounded real functions ϕ on Ω such that

$$\| \phi \|_{L_\infty(\Omega)} = ess \sup_{\mathbf{x}\in\Omega} | \phi(\mathbf{x}) | < \infty. \tag{1.3.1}$$

These spaces are the sets of equivalence classes of functions where any two functions in an equivalence class differ only on a set of measure zero. Note that for $p = 2$, $L^2(\Omega)$ is a Hilbert space.

Let $C^m(\Omega)$ represent the spaces of functions with continuous derivatives up to and including order m in Ω, where m is any non negative integer. $C_0^m(\Omega)$ is the space of all $C^m(\Omega)$ functions having compact support in Ω and $C_0^\infty(\Omega)$ contains all infinitely differentiable functions with compact support in Ω.

We are now in a position to introduce the concept of Sobolev spaces which form an important tool in defining the variational formulation for the problem. For any non-negative integer m and for any p with $1 \leq p \leq \infty$, Sobolev space $W^{m,p}(\Omega)$ is a linear space consists of equivalence class of functions in $L^p(\Omega)$ whose distributional derivatives up to and including order m are also in $L^p(\Omega)$, that is,

$$W^{m,p}(\Omega) = \{\phi : D^\alpha\phi \in L^p(\Omega), 0 \leq |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the Sobolev norm

$$\|\phi\|_{m,p} = \left(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha \phi(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}$$
$$= \left(\sum_{|\alpha|\le m} \|D^\alpha \phi\|^p_{L^p(\Omega)} \right)^{1/p}, \ 1 \le p < \infty.$$

For $p = \infty$, the Sobolev norm is defined as

$$\|\phi\|_{m,\infty} = \max_{|\alpha|\le m} \|D^\alpha \phi\|_{L^\infty(\Omega)}.$$

In case $p = 2$, these spaces are Hilbert spaces, denoted by $H^m(\Omega)$ endowed with norm $\|\phi\|_m = \|\phi\|_{m,2}$ and inner product defined by

$$(\phi, \psi)_{m,\Omega} = \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha \phi(x)\, D^\alpha \psi(x)\, dx, \quad \phi, \psi \in H^m(\Omega). \tag{1.3.2}$$

We also define the seminorm on $W^{m,p}(\Omega)$ space which consists of the L^p-norms of the highest order derivatives as

$$|\phi|_{m,p} = \left(\int_\Omega \sum_{|\alpha|=m} |D^\alpha \phi(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}.$$

The closure of $C_0^m(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$. For $p = 2$, it is denoted as $H_0^m(\Omega)$.

We denote $H^{-m}(\Omega)$ to be the dual space of $H_0^m(\Omega)$ and define it as

$$\| \phi \|_{-m} = \sup \left\{ \frac{(\phi, \psi)}{\| \psi \|_m} : \psi \in H_0^m(\Omega), \ \| \psi \|_m \ne 0 \right\}.$$

For a more concrete discussion on Sobolev spaces, we refer to Adams [3] and Kesavan [53]. For notational convenience, we denote $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces using bold face letters, that is,

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \ \text{ and } \ \mathbf{H}^m = (H^m(\Omega))^d,$$

Note that, in $\mathbf{H}_0^1$, semi-norm $|.|_1$ is equivalent to the norm $\|.\|_1$ and hence the space $\mathbf{H}_0^1$ can be equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^{d} (\partial_j v_i, \partial_j v_i) \right)^{1/2} = \left(\sum_{i=1}^{d} (\nabla v_i, \nabla v_i) \right)^{1/2}.$$

We introduce a few more function spaces, which will be used very frequently in our analysis in subsequent chapters. Given X, a Banach space endowed with norm $\| \cdot \|_X$, let $L^p(0,T;X)$ be the space consists of all strongly measurable functions $\phi : [0,T] \mapsto X$ satisfying $\int_0^T \| \phi(s) \|_X^p \, ds < \infty$ and equipped with norm

$$\| \phi \|_{L^p(0,T;X)} = \left(\int_0^T \| \phi(s) \|_X^p \, ds \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty$$

and for $p = \infty$,

$$\| \phi \|_{L^\infty(0,T;X)} = ess \sup_{t \in [0,T]} \| \phi(t) \|_X.$$

For notational simplicity, we replace $L^p(0,T;X)$ by $L^p(X)$. For a smooth function ϕ, which is a function of $\mathbf{x}$ and t, we will adopt the following notation:

$$\phi_t = \frac{\partial \phi}{\partial t} \quad \text{and} \quad \phi_{tt} = \frac{\partial^2 \phi}{\partial t^2}. \tag{1.3.3}$$

For more detailed description of these spaces, we refer to [23]. Unless otherwise mentioned, C denotes a generic positive constant which has different values at different places of occurrence in the thesis and is independent of the mesh size and may possibly depend on the data.

The system (1.2.3)-(1.2.5) includes the incompressibility condition, that is, velocity vector satisfies the divergence free condition. Hence, we would also need the following function space:

$$\mathcal{V} = \{\phi \in \mathbf{C}_{\mathbf{0}}^\infty(\Omega) : \nabla \cdot \boldsymbol{\phi} = 0\}. \tag{1.3.4}$$

The closure of $\mathcal{V}$ in $\mathbf{L^2}$ and $\mathbf{H}_0^1$ are well known in the study of incompressible spaces. We denote these spaces as $\mathbf{J}$ and $\mathbf{J}_1$, respectively, as

$$\begin{aligned} \mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} &= 0 \text{ in } \Omega, \ \ \boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly}\} \\ \mathbf{J}_1 &= \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\}. \end{aligned}$$

Here, we note that $\mathbf{L^2}$ can be decomposed to $\mathbf{J}$ and $\mathbf{J}^{\perp}$.
As we will be dealing with divergence free spaces, we would like to state a trace theorem on divergence free spaces. In order to achieve this, firstly recall that when Ω is an open bounded set of class C^2, the standard trace theorem on Sobolev spaces confirms the existence of a linear continuous trace operator $\gamma_0 : \mathbf{H}^1 \to L^2(\partial\Omega)$. In this case, for every function $\mathbf{v} \in \mathbf{H}^1$, $\gamma_0\mathbf{v}$ is the restriction of $\mathbf{v}$ to the boundary $\partial\Omega$. The space $\mathbf{H}_0^1$, by definition, is equal to the kernel of γ_0. The range of this trace operator is a dense subspace of $\mathbf{L}^2(\partial\Omega)$, denoted by $\mathbf{H}^{1/2}(\partial\Omega)$. The dual space of $\mathbf{H}^{1/2}(\partial\Omega)$ is denoted by $\mathbf{H}^{-1/2}(\partial\Omega)$, which will be required for the trace theorem. Further, we introduce $\mathbf{H}(div;\Omega)$ as follows:

$$\mathbf{H}(div;\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ div \ \mathbf{v} \in \mathbf{L^2(\Omega)}\}. \tag{1.3.5}$$

The space in (1.3.5) is a Hilbert space with inner product

$$(\mathbf{v}, \mathbf{w})_{\mathbf{H}(div;\Omega)} = (\mathbf{v}, \mathbf{w}) + (div \ \mathbf{v}, div \ \mathbf{w}).$$

Now, we are in a position to state the following trace theorem:

Theorem 1.1. *Let Ω be an open bounded set of class C^2. Then, there exists a continuous linear operator $\gamma_n \in \mathcal{L}\big(\mathbf{H}(div;\Omega), \mathbf{H}^{-1/2}(\partial\Omega)\big)$, which is, in fact, an onto mapping and*

$$\gamma_n\mathbf{v} = \mathbf{v} \cdot \mathbf{n}/_{\partial\Omega}, \tag{1.3.6}$$

where $\mathbf{n}$ is the unit normal vector to the boundary $\partial\Omega$. The following generalized Stokes *formula is true for all* $\mathbf{v} \in \mathbf{H}(div; \Omega)$ *and* $\mathbf{w} \in \mathbf{H}^1$

$$(\mathbf{v}, \nabla \mathbf{w}) + (\nabla \cdot \mathbf{v}, \mathbf{w}) = \langle \gamma_n \mathbf{v}, \gamma_0 \mathbf{w} \rangle,$$

where $\langle \cdot, \cdot \rangle$ *represents the duality pairing between* $\mathbf{H}^{1/2}(\partial\Omega)$ *and* $\mathbf{H}^{-1/2}(\partial\Omega)$.

We would like to refer to Temam ([95], pp 9) for a proof and more complete and rigorous discussion on divergence free spaces.

In order to deal with the nonlinear term present in our problem, we need Sobolev inequalities. Hence, in the following lemma, we mention a few Sobolev inequalities which will be used very frequently in future analysis.

Lemma 1.1. *For any open set* Ω *in* $\mathbb{R}^d$, $d = 2,\ 3$ *and* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$

$$\|\mathbf{v}\|_{L^4} \leq \begin{cases} 2^{\frac{1}{2}} \|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}} & \textit{if } d = 2; \\ 2^{\frac{1}{2}} \|\mathbf{v}\|^{\frac{1}{4}} \|\nabla \mathbf{v}\|^{\frac{3}{4}} & \textit{if } d = 3. \end{cases}$$

For a proof, we refer [95] (page no. 291 and 296).

From time to time, we also make use of the following result:

Lemma 1.2. *Let* $\mathbf{v} \in \mathbf{J}_1$ *and* $\mathbf{w} \in \mathbf{H}_0^1$. *Then,*

$$(\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{w}) = 0.$$

Proof. It is sufficient to prove for $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in (\mathbf{C_0^\infty}(\Omega))^2$. Then, the result follows using the density argument.

$$\int_\Omega \mathbf{v}_j D_j \mathbf{w}_i \mathbf{w}_i d\mathbf{x} = \int_\Omega \mathbf{v}_j D_j \left(\frac{\mathbf{w}_i^2}{2} \right) d\mathbf{x} = -\frac{1}{2} \int_\Omega D_j \mathbf{v}_j (\mathbf{w}_i^2) d\mathbf{x}. \tag{1.3.7}$$

Hence, from (1.3.7), we observe that

$$(\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{w}) = -\frac{1}{2} \sum_{j=1}^{2} \int_\Omega div\, \mathbf{v} (\mathbf{w}_i^2) d\mathbf{x} = 0 \tag{1.3.8}$$

and this completes the proof. $\square$

We also make use of the following inequalities. For more details, one may refer to [23], [39].

(i) **Young's Inequality:** For all $a, b \geq 0,\ 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{\epsilon a^p}{p} + \frac{b^q}{q\epsilon^{q/p}}, \ \ \epsilon > 0. \tag{1.3.9}$$

(ii) **Hölder's Inequality:** For $\phi \in L^p(\Omega),\ \psi \in L^q(\Omega)$ with $1 < p,\ q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\Omega \phi\, \psi dx \leq \left(\int_\Omega |\phi(x)|^p\, dx\right)^{1/p} \left(\int_\Omega |\psi(x)|^q\, dx\right)^{1/q} = \|\phi\|_p \|\psi\|_q. \tag{1.3.10}$$

For $p = q = 2$, the above inequality is known as Cauchy-Schwarz's inequality.

(iii) **Discrete Cauchy-Schwarz's Inequality:** Let $\phi_i, \psi_i,\ i = 1, 2, \ldots, n$, be positive real numbers. Then

$$\sum_{i=1}^{n} \phi_i\, \psi_i \leq \left(\sum_{i=1}^{n} {\phi_i}^2\right)^{1/2} \left(\sum_{i=1}^{n} {\psi_i}^2\right)^{1/2}. \tag{1.3.11}$$

(iv) **General Hölder's inequality:** Let $1 \leq p_1,\ p_2,\ p_3 \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, and assume $\phi_i \in L^{p_i}$ for $1 \leq i \leq 3$. Then,

$$\int_\Omega |\phi_1\ \phi_2\ \phi_3| dx \leq \prod_{i=1}^{3} \|\phi_i\|_{L^{p_i}(\Omega)}. \tag{1.3.12}$$

Below, we present without proof, the two versions of Gronwall's lemma. For a proof, we refer to [16], [37] and [92].

Lemma 1.3. (Gronwall's Lemma) *Let $\mu(t)$ be a continuous function and $\nu(t)$ a non-negative function on the interval $t_0 \leq t \leq t_0 + a$. If a continuous function $\gamma(t)$ has the property*

$$\gamma(t) \leq \mu(t) + \int_{t_0}^{t} \gamma(s)\, \nu(s)\, ds, \ \ \textit{for}\ \ t \in [t_0, t_0 + a], \tag{1.3.13}$$

then

$$\gamma(t) \leq \mu(t) + \int_{t_0}^{t} \mu(s)\,\nu(s)\ \exp\left[\int_{s}^{t} \nu(\tau)\,d\tau\right] ds, \quad for\ \ t \in [t_0, t_0 + a].$$

In particular, when $\mu(t) = C$ *a non-negative constant, then we have*

$$\gamma(t) \leq C\ \exp\left[\int_{t_0}^{t} \nu(s)\,ds\right], \ \ for\ \ t \in [t_0, t_0 + a].$$

Next, we present the discrete version of Gronwall's lemma which will be used very frequently in the subsequent chapters. For a proof, see Pani *et al.* [87].

Lemma 1.4. (Discrete Gronwall's Lemma) *Let* $\{\phi_n\}$ *be a sequence of nonnegative numbers satisfying*

$$\phi_n \leq \psi_n + \sum_{j=0}^{n-1} \eta_j\,\phi_j \qquad\qquad for\ \ n \geq 1,$$

where $\{\psi_n\}$ *is a non decreasing sequence and* η_j*'s are non negative. Then*

$$\phi_n \leq \psi_n\ \ exp\left(\sum_{j=0}^{n-1} \eta_j\right) \qquad\qquad for\ \ n \geq 1.$$

We present below a lemma which will play an important role in Chapter **2** for proving the continuity of velocity vector.

Lemma 1.5. *Let* V*,* H *and* V^* *be three Hilbert spaces with* $V \subset H = H^* \subset V^*$*,* V^* *is dual of* V *and each inclusion is dense and continuous. If a function* $\phi \in L^2(0,T;V)$ *and its derivative* $\phi_t \in L^2(0,T;V^*)$*, then* ϕ *is almost everywhere equal to a function continuous from* $[0,T]$ *into* H *and we have the following equality, which holds in the scalar distribution sense on* $(0,T)$*:*

$$\frac{d}{dt}\|\phi\|^2 = 2\langle\phi_t, \phi\rangle, \tag{1.3.14}$$

where $\langle \cdot, \cdot \rangle$ *is the duality pairing between* V^* *and* V*.*

For a proof of Lemma 1.5, we refer to [95] (Chapter III, Lemma 1.2, pp 260–261).
In proving existence of weak solution (by using Faedo Galerkin method), we need the following compactness theorem.

Theorem 1.2. *Let* X_0*,* X *and* X_1 *be reflexive Banach spaces with*

$$X_0 \subset X \subset X_1,$$

the injection being continuous. Moreover, let the injection of X_0 *into* X *be compact. Let* ϕ_m *be a bounded sequence in* $L^{c_1}(0, T; X_0)$*. Assume that* $\{\partial\phi_m/\partial t\}$ *is bounded in* $L^{c_2}(0, T; X_1)$*, where* $1 < c_1, c_2 < \infty$*. Then, there exists a subsequence* $\phi_{m'}$ *of* ϕ_m *converging in* $L^{c_1}(0, T; X)$*.*

For a proof of Theorem 1.2, see [18] (pp 69).
Further, we need the following two lemmas which will help us in recovering p from weak formulation obtained from the divergence free space. ([95], Chapter I, Proposition 1.1 and 1.2).

Lemma 1.6. *Let* Ω *be an open set of* $\mathbb{R}^d$ $(d = 2$ *(or* $3))$ *and let* $\mathbf{v} = \{v_1, v_2\}$ *(or* $\mathbf{v} = \{v_1, v_2, v_3\}$*), where* v_i*,* $i = 1,\ 2$ *(or* $1 \leq i \leq 3$*) is a distribution. Then, a necessary and sufficient condition that*

$$\mathbf{v} = \nabla p,$$

for some distribution p*, is that*

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \ \forall \mathbf{w} \in \mathcal{V},$$

where $\mathcal{V}$ is defined in (1.3.4) and $\langle \cdot, \cdot \rangle$ is a duality pairing between $\mathbf{V}^*$, the dual of $\mathbf{V}$ and $\mathbf{V}$.

Lemma 1.7. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^2$ and $\mathbf{v}$ be a distribution, having all its first order derivatives in $\mathbf{L^2(\Omega)}$. Then, $\mathbf{v} \in \mathbf{L^2(\Omega)}$ and*

$$\|\mathbf{v}\|_{\mathbf{L^2}/\mathbb{R}} \leq C(\Omega)\|\nabla\mathbf{v}\|.$$

In case, all its first-order derivatives are in $\mathbf{H}^{-1}(\Omega)$, then $\mathbf{v} \in \mathbf{L^2(\Omega)}$ and

$$\|\mathbf{v}\|_{\mathbf{L^2}/\mathbb{R}} \leq C(\Omega)\|\nabla\mathbf{v}\|_{\mathbf{H}^{-1}(\Omega)}.$$

We also introduce the following theorem, which will help us in proving the existence and uniqueness of the solutions for fully discrete schemes. For its proof, see [53] (page no. 219).

Theorem 1.3. *(Brouwer's point theorem): Let $\mathbf{H}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and $\|\cdot\|$. Let $g : \mathbf{H} \to \mathbf{H}$ be a continuous function. If there exists $R > 0$ such that $(g(z), z) > 0$ $\forall z$ with $\|z\| = R$, then there exists $z^* \in \mathbf{H}$ such that $\|z\| \leq R$ and $g(z^*) = 0$.*

1.4 Literature Review

Before introducing the finite element Galerkin analysis related to the equations of motion arising in the Kelvin-Voigt model of viscoelastic fluids, we present a brief literature survey relevant to this problem. As we have already mentioned that our problem is obtained by a perturbation of Navier Stokes equations, that is, it contains an additional term $\kappa\frac{\partial}{\partial t}\mathbf{D}$, we would like to briefly describe the related literature for Navier Stokes equations before describing the literature available for our problem.

1.4.1 Existence and Uniqueness Results

The pioneering results on existence and uniqueness of solutions to the Navier-Stokes equations by Leray [69]-[70], have resulted in an extensive literature on this subject. In [70], Leray has proved the existence of a global (in time) weak solution and a local strong solution for the Cauchy problem corresponding to Navier-Stokes equations. In [47], using Galerkin method, Hopf has proved the existence of a global weak solution, called as Leray-Hopf solution. In an attempt to prove uniqueness, Ladyzhenskaya and Kiselev [54] have proved the unique solvability of initial-boundary value problem for the Navier-Stokes system in an arbitrary three dimensional domain under the smallness condition on data. In this paper, they have made a remark on Hopf's work [47] that the class of Leray-Hopf solutions is too wide to prove uniqueness. In [57], Ladyzhenskaya has proved a remarkable result on existence of a strong solution to the initial-boundary value problem for the two dimensional Navier-Stokes equations. During the same time, Lions and Prodi [71] have shown the existence of a unique weak solution to two dimensional Navier-Stokes equations which is stronger than the Larey's weak solution and weaker than the Ladyzhenskaya's solution. For more references, see Ladyzhenskaya *et al* [34], [56], [58], [59], [60], [61], [62], [63] and Temam [95]. However, the existence of a unique global weak solution for three dimensional Navier-Stokes equations remains a millennium open problem ([24], [64]).

The theory on linear viscoelastic fluids includes mainly the study of three basic linear viscoelastic fluid models, namely; Oldroyd model, Kelvin-Voigt model and Maxwell model, proposed by Oldroyd [74]-[75], Kelvin [52], Voigt [96]-[97] and Maxwell [72], respectively. For literature related to the analysis on existence and uniqueness of solutions of Maxwell fluid, we refer to [77]-[78], for Oldroyd fluid refer to [77]-[80], [83], [86] and for Kelvin-Voigt, see [48], [49], [50], [51], [55], [76], [77], [78], [79], [80], [82], [83], [84], [85], [86], [106].

We discuss below the literature related to solvability of the Kelvin-Voigt model. Based on the analysis of Ladyzhenskaya for the solvability of Navier-Stokes equations, Oskolkov [79] has proved the global existence of unique 'almost' classical solution in finite time interval for the initial and boundary value problem for Kelvin-Voigt fluid (1.2.3)-(1.2.5). The proof is based on a *priori* estimates, Faedo-Galerkin method and weak with weak* compactness arguments. These results are for finite time as the constant in a *priori* estimates has an exponential growth in time. In [86], the investigation on classical global solvability on $[0, \infty)$

has been established for initial-boundary value problems corresponding to Kelvin-Voigt model. Subsequently, the authors of [49] have discussed the existence of global attractor. In [82], Oskolkov has established existence of unique almost classical solutions on the entire semiaxis $\mathbb{R}^+$, that is, on an infinite time interval along with existence of classical time T-periodic solutions. For a complete summary of Oskolkov's work, we refer to his dissertation [77]. Since all the above results for the velocity are proved on a divergence free space, for numerical approximation, it is difficult to construct finite dimensional spaces which include divergence free condition. Therefore in [83], Oskolkov has discussed penalization of divergence free condition and derived a slightly compressible Kelvin-Voigt model. Further, it has been shown that as the penalizing parameter tends to zero, the solution of perturbed Kelvin-Voigt equation converges to the solution of Kelvin-Voigt model.

Recently, Kaya and Çelebi [51] have discussed the existence of a unique weak solution to g-Kelvin-Voigt equations for two-dimensional domains using Faedo Galerkin method. In this case, g is a smooth and real valued function defined on a bounded domain in $\mathbb{R}^2$. It is known that for uniform thickness g-Kelvin-Voigt model turns out to be to Kelvin-Voigt model in two dimensions. In [48], Kalantarov and Titi have studied the long-term dynamics of the three-dimensional Kelvin-Voigt model of viscoelastic incompressible fluids and have proved an estimate of the fractal and Hausdorff dimensions of the global attractor of the dynamical system generated by (1.2.3)-(1.2.5). Further, they have established that the weak solutions of the Kelvin-Voigt equations converge, in the appropriate norm, to the weak solutions of the inviscid simplified Bardina model, as the viscosity coefficient $\nu \to 0$. For related theoretical results in Oldroyd model, we refer Oskolkov [77], [79], [83] and [86]. In this dissertation, we have proved the existence and uniqueness result for the solution pair $(\mathbf{u}, p)$ of weak formulation of (1.2.3)-(1.2.5) in Chapter **2**. This has been achieved following the proof techniques of Ladyzhenskaya for the Navier-Stokes equations, that is, firstly, we establish a *priori* bounds for the solution by using energy arguments and then with the help of these bounds, Faedo-Galerkin method and standard compactness arguments, we have proved existence of a unique pair of weak solution to the Kelvin-Voigt model.

1.4.2 Semidiscrete Scheme

One of the objective of our work is to apply finite element Galerkin scheme in spatial direction keeping time direction continuous to the equations of motion of the Kelvin-Voigt viscoelastic fluid. In literature, this formulation is called semidiscrete formulation. Before presenting the literature related to semidiscrete analysis for Navier Stokes equations and Kelvin-Voigt model, we provide a brief introduction for the semidiscrete schemes.

The Finite Element Method (FEM) introduced by engineers is a numerical technique for solving problems which are described by static or evolutionary partial differential equations with appropriate initial and boundary conditions or to solve problems that can be formulated as a minimization of energy functional. First, discretize the domain into a finite number of triangles, rectangles or quadrilaterals known as elements by introducing a suitable discretization parameter. Then construct a finite element space consisting of piecewise polynomials on each element. Next, with the help of appropriate basis functions for the finite dimensional space, obtain a system of algebraic equations which may be linear or nonlinear. Then solve the system to obtain an approximate solution to the original problem. This happens to be a procedure for elliptic problems. For evolution equations, as our case is, we discretize the problem only in the spatial direction keeping the time variable continuous. Hence, this discretization scheme is known as semidiscrete scheme. Later on, a fully discrete scheme is also discussed, where a discretization in the direction of time variable is also performed. Depending upon the finite element space, a *priori* error estimates for the semidiscrete scheme is then developed.

For the finite element analysis of Navier Stokes equations, there is a great deal of literature available, for example, see Bernardi *et al* [5, 8, 9, 10], Gunzburger [38], Heywood *et al* [43]-[45], Rannacher [90]-[91], Girault *et al.* [28], Wang *et al* [98], Glowinski *et al.* [29], [30], [31], [32], [33], Taylor *et al.* [94] and Temam [95] and the references therein.

In [43], Heywood *et al.* have applied conforming and nonconforming finite element schemes for the Navier-Stokes equations and proved optimal error bounds for the velocity and the pressure under realistic assumptions on the initial data. The proof technique is based on discrete modified Stokes operator as intermediate solution, energy type estimates and parabolic duality arguments. To take care of the behaviour at $t = 0$ and ∞, they have introduced te^{zt} type weight function and have achieved both a *priori* bounds and error estimates without nonlocal compatibility conditions on the initial data. Since, the constants in error estimates contains e^{CT}, these estimates are termed as local. Subsequently, in a series of papers, Heywood and Rannacher [44]-[45], [90] have introduced stability concept and using the assumption that the original solution is stable, they have proved energy estimates which are valid uniformly for all time.

An extensive amount of work has been done for semidiscrete analysis of Oldroyd viscoelastic fluid model. For the related literature, we refer to Goswami *et al.* [36], Cannon *et al.* [15], He *et al* [41]-[42], Pani *et al* [88]-[89] and Wang *et al* [101]-[102]. He *et al.* [41] have obtained optimal error estimates for velocity in $L^\infty(\mathbf{H}^1)$-norm and for pressure in $L^\infty(L^2)$-norm and suboptimal for velocity in $L^\infty(\mathbf{L^2})$-norm. Later, Pani *et al.* [88] have obtained an improvement over results in [41], that is, they have obtained optimal error estimates for velocity in both $L^\infty(\mathbf{H}^1)$ and $L^\infty(\mathbf{L^2})$-norms and for pressure $L^\infty(L^2)$-norm which are valid uniformly in time $t > 0$ for $f = 0$ with exponential decaying property. Their error analysis is based on new regularity results, Stokes-Volterra projection and duality argument. In [36], Deepjyoti *et al.* have established optimal error bounds for velocity and pressure with $f \in L^\infty(\mathbf{L^2})$ and have obtained an improvement of results observed in [41], where the uniform error estimate for velocity is not optimal in $L^\infty(\mathbf{L^2})$.

For the earlier results on the numerical approximations to the solutions of the problem (1.2.3)-(1.2.5), we refer to Oskolkov [81]. Under the condition that solution is asymptotically stable as $t \to \infty$, the authors of [81] have established the convergence of spectral Galerkin approximations for the semi axis $t \geq 0$. Recently, Pani *et al* [21] have applied modified nonlinear spectral Galerkin method and discussed convergence results. Here, for the spectral Galerkin scheme, existence of a unique discrete solution to the semidiscrete scheme is proved and existence of a discrete global attractor is established. Further, optimal error estimates in $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}_0^1)$-norms are proved. Finally, a modified nonlinear

Galerkin method is developed and optimal error bounds are derived.

There is hardly any literature devoted to the analysis of the Galerkin finite element methods for the problem (1.2.3)-(1.2.5), and hence, the present semidiscrete analysis is a step towards achieving this objective. In Chapter **3** of this dissertation, the main contribution for semidiscrete analysis includes proof of regularity results for the solution which are valid uniformly in time, even for 3-D domain and establishment of the exponential decay property for the exact solution. Also optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}_0^1)$-norm and to the pressure in $L^\infty(L^2)$-norm reflecting the exponential decay property in time have been obtained.

1.4.3 Fully Discrete Schemes

In the next step of the dissertation, we consider complete discretization of the problem, that is, discretization in both space and time directions. In fully discrete scheme, we employ a difference scheme for time discretization and a finite element method in spatial direction. For time discretization, we consider a uniform subdivision of time interval such that $0 = t_0 < t_1 < \cdots < t_N = T$, $[0, T]$ with $\Delta t = t_n - t_{n-1}$ and replace the time derivative by a difference quotient. At each time step $t_k = \Delta k, 0 \leq k \leq N$, we solve a semidiscrete problem. The error in time discretization has the same order as the order of error in difference scheme.

Before proceeding towards a brief introduction of literature relevant to our problem, yet again we would like to present literature dealing with fully discrete schemes for Navier-Stokes model. The literature for fully discrete schemes applied to the Navier-Stokes equation is quite vast, Heywood *et al.* [46], Rannacher [90], Baker *et al.* [6], Glowinski *et al* [29]-[32], Gunzburger [38], Girault and Raviart [28], Temam [95] and He [40], to mention a few.

Rannacher [90] has established first-order error estimates for explicit forward and implicit backward Euler schemes for the Navier-Stokes equations. Later in [46], Heywood *et al.* have applied second order Crank-Nicolson scheme and has obtained optimal error estimates

which are local in time under some realistic assumptions on the regularity of the solution. We refer to Baker *et al.* [6] for higher order error estimates for time discretization under smooth assumption on the exact solution . In [40], the author has studied the stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the two-dimensional non stationary Navier-Stokes equations with a non smooth initial data. The time discretization has been done by applying an implicit Crank-Nicolson scheme for the linear terms and an explicit Adams-Bashforth scheme for the nonlinear term. Further, it has been shown that the scheme is unconditionally stable for a non smooth initial data satisfying the divergence free condition and obtained some error estimates for the discrete velocity and pressure.
For the related literature on the time discretization of equations of motion arising in the viscoelastic model of Oldroyd type, see Akhmatov *et al.* [4], Pani *et al.* [89], Goswami [35], Wang *et al* [99], [100], [101].
There is hardly any literature, we have come across for the full discretization of the Kelvin-Voigt model. Hence, in Chapter **4**, an attempt is made to study two fully discrete schemes for the Kelvin-Voigt viscoelastic fluid model. The two schemes are based on backward Euler method and second order backward difference scheme, respectively, in time direction and a finite element method in space direction. Optimal error estimates exhibiting the exponential decay property in time are derived using a *priori* bounds for discrete solutions of two schemes. Error analysis of linearized backward Euler method applied to (1.2.3)-(1.2.5) (with $\mathbf{f} = \mathbf{0}$) for time discretization is also discussed. We verify our theoretical results with the help of computational results. This forms a part of Chapter **4** of our dissertation.

1.4.4 Two-Grid Schemes

It is observed that, numerically handling a nonlinear system is much more difficult than handling a linear system. The most common choice for solving a nonlinear system would be its linearization by Newton's method. Two grid schemes are based on this linearization concept. In the first step, we solve the nonlinear system on a coarse mesh $\mathcal{T}_H$ and obtain an approximate solution, say u_H. In the second step, we linearize the nonlinear system around u_H and compute the solution of linearized system, say u_h, on fine mesh $\mathcal{T}_h$, $h < H$. Then, analytically, we obtain that with an appropriate choice of h and H, the error $\|u - u_h\|$ is of the same order as $\|u - \tilde{u}_h\|$, where $\tilde{u}_h$ is the solution of nonlinear system on fine mesh h. We present below a brief discussion of literature related to this method. The two grid method have been extensively studied for elliptic problems by Xu [104]-[105] and Niemistö [73], for steady state Navier Stokes equations by Layton [65], Layton and Tobiska [68], Layton and Lenferink [66]-[67], Girault and Lions [26], Dai *et al* [20], for semidiscrete transient Navier-Stokes equations by Girault and Lions [27], Abboud *et al.* [1]-[2], Frutos *et al.* [22] and for fully discrete transient Navier-Stokes equations by Abboud *et al.* [1]-[2] and Frutos *et al* [22]. Girault *et al* [26] in their work on steady state Navier Stokes equations have obtained optimal $L^\infty(\mathbf{H}^1)$ and $L^\infty(\mathbf{L^2})$-norms estimates with a choice $h = H^{3/2}$, where h is the spatial mesh size of finer mesh and H corresponds to the discretization parameter in a coarse mesh. They have also applied two-grid method to the transient Navier Stokes equations and have obtained optimal error estimates in $L^\infty(\mathbf{H}^1)$ and suboptimal error estimates for $L^\infty(\mathbf{L^2})$-norms with a choice of $h = H^2$ (see [27]). Recently, Frutos *et al* [22] have worked out the two-grid mixed-finite element schemes for the spatial discretization of the incompressible Navier Stokes equations. They have applied mixed-finite elements of first, second and third order, that is, the mini-element, the quadratic and cubic Hood-Taylor elements for spatial discretization and backward Euler method and two step backward difference scheme for time discretization and recovered the rate of convergence of the fine mesh in the $\mathbf{H}^1$-norm by taking $H = h^{1/2}$, which is an improvement over $H = h^{2/3}$ obtained in [2].

There are hardly any results available for the two-grid discretization for Kelvin-Voigt model and hence, we make an attempt to study this problem in this discretization. The two level algorithm, which is used in the dissertation, is based on three steps. In step 1, we solve nonlinear system on coarse mesh with mesh size H and compute solution u_H. In step 2, we linearize the nonlinear system about u_H based on Newton's iteration and solve the linear system on fine mesh and finally, in step 3, we solve same linear problem with different right hand side on fine mesh. In Chapter **5**, we have established optimal error estimates and have recovered an error of order h^2 in $L^\infty(\mathbf{L}^2)$-norm and h in $L^\infty(\mathbf{H}^1)$-norm provided $h = \mathbb{O}(H^{2-\epsilon})$, where ϵ is arbitrary small for two dimensions and $\epsilon = \frac{1}{2}$ for three dimensions. For time discretization, we apply the first order accurate backward Euler method and second order backward difference scheme to the two level semidiscrete problem and arrive at optimal error estimates for completely discrete schemes. We have also worked out numerical examples to support our theoretical estimates.

1.5 Chapterwise Description

The thesis is organized as follows:

- In Chapter **2**, we prove the wellposedness of weak formulation of problem (1.2.3)-(1.2.5) (with $\mathbf{f} = \mathbf{0}$), that is, we establish existence of a unique weak solution. Further, we derive some a *priori* estimates for the weak solution based on the energy arguments.

- In Chapter **3**, we discretize the space keeping the time derivative continuous and establish some new regularity results, exhibiting the exponential decay property of the semidiscrete solution. Then, we proceed to obtain optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}_0^1)$-norm and to the pressure in $L^\infty(L^2)$-norm which also reflect the exponential decay property in time. We have made use of exponential weights for the derivation of the new regularity results. These weights also become crucial in establishing the optimal error estimates. In order to derive optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm, we first split the error by using a Galerkin approximation to a linearized Kelvin-Voigt model and then introduce a new auxiliary operator through a modification of the Stokes operator. Now making use of estimates derived for the auxiliary operator and the error estimates due to the linearized model, we recover the optimality of $L^\infty(\mathbf{L}^2)$ error estimates for the velocity. Finally, with the help of uniform inf-sup condition and error estimates for the velocity, we derive optimal error estimates for the pressure. Special care has been taken to preserve the exponential decay property even for the error estimates.

- In Chapter **4**, we study the fully discrete approximation of (1.2.3), that is, in this case, we discretize the problem in both space and time directions. Discretization in time has been done by means of a first order backward Euler method and a second order two step backward difference method. Also, we discuss the existence and uniqueness of the solutions of the above schemes and obtain a *priori* bounds for the fully discrete solutions. Then, the error estimates involving $\mathbf{H}^1$ and L^2-norms are derived which are valid uniformly for all time $t \geq 0$. We also present a brief description of linearized backward Euler method applied to (1.2.3). Finally, we provide some numerical results to verify our theoretical results and exhibit the exponential decay property of the fully discrete solution.

- In Chapter **5**, we employ a two level method based on Newton's iteration for resolving the nonlinearity present in our problem. Here, in the first step, we semi-discretize the nonlinear problem on a coarse grid, with mesh size H. In the second and third steps, we linearize the problem in the neighbourhood of the velocity $\mathbf{u}_H$ obtained at step one and then, semidiscretize on a fine grid with mesh size h. First section involves the derivation of a *priori* estimates of linearized semidiscrete solutions. Now, with the help of these a *priori* estimates and already defined Sobolev-Stokes projection (see Chapter **3**), we establish the error estimates for the two-grid semidiscrete approximations to the problem (1.2.3)-(1.2.5). Then, we introduce full discretization of two-grid semidiscretize approximations using backward Euler method and second order backward difference scheme and prove error estimates to conclude the order of convergence. We also work out a few numerical experiments to support the theoretical error estimates.
- In Chapter **6**, we conclude the thesis by providing a summary and a write up on possible future work.

Chapter 2

Existence, Uniqueness and Regularity Results

2.1 Introduction

In this chapter, we provide the existence and uniqueness result for weak solution of the initial and boundary value problem (1.2.3)-(1.2.5) (with $\mathbf{f} = 0$), representing the motion of the Kelvin-Voigt fluid. We employ energy method to derive a *priori* estimates which exhibit exponential decay property of weak solution. These estimates along with Faedo-Galerkin method and standard compactness arguments enable us to prove the convergence of Galerkin solutions.

Following the proof techniques of Ladyzhenskaya [63] for Navier-Stokes equations, Oskolkov [79] has proved the global existence of unique 'almost' classical solution in finite time interval for the initial and boundary value problem for Kelvin-Voigt fluid (1.2.3)-(1.2.5). The author has established the results with the help of a *priori* bounds, Faedo-Galerkin method and weak and weak* compactness arguments. The constants of a *priori* bounds have an exponential growth in time resulting into existence of a unique solution for a finite time interval. Further, efforts have been made by [85] and [86] to prove the existence and uniqueness of a solution on the entire semiaxis $\mathbb{R}^+$. In [83], Oskolkov has penalized

the divergence free condition and has obtained a slightly compressible Kelvin-Voigt model. Further, it has been proved that as the penalizing parameter tends to zero, the solution of perturbed Kelvin-Voigt equation converges to solution of Kelvin-Voigt model.
We now give a section wise description for this chapter. In Section **2.2**, firstly, we state the basic assumptions which will be needed in this chapter and in the subsequent chapters. Then, we discuss the weak formulation of (1.2.3)-(1.2.5) (with $\mathbf{f} = 0$). Section **2.3** establishes the existence and uniqueness result for the weak formulation. Finally, in Section **2.4**, we prove the regularity results for the weak solution defined in Section **2.2**.

2.2 Weak Formulation

For the sake of continuity, we recall the following divergence free spaces to define the weak formulation of (1.2.3)-(1.2.5):

$$\mathbf{J}_1 = \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\}$$

and

$$\mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} = 0 \ \text{ in } \ \Omega, \ \ \boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0 \ \text{ holds weakly}\},$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$. Further, let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.
Throughout this thesis, we make the following assumptions:
(**A1**). For $\mathbf{g} \in \mathbf{L^2}$, let $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ be the unique pair of solutions to the steady state Stokes problem, see [95],

$$-\Delta \mathbf{v} + \nabla q = \mathbf{g},$$
$$\nabla \cdot \mathbf{v} = 0 \ \ \text{in} \ \ \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0$$

satisfying the following regularity result:

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \leq C\|\mathbf{g}\|. \tag{2.2.1}$$

Setting

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator, (**A1**) implies

$$\|\mathbf{v}\|_2 \leq C\|\tilde{\Delta}\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.2.2}$$

The following Poincaré inequality [43] holds true:

$$\|\mathbf{v}\|^2 \leq \lambda_1^{-1}\|\nabla\mathbf{v}\|^2 \;\; \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.2.3}$$

where λ_1^{-1}, is the least eigenvalue of the Stokes operator $\tilde{\Delta}$, which is the best possible positive constant, depending on the domain Ω. We again note that

$$\|\nabla\mathbf{v}\|^2 \leq \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \;\; \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.2.4}$$

Note that, for each $\mathbf{v} \in \mathbf{V}$, where $\mathbf{V}$ is a Hilbert space,

$$\mathbf{w} \mapsto (\nabla\mathbf{v}, \nabla\mathbf{w}) \in \mathbb{R}$$

is linear and continuous in $\mathbf{V}$ and hence, we can introduce $\mathcal{L}\mathbf{v} \in \mathbf{V}^*$ such that

$$\langle \mathcal{L}\mathbf{v}, \mathbf{w}\rangle = (\nabla\mathbf{v}, \nabla\mathbf{w}), \;\; \forall \mathbf{w} \in \mathbf{V}. \tag{2.2.5}$$

Therefore, from (2.2.5), we can conclude that $\tilde{\Delta} : \mathbf{J}_1 \to \mathbf{J}_1^*$ is linear and continuous.
(**A2**). There exists a positive constant M, such that the initial velocity $\mathbf{u}_0$ satisfies

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1 \quad \text{with} \quad \|\mathbf{u}_0\|_2 \leq M.$$

With all the above preparation, let us introduce the weak formulation of (1.2.3)-(1.2.5) with $\mathbf{f} = 0$: find a pair of functions $\{\mathbf{u}(t), p(t)\} \in \mathbf{H}_0^1 \times L^2/\mathbb{R}, \ \ t > 0$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa(\nabla \mathbf{u}_t, \nabla \boldsymbol{\phi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\phi}) = (p, \nabla \cdot \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1, \qquad (2.2.6)$$
$$(\nabla \cdot \mathbf{u}, \chi) = 0 \qquad \forall \chi \in L^2$$

with $\mathbf{u}(0) = \mathbf{u}_0$.
Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa(\nabla \mathbf{u}_t, \nabla \boldsymbol{\phi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1, \ \ t > 0 \qquad (2.2.7)$$

with $\mathbf{u}(0) = \mathbf{u}_0$.
(**A3**). The space pair $\{\mathbf{H}_0^1 \times L^2/\mathbb{R}\}$ satisfies the following inf-sup condition: for every $\chi \in L^2$, there exists a non-trivial function $\boldsymbol{\phi} \in \mathbf{H}_0^1$ and a positive constant K, such that,

$$|(\chi, \nabla \cdot \boldsymbol{\phi})| \geq K \|\nabla \boldsymbol{\phi}\| \|\chi\|_{L^2/\mathbb{R}}.$$

In the remaining part of this chapter, we adopt the following notation: For any given function ϕ, define

$$\hat{\phi} = e^{\alpha t} \phi.$$

Remark 2.2.1. *In Section* **2.3***, we first of all establish the existence and uniqueness of solution of (2.2.7). Later in page no. 41, we prove the equivalence of (2.2.6) and (2.2.7).*

2.3 Existence and Uniqueness of Weak Solution

In this section, we prove the existence and uniqueness results for the pair $(\mathbf{u}, p)$ of weak solution to the problem (1.2.3)-(1.2.5).
Below, we present a theorem which indicates existence of a unique weak solution to (2.2.7).

Theorem 2.1. *Assume that* $\mathbf{u}_0 \in \mathbf{J}_1$. *Then, there exists a unique solution* $\mathbf{u}$ *satisfying (2.2.7) such that* $\mathbf{u}(0) = \mathbf{u}_0$. *Further,* $\mathbf{u}$ *satisfies*

$$\mathbf{u} \in L^{\infty}(0,T;\mathbf{J}) \cap L^2(0,T;\mathbf{J}_1) \;\; \textit{and} \;\; \mathbf{u}_t \in L^2(0,T;\mathbf{J}_1) \; \textit{for} \; 0 < T < \infty. \tag{2.3.1}$$

In order to prove existence of solution, we apply the Faedo-Galerkin method, which is discussed now. Since $\mathbf{J}_1$ is a separable Hilbert space, there exists an orthogonal basis, say, $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_m, \cdots\}$ in $\mathbf{J}_1$.
Let $\mathbf{V}_m := \overline{span\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_m\}}$ be a finite dimensional subspace of $\mathbf{J}_1$.
Now, set

$$\mathbf{u}_m(t) = \sum_{i=1}^{m} h_{im}(t)\boldsymbol{\phi}_i \tag{2.3.2}$$

as a solution of

$$(\mathbf{u}_{mt}, \boldsymbol{\phi}_j) + \kappa(\nabla \mathbf{u}_{mt}, \nabla \boldsymbol{\phi}_j) + \nu(\nabla \mathbf{u}_m, \nabla \boldsymbol{\phi}_j) + (\mathbf{u}_m \cdot \nabla \mathbf{u}_m, \boldsymbol{\phi}_j) = 0 \quad \boldsymbol{\phi}_j \in \mathbf{V}_m \tag{2.3.3}$$

for $t \in [0,T]$ and $1 \leq j \leq m$ with initial condition

$$\mathbf{u}_m(0) = \mathbf{u}_{0m}, \tag{2.3.4}$$

where $\mathbf{u}_{0m}$ is the orthogonal projection of $\mathbf{u}_0$ in $\mathbf{J}$. Note that $\{\mathbf{u}_{0m}\}$ converges strongly to $\mathbf{u}_0$ in $\mathbf{J}$.
Substituting (2.3.2) in (2.3.3) and denoting $A = (\nabla \boldsymbol{\phi}_i, \nabla \boldsymbol{\phi}_j)$, $B = (\boldsymbol{\phi}_i, \boldsymbol{\phi}_j)$, we write

$$(B + \kappa A)\mathbf{h}_m^{'} + \nu A \mathbf{h}_m = F(\mathbf{h}_m), \tag{2.3.5}$$

where $\mathbf{h}_m = (h_{1,m}, h_{2,m} \cdots, h_{m,m})^T$ and $F_j = \sum_{i,l}^{m} (\boldsymbol{\phi}_i \cdot \nabla \boldsymbol{\phi}_l, \boldsymbol{\phi}_j) h_{im} h_{lm}$.
Here $B + \kappa A$ is invertible. This leads to a nonlinear system of ordinary differential equa-

tions. By Picard's theorem, there exists a unique solution $\mathbf{h}_m$ to (2.3.2) in $(0, t_m^*)$, for some $t_m^* > 0$. This in turn implies the existence of a unique solution $\mathbf{u}_m(t)$ to (2.3.3) in $(0, t_m^*)$. In order to prove the existence of a unique solution for all $t > 0$, we need to appeal to a continuation argument, that is, need to obtain a *priori* bound of $\mathbf{u}_m$ in $L^\infty(\mathbf{L^2})$-norm. The estimates related to $\mathbf{u}_m$ appearing in the following lemma will be used for the continuation requirement.

Lemma 2.1. *For $\mathbf{u}_0 \in \mathbf{J}_1$, the solution $\mathbf{u}_m$ of (2.3.3) satisfies*

$$\|\mathbf{u}_m(t)\|^2 + \kappa\|\nabla\mathbf{u}_m(t)\|^2 + 2\nu\int_0^t \|\nabla\mathbf{u}_m(s)\|^2 ds \le (\|\mathbf{u}_0\|^2 + \kappa\|\nabla\mathbf{u}_0\|^2), \;\; t > 0.$$

Proof. Multiply (2.3.3) by h_{mj}, sum it up from $j = 1$ to m and use Lemma 1.2 ($(\mathbf{u}_m \cdot \nabla\mathbf{u}_m, \mathbf{u}_m) = 0$) to obtain

$$\frac{d}{dt}(\|\mathbf{u}_m\|^2 + \kappa\|\nabla\mathbf{u}_m\|^2) + 2\nu\|\nabla\mathbf{u}_m\|^2 \le 0. \tag{2.3.6}$$

Integrate (2.3.6) from 0 to t with respect to time to arrive at

$$\begin{aligned}\|\mathbf{u}_m(t)\|^2 + \kappa\|\nabla\mathbf{u}_m(t)\|^2 + 2\nu\int_0^t \|\nabla\mathbf{u}_m(s)\|^2 ds &\le (\|\mathbf{u}_{0m}\|^2 + \kappa\|\nabla\mathbf{u}_{0m}\|^2) \\ &\le (\|\mathbf{u}_0\|^2 + \kappa\|\nabla\mathbf{u}_0\|^2)\end{aligned}$$

and this completes the rest of the proof. □

We will also need the following estimate of $\mathbf{u}_m$ which will play a crucial role in applying the compactness theorem presented in Chapter **1** (Theorem 1.2).

Lemma 2.2. *For $\mathbf{u}_0 \in \mathbf{J}_1$, the solution $\mathbf{u}_m$ of (2.3.3) satisfies*

$$\int_0^t (\|\mathbf{u}_{mt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{mt}(s)\|^2)ds + \nu\|\nabla\mathbf{u}_m(t)\|^2 ds \;\; \le M, \; t > 0. \tag{2.3.7}$$

Proof. Multiply (2.3.3) by h'_{mj} and summing it up from $j = 1$ to m, we obtain

$$\|\mathbf{u}_{mt}\|^2 + \kappa\|\nabla\mathbf{u}_{mt}\|^2 + \nu\frac{d}{dt}\|\nabla\mathbf{u}_m\|^2 = -(\mathbf{u}_m.\nabla\mathbf{u}_m, \mathbf{u}_{mt}) = I^m. \tag{2.3.8}$$

To estimate I^m, we use the generalized Hölder's inequality (1.3.12) and Sobolev's inequalities (Lemma 1.1) along with (2.2.3). Then, an application of Young's inequality (1.3.9) yields

$$|I^m| \leq C\|\nabla \mathbf{u}_m\|^4 + \frac{\kappa}{2}\|\mathbf{u}_{mt}\|^2. \tag{2.3.9}$$

Plug (2.3.9) in (2.3.8) and integrate the resulting equation with respect to time to obtain

$$\int_0^t (\|\mathbf{u}_{mt}(s)\|^2 + \kappa\|\nabla \mathbf{u}_{mt}(s)\|^2)ds + \nu\|\nabla \mathbf{u}_m(t)\|^2 ds \leq \nu\|\nabla \mathbf{u}_0\|^2 + \int_0^t \|\nabla \mathbf{u}_m(s)\|^4 ds.$$

A use of estimates from Lemma 2.1 leads to the desired result and this completes the proof.
□

Proof of Theorem 2.1. From Lemmas 2.1 and 2.2, we obtain

$$\text{sequence } \{\mathbf{u}_m\} \text{ is bounded uniformly in } L^\infty(0,T;\mathbf{J}) \cap L^2(0,T;\mathbf{J}_1) \tag{2.3.10}$$

and

$$\text{sequence } \{\mathbf{u}_{mt}\} \text{ is bounded uniformly in } L^2(0,T;\mathbf{J}_1). \tag{2.3.11}$$

Since, the sequence $\{\mathbf{u}_m\}$ is uniformly bounded in $L^\infty(0,T;\mathbf{J}) \cap L^2(0,T;\mathbf{J}_1)$, there is a subsequence, again denoted by $\{\mathbf{u}_m\}$, for notational convenience, such that

$$\left.\begin{aligned} \mathbf{u}_m &\to \mathbf{u} \text{ in } L^2(0,T;\mathbf{J}_1) \text{ weakly} \\ \mathbf{u}_m &\to \mathbf{u} \text{ in } L^\infty(0,T;\mathbf{J}) \text{ weak}^*. \end{aligned}\right\} \tag{2.3.12}$$

From (2.3.10) and (2.3.11), we observe that

$$\mathbf{u}_{mt} \in L^2(0,T;\mathbf{J}_1^*), \tag{2.3.13}$$

where $\mathbf{J}_1^*$ is the dual space of $\mathbf{J}_1$.

Hence, using (2.3.13) and Theorem 1.2, we obtain

$$\mathbf{u}_m \to \mathbf{u} \text{ in } L^2(0,T;\mathbf{J}) \text{ strongly.} \tag{2.3.14}$$

The results in (2.3.12) and (2.3.14) are used in passing the limit in (2.3.3) and (2.3.4). In order to pass the limit, we consider a function ψ, which is continuously differentiable on $[0,T]$, such that $\psi(T)=0$.
Multiplying (2.3.3) by $\psi(t)$ and integrating with respect to time, we write

$$\begin{aligned}\int_0^T(\mathbf{u}_{mt}(t),\boldsymbol{\phi}_j)\psi(t)dt+\kappa\int_0^T(\nabla\mathbf{u}_{mt}(t),\nabla\boldsymbol{\phi}_j)\psi(t)dt+\nu\int_0^T(\nabla\mathbf{u}_m(t),\nabla\boldsymbol{\phi}_j)\psi(t)dt\\+\int_0^T(\mathbf{u}_m(t)\cdot\nabla\mathbf{u}_m(t),\boldsymbol{\phi}_j)\psi(t)dt=0.\end{aligned} \tag{2.3.15}$$

Now, we apply the integration by parts in the first term on the left hand side of (2.3.15) and arrive at

$$\int_0^T(\mathbf{u}_{mt}(t),\boldsymbol{\phi}_j)\psi(t)dt=-\int_0^T(\mathbf{u}_m(t),\psi_t(t)\boldsymbol{\phi}_j)dt-(\mathbf{u}_m(0),\boldsymbol{\phi}_j)\psi(0). \tag{2.3.16}$$

Similarly, again an application of integration by parts in the second term on the right hand side of (2.3.15) yields

$$\int_0^T(\nabla\mathbf{u}_{mt}(t),\nabla\boldsymbol{\phi}_j)\psi(t)dt=-\int_0^T(\nabla\mathbf{u}_m(t),\psi_t(t)\nabla\boldsymbol{\phi}_j)dt-(\nabla\mathbf{u}_m(0),\nabla\boldsymbol{\phi}_j)\psi(0). \tag{2.3.17}$$

Substituting (2.3.16) and (2.3.17) in (2.3.15), we observe that

$$\begin{aligned}-\int_0^T(\mathbf{u}_m(t),\psi_t(t)\boldsymbol{\phi}_j)dt-\kappa\int_0^T(\nabla\mathbf{u}_m(t),\psi_t(t)\nabla\boldsymbol{\phi}_j)dt+\nu\int_0^T(\nabla\mathbf{u}_m(t),\psi(t)\nabla\boldsymbol{\phi}_j)dt\\+\int_0^T(\mathbf{u}_m(t)\cdot\nabla\mathbf{u}_m(t),\boldsymbol{\phi}_j)\psi(t)dt=(\mathbf{u}_m(0),\boldsymbol{\phi}_j)\psi(0)+\kappa(\nabla\mathbf{u}_m(0),\nabla\boldsymbol{\phi}_j)\psi(0).\end{aligned} \tag{2.3.18}$$

For the purpose of passing the limit, we fix some $m_0>0$ with $m_0\le m$ and $\mathbf{v}\in\mathbf{V}_{m_0}$. Since, the above equation is true for $1\le j\le m$, $m\in\mathbb{N}$. Hence, it will be true for finite linear combination of $\boldsymbol{\phi}_j$ and in particular, for $\mathbf{v} \in \mathbf{V}_{m_0}$.

Clearly, from (2.3.18), we observe that, for all $m \geq m_0$

$$
\begin{aligned}
&-\int_0^T (\mathbf{u}_m(t), \psi_t(t)\mathbf{v})dt - \kappa\int_0^T (\nabla\mathbf{u}_m(t), \psi_t(t)\nabla\mathbf{v})dt + \nu\int_0^T (\nabla\mathbf{u}_m(t), \psi(t)\nabla\mathbf{v})dt \\
&+\int_0^T (\mathbf{u}_m(t)\cdot\nabla\mathbf{u}_m(t), \psi(t)\mathbf{v})dt = (\mathbf{u}_m(0), \mathbf{v})\psi(0) + \kappa(\nabla\mathbf{u}_m(0), \nabla\mathbf{v})\psi(0). \qquad (2.3.19)
\end{aligned}
$$

By an application of (2.3.12), we obtain

$$
\lim_{m\to\infty}\int_0^T (\mathbf{u}_m(t), \psi_t(t)\mathbf{v})dt = \int_0^T (\mathbf{u}(t), \psi_t(t)\mathbf{v})dt, \qquad (2.3.20)
$$

$$
\lim_{m\to\infty}\int_0^T (\nabla\mathbf{u}_m(t), \psi_t(t)\nabla\mathbf{v})dt = \int_0^T (\nabla\mathbf{u}(t), \psi_t(t)\nabla\mathbf{v})dt \qquad (2.3.21)
$$

and

$$
\lim_{m\to\infty}\int_0^T (\nabla\mathbf{u}_m(t), \psi(t)\nabla\mathbf{v})dt = \int_0^T (\nabla\mathbf{u}(t), \psi(t)\nabla\mathbf{v})dt \qquad (2.3.22)
$$

Next, we present a lemma, which will enable us to pass the limit in nonlinear term. The proof can be obtained in [95] (Lemma 3.2, Chapter III, page no 289).

Lemma 2.3. *If a sequence of functions* $\{\boldsymbol{\psi}_m\}$ *converges weakly to* $\boldsymbol{\psi}$ *in* $L^2(0,T;\mathbf{J}_1)$ *and strongly in* $L^2(0,T;\mathbf{J})$*, then for any vector function* $\mathbf{w}$ *with components in* $C^1(\bar{Q})$*,* $Q = [0,T]\times\Omega$*,*

$$
\int_0^T b(\boldsymbol{\psi}_m(t), \boldsymbol{\psi}_m(t), \mathbf{w}(t))dt \to \int_0^T b(\boldsymbol{\psi}(t), \boldsymbol{\psi}(t), \mathbf{w}(t))dt, \qquad (2.3.23)
$$

where $b(\boldsymbol{\psi}_m(t), \boldsymbol{\psi}_m(t), \mathbf{w}(t)) = (\boldsymbol{\psi}_m(t)\cdot\nabla\boldsymbol{\psi}_m(t), \mathbf{w}(t))$.

By virtue of Lemma 2.3, we write

$$\lim_{m\to\infty}\int_0^T (\mathbf{u}_m(t)\cdot\nabla\mathbf{u}_m(t),\psi(t)\mathbf{v})dt = \int_0^T (\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\psi(t)\mathbf{v})dt. \tag{2.3.24}$$

Also, from definition, we have

$$\lim_{m\to\infty}\mathbf{u}_{0m} = \mathbf{u}_0 \text{ and } \lim_{m\to\infty}\nabla\mathbf{u}_{0m} = \nabla\mathbf{u}_0. \tag{2.3.25}$$

Hence, passing the limit in (2.3.19) and substituting (2.3.20)-(2.3.22), and (2.3.24)-(2.3.25), we obtain

$$\begin{aligned}
&-\int_0^T (\mathbf{u}(t),\psi_t(t)\mathbf{v})dt - \kappa\int_0^T (\nabla\mathbf{u}(t),\psi_t(t)\nabla\mathbf{v})dt + \nu\int_0^T (\nabla\mathbf{u}(t),\psi(t)\nabla\mathbf{v})dt\\
&+\int_0^T (\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\psi(t)\mathbf{v})dt = (\mathbf{u}_0,\mathbf{v})\psi(0) + \kappa(\nabla\mathbf{u}_0,\nabla\mathbf{v})\psi(0).
\end{aligned} \tag{2.3.26}$$

(2.3.26) is true for $\mathbf{v}\in\mathbf{V}_{m_0}$ and $\psi\in C^1([0,T])$ with $\psi(T)=0$. Since m_0 is arbitrary and also $\bigcup_{1\le m}\mathbf{V}_m$ is dense in $\mathbf{J}_1$, (2.3.26) holds true for $\mathbf{v}\in\mathbf{J}_1$. Further, from (2.3.1) and Lemma 1.5, we observe that $\mathbf{u}$ is equal, almost everywhere, to a continuous function from $[0,T]$ to J, that is,

$$\mathbf{u}\in C([0,T];\mathbf{J}). \tag{2.3.27}$$

This shows that $\mathbf{u}$ satisfies (2.2.7) in $L^2(0,T)$. Now, for $\psi=\phi\in C_0^\infty(0,T)$ in (2.3.26), we observe that (2.2.7) is satisfied in the distribution sense.

We are left with proving the fact that $\mathbf{u}$ satisfies the initial condition, that is, $\mathbf{u}(0)=\mathbf{u}_0$. We note that, $\mathbf{u}$ is continuous. Hence, $\mathbf{u}(0)$ makes sense. To prove that weak solution $\mathbf{u}$ of (2.2.7) satisfies the initial condition, we multiply (2.2.7) by ψ. We apply the similar set of operations to the resulting equation as to obtain (2.3.18) from (2.3.3) (with $\mathbf{u}_m$ is replaced by $\mathbf{u}$) and arrive at

$$
\begin{aligned}
&-\int_0^T(\mathbf{u}(t),\psi_t(t)\nabla\boldsymbol{\phi})dt-\kappa\int_0^T(\nabla\mathbf{u}(t),\psi_t(t)\nabla\boldsymbol{\phi})dt+\nu\int_0^T(\nabla\mathbf{u}(t),\psi(t)\nabla\boldsymbol{\phi})dt\\
&+\int_0^T(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\psi(t)\boldsymbol{\phi})dt=(\mathbf{u}(0),\boldsymbol{\phi})\psi(0)+\kappa(\nabla\mathbf{u}(0),\nabla\boldsymbol{\phi})\psi(0). \qquad (2.3.28)
\end{aligned}
$$

A comparison between (2.3.26) and (2.3.28) leads to

$$(\mathbf{u}(0)-\mathbf{u}_0,\boldsymbol{\phi})\psi(0)+\kappa(\nabla(\mathbf{u}(0)-\mathbf{u}_0),\nabla\boldsymbol{\phi})\nabla\psi(0)=0.$$

Without loss of generality, we choose $\psi(0)=1$ to obtain

$$(\mathbf{u}(0)-\mathbf{u}_0,\boldsymbol{\phi})+\kappa(\nabla(\mathbf{u}(0)-\mathbf{u}_0),\nabla\boldsymbol{\phi})=0 \;\; \forall\boldsymbol{\phi}\in\mathbf{J}_1. \qquad (2.3.29)$$

Hence, we note that

$$\mathbf{u}(0)=\mathbf{u}_0 \;\; \text{a. e.} \qquad (2.3.30)$$

and this completes the rest of the proof. □

Following the similar lines of proof as in Lemmas 2.1 and 2.2, we derive from (2.2.7), the following estimates of $\mathbf{u}$, which are used in establishing the existence of p.

Lemma 2.4. *Let the assumptions* (**A1**)-(**A2**) *hold true. Then, the solution* $\mathbf{u}$ *of (2.2.7) satisfies*

$$\|\mathbf{u}(t)\|^2+\kappa\|\nabla\mathbf{u}(t)\|^2+\nu\int_0^t\|\nabla\mathbf{u}(s)\|^2ds\le(\|\mathbf{u}_0\|^2+\kappa\|\nabla\mathbf{u}_0\|^2)\le M, \;\; t>0.$$

and

$$\int_0^t(\|\mathbf{u}_t(s)\|^2+\kappa\|\nabla\mathbf{u}_t(s)\|^2)ds+\nu\|\nabla\mathbf{u}(t)\|^2ds\le M.$$

In the process of recovering p from (2.2.7), firstly, we establish the equivalence between the weak formulations (2.2.6) and (2.2.7). For that purpose, assume that (2.2.6) is satisfied, that is, solution pair $(\mathbf{u}, p) \in (L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega)) \times L^2(0,T;L_0^2(\Omega)))$ satisfies (2.2.6).

Consider the following expression:

$$\langle \nabla p, \boldsymbol{\phi} \rangle = -(p, \nabla \cdot \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \mathcal{V}. \tag{2.3.31}$$

Since $\mathcal{V}$ is dense in $\mathbf{J}_1$, (2.3.31) holds true $\forall \boldsymbol{\phi} \in \mathbf{J}_1$. A use of (2.2.6) and (2.3.31) leads to weak formulation (2.2.7).

Next, we establish (2.2.7) implies (2.2.6), that is, if $\mathbf{u} \in L^\infty(0,T;\mathbf{J}) \cap L^2(0,T;\mathbf{J}_1)$ satisfies (2.2.7), then $\mathbf{u}$ is a solution of (2.2.6). The facts like, *div* $\mathbf{u} = 0$ in the distributional sense, $\mathbf{u}|_{\partial\Omega} = 0$ in the sense of trace, in (2.2.7), are straight forward.

To recover the pressure term, we follow the similar techniques as to recover p in [95] (Chapter III, page no 307). For the sake of completeness, we provide a proof.

Set

$$\tilde{\mathbf{U}} = \int_0^t \mathbf{u}(s)ds \ \text{ and } \ \gamma(t) = \int_0^t \mathbf{u}(s) \cdot \nabla \mathbf{u}(s) ds. \tag{2.3.32}$$

By the virtue of the properties of $\mathbf{u}$, we arrive at

$$\tilde{\mathbf{U}} \in C([0,T];\mathbf{J}_1^*); \quad \gamma(t) \in C([0,T];\mathbf{J}_1^*).$$

Integrating (2.2.7) with respect to time and using (2.3.32), we obtain

$$\left\langle \mathbf{u} - \mathbf{u}_0 - \kappa\tilde{\Delta}\mathbf{u} - \kappa\tilde{\Delta}\mathbf{u}_0 - \nu\tilde{\Delta}\tilde{\mathbf{U}} + \int_0^t \mathbf{u}(s) \cdot \nabla \mathbf{u}(s) ds, \boldsymbol{\phi} \right\rangle = 0 \ \forall \boldsymbol{\phi} \in \mathbf{J}_1. \tag{2.3.33}$$

From the estimates (2.3.27) and (2.3.33), we write

$$\mathbf{u} - \mathbf{u}_0 - \kappa\tilde{\Delta}\mathbf{u} - \kappa\tilde{\Delta}\mathbf{u}_0 - \nu\tilde{\Delta}\tilde{\mathbf{U}} + \int_0^t \mathbf{u}(s) \cdot \nabla \mathbf{u}(s) ds \in C([0,T];\mathbf{J}_1^*). \tag{2.3.34}$$

A use of (2.3.34) and Lemmas 1.6 and 1.7 provides the existence of some function $h(t) \in L^2(\Omega)$ such that $\forall t \in [0,T]$

$$\nabla h(t) = \mathbf{u}(t) - \mathbf{u}_0 - \kappa\tilde{\Delta}\mathbf{u}(t) - \kappa\tilde{\Delta}\mathbf{u}_0 - \nu\tilde{\Delta}\tilde{\mathbf{U}}(t) + \int_0^t \big(\mathbf{u}(s)\cdot\nabla\mathbf{u}(s)\big)ds. \qquad (2.3.35)$$

Observing that the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $\mathbf{H}^{-1}(\Omega)$ (see [95], Remark 1.4(ii), page 15) and $\nabla h \in C([0,T];\mathbf{H}^{-1}(\Omega))$, we conclude that $h(t) \in C\big([0,T];L^2(\Omega)\big)$ and this enables us to differentiate (2.3.35) in t, in the sense of distributions in $\Omega\times(0,T)$ and setting

$$p = \frac{\partial h}{\partial t},$$

we obtain the weak formulation (2.2.6).

It remains to show that $p \in L^2(L^2/\mathbb{R})$. The results presented in the following lemma will be a tool to achieve that goal.

Lemma 2.5. *Let the assumptions* (**A1**)-(**A2**) *hold true. Then, the solution* $\mathbf{u}$ *of (2.2.7) satisfies*

$$\int_0^t \|p(s)\|^2_{L^2/\mathbb{R}}ds \le M, \quad t>0. \qquad (2.3.36)$$

Proof. A use of the Cauchy-Schwarz inequality (1.3.10) and the generalized Hölder's inequality (1.3.12) in (2.2.6) yields

$$(p,\nabla.\boldsymbol{\phi}) \le C\big(\|\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \kappa\|\nabla\mathbf{u}_t\|\|\nabla\boldsymbol{\phi}\| + \nu\|\nabla\mathbf{u}\|\|\nabla\boldsymbol{\phi}\| + \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|\|\boldsymbol{\phi}\|_{L^4}\big). \quad (2.3.37)$$

Using the Sobolev's inequality (Lemma 1.1), (2.2.3) and continuous inf-sup condition (**A3**) in (2.3.37), we obtain

$$\|p\|_{L^2/\mathbb{R}} \le C\big(\|\mathbf{u}_t\| + \kappa\|\nabla\mathbf{u}_t\| + \nu\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}\|^2\big). \qquad (2.3.38)$$

After taking squares on both sides of (2.3.38), we integrate with respect to time to arrive at

$$\int_0^t \|p(s)\|^2_{L^2/\mathbb{R}} ds \le C\int_0^t \big(\|\mathbf{u}_t(s)\|^2 + \kappa\|\nabla\mathbf{u}_t(s)\|^2 + \nu\|\nabla\mathbf{u}(s)\|^2 + \|\nabla\mathbf{u}(s)\|^4\big) ds. \quad (2.3.39)$$

An application of Lemma 2.4 in (2.3.39) leads us to the desired estimate. □

With the help of Lemma 2.5, we observe that $p \in L^2(L^2/\mathbb{R})$.

Now, we proceed to prove uniqueness by contradiction. Let us assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of (2.2.7) and let $\mathbf{e} = \mathbf{u}_1 - \mathbf{u}_2$. Then, the equation in $\mathbf{e}$ is as follows:

$$(\mathbf{e}_t, \boldsymbol{\phi}) + \kappa(\nabla\mathbf{e}_t, \nabla\boldsymbol{\phi}) + \nu(\nabla\mathbf{e}, \nabla\boldsymbol{\phi}) = -(\mathbf{u}_1\cdot\nabla\mathbf{u}_1, \boldsymbol{\phi}) + (\mathbf{u}_2\cdot\nabla\mathbf{u}_2, \boldsymbol{\phi}) \quad (2.3.40)$$
$$\mathbf{e}(0) = 0.$$

Plug $\boldsymbol{\phi} = \mathbf{e}$ in (2.3.40) to obtain

$$\frac{d}{dt}\big(\|\mathbf{e}\|^2 + \kappa\|\nabla\mathbf{e}\|^2\big) + 2\nu\|\nabla\mathbf{e}\|^2 = -2(\mathbf{u}_1\cdot\nabla\mathbf{u}_1, \mathbf{e}) + 2(\mathbf{u}_2\cdot\nabla\mathbf{u}_2, \mathbf{e}) =: \Lambda(\mathbf{e}). \quad (2.3.41)$$

Rewrite the term $\Lambda(\mathbf{e})$ to arrive at

$$\begin{aligned}\Lambda(\mathbf{e}) &= 2((\mathbf{u}_2\cdot\nabla\mathbf{u}_2, \mathbf{e}) - (\mathbf{u}_1\cdot\nabla\mathbf{u}_1, \mathbf{e}))\\ &= 2(-(\mathbf{u}_2\cdot\nabla\mathbf{e}, \mathbf{e}) + (\mathbf{u}_2\cdot\nabla\mathbf{u}_1, \mathbf{e}) - (\mathbf{u}_1\cdot\nabla\mathbf{u}_1, \mathbf{e}))\\ &= 2(-(\mathbf{u}_2\cdot\nabla\mathbf{e}, \mathbf{e}) - (\mathbf{e}\cdot\nabla\mathbf{u}_1, \mathbf{e})). \quad (2.3.42)\end{aligned}$$

The first term in (2.3.42) vanishes because of Lemma 1.2 ($(\mathbf{u}_2\cdot\nabla\mathbf{e}, \mathbf{e}) = 0$). Hence, using generalized Hölder's inequality (1.3.12), Sobolev's inequality (Lemma 1.1), (2.2.3) and Young's inequality (1.3.9), we observe that

$$\begin{aligned}|\Lambda(\mathbf{e})| &\le C\|\nabla\mathbf{u}_1\|\|\nabla\mathbf{e}\|^2 \quad (2.3.43)\\ &\le C(\nu)\|\nabla\mathbf{u}_1\|^2\|\nabla\mathbf{e}\|^2 + \nu\|\nabla\mathbf{e}\|^2.\end{aligned}$$

Substitute (2.3.43) in (2.3.41). Then, integrate with respect to time and use $\mathbf{e}(0) = 0$ to obtain

$$\begin{aligned}\|\mathbf{e}(t)\|^2 + \kappa\|\nabla\mathbf{e}(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{e}(s)\|^2 ds &\le C(\nu)\int_0^t \|\nabla\mathbf{u}_1(s)\|^2\|\nabla\mathbf{e}(s)\|^2 ds \\ &\le C(\nu,\kappa)\int_0^t \|\nabla\mathbf{u}_1(s)\|^2(\|\mathbf{e}(s)\|^2 + \kappa\|\nabla\mathbf{e}(s)\|^2) ds.\end{aligned} \tag{2.3.44}$$

Note that $\mathbf{u}_1 \in L^2(0,T;\mathbf{J}_1)$, that is,

$$\int_0^t \|\nabla\mathbf{u}_1(s)\|^2 ds \le C. \tag{2.3.45}$$

An application of Gronwall's lemma along with (2.3.45) and $\mathbf{e}(0) = 0$ in (2.3.44) leads to

$$\|\mathbf{e}(t)\|^2 + \kappa\|\nabla\mathbf{e}(t)\|^2 \le 0. \tag{2.3.46}$$

Therefore, $\mathbf{e}(t) = 0$. This leads to a contradiction. Hence, the solution is unique. □

The next section would deal with some regularity results for the weak solution of (1.2.3)-(1.2.5) (with $\mathbf{f} = 0$), which will be used very often in the subsequent chapters.

2.4 A Priori Estimates

In this section, we derive some *a priori* bounds for the solutions of the problem (1.2.3)-(1.2.5) which reflect exponential decay behavior in time. First of all, we state the main theorem of this section.

Theorem 2.2. *Let the assumptions* (**A1**) *and* (**A2**) *hold true. Then, there exists a positive constant K depending on M, λ_1, α, κ and ν such that for $0 \le \alpha < \dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$ the following estimate holds true, for $t > 0$*

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_2^2 + \|p(t)\|_{H^1/\mathbb{R}}^2 + \int_0^t e^{2\alpha s}\big(\|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_t(s)\|_2^2 + \|p(s)\|_{H^1/\mathbb{R}}^2\big) ds \le Ke^{-2\alpha t}.$$

The proof can be established using the following series of lemmas.

Lemma 2.6. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$*, and let the assumptions* (**A1**)-(**A2**) *hold true. Then, the solution* $\mathbf{u}$ *of (2.2.7) satisfies*

$$\begin{aligned}\|\mathbf{u}(t)\|^2 &+ \kappa\|\nabla\mathbf{u}(t)\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2\,ds \\ &\le e^{-2\alpha t}(\|\mathbf{u}_0\|^2 + \kappa\|\nabla\mathbf{u}_0\|^2) \le M_0 e^{-2\alpha t}, \quad t > 0.\end{aligned}$$

where $\beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) > 0$*, and* $M_0 = (1+\kappa)M^2$*.*

Proof. Setting $\hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t)$ for some $\alpha \ge 0$, we rewrite (2.2.7) as

$$\begin{aligned}(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}},\boldsymbol{\phi}) + \kappa(\nabla\hat{\mathbf{u}}_t, \nabla\boldsymbol{\phi}) - \kappa\alpha(\nabla\hat{\mathbf{u}}, \nabla\boldsymbol{\phi}) \\ + \nu(\nabla\hat{\mathbf{u}}, \nabla\boldsymbol{\phi}) + e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) = 0 \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1.\end{aligned} \tag{2.4.1}$$

Choose $\boldsymbol{\phi} = \hat{\mathbf{u}}$ in (2.4.1). Using Lemma 1.2, $(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{u}}) = 0$ and (2.2.3), we obtain

$$\frac{d}{dt}(\|\hat{\mathbf{u}}\|^2 + \kappa\|\nabla\hat{\mathbf{u}}\|^2) + 2\beta\|\nabla\hat{\mathbf{u}}\|^2 \le 0. \tag{2.4.2}$$

Integrate (2.4.2) from 0 to t with respect to time and use the assumption (**A2**) to complete the rest of the proof. □

Lemma 2.7. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$

$$\|\nabla\mathbf{u}(t)\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2\,ds \le K e^{-2\alpha t}$$

holds true, where $\beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) > 0$*.*

Proof. Using the Stokes operator $\tilde{\Delta}$, we rewrite (2.4.1) as

$$(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) - \kappa(\tilde{\Delta}\hat{\mathbf{u}}_t, \boldsymbol{\phi}) + \kappa\alpha(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) - \nu(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) = -e^{-\alpha t}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1. \tag{2.4.3}$$

With $\boldsymbol{\phi} = -\tilde{\Delta}\hat{\mathbf{u}}$ in (2.4.3), we note that

$$-(\hat{\mathbf{u}}_t, \tilde{\Delta}\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2,$$

and hence (2.4.3) becomes

$$\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) + 2\nu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 - 2\alpha(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) = 2e^{-\alpha t}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}). \tag{2.4.4}$$

To estimate the term on the right hand side of (2.4.4), a use of the Hölder's inequality yields

$$|I| = 2|e^{-\alpha t}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})| \leq e^{-\alpha t}\|\hat{\mathbf{u}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|\tilde{\Delta}\hat{\mathbf{u}}\|_{L^2}. \tag{2.4.5}$$

Using the Sobolev inequality for 3D, that is, when $d = 3$, (see Lemma 1.1) given by

$$\|\boldsymbol{\phi}\|_{L^4(\Omega)} \leq C\|\boldsymbol{\phi}\|^{\frac{1}{4}}\|\nabla\boldsymbol{\phi}\|^{\frac{3}{4}}, \quad \boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega), \tag{2.4.6}$$

we arrive at

$$\begin{aligned} |I| &\leq 2e^{-\alpha t}\|\hat{\mathbf{u}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|\tilde{\Delta}\hat{\mathbf{u}}\|, \\ &\leq 2e^{-\alpha t}(\|\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{u}}\|^{\frac{3}{4}}\|)(\|\nabla\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{3}{4}})\|\|\tilde{\Delta}\hat{\mathbf{u}}\|, \\ &\leq Ce^{-\alpha t}\|\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{7}{4}}. \end{aligned} \tag{2.4.7}$$

Applying Young's inequality $ab \leq \frac{a^p}{p\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a, b \geq 0$, $\epsilon > 0$ with $p = 8$ and $q = \frac{8}{7}$ (see

(1.3.9)), we obtain

$$|I| \leq C\frac{\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8}{8\epsilon^7} + \frac{7}{8}\epsilon\|\tilde{\Delta}\hat{\mathbf{u}}\|^2. \tag{2.4.8}$$

Choosing $\epsilon = \frac{8\nu}{7}$, we find that

$$|I| \leq C\left(\frac{8\nu}{7}\right)^{-7}\frac{\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8}{8} + \nu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2. \tag{2.4.9}$$

Substitute (2.4.9) in (2.4.4) to arrive at

$$\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) - 2\alpha\big(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2\big) + \nu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 \leq C(\nu)\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8. \tag{2.4.10}$$

A use of (2.2.4) in (2.4.10) and an integration with respect to time from 0 to t yields

$$\begin{aligned}\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + \beta\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}\|^2 ds \leq & \|\nabla\mathbf{u}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_0\|^2 \\ & + C(\nu)\int_0^t \|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8 ds.\end{aligned} \tag{2.4.11}$$

Using Lemma 2.6, we bound

$$\int_0^t \|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8 ds \leq K. \tag{2.4.12}$$

Substitute (2.4.12) in (2.4.11) to complete the rest of the proof. □

Lemma 2.8. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa,\nu,\lambda_1,\alpha,M)$ *such that for all* $t > 0$,

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_t(s)\|^2 + \kappa\|\nabla\mathbf{u}_t(s)\|^2)ds + \|\nabla\mathbf{u}(t)\|^2 \leq Ke^{-2\alpha t}.$$

Proof. Rewrite (2.2.7) as

$$(\mathbf{u}_t,\boldsymbol{\phi}) - \kappa(\tilde{\Delta}\mathbf{u}_t,\boldsymbol{\phi}) - \nu(\tilde{\triangle}\mathbf{u},\boldsymbol{\phi}) + (\mathbf{u}.\nabla\mathbf{u},\boldsymbol{\phi}) = 0 \quad \forall\boldsymbol{\phi}\in\mathbf{J}_1. \tag{2.4.13}$$

On multiplying (2.4.13) by $e^{\alpha t}$ and substituting $\boldsymbol{\phi} = e^{\alpha t}\mathbf{u}_t$, we arrive at

$$e^{2\alpha t}(\|\mathbf{u}_t\|^2 + \kappa\|\nabla\mathbf{u}_t\|^2) + \nu e^{2\alpha t}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 = -e^{2\alpha t}(\mathbf{u}.\nabla\mathbf{u}, \mathbf{u}_t). \tag{2.4.14}$$

To estimate the nonlinear term on the right hand side of (2.4.14), first we consider the term $(\mathbf{u}.\nabla\mathbf{u}, \mathbf{w})$. A use of the generalized Hölder's inequality (1.3.12) and Sobolev inequality (Lemma 1.1) with (2.2.2) yields:

$$\begin{aligned}(\mathbf{u}.\nabla\mathbf{u}, \mathbf{w}) &\leq C\|\mathbf{u}\|_{\mathbf{L}^4}\, \|\nabla\mathbf{u}\|_{\mathbf{L}^4}\, \|\mathbf{w}\| \\ &\leq C\|\nabla\mathbf{u}\|\, \|\mathbf{u}\|_2\, \|\mathbf{w}\| \\ &\leq C\|\nabla\mathbf{u}\|\, \|\tilde{\Delta}\mathbf{u}\|\, \|\mathbf{w}\|.\end{aligned} \tag{2.4.15}$$

We note that,

$$(\mathbf{u}.\nabla\mathbf{u}, \mathbf{u}_t) \leq C\|\nabla\mathbf{u}\|\, \|\tilde{\Delta}\mathbf{u}\|\, \|\mathbf{u}_t\|. \tag{2.4.16}$$

Integration of (2.4.14) with respect to time from 0 to t along with a use of (2.4.16) and Young's inequality yields

$$\begin{aligned}\int_0^t e^{2\alpha s}(\|\mathbf{u}_t(s)\|^2 + \kappa\|\nabla\mathbf{u}_t(s)\|^2)ds + \nu e^{2\alpha t}\|\nabla\mathbf{u}\|^2 \leq & C\big(\|\nabla\mathbf{u}(0)\|^2 + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2 ds \\ & + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2\|\tilde{\Delta}\mathbf{u}(s)\|^2 ds\big).\end{aligned}$$

Again, a use of a priori bounds for $\mathbf{u}$ obtained from the Lemmas 2.6 and 2.7, would provide us the desired result. □

Lemma 2.9. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$,

$$\|\mathbf{u}_t(t)\|^2 + \kappa\|\nabla\mathbf{u}_t(t)\|^2 \leq Ke^{-2\alpha t}.$$

Proof. Substituting $\boldsymbol{\phi} = \mathbf{u}_t$ in (2.2.7), we obtain

$$\|\mathbf{u}_t\|^2 + \kappa\|\nabla\mathbf{u}_t\|^2 = -\nu(\nabla\mathbf{u}, \nabla\mathbf{u}_t) - (\mathbf{u}.\nabla\mathbf{u}, \mathbf{u}_t) = I_1 + I_2, \text{ say.} \tag{2.4.17}$$

To estimate $|I_1|$, we apply the Cauchy-Schwarz inequality and Young's inequality to arrive at

$$|I_1| \leq \frac{\nu}{2\epsilon}\|\nabla\mathbf{u}\|^2 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_t\|^2.$$

Choose $\epsilon = \kappa$ to yield

$$|I_1| \leq \frac{\nu}{2\kappa}\|\nabla\mathbf{u}\|^2 + \frac{\kappa}{2}\|\nabla\mathbf{u}_t\|^2.$$

For I_2, apply (2.4.16) and use Young's inequality to obtain

$$|I_2| \leq C\|\nabla\mathbf{u}\|^2\|\tilde{\Delta}\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{u}_t\|^2.$$

Substitute the bounds for $|I_1|$ and $|I_2|$ in (2.4.17) and use *a priori* estimates from Lemma 2.6 and 2.7 to complete the proof. □

Lemma 2.10. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$,

$$\|\nabla\mathbf{u}_t(t)\|^2 + \frac{\kappa}{2}\|\tilde{\Delta}\mathbf{u}_t(t)\|^2 \leq Ke^{-2\alpha t}.$$

Proof. Setting $\boldsymbol{\phi} = -\tilde{\Delta}\mathbf{u}_t$ in (2.4.13), we obtain

$$\|\nabla\mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2 = -\nu(\tilde{\Delta}\mathbf{u}, \tilde{\Delta}\mathbf{u}_t) + (\mathbf{u}.\nabla\mathbf{u}, \tilde{\Delta}\mathbf{u}_t). \tag{2.4.18}$$

For the nonlinear term, that is, the last term on the right hand side of (2.4.18), we now use (2.4.15) replacing $\mathbf{w}$ by $\tilde{\Delta}\mathbf{u}_t$. Then with the help of the Cauchy-Schwarz inequality and

Young's inequality, we bound right hand side of (2.4.18) and use Lemmas 2.6 and 2.7 to complete the rest of the proof. □

Lemma 2.11. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$,

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_t(s)\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t(s)\|^2)ds + \nu\|\tilde{\Delta}\mathbf{u}(t)\|^2 \le Ke^{-2\alpha t}.$$

Proof. Multiply (2.4.13) by $e^{\alpha t}$ and substitute $\boldsymbol{\phi} = -e^{\alpha t}\tilde{\Delta}\mathbf{u}_t$ to obtain

$$e^{2\alpha t}(\|\nabla \mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2) + \nu e^{2\alpha t}\frac{d}{dt}\|\tilde{\Delta}\mathbf{u}\|^2 = e^{2\alpha t}(\mathbf{u}.\nabla\mathbf{u}, \tilde{\Delta}\mathbf{u}_t). \tag{2.4.19}$$

Now for the nonlinear term, that is, the term on the right hand side of (2.4.19), we now use (2.4.15) replacing $\mathbf{w}$ by $\tilde{\Delta}\mathbf{u}_t$. Then, integrating with respect to time from 0 to t and using Young's inequality, we obtain

$$\begin{aligned}\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2)ds + \nu e^{2\alpha t}\|\tilde{\Delta}\mathbf{u}\|^2 \le\,& C(\kappa)\big(\|\tilde{\Delta}\mathbf{u}(0)\|^2 + \int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}\|^2 ds \\ &+ \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}\|^2\|\tilde{\Delta}\mathbf{u}\|^2 ds\big).\end{aligned}$$

A use of Lemma 2.7 establishes the desired estimate and this completes the rest of the proof. □

Now, we derive the a priori bounds for the pressure p.

Lemma 2.12. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2\,(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$, *the following estimate holds true:*

$$\|p(t)\|^2_{L^2/\mathbb{R}} + \|p(t)\|^2_{H^1/\mathbb{R}} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|p(s)\|^2_{H^1/\mathbb{R}}ds \le Ke^{-2\alpha t}.$$

Proof. A use of the Cauchy-Schwarz inequality (1.3.10) and the generalized Hölder's inequality (1.3.12) in (2.2.6) yields

$$(p, \nabla.\boldsymbol{\phi}) \leq C\big(\|\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \kappa\|\nabla\mathbf{u}_t\|\|\nabla\boldsymbol{\phi}\| + \nu\|\nabla\mathbf{u}\|\|\nabla\boldsymbol{\phi}\| + \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|\|\boldsymbol{\phi}\|_{L^4}\big). \quad (2.4.20)$$

Using the Sobolev's embedding theorem (see [95]), (2.2.3), dividing by $\|\nabla\boldsymbol{\phi}\|$ and applying continuous inf-sup condition (**A3**) in (2.4.20), we obtain

$$\|p\|_{L^2/\mathbb{R}} \leq C\frac{|(p, \nabla.\boldsymbol{\phi})|}{\|\nabla\boldsymbol{\phi}\|} \leq C\big(\|\mathbf{u}_t\| + \kappa\|\nabla\mathbf{u}_t\| + \nu\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}\|^2\big). \quad (2.4.21)$$

An application of Lemmas 2.6 and 2.9 in (2.4.21) yields

$$\|p(t)\|_{L^2/\mathbb{R}} \leq K(\kappa, \nu, \lambda_1, \alpha, M)e^{-\alpha t}. \quad (2.4.22)$$

Using the property of space $\mathbf{J}_1$ (see [95] page no 19, remark 1.9) in (2.2.7), we obtain

$$(\nabla p,\ \boldsymbol{\phi}) = (\mathbf{u}_t - \kappa\tilde{\Delta}\mathbf{u}_t - \nu\tilde{\Delta}\mathbf{u} + \mathbf{u}.\nabla\mathbf{u},\ \boldsymbol{\phi}) \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1. \quad (2.4.23)$$

A use of the Cauchy Schwarz inequality with the generalized Hölder's inequality in (2.4.23) yields

$$|(\nabla p,\ \boldsymbol{\phi})| \leq C(\kappa, \nu)\big(\|\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \|\tilde{\Delta}\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \|\tilde{\Delta}\mathbf{u}\|\|\boldsymbol{\phi}\| + \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\boldsymbol{\phi}\|\big). \quad (2.4.24)$$

Applying the Sobolev embedding theorem (see [95]), in (2.4.24) with (2.2.2) and dividing by $\|\boldsymbol{\phi}\|$, we obtain

$$\|\nabla p\| \leq C(\kappa, \nu)\big(\|\mathbf{u}_t\| + \|\tilde{\Delta}\mathbf{u}_t\| + \|\tilde{\Delta}\mathbf{u}\| + \|\nabla\mathbf{u}\|\ \|\tilde{\Delta}\mathbf{u}\|\big). \quad (2.4.25)$$

A use of Lemmas 2.6, 2.7, 2.9 and 2.10 in (2.4.25) yields

$$\|p(t)\|_{H^1/\mathbb{R}} \leq Ke^{-\alpha t}. \quad (2.4.26)$$

Taking squares on both the sides of (2.4.25), multiplying by $e^{2\alpha t}$ and integrating from 0 to t with respect to time, we obtain

$$\int_0^t e^{2\alpha s}\|\nabla p(s)\|^2 ds \leq C(\kappa,\nu)\Big(\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_t(s)\|^2 + \|\tilde{\Delta}\mathbf{u}_t(s)\|^2\big) ds + \int_0^t e^{2\alpha s}\big(\|\tilde{\Delta}\mathbf{u}(s)\|^2 + \|\nabla\mathbf{u}(s)\|^2\ \|\tilde{\Delta}\mathbf{u}\|^2\big) ds\Big). \tag{2.4.27}$$

Using Lemmas 2.6, 2.7, 2.8 and 2.11, we arrive at

$$\int_0^t e^{2\alpha s}\|\nabla p(s)\|^2 ds \leq K. \tag{2.4.28}$$

A use of (2.4.22), (2.4.26) and (2.4.28) would lead us to the desired result. □

Proof of Theorem 2.2. The proof of Theorem 2.2 follows by combining the estimates obtained in Lemmas 2.6-2.12. □

Chapter 3

Semidiscrete Galerkin Method

3.1 Introduction

In this chapter, we study the semidiscrete finite element Galerkin approximations to the equations of motion arising in Kelvin-Voigt model of viscoelastic fluid flow. The main results of this chapter consist of proving regularity results for the semidiscrete solution, which are valid uniformly in time and even for 3-D domain, establishing the exponential decay property for the semidiscrete solution and obtaining optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}^1)$-norm and to the pressure in $L^\infty(L^2)$-norm which also reflect the exponential decay property in time.

We have made use of exponential weights for the derivation of the new regularity results. These weights also become crucial in establishing the error bounds. In order to derive optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm, we first split the error by using a Galerkin approximation to a linearized Kelvin-Voigt model and then introduce a new auxiliary operator through a modification of the Stokes operator, named as Sobolev-Stoke's operator. Now making use of estimates derived for the auxiliary operator and the error estimates due to the linearized model, we recover the optimality of $L^\infty(\mathbf{L}^2)$ error estimates for the velocity. Finally, with the help of uniform inf-sup condition and error estimates for the velocity, we derive optimal error estimates for the pressure. Special care has been taken to preserve the exponential decay property even for the error estimates.

The literature for the numerical approximation of (1.2.3)-(1.2.5) is limited. In [81], the authors have discussed the convergence of spectral Galerkin approximation for the semi axis $t \geq 0$ under the assumption that solution is asymptotically stable. Recently, Pani *et al* have employed a modified nonlinear spectral Galerkin method. They have proved the existence of a unique discrete solution for the semidiscrete spectral Galerkin scheme and have established existence of a discrete global attractor. Further, they have obtained $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}^1)$-norms optimal error estimates. Then, they have applied a modified spectral Galerkin scheme and have derived optimal error bounds (see [21]).

The remaining part of this chapter includes the following sections: In Section **3.2**, we present semidiscrete scheme and discuss existence and uniqueness of semidiscrete solution. In Section **3.3**, we explore a few a *priori* bounds for the semidiscrete approximations of (1.2.3)-(1.2.5) (with $\mathbf{f} = \mathbf{0}$) which will be required for the analysis in subsequent chapters. In Section **3.4**, we introduce a new auxiliary operator through a modification of the Stokes operator and establish estimates for auxiliary operator. Then, we obtain optimal error estimates for the velocity with the help of previous estimates derived for auxiliary operator. Section **3.5** considers the error analysis for pressure term involving the error estimates for velocity.

3.2 Semidiscrete Scheme

From now onwards, let $h > 0$ be the discretization parameter in space and for each h, let τ_h be a corresponding regular (or non-degenerate) family of triangulations of polygonal domain $\bar{\Omega}$, consisting of closed subsets K, triangles or quadrilaterals in two dimensions. The decompositions τ_h are assumed to be "face to face" and to satisfy a "uniform size" condition", that is, any two elements of τ_h are either disjoint or share a vertex or an entire side. Each element of τ_h contains a circle of radius $k_1 h$ and is contained in a circle of radius

$k_2 h$ (see [11], [17]).

Further, let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, such that, there exist operators i_h and j_h satisfying the following approximation properties:

(**B1**). For each $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there exist approximations $i_h\mathbf{w} \in \mathbf{J}_h$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \le K_0 h^2 \|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/\mathbb{R}} \le K_0 h \|q\|_{H^1/\mathbb{R}}.$$

For defining the Galerkin approximations, for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$, set

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi})$$

and

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v} \cdot \nabla \boldsymbol{\phi}, \mathbf{w}).$$

When $\mathbf{v} \in \mathbf{J}_1$, $\mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$, using Lemma 1.2, we obtain

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = (\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}).$$

Note that the operator $b(\cdot, \cdot, \cdot)$ preserves the antisymmetric properties of the original non-linear term, i.e.,

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{3.2.1}$$

The discrete analogue of the weak formulation (2.2.6) is as follows: find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,$$

$$(\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h, \tag{3.2.2}$$

where $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.

In order to consider a suitable approximation of $\mathbf{J}_1$, we introduce the discrete incompressibility condition in $\mathbf{H}_h$ and call the resulting subspace as $\mathbf{J}_h$. Thus, $\mathbf{J}_h$ is defined as

$$\mathbf{J}_h = \{\mathbf{v}_h \in \mathbf{H}_h : (\chi_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \chi_h \in L_h\}.$$

Note that, the space $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. We now define the finite dimensional problem: find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.2.3}$$

Since $\mathbf{J}_h$ is finite dimensional, the problem (3.2.3) leads to a system of nonlinear ordinary differential equations. A use of Picard's theorem yields existence of a unique local solution in an interval $[0, t^*)$, for some $t^* > 0$. For continuation of solution beyond t^*, we need to establish an $L^\infty(\mathbf{L^2})$ bound for the approximate solution $\mathbf{u}_h$. Setting $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (3.2.3), we obtain

$$\frac{d}{dt}(\|\mathbf{u}_h\|^2 + \kappa \|\nabla \mathbf{u}_h\|^2) + 2\nu \|\nabla \mathbf{u}_h\|^2 = 0.$$

On integration with respect to the temporal variable t, we find that

$$\|\mathbf{u}_h(t)\|^2 + \kappa \|\nabla \mathbf{u}_h(t)\|^2 \le C(\nu)(\|\mathbf{u}_{0h}\|^2 + \kappa \|\nabla \mathbf{u}_{0h}\|^2) \le C \quad \forall t \ge 0,$$

provided $\|\nabla \mathbf{u}_{0h}\| \le C\|\nabla \mathbf{u}_0\|$. This is indeed true, which we shall see later on. This shows the global existence of a unique Galerkin approximation $\mathbf{u}_h$ for all $t > 0$.

Once we compute $\mathbf{u}_h(t) \in J_h$, the approximation $p_h(t) \in L_h$ to the pressure $p(t)$ can be found out by solving the following system

$$(p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.2.4}$$

For the solvability of the above system (3.2.4), we note that the right hand side defines a linear functional ℓ on $\mathbf{H}_h$, i.e., $\boldsymbol{\phi}_h \mapsto \ell(\phi_h)$. By construction $\ell(\boldsymbol{\phi}_h) = 0$, for all $\boldsymbol{\phi}_h \in \mathbf{J}_h$. It is now easy to check that this condition implies existence of $p_h \in L_h$, see [28]. Uniqueness is obtained on the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (q_h, \nabla \cdot \boldsymbol{\phi}_h) = 0, \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L_h/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

Furthermore, the pair $(\mathbf{H}_h, L_h/N_h)$ satisfies a uniform inf-sup condition:
(**B2**). For every $q_h \in L_h$, there exist a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H_h}$ and a positive constant K_1, independent of h, such that,

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq K_1 \|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L_h/N_h}.$$

As a consequence of conditions (**B1**)-(**B2**), we have the following properties of the L^2 projection $P_h : \mathbf{L^2} \to \mathbf{J_h}$. For $\boldsymbol{\phi} \in \mathbf{J}_1$, we note that, see [28], [43],

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h\|\nabla P_h \boldsymbol{\phi}\| \leq Ch\|\nabla \boldsymbol{\phi}\|, \tag{3.2.5}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h \boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta} \boldsymbol{\phi}\|. \tag{3.2.6}$$

We may define the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ through the bilinear form $a(\cdot, \cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.2.7}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h\Delta_h$.
Using Sobolev embedding theorems with Sobolev inequalities, it is a routine calculation to derive the following lemma, see page 360 of [46].

Lemma 3.1. *The trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following estimates:*

$$|b(\boldsymbol{\phi},\boldsymbol{\xi},\chi)| \le C\|\nabla\boldsymbol{\phi}\|^{1/2}\|\tilde{\Delta}_h\boldsymbol{\phi}\|^{1/2}\|\nabla\boldsymbol{\xi}\|\ \|\chi\|, \tag{3.2.8}$$

$$|b(\boldsymbol{\phi},\boldsymbol{\xi},\chi)| \le C\|\nabla\boldsymbol{\phi}\|\|\nabla\boldsymbol{\xi}\|^{1/2}\|\tilde{\Delta}_h\boldsymbol{\xi}\|^{1/2}\|\chi\|, \tag{3.2.9}$$

$$|b(\boldsymbol{\phi},\boldsymbol{\xi},\chi)| \le C\|\boldsymbol{\phi}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\phi}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\xi}\|\|\chi\|^{\frac{1}{2}}\|\nabla\chi\|^{\frac{1}{2}}, \tag{3.2.10}$$

$\forall\ \boldsymbol{\phi},\ \boldsymbol{\xi},\ \chi \in \mathbf{H}_h$.

$$|b(\boldsymbol{\phi},\boldsymbol{\xi},\chi)| \le C\|\nabla\boldsymbol{\phi}\|\|\nabla\boldsymbol{\xi}\|\|\nabla\chi\|, \tag{3.2.11}$$

$\forall\ \boldsymbol{\phi},\ \boldsymbol{\xi},\ \chi \in \mathbf{H}_0^1$.

Moreover, we present below an estimate for trilinear form which can be obtained using the generalized Hölder's inequality, Sobolev embedding theorems and Sobolev's inequalities and will be used in a error analysis of two grid method.

Lemma 3.2. *The trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following estimate:*

$$|b(\boldsymbol{\phi},\boldsymbol{\xi},\chi)| \le C\|\boldsymbol{\phi}\|^{1-\delta}\|\nabla\boldsymbol{\phi}\|^{\delta}\|\nabla\boldsymbol{\xi}\|\|\nabla\chi\|, \tag{3.2.12}$$

where $\delta > 0$ is arbitrarily small for $d = 2$ and $\delta = \frac{1}{2}$ for $d = 3$.
Examples of subspaces $\mathbf{H_h}$ and L_h satisfying assumptions (**B1**) and (**B2**) can be found in [7], [12] and [43]. In the context of non conforming analysis, we would like to refer [43].
Below, we present a few examples of the finite dimensional subspaces $\mathbf{H}_h$ and L_h satisfying

the assumptions (**B1**) and (**B2**).

The first one is introduced by Bercovier-Pironneau.

Example 3.2.1. *([7])*

$$\begin{aligned}\mathbf{H}_h &= \{v_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : v_h|_K \in (P_1(K))^2 \quad \forall K \in \mathcal{T}_{h/2}\}\\ L_h &= \{q_h \in C^0(\bar{\Omega}) : q_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\},\end{aligned}$$

where $\mathcal{T}_{h/2}$ is obtained by dividing each triangle of $\mathcal{T}_h$ into four triangles and let $P_r(K)$ denote the space of all polynomials of degree less than or equal to r.

Next, we consider the Taylor-Hood elements.

Example 3.2.2. *([12])*

$$\begin{aligned}\mathbf{H}_h &= \{v_h \in \mathbf{H}_0^1 : v_h|_K \in (P_2(K))^2 \;\; \forall K \in \mathcal{T}_h\}\\ L_h &= \{q_h \in L^2(\Omega) : q_h|_K \in P_1(K) \;\; \forall K \in \mathcal{T}_h\}\end{aligned}$$

Finally, we present the $P_2 - P_0$ mixed finite element space.

Example 3.2.3. *([12])*

$$\begin{aligned}&\mathbf{V}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \big(C(\bar{\Omega})\big)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\},\\ &W_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \tau_h\},\end{aligned}$$

3.3 A Priori Estimates of Semidiscrete Solution

This section deals with the derivation of a *priori* bounds for the semidiscrete solution $\mathbf{u}_h$ which will be an important part in our fully discrete analysis for the system of equations (1.2.3)-(1.2.5) (with $\mathbf{f} = \mathbf{0}$). The proofs use definition of the discrete Stokes operator $\tilde{\Delta}_h$ presented in (3.2.7) and proceed along the same lines as in the derivation of Theorem 2.2, but for the sake of completeness, we provide the proofs.

Lemma 3.3. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $\mathbf{u}_{0h} = P_h\mathbf{u}_0$*, and the assumptions* (**A1**)–(**A2**) *hold true. Then, the solution* $\mathbf{u}_h$ *of (3.2.3) satisfies*

$$\|\mathbf{u}_h(t)\|^2 + \kappa\|\nabla\mathbf{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_h(s)\|^2 + \|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2)\,ds$$
$$\le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t} \quad t>0,$$

where $\beta = \nu - 2\alpha(\lambda_1^{-1}+\kappa) > 0$*.*

Proof. Setting $\hat{\mathbf{u}}_h(t) = e^{\alpha t}\mathbf{u}_h(t)$ for some $\alpha \ge 0$, we rewrite (3.2.3) as

$$(\hat{\mathbf{u}}_{ht},\boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h) + \kappa(\nabla\hat{\mathbf{u}}_{ht},\nabla\boldsymbol{\phi}_h) - \kappa\alpha(\nabla\hat{\mathbf{u}}_h,\nabla\boldsymbol{\phi}_h) \tag{3.3.1}$$
$$+ \nu(\nabla\hat{\mathbf{u}}_h,\nabla\boldsymbol{\phi}_h) + e^{-\alpha t}b(\hat{\mathbf{u}}_h,\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h) = 0 \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Choose $\boldsymbol{\phi}_h = \hat{\mathbf{u}}_h$ in (3.3.1). Using (3.2.1), $b(\hat{\mathbf{u}}_h,\hat{\mathbf{u}}_h,\hat{\mathbf{u}}_h) = 0$ and from (2.2.3), we find that

$$\frac{d}{dt}(\|\hat{\mathbf{u}}_h\|^2 + \kappa\|\nabla\hat{\mathbf{u}}_h\|^2) + 2\big(\nu - \alpha\big(\kappa + \frac{1}{\lambda_1}\big)\big)\|\nabla\hat{\mathbf{u}}_h\|^2 \le 0. \tag{3.3.2}$$

Integrate (3.3.2) with respect to time from 0 to t to obtain

$$\|\mathbf{u}_h\|^2 + \kappa\|\nabla\mathbf{u}_h\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h(s)\|^2 ds \le e^{-2\alpha t}(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla\mathbf{u}_{0h}\|^2). \tag{3.3.3}$$

Using the discrete Stokes operator $\tilde{\Delta}_h$, we rewrite (3.3.1) as

$$(\hat{\mathbf{u}}_{ht},\boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h) - \kappa(\tilde{\Delta}_h\hat{\mathbf{u}}_{ht},\boldsymbol{\phi}_h) + \kappa\alpha(\tilde{\Delta}_h\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h) \tag{3.3.4}$$
$$- \nu(\tilde{\Delta}_h\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h) = -e^{-\alpha t}b(\hat{\mathbf{u}}_h,\hat{\mathbf{u}}_h,\boldsymbol{\phi}_h).$$

We note that $-(\hat{\mathbf{u}}_{ht}, \tilde{\Delta}_h \hat{\mathbf{u}}_h) = \frac{1}{2}\frac{d}{dt}\|\nabla \hat{\mathbf{u}}_h\|^2$. With $\boldsymbol{\phi}_h = -\tilde{\Delta}_h \hat{\mathbf{u}}_h$, (3.3.4) becomes

$$\frac{d}{dt}(\|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2) + 2(\nu - \kappa\alpha)\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2 \qquad (3.3.5)$$
$$= 2\alpha\|\nabla \hat{\mathbf{u}}_h\|^2 + 2e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \tilde{\Delta}_h \hat{\mathbf{u}}_h).$$

To estimate the nonlinear term on the right hand side of (3.3.5), a use of (3.2.8) yields

$$|I| = |e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \tilde{\Delta}_h \hat{\mathbf{u}}_h)| \leq C\|\nabla \hat{\mathbf{u}}_h\|^{\frac{3}{2}}\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^{\frac{3}{2}}. \qquad (3.3.6)$$

Applying Young's inequality $ab \leq \frac{a^p}{p\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a, b \geq 0$, $\epsilon > 0$ with $p = 4$ and $q = \frac{4}{3}$, we obtain

$$|I| \leq C\frac{\|\nabla \hat{\mathbf{u}}_h\|^6}{4\epsilon^3} + \frac{3\epsilon}{4}\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2. \qquad (3.3.7)$$

Choosing $\epsilon = \frac{4\nu}{3}$, we find that

$$|I| \leq \frac{C}{4}\left(\frac{3}{4\nu}\right)^3 \|\nabla \hat{\mathbf{u}}_h\|^6 + \nu\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2. \qquad (3.3.8)$$

Substitute (3.3.8) in (3.3.5) to arrive at

$$\frac{d}{dt}(\|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2) + (\nu - 2\alpha\kappa)\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2$$
$$\leq C(\nu)\|\nabla \hat{\mathbf{u}}_h\|^6 + 2\alpha\|\nabla \hat{\mathbf{u}}_h\|^2. \qquad (3.3.9)$$

An integration of (3.3.9) with respect to time from 0 to t yields

$$\|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2 + \beta\int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h(s)\|^2 ds \leq \|\nabla \mathbf{u}_{0h}\|^2 \qquad (3.3.10)$$
$$+ \kappa\|\tilde{\Delta}_h \mathbf{u}_{0h}\|^2 + C(\nu, \alpha)\int_0^t \big(\|\nabla \hat{\mathbf{u}}_h(s)\|^6 ds + \|\nabla \hat{\mathbf{u}}_h(s)\|^2\big) ds.$$

Using (3.3.3), we bound

$$\begin{aligned}
\int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^6 ds &= \int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^4 \|\nabla \hat{\mathbf{u}}_h(s)\|^2 ds \\
&\leq C(\kappa)(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2)^2 \int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^2 ds \\
&\leq C(\kappa, \nu, \alpha, \lambda_1)(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2)^3. \qquad (3.3.11)
\end{aligned}$$

Applying estimates from (3.3.3) and (3.3.11) in (3.3.10) and use stability properties of P_h to obtain

$$\begin{aligned}
&\|\nabla \mathbf{u}_h\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h\|^2 + \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\tilde{\Delta}_h \mathbf{u}_h(s)\|^2 ds \leq \Big(\|\nabla \mathbf{u}_{0h}\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_{0h}\|^2 \\
&\quad + C(\kappa, \nu, \alpha, \lambda_1)\big(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2\big)^3 + C(\kappa, \nu, \alpha, \lambda_1)\big(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2\big) \Big) e^{-2\alpha t} \\
&\quad \leq C(\kappa, \nu, \alpha, \lambda_1, M) e^{-2\alpha t}. \qquad (3.3.12)
\end{aligned}$$

Combine (3.3.3) with (3.3.12) to complete the rest of the proof. □

In the following three lemmas, we derive *a priori* estimates involving time derivatives of the semi-discrete solution.

Lemma 3.4. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$*, such that, for all* $t > 0$*,*

$$\|\mathbf{u}_{ht}(t)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(s)\|^2) ds \leq C e^{-2\alpha t}.$$

Proof. Substituting $\boldsymbol{\phi}_h = \mathbf{u}_{ht}$ in (3.2.3), we obtain

$$\begin{aligned}
\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2 &= -\nu(\nabla \mathbf{u}_h, \nabla \mathbf{u}_{ht}) - b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_{ht}) \\
&= I_1 + I_2, \text{ say.} \qquad (3.3.13)
\end{aligned}$$

To estimate $|I_1|$, we apply Cauchy-Schwarz's inequality and Young's inequality to arrive at

$$|I_1| \leq \frac{\nu}{2\epsilon}\|\nabla \mathbf{u}_h\|^2 + \frac{\epsilon}{2}\|\nabla \mathbf{u}_{ht}\|^2. \tag{3.3.14}$$

Choose $\epsilon = \kappa$ in (3.3.14) to obtain

$$|I_1| \leq C(\nu,\kappa)\|\nabla \mathbf{u}_h\|^2 + \frac{\kappa}{2}\|\nabla \mathbf{u}_{ht}\|^2. \tag{3.3.15}$$

An application of (3.2.8) and Young's inequality yields

$$\begin{aligned}|I_2| &\leq C\|\nabla \mathbf{u}_h\|^{\frac{1}{2}}\|\tilde{\Delta}_h \mathbf{u}_h\|^{\frac{1}{2}}\|\nabla \mathbf{u}_h\|\|\mathbf{u}_{ht}\| \\ &\leq C\|\nabla \mathbf{u}_h\|^3\|\tilde{\Delta}_h \mathbf{u}_h\| + \frac{1}{2}\|\mathbf{u}_{ht}\|^2.\end{aligned} \tag{3.3.16}$$

A use of (3.3.15), (3.3.16) and Lemma 3.3 in (3.3.13) yields

$$\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \tag{3.3.17}$$

Next, substituting $\boldsymbol{\phi}_h = e^{2\alpha t}\mathbf{u}_{ht}$ in (3.2.3), we arrive at

$$e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2) = -\nu e^{2\alpha t}a(\mathbf{u}_h,\mathbf{u}_{ht}) - e^{2\alpha t}b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{u}_{ht}). \tag{3.3.18}$$

Using Cauchy-Schwarz's inequality, (3.2.10), (2.2.3), Young's inequality and integrating from 0 to t with respect to time, we obtain

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(s)\|^2)ds \leq C(\kappa,\nu,\lambda_1)\Big(\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_h(s)\|^2 + \|\nabla \mathbf{u}_h(s)\|^4)ds\Big). \tag{3.3.19}$$

A use of Lemma 3.3 to bound

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h(s)\|^4 ds &\leq C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h(s)\|^2 ds \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}.\end{aligned} \tag{3.3.20}$$

An application of (3.3.20) and Lemma 3.3 in (3.3.19) yields

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}(s)\|^2)ds \leq C(\kappa, \nu, \alpha, \lambda_1, M). \tag{3.3.21}$$

A combination of (3.3.17) and (3.3.21) would lead us to the desired result. □

Lemma 3.5. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$*, such that, for all* $t > 0$*,*

$$\|\mathbf{u}_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Proof. Differentiation of (3.2.3) with respect to time yields

$$\begin{aligned}(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + b(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h)& \\ + b(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) = 0 \ \forall \phi_h \in \mathbf{J}_h \ \ t > 0.&\end{aligned} \tag{3.3.22}$$

Substitute $\boldsymbol{\phi}_h = \mathbf{u}_{htt}$ in (3.3.22) to obtain

$$\|\mathbf{u}_{htt}\|^2 + \kappa\|\nabla\mathbf{u}_{htt}\|^2 = -\nu a(\mathbf{u}_{ht}, \mathbf{u}_{htt}) - b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt}) - b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt}). \tag{3.3.23}$$

An application of Cauchy-Schwarz's inequality, Young's inequality, (3.2.10) and (2.2.3) yields

$$\|\mathbf{u}_{htt}\|^2 + \kappa\|\nabla\mathbf{u}_{htt}\|^2 \leq C(\kappa, \nu, \lambda_1)\big(\|\nabla\mathbf{u}_{ht}\|^2 + \|\nabla\mathbf{u}_h\|^2\|\nabla\mathbf{u}_{ht}\|^2\big). \tag{3.3.24}$$

With the help of estimates obtained from Lemma 3.3 and 3.4, we write

$$\|\mathbf{u}_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(t)\|^2 \leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t}. \tag{3.3.25}$$

Multiply (3.3.24) by $e^{2\alpha t}$ and integrate with respect to time from 0 to t to arrive at

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2+\kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq C(\kappa,\nu,\lambda_1)\Big(\int_0^t e^{2\alpha s}\big(\|\nabla\mathbf{u}_{ht}(s)\|^2 + \|\nabla\mathbf{u}_h(s)\|^2\|\nabla\mathbf{u}_{ht}(s)\|^2\big)ds\Big). \tag{3.3.26}$$

Applying the estimates from Lemmas 3.3 and 3.4, we obtain the desired result, that is,

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2+\kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq C(\kappa,\nu,\alpha,\lambda_1,M). \tag{3.3.27}$$

A use of (3.3.25) and (3.3.27) completes the proof. □

Differentiating (3.3.22) with respect to time and proceeding as in the proofs of Lemmas 3.4 and 3.5, we arrive at following lemma:

Lemma 3.6. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\kappa,\nu,\alpha,\lambda_1,M)$, *such that for all* $t>0$,

$$\|\mathbf{u}_{httt}(t)\|^2+\kappa\|\nabla\mathbf{u}_{httt}(t)\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{httt}(s)\|^2+\kappa\|\nabla\mathbf{u}_{httt}(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Finally, we state below the main results of this chapter, which are related to the optimal error estimates of the velocity and the pressure, the proofs of which are established in the next sections.

Theorem 3.1. *Let the assumptions* (**A1)-(A2)** *and* (**B1)-(B2)** *be satisfied. Further, let the discrete initial velocity* $\mathbf{u}_{0h} = P_h\mathbf{u}_0$. *Then, there exists a positive constant* K *which depends on* κ, ν, λ_1, α *and* M, *such that, for all* $t>0$ *and for* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$, *the following estimate holds true:*

$$\|\mathbf{u}-\mathbf{u}_h\|+h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| \leq Kh^2e^{-\alpha t}.$$

Theorem 3.2. *Under the hypotheses of Theorem 3.1, there exists a positive constant* K

depending on κ, ν, λ_1, α and M, such that, for all $t > 0$, the following holds true:

$$\|(p - p_h)(t)\|_{L^2/N_h} \leq Khe^{-\alpha t}.$$

3.4 Error Estimates for Velocity

In this section, we derive optimal error estimates of the velocity. Since $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$, the weak solution $\mathbf{u}$ satisfies

$$(\mathbf{u}_t, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}, \boldsymbol{\phi}_h) = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) + (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.1}$$

Set $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. Then, from (3.4.1) and (3.2.3), we obtain

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa a(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) = \boldsymbol{\Lambda}(\boldsymbol{\phi}_h) + (p, \nabla \cdot \boldsymbol{\phi}_h), \tag{3.4.2}$$

where $\boldsymbol{\Lambda}(\boldsymbol{\phi}_h) = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h)$. Below, we derive an optimal error estimate of $||\nabla \mathbf{e}(t)||$, for $t > 0$.

Lemma 3.7. *Let assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *be satisfied. With $\mathbf{u}_{0h} = P_h \mathbf{u}_0$, then, there exists a positive constant K depending on λ_1, κ, ν, α and M, such that, for all $t > 0$ and for $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, the following estimate holds true :*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\|^2 + \kappa \|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\|^2 \leq Kh^2 e^{-2\alpha t}.$$

Proof: Choose $\boldsymbol{\phi}_h = e^{\alpha t} P_h \hat{\mathbf{e}} = e^{\alpha t}\big(\hat{\mathbf{e}} + (P_h \hat{\mathbf{u}} - \hat{\mathbf{u}})\big)$ in (3.4.2) to rewrite it as:

$$\begin{aligned}(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) + \kappa a(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) + \nu a(\hat{\mathbf{e}}, \hat{\mathbf{e}}) &= e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}}) + (\hat{p}, \nabla \cdot P_h \hat{\mathbf{e}}) \\ &+ (e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \kappa\, a(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \nu a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}).\end{aligned} \tag{3.4.3}$$

Note that,

$$(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) + \kappa\, a(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{e}}) = \frac{1}{2}\frac{d}{dt}(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2) - \alpha(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2), \tag{3.4.4}$$

and

$$\begin{aligned}(e^{\alpha t}\mathbf{e}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) &= (\hat{\mathbf{e}}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})) - \alpha(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) \\ &= \frac{d}{dt}(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) - (\hat{\mathbf{e}}, \hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t) - \alpha(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}). \end{aligned} \tag{3.4.5}$$

Using (2.2.3), (3.4.4) and (3.4.5) in (3.4.3), we arrive at

$$\begin{aligned}&\frac{d}{dt}(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2) + \big(2\nu - 2\alpha(\kappa + {\lambda_1}^{-1})\big)\,\|\nabla\hat{\mathbf{e}}\|^2 \le 2e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}}) + 2(\hat{p}, \nabla\cdot P_h\hat{\mathbf{e}}) \\ &\quad + 2\frac{d}{dt}\bigg((\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \kappa\, a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\bigg) - 2\bigg((\hat{\mathbf{e}}, \hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t) + \kappa\, a(\hat{\mathbf{e}}, \hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t)\bigg) \\ &\quad - 2\alpha\bigg((\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \kappa\, a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\bigg) + 2\nu a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}). \end{aligned} \tag{3.4.6}$$

Using Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we estimate the last two terms on the right-hand side of (3.4.6) as

$$\begin{aligned}&|2\alpha\big((\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \kappa\, a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\big) + 2\nu a(\hat{\mathbf{e}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})| \\ &\qquad \le C(\alpha, \kappa, \lambda_1, \nu, \epsilon)\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \end{aligned} \tag{3.4.7}$$

Similarly, using Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we can bound

$$2\big|(\hat{\mathbf{e}}, \hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t) + \kappa\, a(\hat{\mathbf{e}}, \hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t)\big| \le C(\kappa, \epsilon)\|\nabla(\hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t)\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \tag{3.4.8}$$

For the second term on the right-hand side of (3.4.6), we use Cauchy-Schwarz's inequality, (3.2.5) and Young's inequality to obtain

$$\begin{aligned}2|(\hat{p}, \nabla\cdot P_h\hat{\mathbf{e}})| &\le 2\|\hat{p} - j_h\hat{p}\|\|\nabla\cdot P_h\hat{\mathbf{e}}\| \le C\|\hat{p} - j_h\hat{p}\|\|\nabla P_h\hat{\mathbf{e}}\| \\ &\le C(\epsilon)\|\hat{p} - j_h\hat{p}\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \end{aligned} \tag{3.4.9}$$

To estimate the first term on the right-hand side of (3.4.6), we rewrite it as

$$2e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}}) = 2e^{-\alpha t}\bigg(b(\hat{\mathbf{e}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}}) - b(\hat{\mathbf{e}},\hat{\mathbf{u}},P_h\hat{\mathbf{e}}) - b(\hat{\mathbf{u}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})\bigg).$$

Using the generalized Hölder's inequality, Agmon's inequality (see [25])

$$\|\mathbf{v}\|_{L^\infty} \leq C\|\nabla\mathbf{v}\|\,\|\tilde{\Delta}\mathbf{v}\|, \quad \mathbf{v}\in\mathbf{H}^2\cap\mathbf{J}_1, \tag{3.4.10}$$

Young's inequality, the Sobolev inequality, (2.2.2) and (3.2.5), we arrive at

$$\begin{aligned}
2e^{-\alpha t}(|b(\hat{\mathbf{u}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})| + |b(\hat{\mathbf{e}},\hat{\mathbf{u}},P_h\hat{\mathbf{e}})|) &\leq 2e^{-\alpha t}\big(\|\hat{\mathbf{u}}\|_{L^\infty}\|\nabla\hat{\mathbf{e}}\|\|P_h\hat{\mathbf{e}}\| + \|\hat{\mathbf{e}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|P_h\hat{\mathbf{e}}\|\big)\\
&\leq 2e^{-\alpha t}\big(\|\nabla\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{e}}\|\|P_h\hat{\mathbf{e}}\| + \|\nabla\hat{\mathbf{e}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|\|\hat{\mathbf{e}}\|\big)\\
&\leq 2e^{-\alpha t}\big(\|\nabla\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{1}{2}} + \|\tilde{\Delta}\hat{\mathbf{u}}\|\big)\|\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{e}}\| \qquad (3.4.11)\\
&\leq Ce^{-2\alpha t}\big(\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\| + \|\tilde{\Delta}\hat{\mathbf{u}}\|^2\big)\|\hat{\mathbf{e}}\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2.
\end{aligned}$$

Moreover, rewrite

$$b(\hat{\mathbf{e}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}}) = -b(\hat{\mathbf{e}},\hat{\mathbf{e}},\hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + b(\hat{\mathbf{e}},\hat{\mathbf{e}},\hat{\mathbf{e}}). \tag{3.4.12}$$

Since the last term on the right hand side of (3.4.12) vanishes because of the antisymmetric property of the trilinear form, we use Lemma 1.2, the genearized Hölder's inequality (1.3.12), the Sobolev inequality (Lemma 1.1), Young's inequality (1.3.9), Lemmas 2.6 and, 3.3 in (3.4.12) to obtain

$$\begin{aligned}
|b(\hat{\mathbf{e}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})| &\leq Ce^{-\alpha t}\|\hat{\mathbf{e}}\|_{L^4}\|\nabla\hat{\mathbf{e}}\|\|\hat{\mathbf{u}} - P_h\hat{\mathbf{u}}\|_{L^4}\\
&\leq Ce^{-\alpha t}\|\nabla\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{e}}\|\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|\\
&\leq C(\|\nabla\hat{\mathbf{u}}\| + \|\nabla\hat{\mathbf{u}}_h\|)\|\nabla\hat{\mathbf{e}}\|\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|\\
&\leq C(\epsilon)\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \qquad (3.4.13)
\end{aligned}$$

Integrating (3.4.6) with respect to time from 0 to t, use bounds (3.4.7)-(3.4.13) with $\epsilon = \frac{2\nu}{5}$, to arrive at

$$\|\hat{\mathbf{e}}(t)\|^2 + \kappa\|\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \leq C(\|\mathbf{e}(0)\|^2 + \|\nabla\mathbf{e}(0)\|^2)$$
$$+ C(\alpha,\kappa,\nu,\lambda_1,M)\Big(\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \int_0^t (\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \|\nabla(\hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t)\|^2$$
$$+ \|\hat{p} - j_h\hat{p}\|^2)ds\Big) + C\int_0^t \big(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2\big)\|\hat{\mathbf{e}}\|^2 ds. \tag{3.4.14}$$

Using (3.2.6) and (**B1**) in (3.4.14), we find that

$$\|\hat{\mathbf{e}}(t)\|^2 + \kappa\|\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \leq Ch^2\Big(\|\mathbf{u}_0\|_2^2 + \|\hat{\mathbf{u}}\|_2^2 + \int_0^t (\|\hat{\mathbf{u}}\|_2^2 + \|\hat{\mathbf{u}}_t\|_2^2 + \|\hat{p}(t)\|^2_{H^1/\mathbb{R}})ds\Big)$$
$$+ C\int_0^t \big(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2\big)\big(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2\big)ds. \tag{3.4.15}$$

Use *a priori* bounds for $\mathbf{u}$, $\mathbf{u}_t$ and p (Theorem 2.2) to bound the first term on the right-hand side of (3.4.15) and then apply the Gronwall's lemma to obtain

$$\|\hat{\mathbf{e}}(t)\|^2 + \kappa\|\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \leq C(\nu,\kappa,\alpha,\lambda_1,M)h^2\exp\Big(\int_0^t (\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|)\, ds\Big).$$

A use of a priori bounds from Lemma 2.7 yields

$$\int_0^t \big(\|\nabla\mathbf{u}\|\ \|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2\big)ds \leq C(M,\kappa,\lambda_1,\nu,\alpha)(1 - e^{-2\alpha t}) \leq C(M,\kappa,\lambda_1,\nu,\alpha) < \infty,$$

and hence, it completes the rest of the proof. □

Note that, Lemma 3.7 provides a suboptimal error estimates for the velocity in $L^\infty(\mathbf{L^2})$-norm. Therefore, in the remaining part of this section we derive an optimal error estimate for the velocity in $L^\infty(\mathbf{L^2})$-norm. We shall achieve this by comparing our solutions with appropriate intermediate solutions and then making use of triangle inequality.

To dissociate the nonlinearity, we first introduce an intermediate solution $\mathbf{v}_h$, which is a finite element Galerkin approximation to a linearized Kelvin-Voigt equation, satisfying

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{v}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi} \in \mathbf{J}_h, \tag{3.4.16}$$

with $\mathbf{v}_h(0) = P_h \mathbf{u}_0$.

Now, we split $\mathbf{e}$ as

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Here, $\boldsymbol{\xi}$ denotes the error due to the approximation using a linearized Kelvin-Voigt equation (3.4.16), whereas $\boldsymbol{\eta}$ represents the error due to the non-linearity in the equation.

Subtracting (3.4.16) from (3.4.1), the equation in $\boldsymbol{\xi}$ can be written as

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) = (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.17}$$

For optimal error estimates of $\boldsymbol{\xi}$ in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms, we again introduce the following auxiliary projection V_h such that $V_h\mathbf{u} : [0, \infty) \to J_h$ satisfying

$$\kappa a(\mathbf{u}_t - V_h\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u} - V_h\mathbf{u}, \boldsymbol{\phi}_h) = (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{3.4.18}$$

where $V_h\mathbf{u}(0) = P_h\mathbf{u}_0$.

With $V_h\mathbf{u}$ defined as above, we now split $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} := (\mathbf{u} - V_h\mathbf{u}) + (V_h\mathbf{u} - \mathbf{v}_h) = \boldsymbol{\zeta} + \boldsymbol{\rho}.$$

To obtain estimates for $\mathbf{e}$, first of all, we derive various estimates of $\boldsymbol{\zeta}$ in Lemmas 3.8, 3.9, 3.10 and 3.11. Then, we proceed to estimate $\|\boldsymbol{\rho}\|$ and $\|\nabla\boldsymbol{\rho}\|$ in Lemma 3.12. Combining these results, we obtain estimates for $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}_0^1)$-norms in Lemma 3.13. Finally, we derive an estimate for $\boldsymbol{\eta}$ to complete the proof of Theorem 3.1.

Lemma 3.8. *Assume that* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *are satisfied. Then, there exists a positive constant* $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ *such that for* $0 \le \alpha < \dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, *the following estimate holds true:*

$$\|\nabla(\mathbf{u}-V_h\mathbf{u})(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}-V_h\mathbf{u})(s)\|^2 ds \le Kh^2 e^{-2\alpha t}.$$

Proof. On multiplying (3.4.18) by $e^{\alpha t}$ with $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$, we find that

$$\kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \nu a(\hat{\boldsymbol{\zeta}}, \boldsymbol{\phi}_h) = (\hat{p}, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.19}$$

Using $e^{\alpha t}\boldsymbol{\zeta}_t = \hat{\boldsymbol{\zeta}}_t - \alpha\hat{\boldsymbol{\zeta}}$ and choosing $\boldsymbol{\phi}_h = P_h\hat{\boldsymbol{\zeta}} = \hat{\boldsymbol{\zeta}} + (P_h\hat{\mathbf{u}} - \hat{\mathbf{u}})$ in (3.4.19), we arrive at

$$\begin{aligned}\kappa\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2(\nu-\kappa\alpha)\|\nabla\hat{\boldsymbol{\zeta}}\|^2 &= 2\kappa\frac{d}{dt}a(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) - 2\kappa\, a(\hat{\boldsymbol{\zeta}}, \frac{d}{dt}(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})) \\ &\quad + 2(\nu-\kappa\alpha)\, a(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + 2(\hat{p} - j_h\hat{p}, \nabla\cdot P_h\hat{\boldsymbol{\zeta}}).\end{aligned} \tag{3.4.20}$$

Integrate (3.4.20) with respect to time from 0 to t and apply (3.2.5) along with Young's inequality to obtain

$$\begin{aligned}\kappa\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + (\nu-\kappa\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2 ds &\le C(\nu,\alpha,\kappa)\Big(e^{2\alpha t}\|\nabla(\mathbf{u}-P_h\mathbf{u})\|^2 + \|\nabla(\mathbf{u}_0 - P_h\mathbf{u}_0)\|^2 \\ &\quad + \int_0^t e^{2\alpha s}\big(\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|^2 ds + \|\nabla(\mathbf{u}-P_h\mathbf{u})\|^2 + \|p - j_h p\|^2\big)ds\Big).\end{aligned} \tag{3.4.21}$$

A use of (3.2.6) with (**B1**) in (3.4.21) yields

$$\begin{aligned}\kappa\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + (\nu-\kappa\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2 ds &\le C(\nu,\alpha,\kappa)h^2\Big(e^{2\alpha t}\|\tilde{\Delta}\mathbf{u}\|^2 + \|\tilde{\Delta}\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s}\|\nabla p\|^2 ds \\ &\quad + \int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\tilde{\Delta}\mathbf{u}\|^2)ds\Big).\end{aligned}$$

We now use a priori bounds for $\mathbf{u}$ and p derived in Lemmas 2.7, 2.11 and 2.12 to complete the proof. □

For the estimation of time derivative, we have the following result.

Lemma 3.9. *Under the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**), *there exists a positive constant* $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ *such that for* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, *the following estimate holds true:*

$$\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}_t(s) - V_h\mathbf{u}_t(s))\|^2 ds \leq Kh^2.$$

Proof. Recall (3.4.19) now with $\boldsymbol{\phi}_h = e^{\alpha t}P_h\boldsymbol{\zeta}_t = e^{\alpha t}\boldsymbol{\zeta}_t + e^{\alpha t}(P_h\mathbf{u}_t - \mathbf{u}_t)$ to find that

$$\begin{aligned} 2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + \nu\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 &= 2\nu\alpha\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2(\hat{p}, e^{\alpha t}\nabla\cdot P_h\boldsymbol{\zeta}_t) \\ &+ 2\kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t)) + 2\nu a(\hat{\boldsymbol{\zeta}}, e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t)). \end{aligned} \tag{3.4.22}$$

An application of the Cauchy-Schwarz inequality, discrete incompressibility condition and (3.2.5) in (3.4.22) yields

$$\begin{aligned} 2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + \nu\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 &\leq 2\nu\alpha\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2\|\hat{p} - j_h\hat{p}\|\|e^{\alpha t}\nabla P_h\boldsymbol{\zeta}_t\| \\ &+ 2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|e^{\alpha t}\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\| + 2\nu\|\nabla\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|. \end{aligned} \tag{3.4.23}$$

Integrating (3.4.23) with respect to time from 0 to t, using Young's inequality, (**B1**) and (3.2.6), Lemmas 2.11, 2.12 and 3.8 and proceeding exactly as in the proof Lemma 3.8, we obtain the desired result. This completes the rest of the proof. □

Below, we discuss the $\mathbf{L^2}$-estimate of $\boldsymbol{\zeta}(t)$.

Lemma 3.10. *Under the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**), *there exists a positive constant* $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ *such that for* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, *the following estimate holds true for* $t > 0$*:*

$$\|\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \leq Kh^4e^{-2\alpha t}.$$

Proof. For L^2 estimate, we recall the Aubin-Nitsche duality argument. Let $(\mathbf{w}, q)$ be the unique solution of the following steady state Stokes system:

$$-\nu\Delta\mathbf{w} + \nabla q = \hat{\boldsymbol{\zeta}} \quad \text{in} \quad \Omega, \tag{3.4.24}$$

$$\nabla\cdot\mathbf{w} = 0 \qquad \text{in} \quad \Omega, \tag{3.4.25}$$

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{3.4.26}$$

From assumption (**A1**), $(\mathbf{w}, q)$ satisfies the following regularity result:

$$\|\mathbf{w}\|_2 + \|q\|_{H^1/\mathbb{R}} \le C\|\hat{\boldsymbol{\zeta}}\|. \tag{3.4.27}$$

Forming L^2-inner product between (3.4.24) and $\hat{\boldsymbol{\zeta}}$ and using discrete incompressibility condition, we obtain

$$\|\hat{\boldsymbol{\zeta}}\|^2 = \nu\, a(\mathbf{w} - P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) - (q - j_h q, \nabla\cdot\hat{\boldsymbol{\zeta}}) + \nu\, a(P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}). \tag{3.4.28}$$

Now, by using (3.4.19) with $\boldsymbol{\phi}_h$ replaced by $P_h\mathbf{w}$ and (3.4.25), the last term in (3.4.28) can be rewritten as

$$\nu\, a(P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) = (\hat{p} - j_h\hat{p}, \nabla\cdot(P_h\mathbf{w} - \mathbf{w})) - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w} - \mathbf{w}) - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w}). \tag{3.4.29}$$

Once again, form L^2-inner product between (3.4.24) and $e^{\alpha t}\boldsymbol{\zeta}_t$, and use this in the last term of (3.4.29) to obtain

$$\kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w}) = \frac{\kappa}{\nu}(\hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\zeta}}_t) - \frac{\alpha\kappa}{\nu}\|\hat{\boldsymbol{\zeta}}\|^2 + \frac{\kappa}{\nu}(q - j_h q, \nabla\cdot e^{\alpha t}\boldsymbol{\zeta}_t). \tag{3.4.30}$$

Substituting (3.4.29) and (3.4.30) in (3.4.28), we obtain

$$\begin{aligned}\|\hat{\boldsymbol{\zeta}}\|^2 + \frac{\kappa}{\nu}\frac{d}{dt}\|\hat{\boldsymbol{\zeta}}\|^2 = \frac{\alpha\kappa}{\nu}\|\hat{\boldsymbol{\zeta}}\|^2 + \nu a(\mathbf{w} - P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) - (q - j_h q, \nabla\cdot\hat{\boldsymbol{\zeta}}) + (\hat{p} - j_h\hat{p}, \nabla\cdot(P_h\mathbf{w} - \mathbf{w}))\\ - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w} - \mathbf{w}) - \frac{\kappa}{\nu}(q - j_h q, e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t).\end{aligned} \tag{3.4.31}$$

Integrate (3.4.31) with respect to time from 0 to t, use (3.2.5) and then apply Cauchy Schwarz's inequality to yield

$$(\nu-\alpha\kappa)\int_0^t \|\hat{\boldsymbol{\zeta}}\|^2 ds+\kappa\|\hat{\boldsymbol{\zeta}}\|^2 \le C(\kappa,\nu,\alpha)\bigg(\|\boldsymbol{\zeta}(0)\|^2+\int_0^t \big(\|\nabla(\mathbf{w}-P_h\mathbf{w})\|\|\nabla\hat{\boldsymbol{\zeta}}\| \\ +\|q-j_hq\|\|\nabla\hat{\boldsymbol{\zeta}}\|+\|\hat{p}-j_h\hat{p}\|\|\nabla(P_h\mathbf{w}-\mathbf{w})\| \\ +\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla(P_h\mathbf{w}-\mathbf{w})\|+\|q-j_hq\|\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\big)ds\bigg).$$

By using (**B1**), (3.2.6) and (3.4.27), we arrive at

$$(\nu-\alpha\kappa)\int_0^t \|\hat{\boldsymbol{\zeta}}\|^2 ds+\kappa\|\hat{\boldsymbol{\zeta}}\|^2 \le C(\kappa,\nu,\alpha)\bigg(h^4\|\tilde{\Delta}\mathbf{u}_0\|^2 \\ +h\int_0^t \big(\|\nabla\hat{\boldsymbol{\zeta}}\|+h\|\nabla\hat{p}\|+\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\big)\|\hat{\boldsymbol{\zeta}}\|ds\bigg). \tag{3.4.32}$$

Since $0\le\alpha<\dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, $(\nu-\alpha\kappa)>0$. Then use Young's inequality appropriately and the estimates from Lemmas 2.12, 3.8 and 3.9 to complete the rest of the proof. □

Lemma 3.11. *Under the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**)*, there exists a positive constant* $K=K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0\le\alpha<\dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$*, the following holds true:*

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}_t(s)\|^2 ds\le Kh^4.$$

The above lemma can be proved in an exactly similar fashion as the proof of Lemma 3.10 with the right hand side of (3.4.24) replaced by $e^{\alpha t}\boldsymbol{\zeta}_t$. but for completeness, we provide a short proof.

Proof. For obtaining the desired estimate of $\boldsymbol{\zeta}_t$, once again we appeal to the Aubin-Nitche's duality argument. Now recall the equation (3.4.19)

$$\kappa a(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h)+\nu a(\boldsymbol{\zeta},\boldsymbol{\phi}_h)=(p,\nabla\cdot\boldsymbol{\phi}_h)\quad\forall\boldsymbol{\phi}_h\in\mathbf{J}_h.$$

In (3.4.24), set $e^{\alpha t}\boldsymbol{\zeta}_t$ in stead of $\hat{\boldsymbol{\zeta}}$ on its right hand side and then form $\mathbf{L^2}$-inner product with $e^{\alpha t}\boldsymbol{\zeta}_t$ to obtain

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 = \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t, q) + \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w}).$$

From (3.4.19) with $\boldsymbol{\phi}_h = e^{\alpha t}P_h\mathbf{w}$, it now follows in a similar manner as in the $\mathbf{L^2}$-estimate of $\boldsymbol{\zeta}$ that

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &= \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (q - j_h q, e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t) - \nu a(e^{\alpha t}\boldsymbol{\zeta}, \mathbf{w})\\ &\quad + e^{\alpha t}(p - j_h p, \nabla\cdot(P_h\mathbf{w} - \mathbf{w})) - \nu a(e^{\alpha t}\boldsymbol{\zeta}, P_h\mathbf{w} - \mathbf{w}). \qquad (3.4.33)\end{aligned}$$

Using (3.4.24) with $\hat{\boldsymbol{\zeta}}$ replaced by $e^{\alpha t}\boldsymbol{\zeta}_t$ in the third term of (3.4.33) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \le& C(\nu, \lambda_1, \alpha, \kappa, M)\big[\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla(\mathbf{w} - P_h\mathbf{w})\| + \|q - j_h q\|\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\| + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\|\\ &+ \|\nabla\hat{\boldsymbol{\zeta}}\|\|\mathbf{w} - P_h\mathbf{w}\| + \|\hat{p} - j_h\hat{p}\|\|\nabla(\mathbf{w} - P_h\mathbf{w})\| + \|q - j_h q\|\|\nabla\hat{\boldsymbol{\zeta}}\|\big]. \qquad (3.4.34)\end{aligned}$$

A use of (3.2.5) with (**B1**) yields

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \le& C(\nu, \lambda_1, \alpha, \kappa, M)\Big(h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\Delta\mathbf{w}\| + h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla q\| + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\|\\ &+ h\|\nabla\hat{\boldsymbol{\zeta}}\|\|\Delta\mathbf{w}\| + h^2\|\nabla\hat{p}\|\|\Delta\mathbf{w}\| + h\|\nabla q\|\|\nabla\hat{\boldsymbol{\zeta}}\|\Big). \qquad (3.4.35)\end{aligned}$$

Using regularity result (3.4.27) now with right hand side $\|e^{\alpha t}\boldsymbol{\zeta}_t\|$, we arrive at

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \le& C(\nu, \lambda_1, \alpha, \kappa, M)\Big(\big(h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\| + h\|\nabla\hat{\boldsymbol{\zeta}}\| \qquad (3.4.36)\\ &+ h^2\|\nabla\hat{p}\| + h\|\nabla\hat{\boldsymbol{\zeta}}\|)\|e^{\alpha t}\boldsymbol{\zeta}_t\|\big) + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\|\Big).\end{aligned}$$

An application of Young's inequality yields

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \le C(\nu,\lambda_1,\alpha,\kappa,M)\big(h^2\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + h^2\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + h^4\|\nabla\hat{p}\|^2 + \|\hat{\boldsymbol{\zeta}}\|^2\big). \tag{3.4.37}$$

Integrating (3.4.37) with respect to time from 0 to t, we obtain

$$\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 ds \le C(\nu,\lambda_1,\alpha,\kappa,M)\bigg(\int_0^t \big(h^2\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + h^2\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + h^4\|\nabla\hat{p}\|^2 + \|\hat{\boldsymbol{\zeta}}\|^2\big)ds\bigg). \tag{3.4.38}$$

A use of Lemmas 3.8, 3.9, 3.10 and 2.12 would lead us to

$$\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 ds \le C(\nu,\lambda_1,\alpha,\kappa,M)h^4, \tag{3.4.39}$$

and this completes the rest of the proof. □

Since $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{\rho}$ and the estimates of $\boldsymbol{\zeta}$ are already known, it suffices to derive estimate of $\boldsymbol{\rho}$ to obtain an estimate for $\boldsymbol{\xi}$.

Lemma 3.12. *Under the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**)*, there exists a positive constant* $K = K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0 \le \alpha < \dfrac{\nu}{2(1+\kappa\lambda_1)}$*, the following estimate holds true:*

$$(\|\boldsymbol{\rho}\|^2 + \kappa\|\nabla\boldsymbol{\rho}\|^2) + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\rho}(s)\|^2 ds \le C(\nu,\lambda_1,\alpha,\kappa,M)h^4 e^{-2\alpha t}.$$

Proof. Subtracting (3.4.18) from (3.4.17), we find that

$$(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h) + \nu a(\boldsymbol{\rho},\boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.40}$$

Replace $\boldsymbol{\phi}_h$ by $e^{\alpha t}\hat{\boldsymbol{\rho}}$ in (3.4.40) to obtain

$$(e^{\alpha t}\boldsymbol{\rho}_t,\hat{\boldsymbol{\rho}}) + \kappa\, a(e^{\alpha t}\boldsymbol{\rho}_t,\hat{\boldsymbol{\rho}}) + \nu\|\nabla\hat{\boldsymbol{\rho}}\|^2 = -(e^{\alpha t}\boldsymbol{\zeta}_t,\hat{\boldsymbol{\rho}}) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.41}$$

A use of Cauchy-Schwarz's inequality, (2.2.3) along with Young's inequality in (3.4.41) yields

$$\frac{d}{dt}(\|\hat{\boldsymbol{\rho}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2) + 2\beta\|\nabla\hat{\boldsymbol{\rho}}\|^2 \leq C(\kappa,\alpha,\lambda_1)\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2. \tag{3.4.42}$$

Integrating (3.4.42) with respect to time from 0 to t, we arrive at

$$\|\hat{\boldsymbol{\rho}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2 + 2\beta\int_0^t \|\nabla\hat{\boldsymbol{\rho}}\|^2 ds \leq C(\kappa,\alpha,\lambda_1)\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 ds. \tag{3.4.43}$$

The desired result follows after a use of Lemma 3.11 in (3.4.43). □

We now derive an estimate of $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}_0^1)$-norms.

Lemma 3.13. *Let the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *be satisfied. Then, there exists a positive constant* $K = K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, *the following estimate holds true:*

$$\|\boldsymbol{\xi}(t)\|^2 + \kappa\|\nabla\boldsymbol{\xi}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\xi}(s)\|^2 ds \leq C(\nu,\lambda_1,\alpha,\kappa,M)h^4 e^{-2\alpha t}.$$

A use of the triangle inequality together with the Lemmas 3.8, 3.10 and 3.12 would provide us the result. Now, we derive the proof of the main Theorem 3.1.

Proof of Theorem 3.1.

Since $e = \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v_h}) + (\mathbf{v_h} - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}$ and the estimate of $\boldsymbol{\xi}$ is known from Lemma 3.13, we are left only with the estimate for $\boldsymbol{\eta}$. Subtracting (3.4.16) from (3.2.3), we obtain

$$(\boldsymbol{\eta}_t,\boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\eta}_t,\boldsymbol{\phi}_h) + \nu a(\boldsymbol{\eta},\boldsymbol{\phi}_h) = b(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h) - b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.4.44}$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t}\boldsymbol{\eta}$ and use (2.2.3) to find that

$$\frac{1}{2}\frac{d}{dt}(\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2) + \big(\nu - \alpha(\kappa + \frac{1}{\lambda_1})\big)\|\nabla\hat{\boldsymbol{\eta}}\|^2 = e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}), \tag{3.4.45}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h).$$

To estimate the right hand side term of (3.4.45), we note that

$$e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}) = e^{-\alpha t}\big(- b(\hat{\mathbf{e}}, \hat{\mathbf{u}}_h, \hat{\boldsymbol{\eta}}) + b(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}, \hat{\mathbf{e}})\big).$$

A use of Hölder's inequality with (2.2.3), Agmon's inequality (3.4.10), and the discrete Sobolev inequality (see Lemma 4.4 in [43]) yields

$$\begin{aligned} e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| &\le Ce^{-\alpha t}\big(\|\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{u}}_h\|_{L^6}\|\hat{\boldsymbol{\eta}}\|_{L^3} + \|\hat{\mathbf{u}}\|_{L^\infty}\|\nabla\hat{\boldsymbol{\eta}}\|\|\hat{\mathbf{e}}\|\big) \\ &\le C\big(e^{-\alpha t}\|\tilde{\Delta}_h\mathbf{u}_h\|\|\nabla\hat{\boldsymbol{\eta}}\|\|\hat{\mathbf{e}}\| + \|\nabla\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\nabla\hat{\boldsymbol{\eta}}\|\|\ \hat{\mathbf{e}}\|\big) \\ &\le C\big(\epsilon)e^{-2\alpha t}(\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 + \|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|\|)\big)\|\hat{\mathbf{e}}\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \end{aligned}$$

Since $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$, we obtain

$$e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| \le C(\epsilon)e^{-2\alpha t}(\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 + \|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|)(\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2) + \epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \quad (3.4.46)$$

Using (3.4.46) in (3.4.45), we arrive at

$$\begin{aligned} \frac{d}{dt}(\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2) + (\beta+\nu)\|\nabla\hat{\boldsymbol{\eta}}\|^2 &\le C(\epsilon)e^{-2\alpha t}\big((\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2)\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 \\ &+ (\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2)\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|\big) + 2\epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \quad (3.4.47) \end{aligned}$$

With choice of $\epsilon = \frac{\nu}{2}$, integration of (3.4.47) with respect to time from 0 to t yields

$$\begin{aligned} \|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2 ds &\le C(\nu)\big(\int_0^t \|\hat{\boldsymbol{\xi}}\|^2(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds \\ &+ \int_0^t \|\hat{\boldsymbol{\eta}}\|^2(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds\big). \quad (3.4.48) \end{aligned}$$

A use of Lemmas 2.7, 3.3 and 3.13 in the first term of the right side of (3.4.48) to obtain

$$\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2 ds \le C(\nu, \lambda_1, \alpha, \kappa, M)h^4 e^{-2\alpha t}$$
$$+ \int_0^t \|\hat{\boldsymbol{\eta}}\|^2(\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds. \qquad (3.4.49)$$

An application of Gronwall's lemma yields

$$\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}(s)\|^2\, ds \le Kh^4 \exp\left(\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)\, ds\right) \qquad (3.4.50)$$

Once again, with the help of Lemmas 2.7 and 3.3, we obtain

$$\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)\, ds \le K(\kappa, \nu, \alpha, \lambda_1, M)(1 - e^{-2\alpha t}) \le K. \qquad (3.4.51)$$

Using (3.4.51) in (3.4.50), we derive estimate for $\boldsymbol{\eta}$ as

$$\|\eta\|^2 + \kappa\|\nabla\boldsymbol{\eta}\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\eta}(s)\|^2\, ds \le Kh^4 e^{-2\alpha t}. \qquad (3.4.52)$$

A use of triangle inequality along with (3.4.52) and Lemma 3.13 completes the optimal $L^\infty(\mathbf{L^2})$-estimate of the velocity. For the rest part of proof of Theorem 3.1, we now appeal to Lemma 3.7 to complete the proof. □

3.5 Error Estimate for Pressure

In this section, we derive optimal error estimates for the Galerkin approximation p_h of the pressure p. The main result Theorem 3.2 follows from Lemmas 3.14, 3.15 and the approximation property for j_h.

From (**B2**), we note that

$$\|(j_h p - p_h)(t)\|_{L^2/N_h} \le C \sup_{\boldsymbol{\phi}_h \in \mathbf{H}_h/\{0\}} \left\{\frac{(j_h p - p_h, \nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\}. \qquad (3.5.1)$$

Using (3.5.1), we obtain

$$
\begin{aligned}
\|(j_hp-p_h)(t)\|_{L^2/N_h} &\le C \sup_{\boldsymbol{\phi}_h\in \mathbf{H}_h/\{0\}} \left\{ \frac{(j_hp-p,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|} + \frac{(p-p_h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|} \right\} \\
&\le C\Bigg(\|j_hp-p\| + \sup_{\boldsymbol{\phi}_h\in \mathbf{H}_h/\{0\}} \left\{ \frac{(p-p_h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|} \right\} \Bigg). \qquad (3.5.2)
\end{aligned}
$$

Since the estimate of the first term on the right hand side of (3.5.2) follows from (**B1**), it is sufficient to estimate the second term. Subtracting (3.2.4) from (3.4.1), we find that

$$
(p-p_h,\nabla\cdot\boldsymbol{\phi}_h) = (\mathbf{e}_t,\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t,\boldsymbol{\phi}_h) + \nu a(\mathbf{e},\boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,
$$

where

$$
-\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_h) - b(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h) = -b(\mathbf{e},\mathbf{e},\boldsymbol{\phi}_h) + b(\mathbf{u},\mathbf{e},\boldsymbol{\phi}_h) + b(\mathbf{e},\mathbf{u},\boldsymbol{\phi}_h).
$$

Using the generalized Hölder's inequality (1.3.12), Sobolev's inequality (Lemma 1.1) and Lemma 3.7, we obtain

$$
|\Lambda_h(\boldsymbol{\phi}_h)| \le C(\|\nabla\mathbf{u}\| + \|\mathbf{e}\|_{\mathbf{L}^4})\|\nabla\mathbf{e}\|\|\nabla\boldsymbol{\phi}_h\| \le C\|\nabla\mathbf{e}\|\|\nabla\boldsymbol{\phi}_h\|. \qquad (3.5.3)
$$

Thus,

$$
(p-p_h,\nabla\cdot\boldsymbol{\phi}_h) \le C(\nu,\kappa)\big(\|\mathbf{e}_t\| + \|\nabla\mathbf{e}_t\| + \|\nabla\mathbf{e}\|\big)\|\nabla\boldsymbol{\phi}_h\|.
$$

The results obtained can be stated as:

Lemma 3.14. *For all $t>0$, the semidiscrete Galerkin approximation p_h of the pressure p satisfies*

$$
\|(p-p_h)(t)\|_{L^2/N_h} \le C\big(\|\mathbf{e}_t\| + \|\nabla\mathbf{e}_t\| + \|\nabla\mathbf{e}\|\big). \qquad (3.5.4)
$$

From Lemma 3.7, the estimate $\|\nabla\mathbf{e}\|$ is known. We now derive bounds for $\|\mathbf{e}_t\|$ and $\|\nabla\mathbf{e}_t\|$.

Lemma 3.15. *For all $t > 0$ and for $0 \le \alpha < \dfrac{\nu\lambda_1}{2\big(1+\kappa\lambda_1\big)}$, the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ in the velocity satisfies*

$$\|\mathbf{e}_t(t)\|^2 + \kappa\|\nabla\mathbf{e}_t(t)\|^2 \le Ch^2e^{-2\alpha t}. \tag{3.5.5}$$

Proof. From (3.2.3) and (3.4.1), we obtain

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) = \Lambda_h(\boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h),\ \ \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.5.6}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h).$$

Choosing $\boldsymbol{\phi}_h = P_h\mathbf{e}_t = \mathbf{e}_t + (P_h\mathbf{u}_t - \mathbf{u}_t)$ in (3.5.6), we arrive at

$$\begin{aligned}(\mathbf{e}_t, \mathbf{e}_t) + \kappa\, a(\mathbf{e}_t, \mathbf{e}_t) &= -\nu a(\mathbf{e}, \mathbf{e}_t) + \Lambda_h(P_h\mathbf{e}_t) + (p, \nabla\cdot P_h\mathbf{e}_t)\\ &\quad + (\mathbf{e}_t, \mathbf{u}_t - P_h\mathbf{u}_t) + \kappa\, a(\mathbf{e}_t, \mathbf{u}_t - P_h\mathbf{u}_t) + \nu a(\mathbf{e}, \mathbf{u}_t - P_h\mathbf{u}_t).\end{aligned} \tag{3.5.7}$$

In order to estimate $(p, \nabla\cdot P_h\mathbf{e}_t)$, a use of the discrete incompressible condition with (3.2.5) yields

$$|(p, \nabla\cdot P_h\mathbf{e}_t)| = |(p - j_hp, \nabla\cdot P_h\mathbf{e}_t)| \le \|p - j_hp\|\|\nabla\mathbf{e}_t\|. \tag{3.5.8}$$

Now using Cauchy-Schwarz's inequality in (3.5.7), we arrive at

$$\begin{aligned}\|\mathbf{e}_t\|^2 + \kappa\|\nabla\mathbf{e}_t\|^2 &\le \nu\|\nabla\mathbf{e}\|\|\nabla\mathbf{e}_t\| + |\Lambda_h(P_h\mathbf{e}_t)| + \|p - j_hp\|\|\nabla.(P_h\mathbf{e}_t)\|\\ &\quad + \|\mathbf{e}_t\|\|\mathbf{u}_t - P_h\mathbf{u}_t\| + \kappa\|\nabla\mathbf{e}_t\|\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\| + \nu\|\nabla\mathbf{e}\|\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|.\end{aligned} \tag{3.5.9}$$

Using (3.5.3) and (3.2.5), we obtain

$$|\Lambda_h(P_h\mathbf{e}_t)| \le C\|\nabla\mathbf{e}\|\|\nabla\mathbf{e}_t\|. \tag{3.5.10}$$

Substitute (3.5.8), (3.5.10) in (3.5.9) and use Young's inequality to arrive at

$$\|\mathbf{e}_t\|^2 + \kappa\|\nabla\mathbf{e}_t\|^2 \leq C(\nu,\kappa)\big(\|\nabla\mathbf{e}\|^2 + (\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|^2 + \|p - j_hp\|^2 + \|\mathbf{u}_t - P_h\mathbf{u}_t\|^2)\big).$$

Using (3.2.6) and (**B1**), we now obtain

$$\|\mathbf{e}_t\|^2 + \kappa\|\nabla\mathbf{e}_t\|^2 \leq C(\nu,\kappa)\big(\|\nabla\mathbf{e}\|^2 + h^2(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\nabla p\|^2 + \|\nabla\mathbf{u}_t\|^2)\big)$$

An application of Lemma 2.10, 2.12 and Lemma 3.7 would lead us to the result. This completes the rest of the proof. □

Proof of Theorem 3.2. The proof of Theorem 3.2 now follows from Lemma 3.15 and the approximation property (**B1**) of j_h. □

Chapter 4

Fully Discrete Schemes

4.1 Introduction

In Chapter 3, only semidiscrete approximations for (1.2.3)-(1.2.5) (with $\mathbf{f} = 0$) are discussed, keeping the time variable continuous. In this chapter, we have considered temporal discretization of the semidiscrete Galerkin approximations (3.2.2)-(3.2.3) with two fully discrete schemes: first order backward Euler method and second order backward difference scheme. After establishing the wellposedness of discrete solutions using a *priori* bounds, we prove error estimates which involves energy technique. We also analyse briefly, the proof of linearized backward Euler scheme applied to (3.2.2)-(3.2.3) for time discretization.
Before proceeding to define the backward Euler and second order backward difference approximations to the semidiscrete solution of (3.2.2) (or 3.2.3), we would like to present a glimpse of the literature involving fully discrete approximations of the viscoelastic fluid problem. We refer to [4], [35], [89], [99], [100], [101]. for the time discretization of viscoelastic model of Oldroyd type.
Interestingly, there is hardly any work devoted to the time discretization of (1.2.3)-(1.2.5). In [21] and [81], authors have discussed only semidiscrete approximations for (1.2.3)-(1.2.5), keeping the time variable continuous. In this chapter, we have discussed both backward Euler method and two step backward difference scheme for the time discretization and

have derived optimal error estimates. We have also discussed briefly, the proof of linearized backward Euler method applied to (1.2.3)-(1.2.5) (with $\mathbf{f} = 0$) for time discretization. More precisely, we have

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the backward Euler or linearized backward Euler method.

Further, we have proved the following result for a second order backward difference scheme:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k^2) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k^{2-\gamma}),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the second order backward difference scheme and

$$\gamma = \begin{cases} 0 & \text{if } n \geq 2; \\ 1 & \text{if } n = 1. \end{cases}$$

The remaining part of this chapter includes the following sections: In Section **4.2**, we deal with a *priori* bounds of discrete solution of the backward Euler method. Then, with the help of these estimates, we establish the existence and uniqueness of fully discrete backward Euler approximations and derive the optimal error estimates. In Section **4.3**, we introduce the linearized backward Euler method and provide a brief error analysis. Section **4.4** considers the derivation of a *priori* bounds for the fully discrete backward difference approximations which in turn help us in proving the error estimates. In the final Section

4.5, we work out a few computational results to support our theoretical estimates.

4.2 Backward Euler Method

In this section, we consider a backward Euler method for time discretization of the finite element Galerkin approximation (3.2.2).

From now on, let us assume that $\{t_n\}_{n=0}^N$ be a uniform partition of $[0,T]$, and $t_n = nk$, with time step $k > 0$. For a sequence $\{\boldsymbol{\phi}^n\}_{n\geq 0} \in \mathbf{J}_h$ defined on $[0,T]$, set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$ and $\bar{\partial}_t \boldsymbol{\phi}^n = \frac{(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1})}{k}$.

The backward Euler method applied to (3.2.2) determines a sequence of functions $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P^n\}_{n\geq 1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = (P^n, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \tag{4.2.1}$$

$$(\nabla \cdot \mathbf{U}^n, \chi_h) = 0 \qquad \forall \chi_h \in L_h,$$

$$\mathbf{U}^0 = \mathbf{u}_{0h}.$$

Equivalently, for $\boldsymbol{\phi}_h \in \mathbf{J}_h$, we seek $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{4.2.2}$$

$$\mathbf{U}^0 = \mathbf{u}_{0h}.$$

To study the issue of the existence and uniqueness of the discrete solutions $\{\mathbf{U}^n\}_{n\geq 1}$, we derive a *priori* bounds for the solution $\{\mathbf{U}^n\}_{n\geq 1}$.

Lemma 4.1. *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, *choose* k_0 *such that for* $0 < k \leq k_0$,

$$\frac{\nu k \lambda_1}{\kappa\lambda_1 + 1} + 1 > e^{\alpha k}. \tag{4.2.3}$$

Then the discrete solution $\mathbf{U}^N$, $N \geq 1$ *of (4.2.2) satisfies*

$$(\|\mathbf{U}^N\|^2+\kappa\|\nabla\mathbf{U}^N\|^2)+2\beta_1 e^{-2\alpha t_N}\ k\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2 \le e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2+\kappa\|\nabla\mathbf{U}^0\|^2),$$

where

$$\beta_1=\Bigg(e^{-\alpha k}\nu-2\big(\frac{1-e^{-\alpha k}}{k}\Big)\big(\kappa+\frac{1}{\lambda_1})\Bigg)>0. \tag{4.2.4}$$

Proof. Multiplying (4.2.2) by $e^{\alpha t_n}$ and setting $\hat{\mathbf{U}}^n=e^{\alpha t_n}\mathbf{U}^n$, we obtain

$$\begin{aligned} e^{\alpha t_n}\Bigg((\bar{\partial}_t\mathbf{U}^n,\boldsymbol{\phi}_h)+\kappa a(\bar{\partial}_t\mathbf{U}^n,\boldsymbol{\phi}_h)\Bigg)+\nu a(\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)\\ +e^{-\alpha t_n}b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)=0\ \ \forall\boldsymbol{\phi}_h\in\mathbf{J}_h. \end{aligned} \tag{4.2.5}$$

Note that,

$$e^{\alpha t_n}\bar{\partial}_t\mathbf{U}^n=e^{\alpha k}\bar{\partial}_t\hat{\mathbf{U}}^n-\Bigg(\frac{e^{\alpha k}-1}{k}\Bigg)\hat{\mathbf{U}}^n. \tag{4.2.6}$$

Using (4.2.6) in (4.2.5) and multiplying the resulting equation by $e^{-\alpha k}$, we obtain

$$\begin{aligned} (\bar{\partial}_t\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)+\kappa a(\bar{\partial}_t\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)-\Bigg(\frac{1-e^{-\alpha k}}{k}\Bigg)(\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)+e^{-\alpha k}\nu a(\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)\\ -\kappa\Bigg(\frac{1-e^{-\alpha k}}{k}\Bigg)a(\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)+e^{-\alpha t_{n+1}}b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)=0. \end{aligned} \tag{4.2.7}$$

Note that,

$$(\bar{\partial}_t\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n)\ge\frac{1}{2k}(\|\hat{\mathbf{U}}^n\|^2-\|\hat{\mathbf{U}}^{n-1}\|^2)=\frac{1}{2}\bar{\partial}_t\|\hat{\mathbf{U}}^n\|^2. \tag{4.2.8}$$

Substituting $\boldsymbol{\phi}_h=\hat{\mathbf{U}}^n$ in (4.2.7) and using (2.2.3) along with (3.2.1) yields

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2)+\Bigg(e^{-\alpha k}\nu-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1})\Bigg)\|\nabla\hat{\mathbf{U}}^n\|^2\le 0. \tag{4.2.9}$$

Note that, the coefficient of the second term on the left hand side is greater than β_1. With

$0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, choose $k_0 > 0$ such that for $0 < k \leq k_0$

$$\frac{\nu k\lambda_1}{1+\kappa\lambda_1} + 1 > e^{\alpha k}.$$

Then, for $0 < k \leq k_0$, the coefficient β_1 (see (4.2.4)) of the second term on the left hand side of (4.2.9) becomes positive. Multiplying (4.2.9) by $2k$ and summing over $n = 1$ to N, we obtain

$$\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + 2\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2. \tag{4.2.10}$$

Divide (4.2.10) by $e^{2\alpha t_N}$ to complete the rest of the proof. □

Now, we are ready to prove the following existence and uniqueness result.

Theorem 4.1. *Given $\mathbf{U}^{n-1}$, the discrete problem (4.2.2) has a unique solution $\mathbf{U}^n$, $n \geq 1$.*

Proof. Given $\mathbf{U}^{n-1}$, define a function $\mathbb{F} : \mathbf{J}_h \to \mathbf{J}_h$ for a fixed $'n'$ by

$$\begin{aligned}(\mathbb{F}(\mathbf{v}), \boldsymbol{\phi}_h) &= (\mathbf{v}, \boldsymbol{\phi}_h) + \kappa(\nabla\mathbf{v}, \nabla\boldsymbol{\phi}_h) + k\nu(\nabla\mathbf{v}, \nabla\boldsymbol{\phi}_h) \\ &\quad + k\ b(\mathbf{v}, \mathbf{v}, \boldsymbol{\phi}_h) - (\mathbf{U}^{n-1}, \boldsymbol{\phi}_h) - \kappa(\nabla\mathbf{U}^{n-1}, \nabla\boldsymbol{\phi}_h).\end{aligned} \tag{4.2.11}$$

Define a norm on $\mathbf{J}_h$ as

$$|||\mathbf{v}||| = (\|\mathbf{v}\|^2 + \kappa\|\nabla\mathbf{v}\|^2)^{\frac{1}{2}}. \tag{4.2.12}$$

We can easily show that $\mathbb{F}$ is continuous. Now, after substituting $\boldsymbol{\phi}_h = \mathbf{v}$ in (4.2.11), we use (3.2.1), (4.2.12), Cauchy-Schwarz's inequality and Young's inequality to arrive at

$$(\mathbb{F}(\mathbf{v}), \mathbf{v}) \geq (|||\mathbf{v}||| - |||\mathbf{U}^{n-1}|||)|||\mathbf{v}|||.$$

Choose R such that $|||\mathbf{v}||| = R$ and $R - |||\mathbf{U}^{n-1}||| > 0$ and hence,

$$(\mathbb{F}(\mathbf{v}), \mathbf{v}) > 0.$$

A use of Theorem 1.3 would provide us the existence of $\{\mathbf{U}^n\}_{n\geq 1}$.

Now, to prove uniqueness, set $\mathbf{E}^n = \mathbf{U}_1^n - \mathbf{U}_2^n$, where $\mathbf{U}_1^n$ and $\mathbf{U}_2^n$ are the solutions of (4.2.2). Note that,

$$(\bar{\partial}_t \mathbf{E}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{E}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{E}^n, \boldsymbol{\phi}_h) = b(\mathbf{U}_2^n, \mathbf{U}_2^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_1^n, \mathbf{U}_1^n, \boldsymbol{\phi}_h). \tag{4.2.13}$$

Using $\boldsymbol{\phi}_h = \hat{\mathbf{E}}^n$ and proceeding as in the derivation of (4.2.9), we obtain

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{E}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^n\|^2) + \left(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\right)\|\nabla\hat{\mathbf{E}}^n\|^2 \leq e^{-\alpha k} e^{\alpha t_n}\Lambda_1^n(\hat{\mathbf{E}}^n), \tag{4.2.14}$$

where,

$$\Lambda_1^n(\hat{\mathbf{E}}^n) = -b(\mathbf{U}_1^n, \mathbf{U}_1^n, \hat{\mathbf{E}}^n) + b(\mathbf{U}_2^n, \mathbf{U}_2^n, \hat{\mathbf{E}}^n).$$

Note that,

$$\begin{aligned} e^{\alpha t_n}\Lambda_1^n(\hat{\mathbf{E}}^n) &= e^{-\alpha t_n}|b(\hat{\mathbf{U}}_1^n, \hat{\mathbf{U}}_1^n, \hat{\mathbf{E}}^n) - b(\hat{\mathbf{U}}_2^n, \hat{\mathbf{U}}_2^n, \hat{\mathbf{E}}^n)| \\ &= e^{-\alpha t_n}|b(\hat{\mathbf{E}}^n, \hat{\mathbf{U}}_1^n, \hat{\mathbf{E}}^n) + b(\hat{\mathbf{U}}_2^n, \hat{\mathbf{E}}^n, \hat{\mathbf{E}}^n)|. \end{aligned} \tag{4.2.15}$$

A use of (3.2.1), (3.2.10) and (2.2.3) in (4.2.15) yields

$$\begin{aligned} e^{\alpha t_n}|\Lambda_1^n(\hat{\mathbf{E}}^n)| &\leq C(\lambda_1)e^{-\alpha t_n}\|\hat{\mathbf{E}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{E}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}_1^n\|\|\nabla\hat{\mathbf{E}}^n\| \\ &\leq C(\lambda_1)e^{-\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|\|\nabla\hat{\mathbf{E}}^n\|^2. \end{aligned} \tag{4.2.16}$$

Using (4.2.16), $\mathbf{E}^0 = 0$, Young's inequality in (4.2.14), multiplying by $2k$, summing over $n = 1$ to N and applying the bounds of Lemma 4.1, we arrive at

$$
\begin{aligned}
\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2 \leq & C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N-1} e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|^2\|\nabla\hat{\mathbf{E}}^n\|^2 \\
& + C(\nu,\lambda_1)ke^{-\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{U}}_1^N\|^2\|\nabla\hat{\mathbf{E}}^N\|^2. \qquad (4.2.17)
\end{aligned}
$$

From (4.2.17), we note that

$$
\begin{aligned}
\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2 \leq & C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=0}^{N-1} e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|^2\|\nabla\hat{\mathbf{E}}^n\|^2 \\
& + C(\nu,\lambda_1,\kappa,M)ke^{-\alpha k}(\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2). \qquad (4.2.18)
\end{aligned}
$$

Since, $(1 - C(\nu,\lambda_1,\kappa,M)ke^{-\alpha k})$ can be made positive for small k, an application of the discrete Gronwall's lemma and Lemma 4.1 in (4.2.18) yields

$$
\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2 \leq 0 \qquad (4.2.19)
$$

and this provides the uniqueness of the solutions $\{\mathbf{U}^n\}_{n\geq 1}$. $\square$

Next, we obtain the $\mathbf{H}^1$ and L^2- norm estimates for the error $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$ and the L^2- norm estimate for the error $\rho^n = P^n - p_h(t_n) = P^n - p_h^n$. The following theorem provides a bound on the error $\mathbf{e}^n$:

Theorem 4.2. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \leq k_0$, *(4.2.3) is satisfied. For some fixed* $h > 0$, *let* $\mathbf{u}_h(t)$ *satisfies (3.2.3). Then, there exists a positive constant* C, *independent of* k, *such that for* $n = 1, 2, \cdots, N$

$$
\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + \beta_1 ke^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_i\|^2 \leq Ck^2e^{-2\alpha t_n} \qquad (4.2.20)
$$

and

$$
\|\bar{\partial}_t\mathbf{e}^n\|^2 + \|\bar{\partial}_t\nabla\mathbf{e}^n\|^2 \leq Ck^2e^{-2\alpha t_n}. \qquad (4.2.21)
$$

Proof. Consider (3.2.3) at $t = t_n$ and subtract it from (4.2.2) to obtain

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) \tag{4.2.22}$$
$$= (\sigma_1^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_1^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,$$

where $\sigma_1^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$ and $\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h)$. Multiplying (4.2.22) by $e^{\alpha t_n}$, we arrive at

$$(e^{\alpha t_n}\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n}\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) \tag{4.2.23}$$
$$= (e^{\alpha t_n}\sigma_1^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n}\sigma_1^n, \boldsymbol{\phi}_h) + e^{\alpha t_n}\Lambda_h(\boldsymbol{\phi}_h).$$

Note that,

$$e^{\alpha t_n}\bar{\partial}_t \mathbf{e}^n = e^{\alpha k}\bar{\partial}_t \hat{\mathbf{e}}^n - (\frac{e^{\alpha k}-1}{k})\hat{\mathbf{e}}^n. \tag{4.2.24}$$

Using (4.2.24) in (4.2.23) and dividing the resulting equation by $e^{\alpha k}$, we obtain

$$(\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - (\frac{1-e^{-\alpha k}}{k})(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - (\frac{1-e^{-\alpha k}}{k})\kappa a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) \tag{4.2.25}$$
$$+ \nu e^{-\alpha k} a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) = e^{-\alpha k}(e^{\alpha t_n}\sigma_1^n, \boldsymbol{\phi}_h) + e^{-\alpha k}\kappa a(e^{\alpha t_n}\sigma_1^n, \boldsymbol{\phi}_h) + e^{-\alpha k}e^{\alpha t_n}\Lambda_h(\boldsymbol{\phi}_h).$$

Substitute $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ in (4.2.25). A use of (2.2.3) yields

$$\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2\big) + \bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^n\|^2 \tag{4.2.26}$$
$$= e^{-\alpha k}(e^{\alpha t_n}\sigma_1^n, \hat{\mathbf{e}}^n) + e^{-\alpha k}\kappa a(e^{\alpha t_n}\sigma_1^n, \hat{\mathbf{e}}^n) + e^{-\alpha k}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n).$$

On multiplying (4.2.26) by $2k$ and summing over $n = 1$ to N, we observe that

$$\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + 2k\bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \tag{4.2.27}$$
$$\le 2ke^{-\alpha k}\sum_{n=1}^{N}(e^{\alpha t_n}\sigma_1^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k}\sum_{n=1}^{N}\kappa a(e^{\alpha t_n}\sigma_1^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n)$$
$$= I_1^N + I_2^N + I_3^N, \text{ say.}$$

Using Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we estimate I_1^N as:

$$\begin{aligned} |I_1^N| &\leq 2ke^{-\alpha k}\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma_1^n\|\|\hat{\mathbf{e}}^n\| \\ &\leq C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma_1^n\|^2 + \frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \end{aligned} \tag{4.2.28}$$

Now, using the Taylor series expansion of $\mathbf{u}_h$ around t_n in the interval (t_{n-1}, t_n), we observe that

$$\|e^{\alpha t_n}\sigma_1^n\|^2 \leq e^{2\alpha t_n}\frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}(t_n - s)\|\mathbf{u}_{htt}(s)\|ds\right)^2. \tag{4.2.29}$$

An application of Cauchy-Schwarz's inequality in (4.2.29) yields

$$\begin{aligned} \|e^{\alpha t_n}\sigma_1^n\|^2 &\leq \frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2ds\right)\left(\int_{t_{n-1}}^{t_n}(t_n - s)^2ds\right) \\ &= \frac{k}{3}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(t)\|^2\,dt \end{aligned} \tag{4.2.30}$$

and hence, using (4.2.30), we write

$$k\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma_1^n\|^2 \leq \frac{k^2}{3}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2\,ds. \tag{4.2.31}$$

With the help of Lemma 3.5, we note that

$$\begin{aligned} k\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma_1^n\|^2 &\leq \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_{n-1}}\|\mathbf{u}_{htt}(s)\|^2\,ds \\ &\leq \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|^2\,ds \\ &= \frac{k^2}{3}e^{2\alpha k}\int_{0}^{t_N}e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|^2\,ds \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-2\alpha t_{N-1}}. \end{aligned} \tag{4.2.32}$$

Similarly, we obtain

$$k\sum_{n=1}^{N}\|e^{\alpha t_n}\nabla\sigma_1^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-2\alpha t_{N-1}}. \tag{4.2.33}$$

Using (4.2.32) in (4.2.28), we find that

$$|I_1^N| \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + \frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{4.2.34}$$

Following the similar steps as for bounding $|I_1^N|$ and using (4.2.33), we obtain

$$|I_2^N| \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + \frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{4.2.35}$$

To estimate I_3^N, we note that

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &= b(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n-\mathbf{u}_h^n,\mathbf{U}^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_h^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &= -b(\mathbf{u}_h^n,\mathbf{e}^n,\boldsymbol{\phi}_h) - b(\mathbf{e}^n,\mathbf{U}^n,\boldsymbol{\phi}_h).\end{aligned} \tag{4.2.36}$$

Hence, we find that

$$e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| = e^{-\alpha t_n}\big| - b(\hat{\mathbf{e}}^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)\big|. \tag{4.2.37}$$

The first term of (4.2.36) vanish because of (3.2.1). A use of (3.2.10) in (4.2.37) yields

$$\begin{aligned}e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| &\le Ce^{-\alpha t_n}\|\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n\|\|\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\\ &\le Ce^{-\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|\|\hat{\mathbf{e}}^n\|\|\nabla\hat{\mathbf{e}}^n\|.\end{aligned} \tag{4.2.38}$$

Using Young's inequality, we arrive at

$$
\begin{aligned}
|I_3^N| \leq & C(\nu)\sum_{n=1}^{N-1} ke^{-\alpha k} e^{-2\alpha t_n} \|\nabla \hat{\mathbf{U}}^n\|^2 \ \|\hat{\mathbf{e}}^n\|^2 + C(\nu) ke^{-\alpha k} e^{-2\alpha t_N} \|\nabla \hat{\mathbf{U}}^N\|^2 \ \|\hat{\mathbf{e}}^N\|^2 \\
& + \frac{\nu}{3} ke^{-\alpha k} \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (4.2.39)
\end{aligned}
$$

An application of Lemma 4.1 to estimate the second term on the right hand side of (4.2.39) yields

$$
\begin{aligned}
|I_3^N| \leq C(\nu) & \sum_{n=1}^{N-1} ke^{-\alpha k} e^{-2\alpha t_n} \|\nabla \hat{\mathbf{U}}^n\|^2 \ \|\hat{\mathbf{e}}^n\|^2 + C(\nu,\kappa,M) ke^{-\alpha k} e^{-2\alpha t_N} \ \|\hat{\mathbf{e}}^N\|^2 \\
& + \frac{\nu}{3} ke^{-\alpha k} \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (4.2.40)
\end{aligned}
$$

A use of (4.2.34), (4.2.35) and (4.2.40) in (4.2.27) with $\mathbf{e}^0 = 0$ yields

$$
\begin{aligned}
& \|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M) k^2 \qquad (4.2.41) \\
& \quad + C(\nu) ke^{-\alpha k} \sum_{n=0}^{N-1} e^{-2\alpha t_n} \|\nabla \hat{\mathbf{U}}^n\|^2 \|\hat{\mathbf{e}}^n\|^2 + C(\nu,\kappa,M) ke^{-\alpha k} (\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2).
\end{aligned}
$$

Now choose $k_0 > 0$ such that for $0 < k < k_0$, $(1 - C(\nu,\kappa,M)ke^{-\alpha k}) > 0$ and (4.2.3) is satisfied. Then, an application of the discrete Gronwall's lemma yields

$$
\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M) k^2 \exp\left(k \sum_{n=0}^{N-1} \|\nabla \hat{\mathbf{U}}^n\|^2 \right). \qquad (4.2.42)
$$

With the help of Lemma 4.1, we bound

$$
k \sum_{n=0}^{N-1} \|\nabla \hat{\mathbf{U}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M). \qquad (4.2.43)
$$

Using (4.2.43) in (4.2.42), we arrive at

$$
\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M) k^2. \qquad (4.2.44)
$$

For $0 < k \leq k_0$, the coefficient of the third term on the left-hand side of (4.2.44), becomes positive. Dividing (4.2.44) by $e^{2\alpha t_N}$, we obtain (4.2.20).

Next, we take $\boldsymbol{\phi}_h = \bar{\partial}_t \mathbf{e}^n$ in (4.2.22) and obtain

$$\|\bar{\partial}_t \mathbf{e}^n\|^2 + \kappa \|\nabla \bar{\partial}_t \mathbf{e}^n\|^2 = -\nu a(\mathbf{e}^n, \bar{\partial}_t \mathbf{e}^n) + (\sigma_1^n, \bar{\partial}_t \mathbf{e}^n) + \kappa a(\sigma_1^n, \bar{\partial}_t \mathbf{e}^n) + \Lambda_h(\bar{\partial}_t \mathbf{e}^n). \quad (4.2.45)$$

Using (4.2.36), (3.2.9), (3.2.10) and (2.2.3), we observe that

$$\begin{aligned} |\Lambda_h(\boldsymbol{\phi}_h)| &= |b(\mathbf{e}^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{e}^n, \boldsymbol{\phi}_h)| \\ &\leq C(\lambda_1) \Big(\|\nabla \mathbf{u}_h^n\|^{\frac{1}{2}} \|\tilde{\Delta}_h \mathbf{u}_h^n\|^{\frac{1}{2}} + \|\mathbf{U}^n\|^{\frac{1}{2}} \|\nabla \mathbf{U}^n\|^{\frac{1}{2}} \Big) \|\nabla \mathbf{e}^n\| \|\nabla \boldsymbol{\phi}_h\|. \end{aligned}$$

With the help of Lemmas 3.3 and 4.1, we bound

$$|\Lambda_h(\boldsymbol{\phi}_h)| \leq C(\kappa, \nu, \alpha, \lambda_1, M) \|\nabla \mathbf{e}^n\| \|\nabla \boldsymbol{\phi}_h\|. \quad (4.2.46)$$

A use of Cauchy-Schwarz's inequality, Young's inequality and (4.2.46) in (4.2.45) yields

$$\|\bar{\partial}_t \mathbf{e}^n\|^2 + \kappa \|\bar{\partial}_t \nabla \mathbf{e}^n\|^2 \leq C(\kappa, \alpha, \nu, \lambda_1, M) \Big(\|\nabla \mathbf{e}^n\|^2 + \|\nabla \sigma_1^n\|^2 \Big). \quad (4.2.47)$$

To estimate the second term in the right hand side of (4.2.47), we note from (4.2.30) and Lemma 3.5 that

$$\begin{aligned} \|e^{\alpha t_n} \nabla \sigma_1^n\|^2 &\leq \frac{k}{3} \int_{t_{n-1}}^{t_n} e^{2\alpha t_n} \|\nabla \mathbf{u}_{htt}(s)\|^2 \, ds \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M) k e^{2\alpha t_n} \int_{t_{n-1}}^{t_n} e^{-2\alpha s} ds \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M) k^2 e^{2\alpha k^*}, \end{aligned} \quad (4.2.48)$$

for $k^* \in (0, k)$. In view of (4.2.20) and (4.2.48), (4.2.47) implies (4.2.21). This completes the rest of the proof. $\square$.

It remains to prove the error estimate for the pressure P^n. Consider (3.2.2) at $t = t_n$ and subtract it from (4.2.1) to obtain

$$(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h) = (\bar{\partial}_t\mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) - (\sigma_1^n, \boldsymbol{\phi}_h) - \kappa a(\sigma_1^n, \boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h).$$

Using Cauchy-Schwarz's inequality, (2.2.3) and (4.2.46), we obtain

$$(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h) \le C(\kappa,\nu,\lambda_1)\big(\|\bar{\partial}_t\nabla\mathbf{e}^n\| + \|\nabla\mathbf{e}^n\| + \|\nabla\sigma_1^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{4.2.49}$$

A use of Theorem 4.2 and (4.2.48) in (4.2.49) would lead us to the desired result, that is

$$\|\boldsymbol{\rho}^n\| \le C(\kappa,\nu,\alpha,\lambda_1,M)k\ e^{-\alpha t_n}. \tag{4.2.50}$$

Remark 4.2.1. *Note that in the estimate of I_3^N, that is, the estimate (4.2.40), we have used Lemma 4.1 to bound only $\|\hat{\mathbf{U}}^N\|$ for the second term on the right hand side of (4.2.40). But we could have bounded $\|\hat{\mathbf{U}}^n\|,\ n = 1,\cdots,N-1$ using again Lemma 4.1, but that would have resulted in exponential dependence of CT in the final estimate.*

4.3 Linearized Backward Euler Method

The backward Euler method applied to (3.2.2) gives rise to a non linear system at $t = t_n$. Here, we introduce a linearized version of this method which solves a system of linear equations at each time step. The linearized backward Euler method is as follows: find a sequence of functions $\{\mathbf{U}^n\}_{n\ge1} \in \mathbf{H}_h$ and $\{P^n\}_{n\ge1} \in L_h$ as solutions of the following recursive linear algebraic equations:

$$(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = (P^n, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h, \tag{4.3.1}$$

$$(\nabla\cdot\mathbf{U}^n, \chi_h) = 0 \qquad \forall\chi_h \in L_h,$$

$$\mathbf{U}^0 = \mathbf{u}_{0h}.$$

Equivalently, for $\boldsymbol{\phi}_h \in \mathbf{J}_h$, we seek $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{4.3.2}$$
$$\mathbf{U}^0 = \mathbf{u}_{0h}.$$

The proof for the linearized backward Euler method proceeds along the same lines as in the derivation of Theorem 4.2. Here, the equation in error $\mathbf{e}^n$ is:

$$\begin{aligned}(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) = (\sigma_1^n, \boldsymbol{\phi}_h) \\ + \kappa a(\sigma_1^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,\end{aligned} \tag{4.3.3}$$

where $\sigma_1^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$ and $\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)$. Note that, the difference here is only in the nonlinear term.

Again, with the help of similar applications as in the proof of Theorem 4.2, we arrive at

$$\begin{aligned}&\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + 2\Big(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big) k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \\ &\quad \leq 2ke^{-\alpha k} \sum_{n=1}^{N} (e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k} \sum_{n=1}^{N} \kappa a(e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n) \\ &\quad = I_1^N + I_2^N + I_3^N, \text{ say.}\end{aligned} \tag{4.3.4}$$

The first two terms in the right hand side of (4.3.4) are bounded by (4.2.34) and (4.2.35). Hence, we need to estimate the third term. In this case, we write

$$\begin{aligned}e^{\alpha t_n} |\Lambda_h(\boldsymbol{\phi}_h)| =& e^{\alpha t_n} |b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1} - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ =& e^{\alpha t_n} |b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ =& e^{\alpha t_n} | - b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ & - b(\mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h)|.\end{aligned} \tag{4.3.5}$$

A use of (3.2.1) along with (2.2.3) and (3.2.10) in (4.3.5) with $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ yields

$$\begin{aligned} e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| \le & e^{\alpha t_n}|b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \hat{\mathbf{e}}^n) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \hat{\mathbf{e}}^n)| \\ \le & C(\lambda_1)e^{\alpha t_n}\big(\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|\|\nabla\mathbf{U}^n\|\|\nabla\hat{\mathbf{e}}^n\| \\ & + \|\nabla\mathbf{e}^{n-1}\|\|\nabla\mathbf{U}^n\|\|\nabla\hat{\mathbf{e}}^n\|\big). \end{aligned} \tag{4.3.6}$$

Hence, we observe that

$$\begin{aligned} |I_3^N| \le \quad & 2ke^{-\alpha k}\sum_{n=1}^N e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| \le C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^N\big(e^{\alpha t_n}\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| \times \\ & \|\nabla\mathbf{U}^n\|\|\nabla\hat{\mathbf{e}}^n\| + e^{\alpha t_n}\|\nabla\mathbf{U}^n\|\|\nabla\mathbf{e}^{n-1}\|\|\nabla\hat{\mathbf{e}}^n\|\big) = |I_4^N| + |I_5^N|, \text{ say.} \end{aligned} \tag{4.3.7}$$

Note that, a use of Taylor's series expansion of $u_h(t)$ at t_n in the interval (t_{n-1}, t_n) yields

$$\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| = \|\int_{t_{n-1}}^{t_n}\nabla\mathbf{u}_{ht}(s)ds\|. \tag{4.3.8}$$

With the help of Lemma 3.4 and mean value theorem, we observe that

$$\begin{aligned} \|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| \le & \int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{ht}(s)\|\,ds \\ \le & C(\kappa,\nu,\alpha,\lambda_1,M)\int_{t_{n-1}}^{t_n}e^{-\alpha s}ds \\ \le & C(\kappa,\nu,\alpha,\lambda_1,M)e^{-\alpha t_n}\frac{1}{\alpha}\left(e^{\alpha k} - 1\right) \\ \le & C(\kappa,\nu,\alpha,\lambda_1,M)ke^{\alpha k^*}, \end{aligned} \tag{4.3.9}$$

where $k^* \in (0, k)$.

Using Young's inequality, we bound $|I_4^N|$ as

$$|I_4^N| \le C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^N e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2 + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^N\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{4.3.10}$$

With the help of (4.3.9) and Lemma 4.1 in (4.3.10), we observe that

$$\begin{aligned}|I_4^N| &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\big(k\sum_{n=1}^{N}e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2\big)+\frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2+\frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (4.3.11)\end{aligned}$$

A use of Young's inequality yields

$$\begin{aligned}|I_5^N| &= C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\|\nabla\mathbf{U}^n\|\|\nabla\mathbf{e}^{n-1}\|\|\nabla\hat{\mathbf{e}}^n\|\\ &\le C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{-2\alpha t_{n-1}}\|\nabla\hat{\mathbf{U}}^n\|^2\|\nabla\hat{\mathbf{e}}^{n-1}\|^2+\frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\ &\le C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=0}^{N-1}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^{n+1}\|^2\|\nabla\hat{\mathbf{e}}^n\|^2+\frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (4.3.12)\end{aligned}$$

Substitute (4.2.34)-(4.2.35) and (4.3.11)-(4.3.12) in (4.3.4). As in the estimate of (4.2.41), we now apply Gronwall's lemma to complete the rest of the proof. □

Now a use of Theorems 3.1, 3.2, 4.2 and (4.2.50) completes the proof of the following Theorem.

Theorem 4.3. *Under the assumptions of Theorems 3.1 and 4.2, the following holds true:*

$$\|\mathbf{u}(t_n)-\mathbf{U}^n\|_j \le Ce^{-\alpha t_n}(h^{2-j}+k) \quad j=0,\,1$$

and

$$\|(p(t_n)-P^n)\| \le Ce^{-\alpha t_n}(h+k).$$

4.4 Second Order Backward Difference Scheme

Since the backward Euler method is only first order accurate, we now try to obtain a second order accuracy by employing a second order backward difference scheme. Setting

$$D_t^{(2)}\mathbf{U}^n = \frac{1}{2k}(3\mathbf{U}^n - 4\mathbf{U}^{n-1} + \mathbf{U}^{n-2}), \tag{4.4.1}$$

we obtain the second order backward difference applied to (3.2.2) as follows: find a sequence of functions $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P^n\}_{n\geq 1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$\begin{aligned} (D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ = (P^n, \nabla \cdot \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) \\ = (P^1, \nabla \cdot \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{U}^n, \chi_h) = 0 \, \forall \chi_h \in L_h, \\ \mathbf{U}^0 = \mathbf{u}_{0h}. \end{aligned} \tag{4.4.2}$$

Equivalently, find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ to be the solutions of

$$\begin{aligned} (D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall n \geq 2 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) = 0 \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \\ \mathbf{U}^0 = \mathbf{u}_{0h}. \end{aligned} \tag{4.4.3}$$

The results of this section are based on the identity which is obtained by a modification of a similar identity in [2]:

$$2e^{2\alpha t_n}(a^n, 3a^n - 4a^{n-1} + a^{n-2}) = \|\hat{a}^n\|^2 - \|\hat{a}^{n-1}\|^2 + (1 - e^{2\alpha k})(\|\hat{a}^n\|^2 + \|\hat{a}^{n-1}\|^2) \quad (4.4.4)$$
$$+ \|\delta^2 \hat{a}^{n-1}\|^2 + \|2\hat{a}^n - e^{\alpha k}\hat{a}^{n-1}\|^2 - \|2\hat{a}^{n-1} - e^{\alpha k}\hat{a}^{n-2}\|^2,$$

where

$$\delta^2 \hat{a}^{n-1} = e^{\alpha k}\hat{a}^n - 2\hat{a}^{n-1} + e^{\alpha k}\hat{a}^{n-2}.$$

Next, we discuss the decay properties for the solution of (4.4.3).

Lemma 4.2. *With* $0 \leq \alpha < \dfrac{\nu \lambda_1}{2(1 + \lambda_1 \kappa)}$*, choose* k_0 *small so that for* $0 < k \leq k_0$

$$\frac{\nu k \lambda_1}{\kappa \lambda_1 + 1} + 1 > e^{2\alpha k}. \quad (4.4.5)$$

Then, the discrete solution $\mathbf{U}^N$*,* $N \geq 1$ *of (4.4.3) satisfies the following a priori bound:*

$$(\|\mathbf{U}^N\|^2 + \kappa\|\nabla \mathbf{U}^N\|^2) + e^{-2\alpha t_N}\ k\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla \mathbf{U}^n\|^2 \leq C(\kappa, \nu, \alpha, \lambda_1)e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla \mathbf{U}^0\|^2).$$

Proof. Multiply (4.4.3) by $e^{\alpha t_n}$ and substitute $\boldsymbol{\phi}_h = \hat{\mathbf{U}}^n$. Then, using identity (4.4.4), we obtain

$$\begin{aligned}
&\frac{1}{4}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla \hat{\mathbf{U}}^n\|^2) + \nu\|\nabla \hat{\mathbf{U}}^n\|^2 + \left(\frac{1 - e^{2\alpha k}}{4k}\right)\Big(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla \hat{\mathbf{U}}^n\|^2\Big) \\
&\quad + \left(\frac{1 - e^{2\alpha k}}{4k}\right)\Big(\|\hat{\mathbf{U}}^{n-1}\|^2 + \kappa\|\nabla \hat{\mathbf{U}}^{n-1}\|^2\Big) + \frac{1}{4k}\|\delta^2 \hat{\mathbf{U}}^{n-1}\|^2 + \frac{1}{4k}\kappa\|\delta^2 \nabla \hat{\mathbf{U}}^{n-1}\|^2 \\
&\quad + \frac{1}{4k}\Big((2\hat{\mathbf{U}}^n - e^{\alpha k}\hat{\mathbf{U}}^{n-1})^2 - (2\hat{\mathbf{U}}^{n-1} - e^{\alpha k}\hat{\mathbf{U}}^{n-2})^2\Big) + \frac{\kappa}{4k}\big((2\nabla \hat{\mathbf{U}}^n - e^{\alpha k}\nabla \hat{\mathbf{U}}^{n-1})^2 \\
&\quad - (2\nabla \hat{\mathbf{U}}^{n-1} - e^{\alpha k}\nabla \hat{\mathbf{U}}^{n-2})^2\big) = 0. \qquad (4.4.6)
\end{aligned}$$

Note that, the fifth and sixth terms on the left hand side of (4.4.6) are non-negative. Therefore, we have dropped these terms. Further, observe that

$$\sum_{n=2}^{N}(\|\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{n-1}\|^2)=(\|\hat{\mathbf{U}}^{1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{1}\|^2)-(\|\hat{\mathbf{U}}^{N}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{N}\|^2)$$
$$+\sum_{n=2}^{N}(\|\hat{\mathbf{U}}^{n}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{n}\|^2). \tag{4.4.7}$$

Multiplying (4.4.6) by $4ke^{-2\alpha k}$, summing over $n=2$ to N, using (2.2.3) and (4.4.7), we obtain

$$\|\hat{\mathbf{U}}^{N}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{N}\|^2+k\Big(4\nu e^{-2\alpha k}-2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}^{n}\|^2$$
$$+\|2e^{-\alpha k}\hat{\mathbf{U}}^{N}-\hat{\mathbf{U}}^{N-1}\|^2+\kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{N}-\nabla\hat{\mathbf{U}}^{N-1}\|^2$$
$$\leq\big(\|\hat{\mathbf{U}}^{1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{1}\|^2\big)+\big((2e^{-\alpha k}\hat{\mathbf{U}}^{1}-\mathbf{U}^0)^2+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{1}-\nabla\mathbf{U}^0)^2\big). \tag{4.4.8}$$

To estimate the first term on the right hand side, we choose $n=1$ in (4.2.9) to obtain

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^{1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{1}\|^2)+\Big(e^{-\alpha k}\nu-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{U}}^{1}\|^2\leq 0. \tag{4.4.9}$$

Since $\dfrac{\nu k\lambda_1}{1+\kappa\lambda_1}+1>e^{2\alpha k}$, the coefficient of second term becomes positive. Therefore, we drop this term and obtain

$$\|\hat{\mathbf{U}}^{1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{1}\|^2\leq C(\|\mathbf{U}^0\|^2+\kappa\|\nabla\mathbf{U}^0\|^2). \tag{4.4.10}$$

Now, we bound the second term on the right hand side of (4.4.8) by using Cauchy Schwarz's inequality, Young's inequality and (4.4.10) as follows:

$$(2e^{-\alpha k}\hat{\mathbf{U}}^{1}-\mathbf{U}^0)^2+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{1}-\nabla\mathbf{U}^0)^2\leq C(\kappa)(\|\mathbf{U}^0\|^2+\|\nabla\mathbf{U}^0\|^2). \tag{4.4.11}$$

Using (4.4.10), (4.4.11) and (4.4.5) in (4.4.8), we complete the rest of the proof. □

Remark 4.4.1. *Existence of solution to (4.4.3) can be proved using the Brouwer fixed point theorem and Lemma 4.2.*

As a consequence of Lemma 4.2, we have the following error estimates.

Theorem 4.4. *Assume that* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and choose* $k_0 \geq 0$ *such that for* $0 < k \leq k_0$, *(4.4.5) is satisfied. Let* $u_h(t)$ *be a solution of (3.2.3) and* $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n)$, *for* $n = 1, 2, \cdots, N$. *Then, for some positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$, *there hold*

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2 \leq Ck^4 e^{-2\alpha t_n} \tag{4.4.12}$$

and for $n = 2, \cdots, N$,

$$\|D_t^2\mathbf{e}^n\|^2 + \kappa\|D_t^2\nabla\mathbf{e}^n\|^2 \leq Ck^4 e^{-2\alpha t_n}. \tag{4.4.13}$$

Proof. The proof for error analysis is on the similar lines as that of Theorem 4.2. This time the equation in $\mathbf{e}^n$ for $n \geq 2$ is

$$(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) = (\sigma_2^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_2^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h), \tag{4.4.14}$$

where σ_2^n and $\Lambda_h(\boldsymbol{\phi}_h)$ are defined by

$$\sigma_2^n = \mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n,\ \Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h).$$

Multiply (4.4.14) by $4ke^{\alpha t_n}$ and substitute $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$. Using identity (4.4.4), we arrive at

$$\begin{aligned} & k\bar{\partial}_t(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + \|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2 + 4k\nu\|\nabla\hat{\mathbf{e}}^n\|^2 \\ & \quad + (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^{n-1}\|^2) \\ & \quad + (2\hat{\mathbf{e}}^n - e^{\alpha k}\hat{\mathbf{e}}^{n-1})^2 - (2\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\hat{\mathbf{e}}^{n-2})^2 + \kappa(2\nabla\hat{\mathbf{e}}^n - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-1})^2 \\ & \quad - \kappa(2\nabla\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-2})^2 \\ & = 4k(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa\, a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\, e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n). \end{aligned} \tag{4.4.15}$$

Summing (4.4.15) over $n = 2$ to N, using (4.4.7), $\mathbf{e}^0 = 0$ and dividing by $e^{2\alpha k}$, we obtain

$$\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2) + (2e^{-\alpha k}\hat{\mathbf{e}}^N - \hat{\mathbf{e}}^{N-1})^2 \\
&\quad + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N - \nabla\hat{\mathbf{e}}^{N-1})^2 + k\left(4\nu e^{-2\alpha k} - 2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\
\leq &\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 + (2e^{-\alpha k}\hat{\mathbf{e}}^1 - \mathbf{e}^o)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^1 - \nabla\mathbf{e}^o)^2 \\
&\quad + 4ke^{-2\alpha k}\sum_{n=2}^{N}(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa e^{-2\alpha k}\sum_{n=2}^{N} a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4ke^{-2\alpha k}\sum_{n=2}^{N} e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n) \\
\leq &C(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2) + I_1^* + I_2^* + I_3^*, \ \textit{say}.
\end{aligned} \tag{4.4.16}$$

Now, with the help of Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we bound $|I_1^*|$ as:

$$\begin{aligned}
|I_1^*| &\leq 4ke^{-2\alpha k}(\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2)^{\frac{1}{2}}(\sum_{n=2}^{N}\|\hat{\mathbf{e}}^n\|^2)^{\frac{1}{2}} \\
&\leq C(\epsilon,\lambda_1)ke^{-2\alpha k}\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.
\end{aligned} \tag{4.4.17}$$

Using $\|\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n\| \leq \frac{(k)^{\frac{3}{2}}}{\sqrt{2}}\int_{t_{n-2}}^{t_n}\|\mathbf{u}_{httt}\|dt$ ([2]), we note that

$$\|e^{\alpha t_n}\sigma_2^n\|^2 \leq \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|^2\,dt. \tag{4.4.18}$$

From (4.4.18), we obtain

$$\begin{aligned}
k\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 &\leq \frac{k^4}{2}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|^2 dt \\
&= \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_{n-2}}\|\mathbf{u}_{httt}(t)\|^2 dt.
\end{aligned} \tag{4.4.19}$$

An application of Lemma 3.6 in (4.4.19) yields

$$
\begin{aligned}
k\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 &\leq \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|^2 dt \\
&\leq k^4 e^{4\alpha k}\int_0^{t_N} e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|^2 dt \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k}e^{-2\alpha t_N}. \qquad (4.4.20)
\end{aligned}
$$

Using (4.4.20) in (4.4.17), we arrive at

$$
|I_1^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (4.4.21)
$$

Similarly, with the help of Cauchy-Schwarz's inequality, Young's inequality and Lemma 3.6, we bound

$$
|I_2^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (4.4.22)
$$

Once again, a use of (4.2.38) yields

$$
\begin{aligned}
|I_3^*| \leq & C(\epsilon)\sum_{n=2}^{N} k e^{-2\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2 \qquad (4.4.23)\\
&+ \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.
\end{aligned}
$$

To bound the first term in the right hand side of (4.4.16), we choose $n=1$ in (4.2.26) and obtain

$$
\begin{aligned}
&\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2\big) + \bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^1\|^2 \qquad (4.4.24)\\
&= e^{-\alpha k}(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + e^{-\alpha k}\kappa a(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + e^{-\alpha k}e^{\alpha k}\Lambda_h(\hat{\mathbf{e}}^1).
\end{aligned}
$$

On multiplying (4.4.24) by $2k$, using Cauchy-Schwarz's inequality, (2.2.3), Young's inequality appropriately with the estimates (4.2.40) (for $n=1$ and $\epsilon=\nu$), we obtain

$$
\begin{aligned}
&\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2+2k\Big(\nu e^{-\alpha k}-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{e}}^1\|^2 \qquad (4.4.25)\\
\le & 2ke^{-\alpha k}(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1)+2ke^{-\alpha k}\kappa a(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1)+2ke^{-\alpha k}e^{\alpha k}\Lambda_h(\hat{\mathbf{e}}^1)\\
\le & Ck^2e^{-2\alpha k}\big(\|e^{\alpha k}\sigma_1^1\|^2+\kappa\|e^{\alpha k}\nabla\sigma_1^1\|^2\big)+\frac{1}{2}(\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2)\\
&+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2+\nu ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}^1\|^2
\end{aligned}
$$

and hence, a use of (4.2.48) with $n=1$ along with (2.2.3) yields

$$
\begin{aligned}
&\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2+k\Big(\nu e^{-\alpha k}-2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{e}}^1\|^2 \qquad (4.4.26)\\
&\le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2.
\end{aligned}
$$

Using (4.4.21)-(4.4.23) with $\epsilon=\frac{2\nu}{3}$, (4.4.26), $\mathbf{e}^o=0$ and bounds from Lemma 4.2 in (4.4.16), we obtain

$$
\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2)+(2e^{-\alpha k}\hat{\mathbf{e}}^N-\hat{\mathbf{e}}^{N-1})^2\\
&+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N-\nabla\hat{\mathbf{e}}^{N-1})^2+2k\Big(\nu e^{-2\alpha k}-\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\
\le & C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=2}^{N}ke^{-2\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2\\
\le & C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2+C(\nu)ke^{-2\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{U}}^N\|^2\|\hat{\mathbf{e}}^N\|^2\\
\le & C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2+Cke^{-2\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2).
\end{aligned}
$$
(4.4.27)

Choose k_0, so that (4.4.5) is satisfied and $(1-Cke^{-2\alpha k})>0$ for $0<k\le k_0$. Then, an

application of the discrete Gronwall's lemma yields

$$\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4 \times \exp(k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}^n\|^2). \quad (4.4.28)$$

The bounds obtained from Lemma 4.2 in (4.4.28) would lead us to (4.4.12), that is,

$$\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4 \quad (4.4.29)$$

and this completes the proof of (4.4.12) for $n \geq 2$.
For $n = 1$, we use (4.4.26) along with the bounds in Lemma 4.1. Then, a choice of k such that $(1 - Cke^{-\alpha k}) > 0$ would lead us to the desired result, that is,

$$\|\mathbf{e}^1\|^2 + \kappa\|\nabla\mathbf{e}^1\|^2 + k\|\nabla\mathbf{e}^1\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4e^{-2\alpha k}. \quad (4.4.30)$$

To arrive at the estimates in (4.4.13), we choose $\boldsymbol{\phi}_h = D_t^{(2)}\mathbf{e}^n$ in (4.4.14) and obtain

$$\begin{aligned}\|D_t^{(2)}\mathbf{e}^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 = &-\nu a(\mathbf{e}^n, D_t^{(2)}\mathbf{e}^n) + (\sigma_2^n, D_t^{(2)}\mathbf{e}^n) \\ &+ \kappa a(\sigma_2^n, D_t^{(2)}\mathbf{e}^n) + \Lambda_h(D_t^{(2)}\mathbf{e}^n).\end{aligned} \quad (4.4.31)$$

It follows from (4.2.46) that

$$|\Lambda_h(D_t^{(2)}\mathbf{e}^n)| \leq C(\kappa,\nu,\alpha,\lambda_1,M)\|\nabla\mathbf{e}^n\|\|D_t^{(2)}\nabla\mathbf{e}^n\|. \quad (4.4.32)$$

Using Cauchy-Schwarz's inequality, Young's inequality, (2.2.3) and (4.4.32) in (4.4.31), we write

$$\|D_t^{(2)}\mathbf{e}^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)\Big(\|\nabla\mathbf{e}^n\|^2 + \|\nabla\sigma_2^n\|^2\Big). \quad (4.4.33)$$

For the second term on the right hand side of (4.4.33), we use (4.4.18) and Lemma 3.6 and arrive at

$$
\begin{aligned}
\|e^{\alpha t_n}\nabla\sigma_2^n\|^2 &\le \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\nabla \mathbf{u}_{htttt}(t)\|^2 dt \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^3\, e^{2\alpha t_n}\int_{t_{n-2}}^{t_n} e^{-2\alpha t}dt \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k^*},
\end{aligned}
\tag{4.4.34}
$$

where $k^* \in (0,k)$. Now with the help of (4.4.12) and (4.4.34), (4.4.33) implies (4.4.13). This completes the rest of the proof. □

Finally, we obtain the error estimates for the pressure P^n. Consider (3.2.2) at $t=t_n$ and subtract it from (4.4.2) to obtain

$$
(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) = (D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h)+\kappa a(D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h)+\nu a(\mathbf{e}^n,\boldsymbol{\phi}_h)-(\sigma_2^n,\boldsymbol{\phi}_h)-\kappa a(\sigma_2^n,\boldsymbol{\phi}_h)-\Lambda_h(\boldsymbol{\phi}_h).
$$

With the help of Cauchy-Schwarz's inequality and (4.2.46), we obtain

$$
(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h)\le C(\kappa,\nu,\lambda_1)\big(\|D_t^2\nabla\mathbf{e}^n\|+\|\nabla\mathbf{e}^n\|+\|\nabla\sigma_2^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{4.4.35}
$$

A use of the Theorem 4.4 and (4.4.34) in (4.4.35) yields

$$
\|\boldsymbol{\rho}^n\|\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\, e^{-\alpha t_n} \text{ for } n\ge 2. \tag{4.4.36}
$$

For $n=1$, we use estimates obtained from backward Euler method. Substitute $n=1$ in (4.2.49) to obtain

$$
(\boldsymbol{\rho}^1,\nabla\cdot\boldsymbol{\phi}_h)\le C(\kappa,\nu,\lambda_1)\big(\|\bar{\partial}_t\nabla\mathbf{e}^1\|+\|\nabla\mathbf{e}^1\|+\|\nabla\sigma_1^1\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{4.4.37}
$$

A use of (4.2.47) with $n=1$ in (4.4.37) yields

$$
(\boldsymbol{\rho}^1,\nabla\cdot\boldsymbol{\phi}_h)\le C(\kappa,\alpha,\nu,\lambda_1,M)\big(\|\nabla\mathbf{e}^1\|+\|\nabla\sigma_1^1\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{4.4.38}
$$

Using bounds obtained from (4.2.48) (for $n = 1$) and (4.4.30) in (4.4.38), we arrive at

$$\|\boldsymbol{\rho}^1\| \leq C(\kappa, \alpha, \nu, \lambda_1, M)k \ e^{-\alpha t_1}. \tag{4.4.39}$$

Theorem 4.5. *Under the assumptions of Theorems 3.1, 4.2 and 4.4, the following holds true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq C(h^{2-j} + k^2)e^{-\alpha t_n} \quad j = 0, 1,$$

and,

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k^{2-\gamma}),$$

where

$$\gamma = \begin{cases} 0 & \textit{if } n \geq 2; \\ 1 & \textit{if } n = 1. \end{cases}$$

Proof of Theorem 4.5. A use of Theorems 3.1, 3.2, 4.4, (4.4.30), (4.4.36) and (4.4.39) would complete the proof. □

4.5 Numerical Experiments

In this section, we provide a few computational results to support our theoretical estimates for the equations of motion arising in the Kelvin-Voigt fluid (1.2.3)-(1.2.5). For space discretization, P_2-P_0 mixed finite element space is used: the velocity space consists of continuous piecewise polynomials of degree less than or equal to 2 and the pressure space consists of piecewise constants, that is, we consider the finite dimensional subspaces $\mathbf{V}_h$ and W_h of $\mathbf{H}_0^1$ and L^2 respectively, which satisfy the approximation properties in (**B1**) and (**B2**), as:

$$\mathbf{V}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \left(C(\bar{\Omega})\right)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\},$$

$$W_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \tau_h\},$$

where τ_h denotes an admissible triangulation of $\bar{\Omega}$ in to closed triangles. In Example 4.5.4, we use P_2-P_1 mixed finite element space. Let $0 = t_0 < t_1 < \cdots < t_N = T$, be a uniform subdivision of the time interval $[0, T]$ with $k = t_n - t_{n-1}$. Below, we discuss the fully discrete finite element formulation of (1.2.3)-(1.2.5) using backward Euler method and second order backward difference scheme.

Fully discrete finite element Galerkin approximation: In this scheme, we discuss the discretization of the time variable by replacing the time derivative by difference quotient. Let k be the time step and $\mathbf{U}^n$ be the approximation of $\mathbf{u}(t)$ in $\mathbf{V}_h$ at $t = t_n = nk$.
The backward Euler approximation to (3.2.2) can be stated as: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned}(\mathbf{U}^n, \mathbf{v}_h) + (\kappa + \nu k)\ a(\mathbf{U}^n, \mathbf{v}_h) + k\ b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) - k\ (P^n, \nabla \cdot \mathbf{v}_h) \\ = (\mathbf{U}^{n-1}, \mathbf{v}_h) + \kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) + k\ (\mathbf{f}(t_n), \mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned} \tag{4.5.1}$$

Similarly, the second order backward difference approximation to (3.2.2) is as follows: given $\mathbf{U}^{n-2}$ and $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned}(3\mathbf{U}^n, \mathbf{v}_h) + (\kappa + 2\nu k)\ a(\mathbf{U}^n, \mathbf{v}_h) + 2k\ b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) - 2k\ (P^n, \nabla \cdot \mathbf{v}_h) \\ = 4(\mathbf{U}^{n-1}, \mathbf{v}_h) + 4\ \kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) - (\mathbf{U}^{n-2}, \mathbf{v}_h) - \kappa\ a(\mathbf{U}^{n-2}, \mathbf{v}_h) \\ + k\ (\mathbf{f}(t_n), \mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned} \tag{4.5.2}$$

Now, we approximate the velocity and pressure by

$$\mathbf{U}^n = \sum_{j=1}^{ng} \begin{pmatrix} \mathbf{u}_j^{nx} \\ \mathbf{u}_j^{ny} \end{pmatrix} \boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x}), \quad P^n = \sum_{j=1}^{ne} p_j^n \phi_j^p(\mathbf{x}), \tag{4.5.3}$$

where $\boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x})$ and $\phi_j^p(\mathbf{x})$ form bases for $\mathbf{V}_h$ and W_h respectively with cardinality ng and ne, respectively. Here, $\mathbf{u}_j^{nx}$ and $\mathbf{u}_j^{ny}$ represent the x and y component of the approximate velocity field, respectively, at time $t = t_n$.
Using (4.5.3), the basis functions for $\mathbf{V}_h$ and W_h in (4.5.1) (respectively (4.5.2)), we obtain nonlinear system which is solved using Newton's method.

Remark 4.5.1. *For verifying the theoretical results, we choose right hand side f in Example 4.5.1 in such a way that the exact solution is known. However, the numerical method works for the case $f = 0$ also (see Example 4.5.2).*

Example 4.5.1. *In this example, we validate the theoretical error estimates obtained in Theorem 4.3. For verifying the convergence rates of the solution obtained numerically, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ of (1.2.3)-(1.2.5) is*

$$u_1 = 2e^t x^2(x-1)^2 y(y-1)(2y-1), \quad u_2 = -2e^t y^2(y-1)^2 x(x-1)(2x-1), \quad p = e^t\ y.$$

We assume the viscosity of the fluid as $\nu = 1$ and retardation as $\kappa = 1$ with $\Omega = (0,1)\times(0,1)$ and time interval $(0, T]$ with $T = 1$. Here, Ω is subdivided into triangles with mesh size h. The theoretical analysis provides a convergence rate $\mathcal{O}(h^2)$ for the velocity in $\mathbf{L^2}$-norm and $\mathcal{O}(h)$ for the pressure in the L^2-norm. Table 4.1 gives the numerical errors and convergence rates obtained on successively refined meshes with time step size $k = h^2$. These results agree with the optimal theoretical convergence rates obtained in Theorem 4.3.

S No	h	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{L^2}}$	Convergence Rate	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{H^1}}$	Convergence Rate	$\|p - P^n\|$	Convergence Rate
1	1/2	0.0266		0.1039		1.0443	
2	1/4	0.0090	1.5653	0.0543	0.9357	0.5484	0.9291
3	1/8	0.0026	1.7790	0.0282	0.9428	0.2815	0.9618
4	1/16	0.0007	1.8938	0.0145	0.9601	0.1424	0.9827

Table 4.1: Numerical errors and Convergence rates with $k = h^2$

Example 4.5.2. *In this example, we demonstrate the exponential decay property of the discrete solution. We choose* $\nu = 1$*,* $\kappa = 1$ *and* $f = 0$ *with* $\mathbf{u}_0 = (2x^2(x-1)^2y(y-1)(2y-1), -2y^2(y-1)^2x(x-1)(2x-1),\ y)$ *in (1.2.3)-(1.2.5). In this case, we replace exact solution* $\mathbf{u}$ *by finite element solution obtained in a refined mesh.*

The order of convergence is shown in Table 4.2. In Figure 4.1, for different values of time t, we plot $\|\mathbf{U}^n\|$ versus time and observe the exponential decay property for velocity.

S No	h	$\|\mathbf{u}-\mathbf{U}^n\|_{\mathbf{L}^2}$	Convergence Rate	$\|\mathbf{u}-\mathbf{U}^n\|_{\mathbf{H}^1}$	Convergence Rate
1	1/2	0.797874×10^{-3}		0.012952	
2	1/4	0.203886×10^{-3}	1.9683	0.006767	0.9366
3	1/8	0.051241×10^{-3}	1.9923	0.003418	0.9850
4	1/16	0.012817×10^{-3}	1.9992	0.001713	0.9964

Table 4.2: Numerical errors and Convergence rates with $k = h^2$

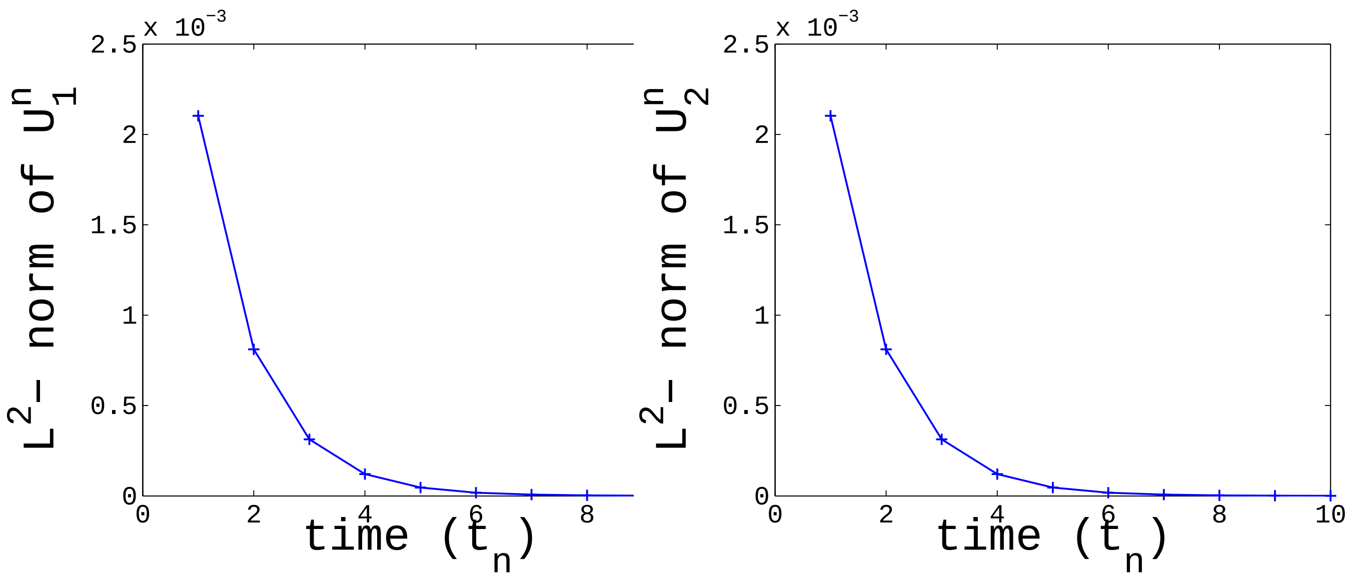

Figure 4.1: Exponential decay property of the approximate solution $\|\mathbf{U}^n\|$.

Example 4.5.3. *In this example the right hand side function f is constructed in such a way that the exact solution pair $(\mathbf{u}, p) = ((u_1, u_2), p)$ is*

$u_1 = 10e^{-t}x^2(x-1)^2y(y-1)(2y-1),\ \ u_2 = -10e^{-t}y^2(y-1)^2x(x-1)(2x-1),\ \ p = e^{-t}y.$

The coefficient of viscosity $\nu = 1$ and retardation time $\kappa = 10^{-2}$ with a square domain $\Omega = (0,1) \times (0,1)$ and time $t = [0,1]$.

The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in $\mathbf{L^2}$-norm, of $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm for velocity and of $\mathcal{O}(h)$ in L^2-norm for pressure. Table 4.3 gives the numerical errors and convergence rates obtained on successively refined meshes for the first order backward Euler method which agrees with the expected theoretical rates of convergence. Table 4.4 contains the errors and convergence rates of the second order two step backward difference method. In Figure 4.2, we plot the order of convergence of velocity in $\mathbf{L^2}$ and $\mathbf{H}^1$-norms and of pressure in L^2- norm for both the methods. These results confirm the theoretical convergence rates obtained in Theorem 4.3 and 4.5.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/2	0.0045761		0.056318		0.136225	
1/4	0.0013220	1.791328	0.024171	1.220311	0.072946	0.901096
1/8	0.0003651	1.856036	0.010997	1.136107	0.037920	0.943847
1/16	0.0000970	1.911519	0.005371	1.033759	0.019201	0.981790

Table 4.3: Errors and Convergence rates for backward Euler method with $k = \mathbb{O}(h^2)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/2	0.0047034		0.043623		0.135666	
1/4	0.0013240	1.828747	0.019412	1.168111	0.073007	0.893959
1/8	0.0003653	1.857587	0.009233	1.072039	0.037925	0.944881
1/16	0.0000970	1.912022	0.004602	1.004511	0.019201	0.981941

Table 4.4: Errors and Convergence rates for backward difference scheme with $k = \mathbb{O}(h)$.

Remark 4.5.2. *Note that, under extra regularity assumptions on the exact solution pair, one can obtain better rates of convergence by using higher order finite element space.*

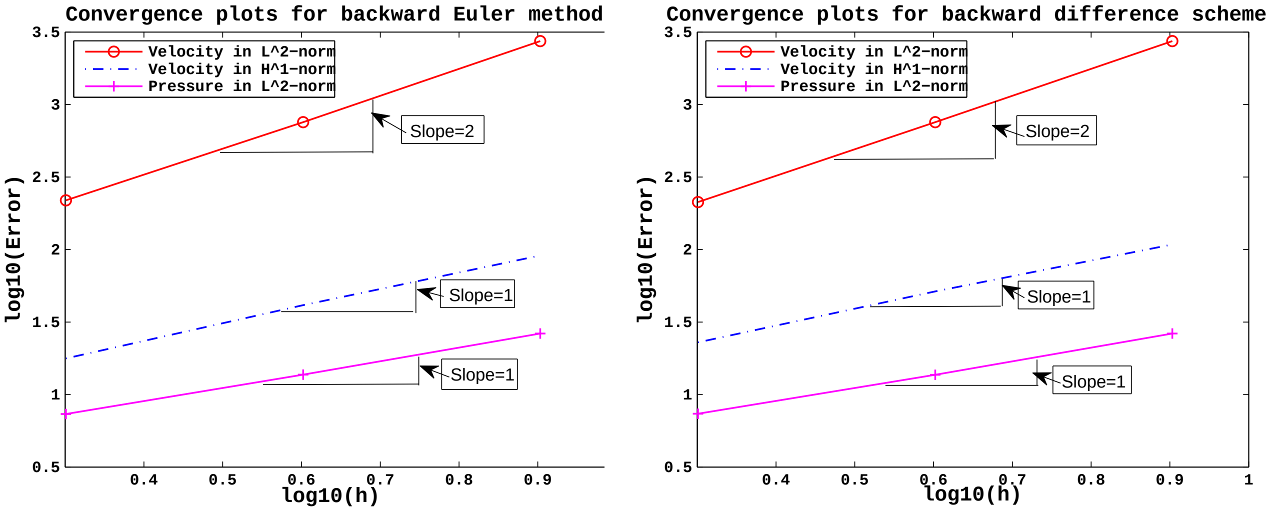

Figure 4.2: Convergence plots for backward Euler method and backward difference scheme

We illustrate this in Example 4.5.4, by choosing an appropriate right hand side function f.

Example 4.5.4. *In this example, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is:*

$$\begin{aligned} u_1 &= te^{-t^2} sin^2(3\pi x)\ sin(6\pi y),\ u_2 = -te^{-t^2} sin^2(3\pi y)\ sin(6\pi x), \qquad (4.5.4) \\ p &= te^{-t}\ sin(2\pi x)\ sin(2\pi y). \end{aligned}$$

We assume that the viscosity of the fluid(ν) is 10^{-2} and the retardation κ is 10^{-4} with $\Omega = (0,1) \times (0,1)$ and time $t = [0,1]$. Here again, $\bar{\Omega}$ is subdivided into triangles with mesh size h. For the problem defined in Example 4.5.4, we have conducted numerical experiments using P_2-P_1 mixed finite element spaces for space discretization, that is, if we choose

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \big(C(\bar{\Omega})\big)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\}, \\ W_h &= \{q \in L^2(\Omega) \cap C(\bar{\Omega}) : q|_K \in P_1(K), K \in \tau_h\}, \end{aligned}$$

we obtain $\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq \mathcal{O}(h^{3-j}),\ j = 0,1$ and $\|(p(t_n) - P^n)\| \leq \mathcal{O}(h^2)$. Since the solution (4.5.4) has extra regularity, we obtained better order of convergence for velocity and pressure as expected [12]. In Tables 4.5 and 4.6, we have shown the convergence rates for backward Euler method and backward difference scheme respectively for $\mathbf{L^2}$ and $\mathbf{H}^1$-norms in velocity and L^2-norm in pressure. In case, we choose $k = \mathcal{O}(h^{3/2})$ for backward Euler method, we observe that convergence rate in comparison with that of the backward difference scheme is lower. Table 4.7 represents the comparison between the errors obtained from backward Euler method and backward difference scheme with $k = \mathcal{O}(h^{3/2})$. Figure 4.3, exhibits the plots between the errors and mesh size h for velocity and pressure for the two fully discrete methods discussed in this chapter.

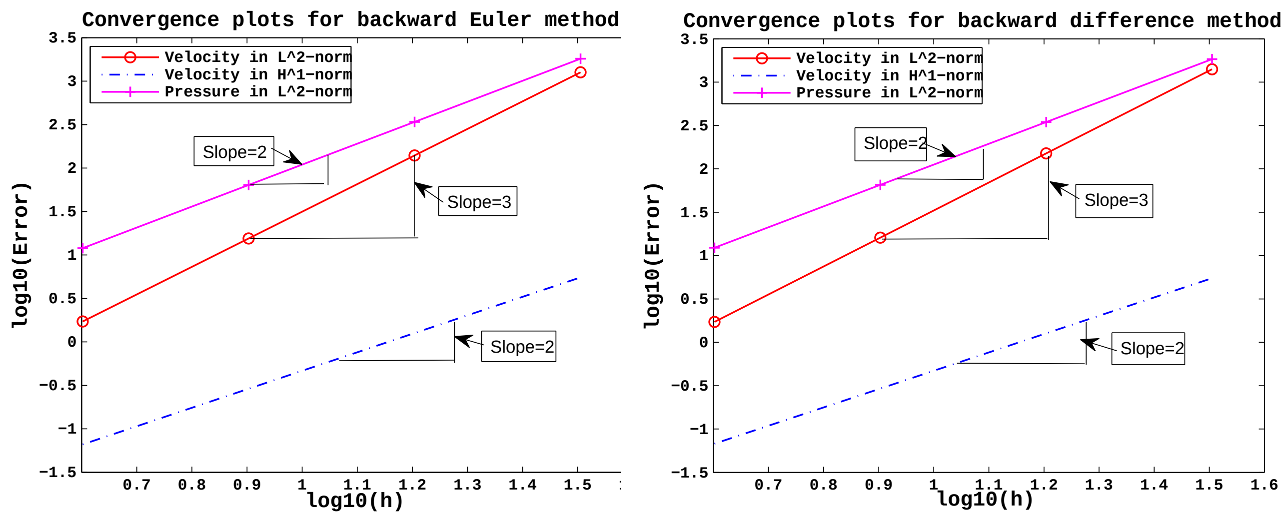

Figure 4.3: Convergence plots for backward Euler method and backward difference scheme

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/4	0.480585		13.147142		0.088453	
1/8	0.085185	2.496114	4.135230	1.668709	0.015389	2.522972
1/16	0.007371	3.530504	0.849642	2.283040	0.002566	2.584331
1/32	0.000709	3.377407	0.163177	2.380417	0.000610	2.072467

Table 4.5: Errors and Convergence rates for backward Euler method with $k = \mathbb{O}(h^3)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/4	0.466215		12.744857		0.085101	
1/8	0.083276	2.485017	4.111723	1.63210	0.015361	2.469883
1/16	0.007224	3.526926	0.850460	2.273426	0.002504	2.616902
1/32	0.000609	3.568176	0.163161	2.381940	0.000596	2.068971

Table 4.6: Errors and Convergence rates for backward difference scheme with $k = \mathbb{O}(h^{3/2})$.

h	Backward Euler velocity in $\mathbf{L^2}$-norm	Backward difference velocity in $\mathbf{L^2}$-norm	Backward Euler pressure in $\mathbf{L^2}$-norm	Backward difference pressure in $\mathbf{L^2}$-norm
1/4	0.514184	0.466215	0.102280	0.085101
1/8	0.084014	0.083276	0.015499	0.015361
1/16	0.008783	0.007224	0.003010	0.002504
1/32	0.001991	0.000609	0.000880	0.000596

Table 4.7: Comparison of errors with $k = \mathcal{O}(h^{3/2})$ between the two schemes.

Chapter 5

Two-grid Method

5.1 Introduction

In this chapter, we study a two level method based on Newton's iteration for a nonlinear system arising from the finite element Galerkin approximations to the equations of motion arising in Kelvin-Voigt viscoelastic fluid flow model. The two level algorithm is based on three steps. In the first step, we solve nonlinear system on coarse mesh $\mathcal{T}_H$ with mesh size H and obtain an approximate solution u_H. In the second step, we linearize the nonlinear system around u_H based on Newton's iteration and solve linear system on fine mesh $\mathcal{T}_h$ with mesh size h and finally, in the third step, we do a correction to the results obtained in the second step by solving the same linear problem with different right hand side on fine mesh.

We have established optimal error estimates and recover an error of the order h^2 in $L^\infty(\mathbf{L}^2)$-norm and h in $L^\infty(\mathbf{H}^1)$-norm for velocity and h for pressure in $L^\infty(L^2)$-norm provided $h = \mathbb{O}(H^{2-\delta})$, where δ is arbitrary small for two dimensions and $\delta = \frac{1}{2}$ for three dimensions, that is, we have shown that the error estimates for the two-grid method is of the same order as that of the error estimates obtained from the direct treatment of the nonlinear system on a fine grid with a choice of $h = \mathbb{O}(H^{2-\delta})$.

For a brief introduction of the literature related to two-grid discretization, we refer to

Xu [104]-[105] and Niemistö [73] for elliptic problems, Layton [65], Layton and Tobiska [68], Layton and Lenferink [66]-[67], Girault and Lions [26], Dai *et al* [20] for steady state Navier Stokes equations, Girault and Lions [27], Abboud *et al.* [1]-[2], Frutos *et al.* [22] for semidiscrete two-grid analysis of transient Navier-Stokes equations, Abboud *et al.* [1]-[2] and Frutos *et al* [22] for fully discrete two-grid analysis of transient Navier-Stokes equations. In [26], Girault *et al* have obtained optimal $L^\infty(\mathbf{H}^1)$ and $L^\infty(\mathbf{L^2})$-norms estimates with a choice $h = H^{3/2}$, where h is the spatial mesh size of finer mesh and H corresponds to a coarse mesh for seady state Navier Stokes equations. They have also worked out two-grid analysis for transient Navier-Stokes equations and established optimal for $L^\infty(\mathbf{H}^1)$-norm and suboptimal for $L^\infty(\mathbf{L^2})$-norm error estimates with $h = H^2$ (see [27]). There are hardly any results available for the two-grid discretization for Kelvin-Voigt model.
The remaining part of this chapter consists of the following sections: In Section **5.2**, we introduce the semidiscrete two-grid formulation and establish some a *priori* bounds to carry out error analysis. With these a *priori* bounds, we establish optimal error estimates for velocity and pressure for the semidiscrete two-grid solution. Sections **5.3** and **5.4** deal with the backward Euler method and second order backward difference scheme, respectively, applied to the semidiscrete two grid system. Finally, in Section **5.5**, we present the results of some numerical examples in order to verify theoretical results.

5.2 Two-Grid Formulation and A *Priori* bounds

In this section, we describe the two-grid algorithm and derive a *priori* bounds for the two-grid approximations.
For the sake of continuity, let us recall the weak formulation of (1.2.3)-(1.2.5) with $\mathbf{f} = 0$ as: find a pair of functions $\{\mathbf{u}(t), p(t)\} \in \mathbf{H}_0^1 \times L^2/\mathbb{R}, \;\; t > 0$, such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa\, a(\mathbf{u}_t, \boldsymbol{\phi}) + \nu\, a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) = (p, \nabla \cdot \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1, \tag{5.2.1}$$
$$(\nabla \cdot \mathbf{u}, \chi) = 0 \qquad \forall \chi \in L^2,$$

with $\mathbf{u}(0) = \mathbf{u}_0$.

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa\, a(\mathbf{u}_t, \boldsymbol{\phi}) + \nu\, a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1, \;\; t > 0, \tag{5.2.2}$$

with $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\mathbf{J}_1 \;\; = \;\; \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\}.$$

Finally, we recall the semidiscrete finite element Galerkin approximations to (5.2.1)-(5.2.2) as: find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) = (p_h, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \tag{5.2.3}$$
$$(\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h,$$

where $\mathbf{H}_h$ and L_h, $0 < h < 1$ are finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively.

Equivalently, find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.2.4}$$

where

$$\mathbf{J}_h = \{\boldsymbol{\phi}_h \in \mathbf{H}_h : (\nabla \cdot \boldsymbol{\phi}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h\}.$$

For applying the two level method, we choose two triangulations of $\bar{\Omega}$: one coarse mesh $\mathcal{T}_H$ with mesh size H and one fine mesh $\mathcal{T}_h$ with mesh size h and $H \geq h$ and define the associated finite element spaces $\mathbf{J}_H$ and $\mathbf{J}_h$.

Now, the two-grid algorithm applied to (5.2.2) can be explained as follows [20]:

Algorithm:

Step 1: Solve the nonlinear system on a coarse mesh $\mathcal{T}_H$: find $\mathbf{u}_H \in \mathbf{J}_H$ such that for each $t > 0$,

$$(\mathbf{u}_{Ht}, \boldsymbol{\phi}_H) + \kappa a(\mathbf{u}_{Ht}, \boldsymbol{\phi}_H) + \nu a(\mathbf{u}_H, \boldsymbol{\phi}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_H) = 0 \;\; \forall \boldsymbol{\phi}_H \in \mathbf{J}_H. \tag{5.2.5}$$

Step 2: Update on fine mesh $\mathcal{T}_h$ with one Newton iteration: find $\mathbf{u}_h^* \in \mathbf{J}_h$ such that

$$\begin{aligned}(\mathbf{u}_{ht}^*, \boldsymbol{\phi}_h)+&\kappa a(\mathbf{u}_{ht}^*, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h^*, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ &+ b(\mathbf{u}_H, \mathbf{u}_h^*, \boldsymbol{\phi}_h) = b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.2.6}$$

Step 3: Correct on fine mesh $\mathcal{T}_h$: find $\mathbf{u}_h \in \mathbf{J}_h$ such that

$$\begin{aligned}(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_H, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ = b(\mathbf{u}_H, \mathbf{u}_h^*, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.2.7}$$

For the work related to steady state Navier Stokes equation with the above mentioned algorithm, we refer to [20].

Now, let us first recall a *priori* estimates of the semidiscrete solution $\mathbf{u}_H$ of **Step 1** (for a proof, see Chapter **3**, Lemmas 3.3 and 3.4 with $\mathbf{u}_h$ replaced by $\mathbf{u}_H$) and then prove some a *priori* error estimates in the seond step for the semidiscrete solution $\mathbf{u}_h^*$ which will be used in our subsequent error analysis.

Lemma 5.1. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $\mathbf{u}_{0H} = P_H\mathbf{u}_0$. *Let the assumptions* (**A1**)–(**A2**) *hold true. Then, the solution* $\mathbf{u}_H$ *of (5.2.5) satisfies*

$$\begin{aligned}\|\mathbf{u}_H(t)\|^2 + \kappa\|\nabla\mathbf{u}_H(t)\|^2 + \kappa\|\tilde{\Delta}_H\mathbf{u}_H(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_H(s)\|^2 + \|\tilde{\Delta}_H\mathbf{u}_H(s)\|^2)\,ds \\ \leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t} \quad t > 0\end{aligned}$$

and

$$\|\mathbf{u}_{Ht}(t)\|^2 + \kappa\|\nabla \mathbf{u}_{Ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{Ht}(s)\|^2 + \kappa\|\nabla \mathbf{u}_{Ht}(s)\|^2)ds \le Ce^{-2\alpha t},$$

where $\beta = \nu - 2\alpha(\lambda_1^{-1} + \kappa) > 0$.

Lemma 5.2. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $\mathbf{u}_{0h}^* = P_h\mathbf{u}_0$. *Let the assumptions* (**A1**)–(**A2**) *hold true. Then, the solution* $\mathbf{u}_h^*$ *of (5.2.6) satisfies*

$$\begin{aligned}\|\mathbf{u}_h^*(t)\|^2 + \kappa\|\nabla \mathbf{u}_h^*(t)\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h^*(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_h^*(s)\|^2 + \|\tilde{\Delta}_h\mathbf{u}_h^*(s)\|^2)\,ds \\ \le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t} \quad t>0,\end{aligned}$$

where $\beta = \nu - 2\alpha(\lambda_1^{-1} + \kappa) > 0$.

Proof. Set $\hat{\mathbf{u}}_h^*(t) = e^{\alpha t}\mathbf{u}_h^*(t)$ for some $\alpha \ge 0$ and rewrite (5.2.6) as

$$\begin{aligned}&(\hat{\mathbf{u}}_{ht}^*, \boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h^*, \boldsymbol{\phi}_h) + \kappa(\nabla\hat{\mathbf{u}}_{ht}^*, \nabla\boldsymbol{\phi}_h) - \kappa\alpha(\nabla\hat{\mathbf{u}}_h^*, \nabla\boldsymbol{\phi}_h) + \nu(\nabla\hat{\mathbf{u}}_h^*, \nabla\boldsymbol{\phi}_h) \\ &\quad + e^{-\alpha t}b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \boldsymbol{\phi}_h) + e^{-\alpha t}b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*, \boldsymbol{\phi}_h) = e^{-\alpha t}b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \qquad (5.2.8)\end{aligned}$$

where $\hat{\mathbf{u}}_H(t) = e^{\alpha t}\mathbf{u}_H(t)$.

Substitute $\boldsymbol{\phi}_h = \hat{\mathbf{u}}_h^*$ in (5.2.8). A use of (3.2.1) ($b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_h^*) = 0$) and (2.2.3) yields

$$\begin{aligned}\frac{d}{dt}(\|\hat{\mathbf{u}}_h^*\|^2 + \kappa\|\nabla\hat{\mathbf{u}}_h^*\|^2) + 2\Big(\nu - \alpha\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{u}}_h^*\|^2 \\ \le -e^{-\alpha t}b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*) + e^{-\alpha t}b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*). \qquad (5.2.9)\end{aligned}$$

We write the first term on the right hand side of (5.2.9) by using (3.2.10) and Young's inequality as follows:

$$\begin{aligned}e^{-\alpha t}|b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*)| &\le Ce^{-\alpha t}\|\hat{\mathbf{u}}_h^*\|^{\frac12}\|\nabla\hat{\mathbf{u}}_h^*\|^{\frac12}\|\nabla\hat{\mathbf{u}}_H\|\|\hat{\mathbf{u}}_h^*\|^{\frac12}\|\nabla\hat{\mathbf{u}}_h^*\|^{\frac12} \\ &\le Ce^{-\alpha t}\|\hat{\mathbf{u}}_h^*\|\|\nabla\hat{\mathbf{u}}_h^*\|\|\nabla\hat{\mathbf{u}}_H\| \\ &\le C(\epsilon)e^{-2\alpha t}\|\hat{\mathbf{u}}_h^*\|^2\|\nabla\hat{\mathbf{u}}_H\|^2 + \epsilon\|\nabla\hat{\mathbf{u}}_h^*\|^2. \qquad (5.2.10)\end{aligned}$$

To estimate the second term on the right hand side of (5.2.9), we apply (3.2.11), Young's inequality and the estimates from Lemma 5.1 and obtain

$$
\begin{aligned}
e^{-\alpha t}|b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*)| &\leq Ce^{-\alpha t}\|\nabla \hat{\mathbf{u}}_H\|^2 \|\nabla \hat{\mathbf{u}}_h^*\| \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M)\|\nabla \hat{\mathbf{u}}_H\| \|\nabla \hat{\mathbf{u}}_h^*\| \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)\|\nabla \hat{\mathbf{u}}_H\|^2 + \epsilon \|\nabla \hat{\mathbf{u}}_h^*\|^2. \qquad (5.2.11)
\end{aligned}
$$

A use of (5.2.10) and (5.2.11) in (5.2.9) yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\hat{\mathbf{u}}_h^*\|^2 + \kappa \|\nabla \hat{\mathbf{u}}_h^*\|^2) + 2\bigg(\nu - \alpha\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla \hat{\mathbf{u}}_h^*\|^2 \\
&\quad \leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)\|\nabla \hat{\mathbf{u}}_H\|^2 + C(\epsilon)e^{-2\alpha t}\|\hat{\mathbf{u}}_h^*\|^2 \|\nabla \hat{\mathbf{u}}_H\|^2 + 2\epsilon \|\nabla \hat{\mathbf{u}}_h^*\|^2. \qquad (5.2.12)
\end{aligned}
$$

Choosing $\epsilon = \frac{\nu}{2}$ and integrating from 0 to t, we arrive at

$$
\begin{aligned}
&\|\hat{\mathbf{u}}_h^*\|^2 + \kappa \|\nabla \hat{\mathbf{u}}_h^*\|^2 + \beta \int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h^*(s)\|^2 ds \leq \|\mathbf{u}_{0h}^*\|^2 + \kappa \|\nabla \mathbf{u}_{0h}^*\|^2 \\
&+ C(\kappa, \nu, \alpha, \lambda_1, M)\int_0^t \|\nabla \hat{\mathbf{u}}_H(s)\|^2 ds + C(\nu)\int_0^t e^{-2\alpha s}\|\nabla \hat{\mathbf{u}}_H(s)\|^2 \|\hat{\mathbf{u}}_h^*(s)\|^2 ds. \qquad (5.2.13)
\end{aligned}
$$

Using the estimates from Lemma 5.1 for the third term on the right hand side of (5.2.13) and applying the Gronwall's lemma 1.3, we write

$$
\begin{aligned}
&\|\hat{\mathbf{u}}_h^*\|^2 + \kappa \|\nabla \hat{\mathbf{u}}_h^*\|^2 + \beta \int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h^*(s)\|^2 ds \\
&\quad \leq C(\kappa, \nu, \alpha, \lambda_1, M)\exp\bigg(\int_0^t \|\nabla \mathbf{u}_H(s)\|^2 ds\bigg). \qquad (5.2.14)
\end{aligned}
$$

Once again, with the help of Lemma 5.1, we obtain

$$
\begin{aligned}
\int_0^t \|\nabla \mathbf{u}_H(s)\|^2 ds &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\int_0^t e^{-2\alpha s} ds \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M)(1 - e^{-2\alpha t}) \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M). \qquad (5.2.15)
\end{aligned}
$$

An application of (5.2.15) in (5.2.14) yields

$$\|\mathbf{u}_h^*\|^2 + \kappa\|\nabla\mathbf{u}_h^*\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h^*(s)\|^2 ds \le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \tag{5.2.16}$$

Using the discrete Stokes operator $\tilde{\Delta}_h$ in (5.2.6), we arrive at

$$\begin{aligned}&(\hat{\mathbf{u}}_{ht}^*,\boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h^*,\boldsymbol{\phi}_h) - \kappa(\tilde{\Delta}_h\hat{\mathbf{u}}_{ht}^*,\boldsymbol{\phi}_h) + \kappa\alpha(\tilde{\Delta}_h\hat{\mathbf{u}}_h^*,\boldsymbol{\phi}_h) - \nu(\tilde{\Delta}_h\hat{\mathbf{u}}_h^*,\boldsymbol{\phi}_h)\\ &+ e^{-\alpha t}b(\hat{\mathbf{u}}_h^*,\hat{\mathbf{u}}_H,\boldsymbol{\phi}_h) + e^{-\alpha t}b(\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_h^*,\boldsymbol{\phi}_h) = e^{-\alpha t}b(\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_H,\boldsymbol{\phi}_h) \ \ \forall\boldsymbol{\phi}_h\in\mathbf{J}_h.\end{aligned} \tag{5.2.17}$$

Observe that,

$$-(\hat{\mathbf{u}}_{ht}^*,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}_h^*\|^2. \tag{5.2.18}$$

With $\boldsymbol{\phi}_h = -\tilde{\Delta}_h\hat{\mathbf{u}}_h^*$ and (5.2.18), (5.2.17) becomes

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}_h^*\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|^2) + 2(\nu-\kappa\alpha)\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|^2 = 2\alpha\|\nabla\hat{\mathbf{u}}_h^*\|^2 + 2e^{-\alpha t}b(\hat{\mathbf{u}}_h^*,\hat{\mathbf{u}}_H,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*)\\ &+ 2e^{-\alpha t}b(\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_h^*,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*) - 2e^{-\alpha t}b(\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_H,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*) = 2\alpha\|\nabla\hat{\mathbf{u}}_h^*\|^2 + I_1 + I_2 + I_3.\end{aligned} \tag{5.2.19}$$

We use (3.2.8) and (3.2.9) to estimate $|I_1|$ and $|I_2|$ and arrive at

$$\begin{aligned}|I_1| + |I_2| &= e^{-\alpha t}\big(|b(\hat{\mathbf{u}}_h^*,\hat{\mathbf{u}}_H,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*)| + |b(\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_h^*,\tilde{\Delta}_h\hat{\mathbf{u}}_h^*)|\big)\\ &\le C\|\nabla\hat{\mathbf{u}}_h^*\|^{\frac{1}{2}}\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{u}}_H\|\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|\\ &\le C\|\nabla\hat{\mathbf{u}}_h^*\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{u}}_H\|\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|^{\frac{3}{2}}.\end{aligned} \tag{5.2.20}$$

An application of Young's inequality $ab \le \frac{a^p}{p\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a,b\ge 0$, $\epsilon>0$ with $p=4$ and $q=\frac{4}{3}$ (see (1.3.9)) to obtain

$$|I_1| + |I_2| \le C\frac{\|\nabla\hat{\mathbf{u}}_h^*\|^2\|\nabla\hat{\mathbf{u}}_H\|^4}{4\epsilon^3} + \frac{3\epsilon}{4}\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^*\|^2. \tag{5.2.21}$$

Choosing $\epsilon = \frac{2\nu}{3}$, we note that

$$|I_1| + |I_2| \leq \frac{C}{4}\left(\frac{3}{4\nu}\right)^3 \|\nabla \hat{\mathbf{u}}_h^*\|^2 \|\nabla \hat{\mathbf{u}}_H\|^4 + \frac{\nu}{2}\|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\|^2. \tag{5.2.22}$$

For estimating $|I_3|$, we apply (3.2.8) and Young's inequality with $\epsilon = \nu$ to arrive at

$$\begin{aligned}
|I_3| &= e^{-\alpha t}|b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*)| \\
&\leq C\|\nabla \hat{\mathbf{u}}_H\|^{\frac{1}{2}} \|\tilde{\Delta}_H \hat{\mathbf{u}}_H\|^{\frac{1}{2}} \|\nabla \hat{\mathbf{u}}_H\| \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\| \\
&\leq C\|\nabla \hat{\mathbf{u}}_H\|^{\frac{3}{2}} \|\tilde{\Delta}_H \hat{\mathbf{u}}_H\|^{\frac{1}{2}} \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\| \\
&\leq \big(\frac{C}{2\nu}\big)\|\nabla \hat{\mathbf{u}}_H\|^3 \|\tilde{\Delta}_H \hat{\mathbf{u}}_H\| + \frac{\nu}{2}\|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\|^2.
\end{aligned} \tag{5.2.23}$$

Using (5.2.22) and (5.2.23) in (5.2.19), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla \hat{\mathbf{u}}_h^*\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\|^2) + (\nu - 2\kappa\alpha)\|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\|^2 \\
&\leq 2\alpha\|\nabla \hat{\mathbf{u}}_h^*\|^2 + C(\nu)\big(\|\nabla \hat{\mathbf{u}}_h^*\|^2 \|\nabla \hat{\mathbf{u}}_H\|^4 + \|\nabla \hat{\mathbf{u}}_H\|^3 \|\tilde{\Delta}_H \hat{\mathbf{u}}_H\|\big).
\end{aligned} \tag{5.2.24}$$

An integration of (5.2.24) with respect to time from 0 to t yields

$$\begin{aligned}
&\|\nabla \hat{\mathbf{u}}_h^*\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*\|^2 + \beta\int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*(s)\|^2 ds \leq \|\nabla \mathbf{u}_{0h}^*\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_{0h}^*\|^2 \\
&+ C(\nu,\alpha)\int_0^t \big(\|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 + \|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 \|\nabla \hat{\mathbf{u}}_H(s)\|^4 + \|\nabla \hat{\mathbf{u}}_H(s)\|^3 \|\tilde{\Delta}_H \hat{\mathbf{u}}_H(s)\|\big) ds.
\end{aligned} \tag{5.2.25}$$

Using (5.2.16) and Lemma 5.1, we bound

$$\int_0^t \big(\|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 + \|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 \|\nabla \hat{\mathbf{u}}_H(s)\|^4 + \|\nabla \hat{\mathbf{u}}_H(s)\|^3 \|\tilde{\Delta}_H \hat{\mathbf{u}}_H(s)\|\big) ds \leq C(\kappa,\nu,\alpha,\lambda_1,M). \tag{5.2.26}$$

An application of (5.2.26) in (5.2.25) leads to

$$\|\nabla \mathbf{u}_h^*\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h^*\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h \mathbf{u}_h^*(s)\|^2 ds \leq C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \tag{5.2.27}$$

A combination of (5.2.16) and (5.2.27) completes the rest of the proof. □

Lemma 5.3. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then there exists a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$*, such that, for all* $t > 0$*,*

$$\|\mathbf{u}_{ht}^*(t)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}^*(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Proof. Substitute $\boldsymbol{\phi}_h = \mathbf{u}_{ht}^*$ in (5.2.6) and observe that

$$\begin{aligned}\|\mathbf{u}_{ht}^*\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*\|^2 &= -\nu(\nabla\mathbf{u}_h^*, \nabla\mathbf{u}_{ht}^*) - b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{u}_{ht}^*) - b(\mathbf{u}_h^*, \mathbf{u}_H, \mathbf{u}_{ht}^*) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}_{ht}^*)\\ &= I_1 + I_2 + I_3 + I_4, \text{ say.} \qquad (5.2.28)\end{aligned}$$

We use Cauchy-Schwarz's inequality and Young's inequality to obtain

$$\begin{aligned}|I_1| &\leq \nu\|\nabla\mathbf{u}_h^*\|\|\nabla\mathbf{u}_{ht}^*\|\\ &\leq \Big(\frac{\nu}{2\epsilon}\Big)\|\nabla\mathbf{u}_h^*\|^2 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_{ht}^*\|^2. \qquad (5.2.29)\end{aligned}$$

To bound $|I_2|$ and $|I_3|$, we apply (3.2.11) and Young's inequality as follows:

$$\begin{aligned}|I_2| + |I_3| &= |b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{u}_{ht}^*)| + |b(\mathbf{u}_h^*, \mathbf{u}_H, \mathbf{u}_{ht}^*)|\\ &\leq C\|\nabla\mathbf{u}_H\|\|\nabla\mathbf{u}_h^*\|\|\nabla\mathbf{u}_{ht}^*\|\\ &\leq \left(\frac{C}{2\epsilon}\right)\|\nabla\mathbf{u}_H\|^2\|\nabla\mathbf{u}_h^*\|^2 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_{ht}^*\|^2. \qquad (5.2.30)\end{aligned}$$

We find estimates for I_4 using (3.2.11) as

$$\begin{aligned}|I_4| &= |b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}_{ht}^*)|\\ &\leq C\|\nabla\mathbf{u}_H\|^2\|\nabla\mathbf{u}_{ht}^*\|\\ &\leq \left(\frac{C}{2\epsilon}\right)\|\nabla\mathbf{u}_H\|^4 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_{ht}^*\|^2. \qquad (5.2.31)\end{aligned}$$

A use of (5.2.29)-(5.2.31) with $\epsilon = \frac{\kappa}{3}$ in (5.2.28) yields

$$\|\mathbf{u}_{ht}^*\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*\|^2 \le C(\kappa,\nu)\big(\|\nabla\mathbf{u}_h^*\|^2 + \|\nabla\mathbf{u}_H\|^2\|\nabla\mathbf{u}_h^*\|^2 + \|\nabla\mathbf{u}_H\|^4\big). \tag{5.2.32}$$

With the help of estimates obtained in Lemmas 5.1 and 5.2, we write

$$\|\mathbf{u}_{ht}^*\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \tag{5.2.33}$$

Next, substitute $\boldsymbol{\phi}_h = e^{2\alpha t}\mathbf{u}_{ht}^*$ in (5.2.6) to obtain

$$\begin{aligned} e^{2\alpha t}(\|\mathbf{u}_{ht}^*\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*\|^2) &= -e^{2\alpha t}\nu a(\mathbf{u}_h^*,\mathbf{u}_{ht}^*) - e^{2\alpha t}b(\mathbf{u}_H,\mathbf{u}_h^*,\mathbf{u}_{ht}^*) \\ &\quad - e^{2\alpha t}b(\mathbf{u}_h^*,\mathbf{u}_H,\mathbf{u}_{ht}^*) + e^{2\alpha t}b(\mathbf{u}_H,\mathbf{u}_H,\mathbf{u}_{ht}^*). \end{aligned} \tag{5.2.34}$$

An application of Cauchy-Schwarz's inequality, (3.2.11) and Young's inequality with integration from 0 to t with respect to time leads to

$$\begin{aligned} \int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}^*(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*(s)\|^2)ds &\le C(\kappa,\nu)\int_0^t e^{2\alpha s}\big(\|\nabla\mathbf{u}_h^*(s)\|^2 + \|\nabla\mathbf{u}_H(s)\|^2\|\nabla\mathbf{u}_h^*(s)\|^2 \\ &\quad + \|\nabla\mathbf{u}_H(s)\|^4\big)ds. \end{aligned} \tag{5.2.35}$$

Note that from Lemma 5.1, we arrive at

$$\begin{aligned} \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_H(s)\|^4 ds &\le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_H(s)\|^2 ds \\ &\le C(\kappa,\nu,\alpha,\lambda_1,M). \end{aligned} \tag{5.2.36}$$

An application of (5.2.36), Lemmas 5.1 and 5.2 in (5.2.35) yields

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}^*(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}^*(s)\|^2)ds \le C(\kappa,\nu,\alpha,\lambda_1,M). \tag{5.2.37}$$

A combination of (5.2.33) and (5.2.37) leads to the desired result and this completes the rest of the proof. $\square$

Lemma 5.4. *Let $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ and let the assumptions $(\mathbf{A1})$–$(\mathbf{A2})$ hold true. Then, there exists a positive constant $C = C(\kappa,\nu,\alpha,\lambda_1,M)$, such that, for all $t>0$,*

$$\|\mathbf{u}^*_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\big(\|\mathbf{u}^*_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}(s)\|^2\big)ds \le Ce^{-2\alpha t}.$$

Proof. Differentiate (5.2.6) with respect to time to arrive at

$$\begin{aligned}(\mathbf{u}^*_{htt},\boldsymbol{\phi}_h) &+ \kappa a(\mathbf{u}^*_{htt},\boldsymbol{\phi}_h) + \nu a(\mathbf{u}^*_{ht},\boldsymbol{\phi}_h) + b(\mathbf{u}^*_{ht},\mathbf{u}_H,\boldsymbol{\phi}_h) + b(\mathbf{u}^*_{h},\mathbf{u}_{Ht},\boldsymbol{\phi}_h) \\ &+ b(\mathbf{u}_H,\mathbf{u}^*_{ht},\boldsymbol{\phi}_h) + b(\mathbf{u}_{Ht},\mathbf{u}^*_{h},\boldsymbol{\phi}_h) = b(\mathbf{u}_{Ht},\mathbf{u}_H,\boldsymbol{\phi}_h) + b(\mathbf{u}_H,\mathbf{u}_{Ht},\boldsymbol{\phi}_h)\ \forall\phi_h\in\mathbf{J}_h,\ \ t>0.\end{aligned} \tag{5.2.38}$$

Substituting $\boldsymbol{\phi}_h = \mathbf{u}^*_{htt}$ in (5.2.38), we obtain

$$\begin{aligned}\|\mathbf{u}^*_{htt}\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}\|^2 &= -\nu a(\mathbf{u}^*_{ht},\mathbf{u}^*_{htt}) - b(\mathbf{u}^*_{ht},\mathbf{u}_H,\mathbf{u}^*_{htt}) - b(\mathbf{u}^*_{h},\mathbf{u}_{Ht},\mathbf{u}^*_{htt}) \\ &- b(\mathbf{u}_H,\mathbf{u}^*_{ht},\mathbf{u}^*_{htt}) - b(\mathbf{u}_{Ht},\mathbf{u}^*_{h},\mathbf{u}^*_{htt}) + b(\mathbf{u}_{Ht},\mathbf{u}_H,\mathbf{u}^*_{htt}) + b(\mathbf{u}_H,\mathbf{u}_{Ht},\mathbf{u}^*_{htt}).\end{aligned} \tag{5.2.39}$$

Using Cauchy-Schwarz's inequality, Young's inequality and (3.2.11), we find that

$$\begin{aligned}\|\mathbf{u}^*_{htt}\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}\|^2 \le & C(\kappa,\nu)\big(\|\nabla\mathbf{u}^*_{ht}\|^2 + \|\nabla\mathbf{u}_H\|^2\|\nabla\mathbf{u}^*_{ht}\|^2 \\ & + \|\nabla\mathbf{u}_{Ht}\|^2\|\nabla\mathbf{u}^*_{h}\|^2 + \|\nabla\mathbf{u}_H\|^2\|\nabla\mathbf{u}_{Ht}\|^2\big).\end{aligned} \tag{5.2.40}$$

With the help of Lemmas 5.1, 5.2 and 5.3 in (5.2.40), we write

$$\|\mathbf{u}^*_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}(t)\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \tag{5.2.41}$$

Multiplying (5.2.40) by $e^{2\alpha t}$ and integrating with respect to time from 0 to t, we arrive at

$$\begin{aligned}\int_0^t e^{2\alpha s}\big(\|\mathbf{u}^*_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}^*_{htt}(s)\|^2\big)ds \le C(\kappa,\nu)\bigg(\int_0^t e^{2\alpha s}\big(\|\nabla\mathbf{u}^*_{ht}(s)\|^2 + \|\nabla\mathbf{u}_H(s)\|^2\times \\ \|\nabla\mathbf{u}^*_{ht}(s)\|^2 + \|\nabla\mathbf{u}_{Ht}(s)\|^2\|\nabla\mathbf{u}^*_{h}(s)\|^2 + \|\nabla\mathbf{u}_H(s)\|^2\|\nabla\mathbf{u}_{Ht}(s)\|^2\big)ds\bigg).\end{aligned} \tag{5.2.42}$$

An application of estimates obtained in Lemmas 5.1, 5.2 and 5.3 in (5.2.42) yields

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}^*_{htt}(s)\|^2+\kappa\|\nabla\mathbf{u}^*_{htt}(s)\|^2)ds \le C(\kappa,\nu,\alpha,\lambda_1,M). \tag{5.2.43}$$

A use of (5.2.41) with (5.2.43) completes the rest of the proof. □

We state below a few more a *priori* bounds for semidiscrete solution $\mathbf{u}^*_h$ in **Step 2** (see (5.2.6)) which will be helpful in our fully discrete error analysis. We will not go in to the details of the proof as it is along the similar lines as in the proofs of previous lemmas.

Lemma 5.5. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there exists a positive constant* $C = C(\kappa,\nu,\alpha,\lambda_1,M)$*, such that, for all* $t > 0$*,*

$$\|\mathbf{u}^*_{httt}(t)\|^2+\kappa\|\nabla\mathbf{u}^*_{httt}(t)\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}^*_{httt}(s)\|^2+\kappa\|\nabla\mathbf{u}^*_{httt}(s)\|^2)ds \le Ce^{-2\alpha t}.$$

□

Next, we present a *priori* estimates in **Step 3** (see (5.2.7)) of the semidiscrete solution $\mathbf{u}_h$. These bounds follow by proceeding as in the proofs of Lemmas 5.2, 5.3, 5.4 and 5.5.

Lemma 5.6. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there exists a positive constant* $C = C(\kappa,\nu,\alpha,\lambda_1,M)$*, such that, for all* $t > 0$*,*

$$\|\mathbf{u}_h(t)\|^2+\kappa\|\nabla\mathbf{u}_h(t)\|^2+\kappa\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2+\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_h(s)\|^2+\|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2)\,ds \le Ce^{-2\alpha t},$$

$$\|\mathbf{u}_{ht}(t)\|^2+\kappa\|\nabla\mathbf{u}_{ht}(t)\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2+\kappa\|\nabla\mathbf{u}_{ht}(s)\|^2)ds \le Ce^{-2\alpha t},$$

$$\|\mathbf{u}_{htt}(t)\|^2+\kappa\|\nabla\mathbf{u}_{htt}(t)\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2+\kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \le Ce^{-2\alpha t}$$

and

$$\|\mathbf{u}_{httt}(t)\|^2 + \kappa\|\nabla \mathbf{u}_{httt}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{httt}\|^2 + \kappa\|\nabla \mathbf{u}_{httt}(s)\|^2)ds \le Ce^{-2\alpha t}.$$

□

The error estimates of velocity in $L^\infty(\mathbf{H}^1)$ and $L^\infty(\mathbf{L^2})$-norms are based on estimates derived for a new auxiliary operator defined through a modification of the Stokes operator. Since $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$, the weak solution $\mathbf{u}$ satisfies

$$(\mathbf{u}_t, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}, \boldsymbol{\phi}_h) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) = (p, \nabla\cdot\boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.2.44}$$

Then, from (5.2.6) and (5.2.44), we obtain

$$\begin{aligned}(\mathbf{u}_t - \mathbf{u}^*_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_t - \mathbf{u}^*_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u} - \mathbf{u}^*_h, \boldsymbol{\phi}_h) = -b(\mathbf{u} - \mathbf{u}^*_h, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ - b(\mathbf{u}_H, \mathbf{u} - \mathbf{u}^*_h, \boldsymbol{\phi}_h) + b(\mathbf{u} - \mathbf{u}_H, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h).\end{aligned} \tag{5.2.45}$$

Similarly, subtracting (5.2.7) from (5.2.6), we arrive at

$$\begin{aligned}&(\mathbf{u}^*_{ht} - \mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}^*_{ht} - \mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}^*_h - \mathbf{u}_h, \boldsymbol{\phi}_h) \\ &= -b(\mathbf{u}^*_h - \mathbf{u}_h, \mathbf{u}_H, \boldsymbol{\phi}_h) - b(\mathbf{u}_H, \mathbf{u}^*_h - \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_H - \mathbf{u}^*_h, \mathbf{u}_H - \mathbf{u}^*_h, \boldsymbol{\phi}_h).\end{aligned} \tag{5.2.46}$$

Using (5.2.45) and (5.2.46), the equation in $\mathbf{u} - \mathbf{u}_h$ can be written as

$$\begin{aligned}(\mathbf{u}_t - \mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_t - \mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ - b(\mathbf{u}_H, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_H - \mathbf{u}, \mathbf{u} - \mathbf{u}^*_h, \boldsymbol{\phi}_h) + b(\mathbf{u} - \mathbf{u}^*_h, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h) \\ + b(\mathbf{u}^*_h - \mathbf{u}, \mathbf{u}^*_h - \mathbf{u}, \boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.2.47}$$

For optimal error estimates of $\mathbf{u} - \mathbf{u}_h$ in $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}^1)$-norms, we recall the following auxiliary projection, called as Sobolev-Stokes projection V_h, such that $V_h\mathbf{u}(t) \in \mathbf{J}_h$ and satisfies

$$\kappa a(\mathbf{u}_t - V_h\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u} - V_h\mathbf{u}, \boldsymbol{\phi}_h) = (p, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.2.48}$$

where $V_h\mathbf{u}(0) = P_h\mathbf{u}_0$, defined in Chapter **3** (see (3.4.18)).
With $V_h\mathbf{u}$ defined as above, we now split $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ as

$$\mathbf{e} := (\mathbf{u} - V_h\mathbf{u}) + (V_h\mathbf{u} - \mathbf{u}_h) := \boldsymbol{\zeta} + \Theta. \tag{5.2.49}$$

Using (5.2.47)-(5.2.49), the equation in Θ becomes

$$\begin{aligned}(\Theta_t, \boldsymbol{\phi}_h) + \kappa a(\Theta_t, \boldsymbol{\phi}_h) + \nu a(\Theta, \boldsymbol{\phi}_h) &= -b(\Theta, \mathbf{u}_H, \boldsymbol{\phi}_h) - b(\mathbf{u}_H, \Theta, \boldsymbol{\phi}_h) \\ -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) - b(\boldsymbol{\zeta}, \mathbf{u}_H, \boldsymbol{\phi}_h) &- b(\mathbf{u}_H, \boldsymbol{\zeta}, \boldsymbol{\phi}_h) + b(\mathbf{u} - \mathbf{u}_H, \mathbf{u}_h^* - \mathbf{u}, \boldsymbol{\phi}_h) \\ +b(\mathbf{u} - \mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h) &+ b(\mathbf{u}_h^* - \mathbf{u}, \mathbf{u}_h^* - \mathbf{u}, \boldsymbol{\phi}_h).\end{aligned} \tag{5.2.50}$$

Observe that, estimates for $\mathbf{e}$ involve estimates of $\boldsymbol{\zeta}$ and Θ and to bound Θ, we need $\mathbf{u}-\mathbf{u}_H$, $\mathbf{u} - \mathbf{u}_h^*$ and $\boldsymbol{\zeta}$ estimates. The following estimates of $\boldsymbol{\zeta}$ and $\mathbf{u} - \mathbf{u}_H$ are already derived in Chapter **3** (see Lemmas 3.8, 3.10, 3.11 and Theorem 3.1), so we state the Thorem without proof.

Theorem 5.1. *Assume that* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *are satisfied. Then, there exists a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$ *such that for* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, *the following estimate holds true:*

$$\|\boldsymbol{\zeta}(t)\|^2 + h^2\|\nabla\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{\zeta}(s)\|^2 + h^2\nabla\boldsymbol{\zeta}(s)\|^2 + \|\boldsymbol{\zeta}_t(s)\|^2)ds \le Ch^4e^{-2\alpha t}.$$

□

Theorem 5.2. *Let the assumptions* (**A1)-(A2)** *and* (**B1)-(B2)** *be satisfied. Further, let the discrete initial velocity* $\mathbf{u}_{0H} = P_H\mathbf{u}_0$. *Then, there exists a positive constant* C *which depends on* κ, ν, λ_1, α *and* M, *such that, for all* $t > 0$ *and for* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, *the* *following estimate holds true:*

$$\|(\mathbf{u}-\mathbf{u}_H)(t)\|^2 + H^2\|\nabla(\mathbf{u}-\mathbf{u}_H)(t)\|^2 + H^2 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}-\mathbf{u}_H)(s)\|^2 ds \le CH^4 e^{-2\alpha t}.$$

□

Now, we are left with derivations of estimate for $\mathbf{u}-\mathbf{u}_h^*$ to obtain estimates of Θ.
In order to derive $\mathbf{u}-\mathbf{u}_h^*$ estimates, we use the Sobolev Stoke's projection to split $\mathbf{u}-\mathbf{u}_h^*$ as:

$$\mathbf{u}-\mathbf{u}_h^* = (\mathbf{u}-V_h\mathbf{u}) + (V_h\mathbf{u}-\mathbf{u}_h^*) := \boldsymbol{\zeta}+\boldsymbol{\rho}. \tag{5.2.51}$$

Once again, with the help of Theorem 5.1, we have derived various estimates of $\boldsymbol{\zeta}$. The following lemma yields the estimates for $\boldsymbol{\rho}$.

Lemma 5.7. *Let the assumptions* **(A1)-(A2)** *and* **(B1)-(B2)** *be satisfied. Then, there exists a positive constant C which depends on $\kappa,\ \nu,\ \lambda_1,\ \alpha$ and M, such that, for all $t>0$ and for $0\le\alpha<\dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, the following estimate holds true:*

$$(\|\boldsymbol{\rho}\|^2+\kappa\|\nabla\boldsymbol{\rho}\|^2) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\rho}(s)\|^2 ds \le C(\kappa,\nu,\lambda_1,\alpha,M)(h^2+H^{(6-2\delta)})e^{-2\alpha t},$$

where $\delta>0$ arbitrary small for two dimensions and $\delta=1/2$ for three dimensions.

Proof. A use of (5.2.48) and (5.2.51) in (5.2.45) yields

$$\begin{aligned}(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h)+\kappa a(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h)+\nu a(\boldsymbol{\rho},\boldsymbol{\phi}_h) &= -b(\boldsymbol{\rho},\mathbf{u}_H,\boldsymbol{\phi}_h)-b(\mathbf{u}_H,\boldsymbol{\rho},\boldsymbol{\phi}_h)\\ &-(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h)-b(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\phi}_h)-b(\mathbf{u}_H,\boldsymbol{\zeta},\boldsymbol{\phi}_h)+b(\mathbf{u}-\mathbf{u}_H,\mathbf{u}_H-\mathbf{u},\boldsymbol{\phi}_h).\end{aligned} \tag{5.2.52}$$

Substituting $\boldsymbol{\phi}_h=e^{\alpha t}\hat{\boldsymbol{\rho}}$ in (5.2.52), using (2.2.3) and (3.2.1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\hat{\boldsymbol{\rho}}\|^2+\kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2)+\Big(\nu-\alpha\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\boldsymbol{\rho}}\|^2 &= -e^{\alpha t}(\boldsymbol{\zeta}_t,\hat{\boldsymbol{\rho}})-e^{-\alpha t}b(\hat{\boldsymbol{\rho}},\hat{\mathbf{u}}_H,\hat{\boldsymbol{\rho}})\\ -e^{-\alpha t}b(\hat{\boldsymbol{\zeta}},\hat{\mathbf{u}}_H,\hat{\boldsymbol{\rho}})-e^{-\alpha t}b(\hat{\mathbf{u}}_H,\hat{\boldsymbol{\zeta}},\hat{\boldsymbol{\rho}})&+e^{-\alpha t}b(\hat{\mathbf{u}}-\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_H-\hat{\mathbf{u}},\hat{\boldsymbol{\rho}}).\end{aligned} \tag{5.2.53}$$

With the help of Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we find that

$$
\begin{aligned}
|e^{\alpha t}(\boldsymbol{\zeta}_t, \hat{\boldsymbol{\rho}})| &\leq C(\lambda_1)\|e^{\alpha t}\boldsymbol{\zeta}_t\|\|\nabla\hat{\boldsymbol{\rho}}\| \\
&\leq C(\lambda_1,\epsilon)\|e^{2\alpha t}\boldsymbol{\zeta}_t\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\rho}}\|^2. \qquad (5.2.54)
\end{aligned}
$$

Using (3.2.9) and Young's inequality in the second term on the right hand side of (5.2.53), we obtain

$$
\begin{aligned}
e^{-\alpha t}|b(\hat{\boldsymbol{\rho}}, \hat{\mathbf{u}}_H, \hat{\boldsymbol{\rho}})| &\leq Ce^{-\alpha t}\|\hat{\boldsymbol{\rho}}\|\|\nabla\hat{\mathbf{u}}_H\|^{\frac{1}{2}}\|\tilde{\Delta}_H\hat{\mathbf{u}}_H\|^{\frac{1}{2}}\|\nabla\hat{\boldsymbol{\rho}}\| \qquad (5.2.55) \\
&\leq C(\epsilon)e^{-2\alpha t}\|\nabla\hat{\mathbf{u}}_H\|\|\tilde{\Delta}_H\hat{\mathbf{u}}_H\|\|\hat{\boldsymbol{\rho}}\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\rho}}\|^2.
\end{aligned}
$$

An application of (3.2.11), Young's inequality and Theorem 5.1 in the third and fourth terms on the right hand side of (5.2.53) yields

$$
\begin{aligned}
|e^{-\alpha t}b(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}}_H, \hat{\boldsymbol{\rho}}) + e^{-\alpha t}b(\hat{\mathbf{u}}_H, \hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\rho}})| &\leq C\|\nabla\hat{\boldsymbol{\zeta}}\|\|\nabla\hat{\mathbf{u}}_H\|\|\nabla\hat{\boldsymbol{\rho}}\| \\
&\leq C(\epsilon)\|\nabla\hat{\boldsymbol{\zeta}}\|^2\|\nabla\hat{\mathbf{u}}_H\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\rho}}\|^2 \\
&\leq C(\kappa,\ \nu,\ \lambda_1,\ \alpha, M,\epsilon)h^2\|\nabla\hat{\mathbf{u}}_H\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\rho}}\|^2. \quad (5.2.56)
\end{aligned}
$$

Using Lemma 3.2, estimates of Theorem 5.2 and Young's inequality in the last term on the right hand side of (5.2.53), we find that

$$
\begin{aligned}
|e^{-\alpha t}b(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H - \hat{\mathbf{u}}, \hat{\boldsymbol{\rho}})| &\leq C\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_H\|^{(1-\delta)}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|^{\delta}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|\|\nabla\hat{\boldsymbol{\rho}}\| \\
&\leq C(\kappa,\nu,\lambda_1,\alpha,M)H^{2(1-\delta)}H^{\delta}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|\|\nabla\hat{\boldsymbol{\rho}}\| \\
&\leq C(\kappa,\nu,\lambda_1,\alpha,M)H^{2-\delta}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|\|\nabla\hat{\boldsymbol{\rho}}\| \\
&\leq C(\kappa,\nu,\lambda_1,\alpha,M,\epsilon)H^{(4-2\delta)}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\rho}}\|^2.
\end{aligned}
$$
(5.2.57)

Substitute estimates from (5.2.54)-(5.2.57) with $\epsilon = \frac{\nu}{4}$ in (5.2.53) and integrate the resulting equation with respect to time from 0 to t to obtain

$$
\begin{aligned}
(\|\hat{\boldsymbol{\rho}}\|^2+\kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2)+\Big(\nu-2\alpha\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\int_0^t\|\nabla\hat{\boldsymbol{\rho}}(s)\|^2ds &\le C(\kappa,\nu,\lambda_1,\alpha,M)\times \\
\int_0^t\Big(\|e^{2\alpha s}\boldsymbol{\zeta}_t(s)\|^2+h^2\|\nabla\hat{\mathbf{u}}_H(s)\|^2+H^{(4-2\delta)}\|\nabla(\hat{\mathbf{u}}-\hat{\mathbf{u}}_H)(s)\|^2\Big)ds \\
+C(\nu)\int_0^t e^{-2\alpha s}\|\nabla\hat{\mathbf{u}}_H(s)\|\|\tilde{\Delta}_H\hat{\mathbf{u}}_H(s)\|\|\hat{\boldsymbol{\rho}}(s)\|^2ds. \qquad (5.2.58)
\end{aligned}
$$

A use of Lemma 5.1 with Theorems 5.1 and 5.2 yields

$$
\begin{aligned}
(\|\hat{\boldsymbol{\rho}}\|^2+\kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2)+\Big(\nu-2\alpha\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\int_0^t\|\nabla\hat{\boldsymbol{\rho}}(s)\|^2ds &\le C(\kappa,\nu,\lambda_1,\alpha,M)(h^2+H^{6-2\delta}) \\
+C(\nu)\int_0^t e^{-2\alpha s}\|\nabla\hat{\mathbf{u}}_H(s)\|\|\tilde{\Delta}_H\hat{\mathbf{u}}_H(s)\|\|\hat{\boldsymbol{\rho}}(s)\|^2ds. \qquad (5.2.59)
\end{aligned}
$$

An application of the Gronwall's lemma along with a *priori* bounds of $\mathbf{u}_H$ from Lemma 5.1 in (5.2.59) yields

$$
(\|\hat{\boldsymbol{\rho}}\|^2+\kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2)+\Big(\nu-2\alpha\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\int_0^t\|\nabla\hat{\boldsymbol{\rho}}(s)\|^2ds \le C(\kappa,\nu,\lambda_1,\alpha,M)(h^2+H^{(6-2\delta)}). \qquad (5.2.60)
$$

Using $0\le\alpha<\dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$, the coefficient of second term on the left hand side becomes positive. Hence, we write

$$
(\|\boldsymbol{\rho}\|^2+\kappa\|\nabla\boldsymbol{\rho}\|^2)+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\rho}(s)\|^2ds \le C(\kappa,\nu,\lambda_1,\alpha,M)(h^2+H^{(6-2\delta)})e^{-2\alpha t}
$$

and this completes the proof. □

Applying estimates of Theorem 5.1 and Lemma 5.7 along with triangle's inequality in (5.2.51) would lead us to the following result:

Theorem 5.3. *Let the assumptions* **(A1)-(A2)** *and* **(B1)-(B2)** *be satisfied. Then, there exists a positive constant C which depends on κ, ν, λ_1, α and M such that for all $t>0$*

and $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$, *the following estimate holds:*

$$\big(\|\mathbf{u}-\mathbf{u}_h^*\|^2+\kappa\|\nabla(\mathbf{u}-\mathbf{u}_h^*)\|^2\big)+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}-\mathbf{u}_h^*)(s)\|^2 ds \leq C(h^2+H^{(6-2\delta)})e^{-2\alpha t},$$

where $\delta > 0$ is same as in (3.2.12).

Now, to derive estimates of $\mathbf{e}$, we are left with the derivation of estimates of Θ.

The next lemma establishes the estimates for Θ.

Lemma 5.8. *Let the assumptions* **(A1)-(A2)** *and* **(B1)-(B2)** *be satisfied. Then, there exists a positive constant C which depends on κ, ν, λ_1, α and M such that for all $t > 0$ and* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$, *the following estimate holds:*

$$\|\Theta\|^2+\kappa\|\nabla\Theta\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\Theta(s)\|^2 ds \leq C\big(h^4+h^2H^{(4-2\delta)}+H^{(10-4\delta)}+H^{(12-4\delta)}\big)e^{-2\alpha t}.$$

Proof. Using (5.2.50), the equation in Θ is

$$\begin{aligned}(\Theta_t,\boldsymbol{\phi}_h)+\kappa a(\Theta_t,\boldsymbol{\phi}_h)+\nu a(\Theta,\boldsymbol{\phi}_h) &= -b(\Theta,\mathbf{u}_H,\boldsymbol{\phi}_h)-b(\mathbf{u}_H,\Theta,\boldsymbol{\phi}_h)\\ -(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h)-b(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\phi}_h)&-b(\mathbf{u}_H,\boldsymbol{\zeta},\boldsymbol{\phi}_h)+b(\mathbf{u}-\mathbf{u}_H,\mathbf{u}_h^*-\mathbf{u},\boldsymbol{\phi}_h)\\ +b(\mathbf{u}-\mathbf{u}_h^*,\mathbf{u}_H-\mathbf{u},\boldsymbol{\phi}_h)&+b(\mathbf{u}_h^*-\mathbf{u},\mathbf{u}_h^*-\mathbf{u},\boldsymbol{\phi}_h). \qquad (5.2.61)\end{aligned}$$

Substituting $\boldsymbol{\phi}_h = e^{2\alpha t}\hat{\Theta}$, using (2.2.3) and (3.2.1) in (5.2.61), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\hat{\Theta}\|^2+\kappa\|\nabla\hat{\Theta}\|^2)+\Big(\nu-\alpha\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\Theta}\|^2 \leq -e^{-\alpha t}b(\hat{\Theta},\hat{\mathbf{u}}_H,\hat{\Theta})\\ &-(e^{\alpha t}\boldsymbol{\zeta}_t,\hat{\Theta})+e^{-\alpha t}\Big(-b(\hat{\boldsymbol{\zeta}},\hat{\mathbf{u}}_H,\hat{\Theta})-b(\hat{\mathbf{u}}_H,\hat{\boldsymbol{\zeta}},\hat{\Theta})+b(\hat{\mathbf{u}}-\hat{\mathbf{u}}_H,\hat{\mathbf{u}}_h^*-\hat{\mathbf{u}},\hat{\Theta})\\ &+b(\hat{\mathbf{u}}-\hat{\mathbf{u}}_h^*,\hat{\mathbf{u}}_H-\hat{\mathbf{u}},\hat{\Theta})+b(\hat{\mathbf{u}}_h^*-\hat{\mathbf{u}},\hat{\mathbf{u}}_h^*-\hat{\mathbf{u}},\hat{\Theta})\Big). \qquad (5.2.62)\end{aligned}$$

Now, to bound the first term in the right hand side of (5.2.62), we use (3.2.10) and Young's inequality to obtain

$$
\begin{aligned}
|e^{-\alpha t}b(\hat{\Theta}, \hat{\mathbf{u}}_H, \hat{\Theta})| &\leq Ce^{-\alpha t}\|\hat{\Theta}\|^{\frac{1}{2}}\|\nabla\hat{\Theta}\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{u}}_H\|\|\hat{\Theta}\|^{\frac{1}{2}}\|\nabla\hat{\Theta}\|^{\frac{1}{2}} \\
&\leq C(\epsilon)e^{-2\alpha t}\|\nabla\hat{\mathbf{u}}_H\|^2\|\hat{\Theta}\|^2 + \epsilon\|\nabla\hat{\Theta}\|^2. \qquad (5.2.63)
\end{aligned}
$$

For the second term on the right hand side of (5.2.62), use Cauchy-Schwarz's inequality, Young's inequality and (2.2.3) to find that

$$
|(e^{\alpha t}\boldsymbol{\zeta}_t, \hat{\Theta})| \leq \|e^{\alpha t}\boldsymbol{\zeta}_t\|\|\hat{\Theta}\| \leq C(\lambda_1, \epsilon)\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 + \epsilon\|\nabla\hat{\Theta}\|^2. \qquad (5.2.64)
$$

A use of Young's inequality, the generalized Hölder's inequality, Sobolev embedding theorems, Sobolev's inequalities and Lemma 5.1 yields

$$
\begin{aligned}
|e^{-\alpha t}\big(-b(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}}_H, \hat{\Theta}) - b(\hat{\mathbf{u}}_H, \hat{\boldsymbol{\zeta}}, \hat{\Theta})\big)| &\leq C\|\nabla\hat{\mathbf{u}}_H\|^{\frac{1}{2}}\|\tilde{\Delta}_H\hat{\mathbf{u}}_H\|^{\frac{1}{2}}\|\hat{\boldsymbol{\zeta}}\|\|\nabla\hat{\Theta}\| \\
&\leq C(\kappa, \nu, \lambda_1, \alpha, M)\|\hat{\boldsymbol{\zeta}}\|\|\nabla\hat{\Theta}\| \\
&\leq C(\kappa, \nu, \lambda_1, \alpha, M, \epsilon)\|\hat{\boldsymbol{\zeta}}\|^2 + \epsilon\|\nabla\hat{\Theta}\|^2. \qquad (5.2.65)
\end{aligned}
$$

With the help of Lemma 3.2, Young's inequality and the estimates obtained in Theorem 5.2, we arrive at

$$
\begin{aligned}
|e^{-\alpha t}&(b(\hat{\mathbf{u}}-\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^* - \hat{\mathbf{u}}, \hat{\Theta}) + b(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H - \hat{\mathbf{u}}, \hat{\Theta}))| \\
&\leq C\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_H\|^{1-\delta}\|\nabla(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)\|^{\delta}\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|\|\nabla\hat{\Theta}\| \\
&\leq C(\kappa, \nu, \lambda_1, \alpha, M)H^{2(1-\delta)}H^{\delta}\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|\|\nabla\hat{\Theta}\| \\
&\leq C(\kappa, \nu, \lambda_1, \alpha, M)H^{(2-\delta)}\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|\|\nabla\hat{\Theta}\| \qquad (5.2.66) \\
&\leq C(\kappa, \nu, \lambda_1, \alpha, M, \epsilon)H^{(4-2\delta)}\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|^2 + \epsilon\|\nabla\hat{\Theta}\|^2.
\end{aligned}
$$

Using (3.2.11), we write

$$
|e^{-\alpha t}b(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}}, \hat{\mathbf{u}}_h^* - \hat{\mathbf{u}}, \hat{\Theta})| \leq C\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|^2\|\nabla\hat{\Theta}\|. \qquad (5.2.67)
$$

A use of Theorem 5.3 and Young's inequality yields

$$
\begin{aligned}
|e^{-\alpha t} b(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}}, \hat{\mathbf{u}}_h^* - \hat{\mathbf{u}}, \hat{\Theta})| &\le C(\kappa,\nu,\lambda_1,\alpha,M)(h + H^{(3-\delta)})\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|\|\nabla\hat{\Theta}\| \\
&\le C(\kappa,\nu,\lambda_1,\alpha,M,\epsilon)(h^2 + H^{6-2\delta})\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|^2 + \epsilon\|\nabla\hat{\Theta}\|^2. \quad (5.2.68)
\end{aligned}
$$

Using (5.2.63)-(5.2.68) in (5.2.62) with $\epsilon = \frac{\nu}{5}$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\|\hat{\Theta}\|^2 + \kappa\|\nabla\hat{\Theta}\|^2) + \Big(\nu - 2\alpha\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\Theta}\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)\Big(\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 + \|\hat{\boldsymbol{\zeta}}\|^2 \\
&\big(h^2 + H^{(4-2\delta)} + H^{(6-2\delta)}\big)\|\nabla(\hat{\mathbf{u}}_h^* - \hat{\mathbf{u}})\|^2\Big) + C(\nu)e^{-2\alpha t}\|\nabla\hat{\mathbf{u}}_H\|^2\|\hat{\Theta}\|^2. \quad (5.2.69)
\end{aligned}
$$

Integrate (5.2.69) with respect to time from 0 to t. Now a use of Theorems 5.1 and 5.3 yields

$$
\begin{aligned}
\|\hat{\Theta}\|^2 + \kappa\|\nabla\hat{\Theta}\|^2 + \Big(\nu - 2\alpha\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\int_0^t \|\nabla\hat{\Theta}(s)\|^2 ds &\le C(\kappa,\nu,\alpha,\lambda_1,M)\Big(h^4 \\
&+ \big(h^2 + H^{(4-2\delta)} + H^{(6-2\delta)}\big)\big(h^2 + H^{(6-2\delta)}\big)\Big) + C(\nu)\int_0^t \|\nabla\hat{\mathbf{u}}_H(s)\|^2\|\hat{\Theta}(s)\|^2 ds \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)\Big(h^4 + h^2H^{(4-2\delta)} + h^2H^{(6-2\delta)} + H^{(10-4\delta)} + H^{(12-4\delta)}\Big) \\
&+ C(\nu)\int_0^t e^{-2\alpha s}\|\nabla\hat{\mathbf{u}}_H(s)\|^2\|\hat{\Theta}(s)\|^2 ds. \quad (5.2.70)
\end{aligned}
$$

An application of Gronwall's lemma in (5.2.70) leads to

$$
\begin{aligned}
\|\hat{\Theta}\|^2 + \kappa\|\nabla\hat{\Theta}\|^2 + \Big(\nu - 2\alpha\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\int_0^t \|\nabla\hat{\Theta}(s)\|^2 ds &\le C(\kappa,\nu,\alpha,\lambda_1,M)\Big(h^4 + h^2H^{(4-2\delta)} \\
&+ H^{(10-4\delta)} + H^{(12-4\delta)}\Big)\exp\Big(\int_0^t \|\nabla\mathbf{u}_H(s)\|^2 ds\Big). \quad (5.2.71)
\end{aligned}
$$

Using Lemma 5.1, we bound

$$
\begin{aligned}
\int_0^t \|\nabla\mathbf{u}_H(s)\|^2 ds &\le C(\kappa,\nu,\alpha,\lambda_1,M)\int_0^t e^{-2\alpha s} ds \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)(1 - e^{-2\alpha t}) \le C(\kappa,\nu,\alpha,\lambda_1,M). \quad (5.2.72)
\end{aligned}
$$

A use of (5.2.72) in (5.2.71) yields the desired estimates, that is,

$$\begin{aligned}\|\Theta\|^2+\kappa\|\nabla\Theta\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\Theta(s)\|^2ds &\leq C(\kappa,\nu,\alpha,\lambda_1,M)\big(h^4+h^2H^{(4-2\delta)}\\ &\quad+h^2H^{(6-2\delta)}+H^{(10-4\delta)}+H^{(12-4\delta)}\big)e^{-2\alpha t}\\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)\big(h^4+h^2H^{(4-2\delta)}+H^{(10-4\delta)}\big)e^{-2\alpha t}. \quad (5.2.73)\end{aligned}$$

This completes the rest of the proof. □

As a consequence of the above lemma, we have following result:

Theorem 5.4. *Let the assumptions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *be satisfied. Then, there exists a positive constant* C*, which depends on* $\kappa,\ \nu,\ \lambda_1,\ \alpha$ *and* M*, such that for* $0\leq\alpha<\dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ *and* $t>0$*, the following estimate holds true:*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2+h^2\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2+h^2e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}-\mathbf{u}_h)(s)\|^2ds\\ \leq C\big(h^4+h^2H^{(4-2\delta)}+H^{(10-4\delta)}\big)e^{-2\alpha t},\end{aligned}$$

where $\delta>0$ is sufficiently small for two dimensions and $\delta=1/2$ for three dimensions.

Proof. An application of Theorem 5.1, Lemma 5.8 and triangle inequality in (5.2.49) completes the proof. □

Remark 5.2.1. *When* $d=2$*, choosing* $h=\mathcal{O}(H^{2-\delta})$ *with* $\delta>0$ *arbitrary small, we obtain optimal estimates for velocity in* $L^\infty(\mathbf{L^2})$ *and* $L^\infty(\mathbf{H}^1)$*-norms, that is,*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|^2+h^2\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|^2+h^2e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}-\mathbf{u}_h)(s)\|^2ds\leq Ch^4e^{-2\alpha t},\ \ t>0.$$

For $d=3$*, we have to choose* $h=\mathcal{O}(H^{3/2})$ *for optimal error estimates in* $L^\infty(\mathbf{L^2})$*-norm.*

Remark 5.2.2. *For optimality in* $L^\infty(\mathbf{H}^1)$*-norm, we can choose* $h=\mathcal{O}(H^2)$ *for* $d=2$ *and* 3.

Next, we derive the error estimates for the two grid approximations p_h^* and p_h of the pressure p.

Recalling (3.5.2) with p_h replaced by p_h^*, we observe that

$$\begin{aligned}\|(j_hp-p_h^*)(t)\|_{L^2/N_h} &\le C \sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(j_hp-p_h^*,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\}\\ &\le C \sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(j_hp-p,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}+\frac{(p-p_h^*,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\}\\ &\le C\left(\|j_hp-p\|+\sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(p-p_h^*,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\}\right). \end{aligned} \tag{5.2.74}$$

The first term on the right hand side of (5.2.74) can be estimated using the property (**B1**). Hence, we are left with obtaining an estimate for the second term. In order to bound the second term on the right hand side of (5.2.74), we consider the equivalent form of (5.2.6), that is, find $\mathbf{u}_h^*(t)\in\mathbf{H}_h$ and $p_h^*(t)\in L_h$ such that $\mathbf{u}_h^*(0)=\mathbf{u}_{0h}$ and for $t>0$

$$\begin{aligned}(\mathbf{u}_{ht}^*,\boldsymbol{\phi}_h)+\kappa a(\mathbf{u}_{ht}^*,\boldsymbol{\phi}_h)+\nu a(\mathbf{u}_h^*,\boldsymbol{\phi}_h)+b(\mathbf{u}_h^*,\mathbf{u}_H,\boldsymbol{\phi}_h)\\ +b(\mathbf{u}_H,\mathbf{u}_h^*,\boldsymbol{\phi}_h)=b(\mathbf{u}_H,\mathbf{u}_H,\boldsymbol{\phi}_h)+(p_h^*,\nabla\cdot\boldsymbol{\phi}_h)\quad\forall\boldsymbol{\phi}_h\in\mathbf{H}_h,\\ (\nabla\cdot\mathbf{u}_h^*,\chi_h)=0\quad\forall\chi_h\in L_h.\end{aligned} \tag{5.2.75}$$

Subtracting (5.2.75) from (5.2.44), we arrive at

$$(p-p_h^*,\nabla\cdot\boldsymbol{\phi}_h)=(\mathbf{e}_t^*,\boldsymbol{\phi}_h)+\kappa\,a(\mathbf{e}_t^*,\boldsymbol{\phi}_h)+\nu\,a(\mathbf{e}^*,\boldsymbol{\phi}_h)+\Lambda^*(\boldsymbol{\phi}_h)\quad\forall\boldsymbol{\phi}_h\in\mathbf{H}_h, \tag{5.2.76}$$

where $\mathbf{e}^*=\mathbf{u}-\mathbf{u}_h^*$ and

$$\Lambda^*(\boldsymbol{\phi}_h)=b(\mathbf{e}^*,\mathbf{u}_H,\boldsymbol{\phi}_h)+b(\mathbf{u}_H,\mathbf{e}^*,\boldsymbol{\phi}_h)-b(\mathbf{u}-\mathbf{u}_H,\mathbf{u}_H-\mathbf{u},\boldsymbol{\phi}_h)=\Lambda_1^*(\boldsymbol{\phi}_h)+\Lambda_2^*(\boldsymbol{\phi}_h). \tag{5.2.77}$$

For $\Lambda_1^*(\boldsymbol{\phi}_h)$, we use (3.2.11) and Lemma 5.1 to obtain

$$\begin{aligned}|\Lambda_1^*(\boldsymbol{\phi}_h)| &\le C\|\nabla\mathbf{e}^*\|\|\nabla\mathbf{u}_H\|\|\nabla\boldsymbol{\phi}_h\|\\ &\le C(\kappa,\nu,\lambda_1,\alpha,M)\|\nabla\mathbf{e}^*\|\|\nabla\boldsymbol{\phi}_h\|.\end{aligned} \tag{5.2.78}$$

We apply Lemma 3.2 and estimates in Theorem 5.2 to bound $\Lambda_2^*(\boldsymbol{\phi}_h)$ as

$$\begin{aligned}|\Lambda_2^*(\boldsymbol{\phi}_h)| = |b(\mathbf{u}-\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi})| &\leq C\|\mathbf{u}-\mathbf{u}_H\|^{(1-\delta)}\|\nabla(\mathbf{u}-\mathbf{u}_H)\|^{\delta}\|\nabla(\mathbf{u}-\mathbf{u}_H)\|\|\nabla\boldsymbol{\phi}_h\| \\ &\leq C(\kappa,\nu,\lambda_1,\alpha,M)H^{2(1-\delta)}H^{1+\delta}\|\nabla\boldsymbol{\phi}_h\| \\ &\leq C(\kappa,\nu,\lambda_1,\alpha,M)H^{3-\delta}\|\nabla\boldsymbol{\phi}_h\|. \end{aligned} \tag{5.2.79}$$

A combination of (5.2.78) and (5.2.79) leads to

$$|\Lambda^*(\boldsymbol{\phi}_h)| \leq C(\kappa,\nu,\lambda_1,\alpha,M)(\|\nabla\mathbf{e}^*\| + H^{3-\delta})\|\nabla\boldsymbol{\phi}_h\|. \tag{5.2.80}$$

Apply Cauchy-Schwarz's inequality in (5.2.76) and then use (5.2.80) to obtain

$$(p-p_h^*, \nabla\cdot\boldsymbol{\phi}_h) \leq C(\kappa,\nu,\lambda_1,\alpha,M)\big(\|\mathbf{e}_t^*\| + \|\nabla\mathbf{e}_t^*\| + \|\nabla\mathbf{e}^*\| + H^{3-\delta}\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{5.2.81}$$

Since, the estimate of $\|\nabla\mathbf{e}^*\|$ is known from Theorem 5.3, we shall proceed to derive estimates for e_t^*.

The following Lemma provides an estimate for e_t^*:

Lemma 5.9. *There exists a positive constant C such that $t > 0$, $\mathbf{e}^* = \mathbf{u} - \mathbf{u}_h^*$ satisfies*

$$\|\mathbf{e}_t^*(t)\|^2 + \kappa\|\nabla\mathbf{e}_t^*(t)\|^2 \leq C(h^2 + H^{6-2\delta})e^{-2\alpha t}. \tag{5.2.82}$$

Proof. With $\mathbf{e}^* = \mathbf{u} - \mathbf{u}_h^*$, we recall (5.2.45) as

$$\begin{aligned}(\mathbf{e}_t^*, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t^*, \boldsymbol{\phi}_h) &= -\nu\, a(\mathbf{e}^*, \boldsymbol{\phi}_h) - b(\mathbf{e}^*, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ &\quad - b(\mathbf{u}_H, \mathbf{e}^*, \boldsymbol{\phi}_h) + b(\mathbf{u}-\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h). \end{aligned} \tag{5.2.83}$$

Substituting $\boldsymbol{\phi}_h = P_h\mathbf{e}_t^* = \mathbf{e}_t^* + (P_h\mathbf{u}_t - \mathbf{u}_t)$ in (5.2.83), we arrive at

$$\begin{aligned}\|\mathbf{e}_t^*\|^2 + \kappa\,\|\nabla\mathbf{e}_t^*\|^2 &= -\nu a(\mathbf{e}^*, \mathbf{e}_t^*) + \nu a(\mathbf{e}^*, \mathbf{u}_t - P_h\mathbf{u}_t) + (\mathbf{e}_t^*, \mathbf{u}_t - P_h\mathbf{u}_t) \\ &\quad + \kappa\, a(\mathbf{e}_t^*, \mathbf{u}_t - P_h\mathbf{u}_t) + (p, \nabla\cdot P_h\mathbf{e}_t^*) - \Lambda^*(P_h\mathbf{e}_t^*). \end{aligned} \tag{5.2.84}$$

Using the discrete incompressible condition along with (3.2.5) yields

$$|(p, \nabla \cdot P_h \mathbf{e}_t^*)| = |(p - j_h p, \nabla \cdot P_h \mathbf{e}_t^*)| \le \|p - j_h p\| \|\nabla \mathbf{e}_t^*\|. \tag{5.2.85}$$

With the help of Cauchy-Schwarz's inequality in (5.2.84) and using (5.2.85), we arrive at

$$\begin{aligned} \|\mathbf{e}_t^*\|^2 + \kappa \|\nabla \mathbf{e}_t^*\|^2 \le \nu \|\nabla \mathbf{e}^*\| \|\nabla \mathbf{e}_t^*\| + \nu \|\nabla \mathbf{e}^*\| \|\nabla (\mathbf{u}_t - P_h \mathbf{u}_t)\| + \|\mathbf{e}_t^*\| \|\mathbf{u}_t - P_h \mathbf{u}_t\| \\ + \kappa \|\nabla \mathbf{e}_t^*\| \|\nabla (\mathbf{u}_t - P_h \mathbf{u}_t)\| + \|p - j_h p\| \|\nabla \mathbf{e}_t^*\| + |\Lambda^*(P_h \mathbf{e}_t^*)|. \end{aligned} \tag{5.2.86}$$

Using (5.2.80) and (3.2.5), we obtain

$$|\Lambda^*(P_h \mathbf{e}_t^*)| \le C(\kappa, \nu, \lambda_1, \alpha, M)(\|\nabla \mathbf{e}^*\| + H^{3-\delta}) \|\nabla \mathbf{e}_t^*\|. \tag{5.2.87}$$

Applying (5.2.87) and Young's inequality in (5.2.86) to arrive at

$$\begin{aligned} \|\mathbf{e}_t^*\|^2 + \kappa \|\nabla \mathbf{e}_t^*\|^2 \; & \le C(\kappa, \nu, \lambda_1, \alpha, M) \Big(\|\nabla \mathbf{e}^*\|^2 + H^{6-2\delta} + \|\mathbf{u}_t - P_h \mathbf{u}_t\|^2 \\ & \quad + \|\nabla (\mathbf{u}_t - P_h \mathbf{u}_t)\|^2 + \|p - j_h p\|^2 \Big). \end{aligned} \tag{5.2.88}$$

A use of (3.2.5) and (**B1**) in (5.2.88) leads to

$$\|\mathbf{e}_t^*\|^2 + \kappa \|\nabla \mathbf{e}_t^*\|^2 \le C(\kappa, \nu, \lambda_1, \alpha, M) \Big(\|\nabla \mathbf{e}^*\|^2 + H^{6-2\delta} + h^2 \big(\|\tilde{\Delta} \mathbf{u}_t\|^2 + \|\nabla p\|^2 + \|\nabla \mathbf{u}_t\|^2 \big) \Big).$$

Applying estimates from Lemmas 2.10, 2.12 and Theorem 5.3 would lead to the desired result. This completes the rest of the proof. □

As a consequence of Lemma 5.9, we obtain the following estimate for pressure:

Theorem 5.5. *Under the hypotheses of Theorem 5.3, there exists a positive constant K depending on κ, ν, λ_1, α and M, such that, for all $t > 0$, the following holds true:*

$$\|(p - p_h^*)(t)\|_{L^2/N_h} \le K(h + H^{3-\delta}) e^{-\alpha t}.$$

Proof. An application of estimates obtained in Lemma 5.9, Theorem 5.3, (5.2.81) and the approximation property (**B1**) of j_h completes the proof. □

The pressure error estimates in **Step 3** can be obtained following the similar techniques used in the proof of pressure estimates in **Step 2**. For the sake of completeness, we present below a short proof.

Theorem 5.6. *Under the hypotheses of Theorem 5.4, there exists a positive constant C depending on κ, ν, λ_1, α and M, such that, for all $t > 0$, the following holds true:*

$$\|(p - p_h)(t)\|_{L^2/N_h} \leq C(h + H^{5-2\delta})e^{-\alpha t}.$$

Proof. From (5.2.74), it follows that

$$\|(j_h p - p_h)(t)\|_{L^2/N_h} \leq C\Bigg(\|j_h p - p\| + \sup_{\boldsymbol{\phi}_h \in \mathbf{H}_h/\{0\}} \Bigg\{\frac{(p - p_h, \nabla \cdot \boldsymbol{\phi}_h)}{\|\nabla \boldsymbol{\phi}_h\|}\Bigg\}\Bigg). \tag{5.2.89}$$

Once again, estimate for the first term on the right hand side of (5.2.89) can be calculated by using property (**B1**) of j_h. Hence, we aim at deriving the estimate of second term.
To achieve this, we consider the following equivalent form of (5.2.7): find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} &(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_H, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ &\quad = b(\mathbf{u}_H, \mathbf{u}_h^*, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \boldsymbol{\phi}_h) + (p_h, \nabla \cdot \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ &\qquad (\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h. \end{aligned} \tag{5.2.90}$$

Subtracting (5.2.90) from (5.2.44), we arrive at

$$(p - p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa a(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.2.91}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{e}, \mathbf{u}_H, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{e}, \boldsymbol{\phi}_h) - b(\mathbf{u}_H - \mathbf{u}, \mathbf{u} - \mathbf{u}_h^*, \boldsymbol{\phi}_h)$$
$$- b(\mathbf{u} - \mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^* - \mathbf{u}, \mathbf{u}_h^* - \mathbf{u}, \boldsymbol{\phi}_h). \qquad (5.2.92)$$

Using (3.2.11), Theorem 5.4 and Lemma 5.1, we obtain bound for the first two terms on the right hand side of (5.2.92) as

$$|b(\mathbf{e}, \mathbf{u}_H, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{e}, \boldsymbol{\phi}_h)| \leq C\|\nabla \mathbf{e}\|\|\nabla \mathbf{u}_H\|\|\nabla \boldsymbol{\phi}_h\|$$
$$\leq C(\kappa, \nu, \lambda_1, \alpha, M)(h^2 + H^{5-2\delta}). \qquad (5.2.93)$$

Next, a use of Lemma 3.2, Theorems 5.2, 5.3 yields

$$|b(\mathbf{u} - \mathbf{u}_H, \mathbf{u}_h^* - \mathbf{u}, \boldsymbol{\phi}_h) + b(\mathbf{u} - \mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}, \boldsymbol{\phi}_h)|$$
$$\leq C\|\mathbf{u} - \mathbf{u}_H\|^{1-\delta}\|\nabla(\mathbf{u} - \mathbf{u}_H)\|^{\delta}\|\nabla(\mathbf{u}_h^* - \mathbf{u})\|\|\nabla \boldsymbol{\phi}_h\| \qquad (5.2.94)$$
$$\leq C(\kappa, \nu, \lambda_1, \alpha, M)H^{2(1-\delta)}H^{\delta}(h + H^{3-\delta})\|\nabla \boldsymbol{\phi}_h\|$$
$$\leq C(\kappa, \nu, \lambda_1, \alpha, M)H^{2-\delta}(h + H^{3-\delta})\|\nabla \boldsymbol{\phi}_h\|.$$

For the last term on the right hand side of (5.2.92), we apply (3.2.11) along with Theorem 5.3 to find that

$$|b(\mathbf{u}_h^* - \mathbf{u}, \mathbf{u}_h^* - \mathbf{u}, \boldsymbol{\phi}_h)| \leq C\|\nabla(\mathbf{u}_h^* - \mathbf{u})\|^2\|\nabla \boldsymbol{\phi}_h\|$$
$$\leq C(\kappa, \nu, \lambda_1, \alpha, M)(h^2 + H^{6-2\delta})\|\nabla \boldsymbol{\phi}_h\|. \qquad (5.2.95)$$

A combination of (5.2.93)-(5.2.95) leads to the bound for $\Lambda_h(\boldsymbol{\phi}_h)$, that is,

$$|\Lambda_h(\boldsymbol{\phi}_h)| \leq C(\kappa, \nu, \lambda_1, \alpha, M)(h^2 + hH^{2-\delta} + H^{5-2\delta})\|\nabla \boldsymbol{\phi}_h\|. \qquad (5.2.96)$$

With the help of Cauchy-Schwarz's inequality and (5.2.96) in (5.2.91), we obtain

$$
\begin{aligned}
(p - p_h, \nabla \cdot \boldsymbol{\phi}_h) \leq C(\kappa, \nu, \lambda_1, \alpha, M)(\|\mathbf{e}_t\| + \|\nabla \mathbf{e}_t\| + \|\nabla \mathbf{e}\| \\
+ h^2 + hH^{2-\delta} + H^{5-2\delta})\|\nabla \boldsymbol{\phi}_h\|.
\end{aligned} \tag{5.2.97}
$$

Since, estimate of $\|\nabla \mathbf{e}\|$ is already known from Theorem 5.4. The following lemma provides estimates for $\mathbf{e}_t$.

Lemma 5.10. *There exists a positive constant C such that $t > 0$, $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ satisfies*

$$
\|\mathbf{e}_t(t)\|^2 + \kappa\|\nabla \mathbf{e}_t(t)\|^2 \leq C(h^2 + H^{10-4\delta})e^{-2\alpha t}.
$$

Proof. We recall (5.2.47) as

$$
(\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa a(\mathbf{e}_t, \boldsymbol{\phi}_h) = -\nu a(\mathbf{e}, \boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h) + (p, \nabla \cdot \boldsymbol{\phi}_h). \tag{5.2.98}
$$

Substitute $\boldsymbol{\phi}_h = P_h \mathbf{e}_t = \mathbf{e}_t + P_h \mathbf{u}_t - \mathbf{u}_t$ in (5.2.98) to obtain

$$
\begin{aligned}
\|\mathbf{e}_t\|^2 + \kappa\|\nabla \mathbf{e}_t\|^2 = (\mathbf{e}_t, \mathbf{u}_t - P_h \mathbf{u}_t) + \kappa a(\mathbf{e}_t, \mathbf{u}_t - P_h \mathbf{u}_t) \\
- \nu a(\mathbf{e}, \mathbf{e}_t) + \nu a(\mathbf{e}, \mathbf{u}_t - P_h \mathbf{u}_t) - \Lambda_h(P_h \mathbf{e}_t) + (p, \nabla \cdot P_h \mathbf{e}_t).
\end{aligned} \tag{5.2.99}
$$

Using (5.2.96) and (3.2.5), we arrive at

$$
|\Lambda_h(P_h \mathbf{e}_t)| \leq C(\kappa, \nu, \lambda_1, \alpha, M)(h^2 + hH^{2-\delta} + H^{5-2\delta})\|\nabla \mathbf{e}_t\|. \tag{5.2.100}
$$

With the help of (5.2.100), Cauchy-Schwarz's inequality and Young's inequality in (5.2.99), we observe that

$$
\begin{aligned}
\|\mathbf{e}_t\|^2 + \kappa\|\nabla \mathbf{e}_t\|^2 \leq C(\kappa, \nu, \lambda_1, \alpha, M)\Big(\|\nabla \mathbf{e}\|^2 + h^4 + h^2 H^{4-2\delta} + H^{10-4\delta} \\
+ h^2\big(\|\tilde{\Delta} \mathbf{u}_t\|^2 + \|\nabla p\|^2 + \|\nabla \mathbf{u}_t\|^2\big)\Big).
\end{aligned} \tag{5.2.101}
$$

A use of Lemmas 2.10, 2.12 and Theorem 5.4 in (5.2.101) leads to the desired estimates. □

Proof of Theorem 5.6. The proof follows using the approximation property (**B1**) of j_h, (5.2.97) and Lemma 5.10. □

5.3 Backward Euler Method

In the previous sections, we have discussed only the semidiscrete Galerkin approximations applied to the continuous two-grid system keeping the time variable continuous. In this section, we present an analysis of the backward Euler method for the time discretization. The backward Euler method applied to (5.2.5)-(5.2.7) is as follows:

Algorithm:

Step 1: Solve nonlinear system on coarse mesh $\mathcal{T}_H$: for $\boldsymbol{\phi}_H \in \mathbf{J}_H$, we seek $\{\mathbf{U}_H^n\}_{n\geq 1} \in \mathbf{J}_H$ such that

$$(\bar{\partial}_t \mathbf{U}_H^n, \boldsymbol{\phi}_H) + \kappa a(\bar{\partial}_t \mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu a(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) = 0 \quad \forall \boldsymbol{\phi}_H \in \mathbf{J}_H, \tag{5.3.1}$$

where $\bar{\partial}_t \mathbf{U}_H^n = \frac{\mathbf{U}_H^n - \mathbf{U}_H^{n-1}}{t}$.

Step 2: Update on fine mesh $\mathcal{T}_h$ with one Newton iteration: find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$\begin{aligned}(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.3.2}$$

Step 3: Correct on fine mesh $\mathcal{T}_h$: find $\mathbf{U}_h^n \in \mathbf{J}_h$ such that

$$\begin{aligned}(\bar{\partial}_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.3.3}$$

Set $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H(t_n)$, $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h^*(t_n)$ and $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h(t_n)$.

Consider (5.2.5)-(5.2.7) at $t = t_n$ and subtract the resulting equations from (5.3.1)-(5.3.3), respectively, to arrive at

Equation for **Step 1:**

$$(\bar{\partial}_t \mathbf{e}_H^n, \boldsymbol{\phi}_H) + \kappa a(\bar{\partial}_t \mathbf{e}_H^n, \boldsymbol{\phi}_H) + \nu a(\mathbf{e}_H^n, \boldsymbol{\phi}_H) \tag{5.3.4}$$
$$= (\sigma_H^n, \boldsymbol{\phi}_H) + \kappa a(\sigma_H^n, \boldsymbol{\phi}_H) + \Lambda_H(\boldsymbol{\phi}_H) \ \forall \boldsymbol{\phi}_H \in \mathbf{J}_H,$$

where $\sigma_H^n = \mathbf{u}_{Ht}^n - \bar{\partial}_t \mathbf{u}_H^n$ and $\Lambda_H(\boldsymbol{\phi}_H) = b(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_H) - b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H)$.

Equation for **Step 2:**

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) \tag{5.3.5}$$
$$= (\sigma^n, \boldsymbol{\phi}_h) + \kappa a(\sigma^n, \boldsymbol{\phi}_h) + \Lambda^*(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,$$

where $\sigma^n = \mathbf{u}_{ht}^{*n} - \bar{\partial}_t \mathbf{u}_h^{*n}$ and

$$\begin{aligned} \Lambda^*(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) \\ &- b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h). \end{aligned} \tag{5.3.6}$$

Similarly, equation in **Step 3** satisfies

$$(\bar{\partial}_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_h^n, \boldsymbol{\phi}_h) \tag{5.3.7}$$
$$= (\sigma_h^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_h^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,$$

where, $\sigma_h^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$ and

$$\begin{aligned} \Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ &- b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h). \end{aligned} \tag{5.3.8}$$

Before proceeding to derive the error estimates for **Step 2**, we recall below the bounds for $\{\mathbf{U}_H^n\}_{n\geq 1}$ and $\{\mathbf{e}_H^n\}_{n\geq 1}$ of **Step 1**, which is required for the error analysis in **Step 2**. For a proof, we refer to Lemma 4.1 and Theorem 4.2 of Chapter **4**.

Theorem 5.7. *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$*, choose* k_0 *so that for* $0 < k \leq k_0$

$$\frac{\nu k\lambda_1}{\kappa\lambda_1+1} + 1 > e^{\alpha k}. \tag{5.3.9}$$

Then the discrete solution $\mathbf{U}_H^n$*,* $n = 1, 2, \cdots, N$ *of (5.3.1) satisfies*

$$(\|\mathbf{U}_H^n\|^2 + \kappa\|\nabla\mathbf{U}_H^n\|^2) + 2\beta_1 e^{-2\alpha t_n}\; k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{U}_H^i\|^2 \leq e^{-2\alpha t_n}(\|\mathbf{U}_H^0\|^2 + \kappa\|\nabla\mathbf{U}_H^0\|^2),$$

where

$$\beta_1 = \left(e^{-\alpha k}\nu - 2\Big(\frac{1-e^{-\alpha k}}{k}\Big)\big(\kappa + \frac{1}{\lambda_1})\right) > 0.$$

Theorem 5.8. *Assume that* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \leq k_0$*, (5.3.9) is satisfied. For some fixed* H*, assume that* $\mathbf{u}_H(t)$ *satisfies (5.2.5). Then, there exists a positive constant* C*, independent of* k*, such that, for* $n = 1, 2, \cdots, N$*, the following holds true:*

$$\|\mathbf{e}_H^n\|^2 + \kappa\|\nabla\mathbf{e}_H^n\|^2 + \beta_1 e^{-2\alpha t_n}\; k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_H^i\|^2 \leq Ck^2 e^{-2\alpha t_n}. \tag{5.3.10}$$

Next, we derive a *priori* bounds for $\mathbf{U}^n$ in **Step 2**, which will be used subsequently in the estimation of error in **Step 2**.

Lemma 5.11. *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$*, choose* k_0 *such that for* $0 < k \leq k_0$*, (5.3.9) is satisfied. Then the discrete solution* $\mathbf{U}^n$*,* $n = 1, 2, \cdots, N$ *of (5.3.2) satisfies*

$$(\|\mathbf{U}^n\|^2 + \kappa\|\nabla\mathbf{U}^n\|^2) + \beta_1 e^{-2\alpha t_n}\; k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 \leq e^{-2\alpha t_n}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2).$$

Proof. Multiplying (5.3.2) by $e^{\alpha t_n}$ and setting $\hat{\mathbf{U}}^n = e^{\alpha t_n}\mathbf{U}^n$, we obtain

$$e^{\alpha t_n}\Big((\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h)\Big) + \nu a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) = e^{-\alpha t_n}\Big(- b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n_H, \boldsymbol{\phi}_h) - b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n_H, \boldsymbol{\phi}_h)\Big) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.3.11}$$

Note that,

$$e^{\alpha t_n}\bar{\partial}_t \mathbf{U}^n = e^{\alpha k}\bar{\partial}_t \hat{\mathbf{U}}^n - \Big(\frac{e^{\alpha k}-1}{k}\Big)\hat{\mathbf{U}}^n. \tag{5.3.12}$$

Using (5.3.12) in (5.3.11) and multiplying the resulting equation by $e^{-\alpha k}$, we obtain

$$(\bar{\partial}_t \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) - \Big(\frac{1-e^{-\alpha k}}{k}\Big)(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + e^{-\alpha k}\nu a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) - \kappa\Big(\frac{1-e^{-\alpha k}}{k}\Big)a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h)$$
$$= -e^{-\alpha k}e^{-\alpha t_n}\big(b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n_H, \boldsymbol{\phi}_h)\big) + e^{-\alpha k}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n_H, \boldsymbol{\phi}_h). \tag{5.3.13}$$

Note that,

$$\begin{aligned}(\bar{\partial}_t \hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) &= \frac{1}{k}(\hat{\mathbf{U}}^n - \hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n)\\ &\geq \frac{1}{2k}(\|\hat{\mathbf{U}}^n\|^2 - \|\hat{\mathbf{U}}^{n-1}\|^2)\\ &= \frac{1}{2}\bar{\partial}_t\|\hat{\mathbf{U}}^n\|^2.\end{aligned} \tag{5.3.14}$$

Substituting $\boldsymbol{\phi}_h = \hat{\mathbf{U}}^n$ in (5.3.13), using (2.2.3), (5.3.14) along with (3.2.1), we arrive at

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2) + \Big(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{U}}^n\|^2$$
$$\leq -e^{-\alpha k}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n) + e^{-\alpha k}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n). \tag{5.3.15}$$

Multiplying (5.3.15) by $2k$ and summing over $n = 1$ to N, we obtain

$$
\begin{aligned}
&\|\hat{\mathbf{U}}^N\|^2+\kappa\|\nabla\hat{\mathbf{U}}^N\|^2+2k\bigg(e^{-\alpha k}\nu-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\bigg)\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2\\
&\leq\|\mathbf{U}^0\|^2+\kappa\|\nabla\mathbf{U}^0\|^2+2ke^{-\alpha k}\sum_{n=1}^{N}e^{-\alpha t_n}\big(b(\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)-b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)\big)\\
&\leq\|\mathbf{U}^0\|^2+\kappa\|\nabla\mathbf{U}^0\|^2+I_1^N+I_2^N. \qquad (5.3.16)
\end{aligned}
$$

Next, we apply (3.2.11) to obtain

$$
|b(\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)|\leq C\|\nabla\hat{\mathbf{U}}^n_H\|^2\|\nabla\hat{\mathbf{U}}^n\|. \qquad (5.3.17)
$$

To derive bound for I_1^N, we use (5.3.17), Young's inequality and Theorem 5.7 to arrive at

$$
\begin{aligned}
|I_1^N|&=|2ke^{-\alpha k}\sum_{n=1}^{N}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)|\\
&\leq Cke^{-\alpha k}\sum_{n=1}^{N}e^{-\alpha t_n}\|\nabla\hat{\mathbf{U}}^n_H\|^2\|\nabla\hat{\mathbf{U}}^n\|\\
&\leq C(\kappa,M,\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n_H\|^2+\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2\\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)(\|\mathbf{U}^0_H\|^2+\kappa\|\nabla\mathbf{U}^0_H\|^2)+\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2. \qquad (5.3.18)
\end{aligned}
$$

An application of (3.2.10) leads to

$$
\begin{aligned}
|b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)|&\leq C\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n_H\|\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\\
&\leq C\|\hat{\mathbf{U}}^n\|\|\nabla\hat{\mathbf{U}}^n\|\|\nabla\hat{\mathbf{U}}^n_H\| \qquad (5.3.19)
\end{aligned}
$$

and a use of (5.3.19) with Young's inequality yields

$$
\begin{aligned}
|I_2^N|&=|2ke^{-\alpha k}\sum_{n=1}^{N}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n_H,\hat{\mathbf{U}}^n)|\\
&\leq C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n_H\|^2\|\hat{\mathbf{U}}^n\|^2+\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2. \qquad (5.3.20)
\end{aligned}
$$

Substituting the estimates (5.3.18), (5.3.20) in (5.3.16) and using Theorem 5.7 with $\epsilon = \frac{\nu}{2}$, we obtain

$$\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \le \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 + C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\mathbf{U}_H^0\|^2$$
$$+ \kappa\|\nabla\mathbf{U}_H^0\|^2\big) + C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{U}}^n\|^2 + C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^N\|^2\|\hat{\mathbf{U}}^N\|^2$$
$$\le \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 + C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\mathbf{U}_H^0\|^2 + \kappa\|\nabla\mathbf{U}_H^0\|^2\big)$$
$$+ C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{U}}^n\|^2 + C(\nu,\kappa,M)ke^{-\alpha k}\big(\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2\big). \tag{5.3.21}$$

For $k_0 > 0$ with $0 < k \le k_0$, $(1 - C(\nu,\kappa,M)ke^{-\alpha k})$ can be made positive. Then, using the discrete Gronwall's lemma in (5.3.21), we arrive at

$$\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)\exp\big(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\big). \tag{5.3.22}$$

An application of Theorem 5.7 yields

$$\exp\big(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\big) \le C(\kappa,\nu,\alpha,\lambda_1,M). \tag{5.3.23}$$

Applying the estimate (5.3.23) in (5.3.22), we obtain

$$\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)$$

and this completes the rest of the proof. □

The following theorem provides error estimates for **Step 2**.

Theorem 5.9. *Assume that* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \le k_0$, *(5.3.9) is satisfied. Further, suppose that* $\mathbf{u}_h^*(t)$ *satisfies (5.2.6) for some fixed* h. *Then, there exists a positive constant* C, *independent of* k, *such that, for* $n = 1, 2, \cdots, N$, *the* *following holds true:*

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + \beta_1 k e^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2 \le Ck^2 e^{-2\alpha t_n}.$$

Proof. Multiply (5.3.5) by $e^{\alpha t_n}$ to arrive at

$$\begin{aligned}(e^{\alpha t_n}\bar{\partial}_t\mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n}\bar{\partial}_t\mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h)& \\ = (e^{\alpha t_n}\sigma^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n}\sigma^n, \boldsymbol{\phi}_h) + e^{\alpha t_n}\Lambda(\boldsymbol{\phi}_h).&\end{aligned} \tag{5.3.24}$$

Note that,

$$e^{\alpha t_n}\bar{\partial}_t\mathbf{e}^n = e^{\alpha k}\bar{\partial}_t\hat{\mathbf{e}}^n - \big(\frac{e^{\alpha k}-1}{k}\big)\hat{\mathbf{e}}^n. \tag{5.3.25}$$

Using (5.3.25) in (5.3.24) and dividing the resulting equation by $e^{\alpha k}$, we obtain

$$\begin{aligned}&(\bar{\partial}_t\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - \big(\frac{1-e^{-\alpha k}}{k}\big)(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - \big(\frac{1-e^{-\alpha k}}{k}\big)\kappa a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) \\ &+ \nu e^{-\alpha k} a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) = e^{-\alpha k}(e^{\alpha t_n}\sigma^n, \boldsymbol{\phi}_h) + e^{-\alpha k}\kappa a(e^{\alpha t_n}\sigma^n, \boldsymbol{\phi}_h) + e^{-\alpha k}e^{\alpha t_n}\Lambda(\boldsymbol{\phi}_h).\end{aligned} \tag{5.3.26}$$

Substitute $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ in (5.3.26). Then, a use of (2.2.3) yields

$$\begin{aligned}&\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2\big) + \bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^n\|^2 \\ &\quad = e^{-\alpha k}(e^{\alpha t_n}\sigma^n, \hat{\mathbf{e}}^n) + e^{-\alpha k}\kappa a(e^{\alpha t_n}\sigma^n, \hat{\mathbf{e}}^n) + e^{-\alpha k}e^{\alpha t_n}\Lambda(\hat{\mathbf{e}}^n).\end{aligned} \tag{5.3.27}$$

On multiplying (5.3.27) by $2k$ and summing over $n = 1$ to N, we observe that

$$\begin{aligned}&\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + 2\bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\ &\le 2ke^{-\alpha k}\sum_{n=1}^{N}(e^{\alpha t_n}\sigma^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k}\sum_{n=1}^{N}\kappa a(e^{\alpha t_n}\sigma^n, \hat{\mathbf{e}}^n) + 2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\Lambda(\hat{\mathbf{e}}^n) \\ &= I_1^N + I_2^N + I_3^N, \text{ say.}\end{aligned} \tag{5.3.28}$$

Using Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality, we estimate I_1^N as:

$$
\begin{aligned}
|I_1^N| &\le 2ke^{-\alpha k}\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma^n\|\|\hat{\mathbf{e}}^n\| \\
&\le C(\epsilon,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma^n\|^2+\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.3.29)
\end{aligned}
$$

Now, using the Taylor's series expansion of $\mathbf{u}_h^*$ around t_n in the interval (t_{n-1},t_n), we arrive at

$$
\|e^{\alpha t_n}\sigma^n\|^2 \le e^{2\alpha t_n}\frac{1}{k^2}\Big(\int_{t_{n-1}}^{t_n}(t_n-s)\|\mathbf{u}_{htt}^*(s)\|ds\Big)^2. \qquad (5.3.30)
$$

An application of Cauchy-Schwarz's inequality in (5.3.30) yields

$$
\begin{aligned}
\|e^{\alpha t_n}\sigma^n\|^2 &\le \frac{1}{k^2}\Big(\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}^*(s)\|^2ds\Big)\Big(\int_{t_{n-1}}^{t_n}(t_n-s)^2ds\Big) \\
&= \frac{k}{3}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}^*(s)\|^2\,ds \qquad (5.3.31)
\end{aligned}
$$

and hence, using (5.3.31), we write

$$
\begin{aligned}
k\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma^n\|^2 &\le \frac{k^2}{3}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}^*(s)\|^2\,ds \\
&= \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_{n-1}}\|\mathbf{u}_{htt}^*(s)\|^2\,ds \\
&\le \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha s}\|\mathbf{u}_{htt}^*(s)\|^2\,ds. \qquad (5.3.32)
\end{aligned}
$$

From (5.3.32) and Lemma 5.4, we note that

$$
\begin{aligned}
k\sum_{n=1}^{N}\|e^{\alpha t_n}\sigma^n\|^2 &\le \frac{k^2}{3}e^{2\alpha k}\int_0^{t_N}e^{2\alpha s}\|\mathbf{u}_{htt}^*(s)\|^2\,ds \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-2\alpha t_{N-1}}. \qquad (5.3.33)
\end{aligned}
$$

Similarly, we obtain

$$k\sum_{n=1}^{N}\|e^{\alpha t_n}\nabla\sigma^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-2\alpha t_{N-1}}. \tag{5.3.34}$$

Using (5.3.33) in (5.3.29), we observe that

$$|I_1^N| \le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^2 + \epsilon k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.3.35}$$

Following steps for bounding $|I_1^N|$, and using (5.3.34), we obtain

$$|I_2^N| \le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^2 + \epsilon k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.3.36}$$

To estimate I_3^N, we note that

$$\begin{aligned}\Lambda^*(\boldsymbol{\phi}_h) &= \Big(b(\mathbf{u}_h^{*n},\mathbf{u}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)\Big) + \Big(b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\\ &\quad - b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\Big) + \Big(b(\mathbf{U}_H^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_H^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h)\Big)\\ &= \Lambda_1(\boldsymbol{\phi}_h) + \Lambda_2(\boldsymbol{\phi}_h) + \Lambda_3(\boldsymbol{\phi}_h). \end{aligned} \tag{5.3.37}$$

We write $\Lambda_1(\boldsymbol{\phi}_h)$ as

$$\begin{aligned}|\Lambda_1(\boldsymbol{\phi}_h)| &= |b(\mathbf{u}_h^{*n},\mathbf{u}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|\\ &= |b(\mathbf{u}_h^{*n},\mathbf{u}_H^n - \mathbf{U}_H^n,\boldsymbol{\phi}_h) + b(\mathbf{u}_h^{*n},\mathbf{U}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|\\ &= |-b(\mathbf{u}_h^{*n},\mathbf{e}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{e}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|. \end{aligned} \tag{5.3.38}$$

Using (3.2.10) and (3.2.11) with $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ in (5.3.38), we observe that

$$e^{\alpha t_n}|\Lambda_1(\hat{\mathbf{e}}^n)| \le Ce^{-\alpha t_n}\big(\|\nabla\hat{\mathbf{u}}_h^{*n}\|\|\nabla\hat{\mathbf{e}}_H^n\|\|\nabla\hat{\mathbf{e}}^n\| + \|\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}_H^n\|\|\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{e}}^n\|^{\frac{1}{2}}\big). \tag{5.3.39}$$

An application of Lemma 5.2 in (5.3.39) yields

$$e^{\alpha t_n}|\Lambda_1(\hat{\mathbf{e}}^n)| \leq C(\kappa,\nu,\alpha,\lambda_1,M)\|\nabla\hat{\mathbf{e}}_H^n\|\|\nabla\hat{\mathbf{e}}^n\| + Ce^{-\alpha t_n}\|\hat{\mathbf{e}}^n\|\|\nabla\hat{\mathbf{e}}^n\|\|\nabla\hat{\mathbf{U}}_H^n\|. \tag{5.3.40}$$

A use of Young's inequality in (5.3.40) yields

$$\begin{aligned} 2ke^{-\alpha k}\sum_{n=1}^{N}|e^{\alpha t_n}\Lambda_1(\hat{\mathbf{e}}^n)| &\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_H^n\|^2 \\ &\quad + C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 + \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \end{aligned} \tag{5.3.41}$$

Similarly, using (3.2.1), (3.2.11) and estimates of $\hat{\mathbf{U}}^n$ from Lemma 5.11, we obtain

$$\begin{aligned} |e^{\alpha t_n}\Lambda_2(\hat{\mathbf{e}}^n)| &= e^{-\alpha t_n}|b(\hat{\mathbf{u}}_H^n,\hat{\mathbf{u}}_h^{*n},\hat{\mathbf{e}}^n) - b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)|, \\ &= e^{-\alpha t_n}|b(\hat{\mathbf{u}}_H^n,\hat{\mathbf{u}}_h^{*n}-\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n) + b(\hat{\mathbf{u}}_H^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n) - b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)| \\ &= e^{-\alpha t_n}|-b(\hat{\mathbf{u}}_H^n,\hat{\mathbf{e}}^n,\hat{\mathbf{e}}^n) - b(\hat{\mathbf{e}}_H^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)| \\ &= e^{-\alpha t_n}|b(\hat{\mathbf{e}}_H^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)| \\ &\leq C\|\nabla\hat{\mathbf{U}}^n\|\|\nabla\hat{\mathbf{e}}_H^n\|\|\nabla\hat{\mathbf{e}}^n\| \\ &\leq C(\kappa,M)\|\nabla\hat{\mathbf{e}}_H^n\|\|\nabla\hat{\mathbf{e}}^n\|. \end{aligned} \tag{5.3.42}$$

We use Young's inequality in (5.3.42) to arrive at

$$2ke^{-\alpha k}\sum_{n=1}^{N}|e^{\alpha t_n}\Lambda_2(\hat{\mathbf{e}}^n)| \leq C(\kappa,M,\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_H^n\|^2 + \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.3.43}$$

Next, to bound $\Lambda_3(\hat{\mathbf{e}}^n)$, we write it as

$$\begin{aligned} |e^{\alpha t_n}\Lambda_3(\hat{\mathbf{e}}^n)| &= e^{-\alpha t_n}|\big(b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{U}}_H^n,\hat{\mathbf{e}}^n) - b(\hat{\mathbf{u}}_H^n,\hat{\mathbf{u}}_H^n,\hat{\mathbf{e}}^n)\big)| \\ &= e^{-\alpha t_n}|\big(b(\hat{\mathbf{u}}_H^n-\hat{\mathbf{U}}_H^n,\hat{\mathbf{u}}_H^n,\hat{\mathbf{e}}^n) + b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{u}}_H^n,\hat{\mathbf{e}}^n) - b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{U}}_H^n,\hat{\mathbf{e}}^n)\big)| \\ &= e^{-\alpha t_n}|\big(b(\hat{\mathbf{e}}_H^n,\hat{\mathbf{u}}_H^n,\hat{\mathbf{e}}^n) + b(\hat{\mathbf{U}}_H^n,\hat{\mathbf{e}}_H^n,\hat{\mathbf{e}}^n)\big)|. \end{aligned} \tag{5.3.44}$$

We use (3.2.11), Lemma 5.1, Theorem 5.7 in (5.3.44) and arrive at

$$
\begin{aligned}
|e^{\alpha t_n}\Lambda_3(\hat{\mathbf{e}}^n)| &\leq C\big(\|\nabla \hat{\mathbf{u}}_H^n\| + \|\nabla \hat{\mathbf{U}}_H^n\|\big)\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}^n\| \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}^n\|. \qquad (5.3.45)
\end{aligned}
$$

An application of Young's inequality in (5.3.45) yields

$$
2ke^{-\alpha k}\sum_{n=1}^{N}|e^{\alpha t_n}\Lambda_3(\hat{\mathbf{e}}^n)| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_H^n\|^2 + \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (5.3.46)
$$

Combining (5.3.41), (5.3.43), (5.3.46), we use estimate of $\hat{\mathbf{e}}_H^n$ from Theorem 5.8 to obtain

$$
\begin{aligned}
|I_3^N| =& 2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\big(|\Lambda_1(\hat{\mathbf{e}}^n) + \Lambda_2(\hat{\mathbf{e}}^n) + \Lambda_3(\hat{\mathbf{e}}^n)|\big) \\
\leq& C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_H^n\|^2 + C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 \\
&+ 3\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 \\
\leq& C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^2 + C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 \\
&+ 3\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (5.3.47)
\end{aligned}
$$

A use of (5.3.35)-(5.3.36) and (5.3.47) with $\epsilon = \frac{\nu}{5}$ in (5.3.28) yields

$$
\begin{aligned}
\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 \\
&+ C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla \hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 + C(\nu)ke^{-\alpha k}\|\nabla \hat{\mathbf{U}}_H^N\|^2\|\hat{\mathbf{e}}^N\|^2. \qquad (5.3.48)
\end{aligned}
$$

Applying estimates of Theorem 5.7 in (5.3.48), we obtain

$$
\begin{aligned}
\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2 \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 \\
&+ C(\nu,\kappa,M)ke^{-\alpha k}\big(\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2\big). \qquad (5.3.49)
\end{aligned}
$$

Similar to the previous cases, choose $k_0 > 0$ such that for $0 < k \le k_0$, $(1-C(\nu,\kappa,M)ke^{-\alpha k}) > 0$. Using discrete Gronwall's lemma in (5.3.49), we obtain

$$
\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\exp(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2). \qquad (5.3.50)
$$

With the help of bounds in Theorem 5.7, we write

$$
\exp(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2) \le C(\kappa,\nu,\alpha,\lambda_1,M). \qquad (5.3.51)
$$

An application of (5.3.51) in (5.3.50) would complete the proof. □

To obtain pressure error estimates, we derive a bound for $\|\bar{\partial}_t\mathbf{e}^n\|$. Substitute $\boldsymbol{\phi}_h = \bar{\partial}_t\mathbf{e}^n$ in (5.3.5) and arrive at

$$
\|\bar{\partial}_t\mathbf{e}^n\|^2 + \kappa\|\nabla\bar{\partial}_t\mathbf{e}^n\|^2 = -\nu a(\mathbf{e}^n,\bar{\partial}_t\mathbf{e}^n) + (\sigma^n,\bar{\partial}_t\mathbf{e}^n) + \kappa a(\sigma^n,\bar{\partial}_t\mathbf{e}^n) + \Lambda^*(\bar{\partial}_t\mathbf{e}^n). \qquad (5.3.52)
$$

Using (5.3.37), we arrive at

$$
\begin{aligned}
\Lambda^*(\boldsymbol{\phi}_h) &= \big(b(\mathbf{u}_h^{*n},\mathbf{u}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)\big) + \big(b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h) \\
&- b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\big) + \big(b(\mathbf{U}_H^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_H^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h)\big) \\
&= \Lambda_1(\boldsymbol{\phi}_h) + \Lambda_2(\boldsymbol{\phi}_h) + \Lambda_3(\boldsymbol{\phi}_h). \qquad (5.3.53)
\end{aligned}
$$

From (5.3.38), (3.2.11), Lemma 5.2 and Theorem 5.7, we obtain

$$\begin{aligned}|\Lambda_1(\boldsymbol{\phi}_h)| =& \ |-b(\mathbf{u}_h^{*n}, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{u}_h^{*n}\| \|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\| \|\nabla \mathbf{U}_H^n\|\big)\|\nabla \boldsymbol{\phi}_h\| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)(\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\|)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.54}$$

Similarly, from (5.3.42), (3.2.11), Lemmas 5.1 and 5.11, we write

$$\begin{aligned}|\Lambda_2(\boldsymbol{\phi}_h)| =& |-b(\mathbf{u}_H^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{u}_H^n\| \|\nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}_H^n\| \|\nabla \mathbf{U}^n\|\big)\|\nabla \boldsymbol{\phi}_h\| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}_H^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.55}$$

With the help of (5.3.45), (3.2.11), Lemma 5.1 and Theorem 5.7, we observe that

$$\begin{aligned}|\Lambda_3(\boldsymbol{\phi}_h)| =& |b(\mathbf{e}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{u}_H^n\| + \|\nabla \mathbf{U}_H^n\|\big)\|\nabla \mathbf{e}_H^n\| \|\nabla \boldsymbol{\phi}_h\| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\|\nabla \mathbf{e}_H^n\| \|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.56}$$

A use of (5.3.54)–(5.3.56) in (5.3.53) implies

$$|\Lambda^*(\boldsymbol{\phi}_h)| \leq C(\kappa, \nu, \alpha, \lambda_1, M)(\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\|)\|\nabla \boldsymbol{\phi}_h\|. \tag{5.3.57}$$

An application of (5.3.31) and estimate in Lemma 5.4 leads to

$$\begin{aligned}\|e^{\alpha t_n}\sigma^n\|^2 &\leq \frac{1}{k^2}\Big(\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}^*(s)\|^2 ds\Big)\Big(\int_{t_{n-1}}^{t_n} (t_n - s)^2 ds\Big) \\ &= \frac{k}{3}\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}^*(s)\|^2\, ds \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M) k e^{2\alpha t_n}\int_{t_{n-1}}^{t_n} e^{-2\alpha s} ds \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M) k^2 e^{2\alpha k^*}, \end{aligned} \tag{5.3.58}$$

for $k^* \in (0,k)$.

Similarly, we observe that

$$\|e^{\alpha t_n}\nabla\sigma^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{2\alpha k^*}. \tag{5.3.59}$$

With the help of Cauchy-Schwarz's inequality, Young's inequality, (2.2.3) and (5.3.57) in (5.3.52), we find that

$$\|\bar{\partial}_t\mathbf{e}^n\|^2 + \kappa\|\nabla\bar{\partial}_t\mathbf{e}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)(\|\nabla\sigma^n\|^2 + \|\nabla\mathbf{e}_H^n\|^2 + \|\nabla\mathbf{e}^n\|^2). \tag{5.3.60}$$

In view of (5.3.59), Theorems 5.8 and 5.9, we arrive at

$$\|\bar{\partial}_t\mathbf{e}^n\|^2 + \kappa\|\nabla\bar{\partial}_t\mathbf{e}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-2\alpha t_n}. \tag{5.3.61}$$

To obtain pressure error estimates, we consider the equivalent form of (5.3.2): Find a sequence of functions $\{\mathbf{U}^n\}_{n\ge1} \in \mathbf{H}_h$ and $\{P^n\}_{n\ge1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$\begin{aligned}(\bar{\partial}_t\mathbf{U}^n,\boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n,\boldsymbol{\phi}_h) + b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)&\\ +b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h) = b(\mathbf{U}_H^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) + (P^n,\nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h,&\\ (\nabla\cdot\mathbf{U}^n,\chi_h) = 0 \quad \forall\chi_h \in L_h.&\end{aligned} \tag{5.3.62}$$

Now, consider (5.2.75) at $t = t_n$ and subtract it from (5.3.62) to write

$$\begin{aligned}(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) &= (\bar{\partial}_t\mathbf{e}^n,\boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{e}^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n,\boldsymbol{\phi}_h)\\ &- (\sigma^n,\boldsymbol{\phi}_h) - \kappa a(\sigma^n,\boldsymbol{\phi}_h) - \Lambda^*(\boldsymbol{\phi}_h),\end{aligned} \tag{5.3.63}$$

where $\boldsymbol{\rho}^n = P^n - p_h^*(t_n)$ and $\Lambda^*(\boldsymbol{\phi}_h)$ is defined in (5.3.53).

A use of Cauchy-Schwarz's inequality, (2.2.3) and (5.3.57) in (5.3.63) yields

$$(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h) \le C(\kappa,\nu,\alpha,\lambda_1,M)\big(\big(\|\bar{\partial}_t\nabla\mathbf{e}^n\| + \|\nabla\mathbf{e}^n\| + \|\nabla\mathbf{e}_H^n\| + \|\nabla\sigma^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \quad (5.3.64)$$

Applying (5.3.59), (5.3.61) and the estimates from Theorems 5.8 and 5.9, we arrive at the desired result, that is,

$$\|\boldsymbol{\rho}^n\| \le C(\kappa,\nu,\alpha,\lambda_1,M)k\ e^{-\alpha t_n}. \quad (5.3.65)$$

The following theorem is an easy consequence of the results obtained in (5.3.65), Theorems 5.3, 5.5 and 5.9.

Theorem 5.10. *Under the assumptions of Theorems 5.3 and 5.9, the following hold true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \le C\bigg(h + H^{(3-\delta)} + k\bigg)e^{-\alpha t_n}$$

and

$$\|p(t_n) - P^n\| \le C(h + H^{3-\delta} + k)e^{-\alpha t_n}.$$

Next, we establish the error estimates for velocity in **Step 3**.

Theorem 5.11. *Assume that* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \le k_0$, *(5.3.9) is satisfied. Let for some fixed* h, *assume that* $\mathbf{u}_h(t)$ *satisfies (5.2.7). Then, there exists a positive constant* C, *independent of* k, *such that, for* $n = 1, 2, \cdots, N$, *the following holds true:*

$$\|\mathbf{e}_h^n\|^2 + \kappa\|\nabla\mathbf{e}_h^n\|^2 + \beta_1 k e^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_h^i\|^2 \le Ck^2 e^{-2\alpha t_n}.$$

Proof. Applying similar set of operations to (5.3.7) as we apply on (5.3.5) to arrive at

(5.3.27) in Theorem 5.9, we obtain

$$\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}_h^n\|^2+\kappa\|\nabla\hat{\mathbf{e}}_h^n\|^2\big)+\Bigg(\nu e^{-\alpha k}-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Bigg)\|\nabla\hat{\mathbf{e}}_h^n\|^2 \qquad (5.3.66)$$
$$= e^{-\alpha k}(e^{\alpha t_n}\sigma_h^n,\hat{\mathbf{e}}_h^n)+e^{-\alpha k}\kappa a(e^{\alpha t_n}\sigma_h^n,\hat{\mathbf{e}}_h^n)+e^{-\alpha k}e^{\alpha t_n}\Lambda(\hat{\mathbf{e}}_h^n).$$

After multiplying (5.3.66) by $2k$, sum over $n=1$ to N and write

$$\|\hat{\mathbf{e}}_h^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2+2\Bigg(\nu e^{-\alpha k}-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Bigg)k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2$$
$$\leq 2ke^{-\alpha k}\sum_{n=1}^{N}(e^{\alpha t_n}\sigma_h^n,\hat{\mathbf{e}}_h^n)+2ke^{-\alpha k}\sum_{n=1}^{N}\kappa a(e^{\alpha t_n}\sigma_h^n,\hat{\mathbf{e}}_h^n)+2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}_h^n)$$
$$= I_1^N+I_2^N+I_3^N, \text{ say.} \qquad (5.3.67)$$

The first two terms in the right hand side of (5.3.67) can be calculated following the similar steps as in the derivation of (5.3.35), (5.3.36) and with the help of estimates in Lemma 5.6. Hence, we arrive at

$$|I_1^N|+|I_2^N|\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^2+2\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.68)$$

Next, we write

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= \big(b(\mathbf{u}_h^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_H^n,\mathbf{U}_h^n,\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &- b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{U}^n,\mathbf{U}_H^n-\mathbf{U}^n,\boldsymbol{\phi}_h)-b(\mathbf{u}_h^{*n},\mathbf{u}_H^n-\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\big)\\ &= \Lambda_h^1(\boldsymbol{\phi}_h)+\Lambda_h^2(\boldsymbol{\phi}_h)+\Lambda_h^3(\boldsymbol{\phi}_h)+\Lambda_h^4(\boldsymbol{\phi}_h). \qquad (5.3.69)\end{aligned}$$

The first term in (5.3.69) can be written as

$$\begin{aligned}|\Lambda_h^1(\boldsymbol{\phi}_h)| &= |b(\mathbf{u}_h^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|\\ &= |b(\mathbf{u}_h^n,\mathbf{u}_H^n-\mathbf{U}_H^n,\boldsymbol{\phi}_h)+b(\mathbf{u}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|\\ &= |-b(\mathbf{u}_h^n,\mathbf{e}_H^n,\boldsymbol{\phi}_h)-b(\mathbf{e}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)|. \qquad (5.3.70)\end{aligned}$$

A use of (3.2.10), (3.2.11) and Lemma 5.6 in (5.3.70) leads to

$$
\begin{aligned}
e^{\alpha t_n}|\Lambda_h^1(\hat{\mathbf{e}}_h^n)| &= e^{-\alpha t_n}|b(\hat{\mathbf{u}}_h^n, \hat{\mathbf{e}}_H^n, \hat{\mathbf{e}}_h^n) + b(\hat{\mathbf{e}}_h^n, \hat{\mathbf{U}}_H^n, \hat{\mathbf{e}}_h^n)| \\
&\leq C\big(\|\nabla \hat{\mathbf{u}}_h^n\|\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\| + \|\hat{\mathbf{e}}_h^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{e}}_h^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{U}}_H^n\|\|\hat{\mathbf{e}}_h^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{e}}_h^n\|^{\frac{1}{2}}\big) \\
&\leq C\big(\|\nabla \hat{\mathbf{u}}_h^n\|\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\| + \|\hat{\mathbf{e}}_h^n\|\|\nabla \hat{\mathbf{U}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\|\big) \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M)\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\| + C\|\hat{\mathbf{e}}_h^n\|\|\nabla \hat{\mathbf{U}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\|. \qquad (5.3.71)
\end{aligned}
$$

Hence, using Young's inequality, we obtain

$$
\begin{aligned}
2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h^1(\hat{\mathbf{e}}_h^n)| &\leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_H^n\|^2 + C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2 \\
&\quad + \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.72)
\end{aligned}
$$

Rewrite the second term of (5.3.69) as

$$
\begin{aligned}
|\Lambda_h^2(\boldsymbol{\phi}_h)| &= |b(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h)| \\
&= |b(\mathbf{u}_H^n - \mathbf{U}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h)| \\
&= |-b(\mathbf{e}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h)|. \qquad (5.3.73)
\end{aligned}
$$

An application of (3.2.1), (3.2.11) and Lemma 5.6 in (5.3.73) yields

$$
\begin{aligned}
e^{\alpha t_n}|\Lambda_h^2(\hat{\mathbf{e}}_h^n)| = e^{-\alpha t_n}|b(\hat{\mathbf{e}}_H^n, \hat{\mathbf{u}}_h^n, \hat{\mathbf{e}}_h^n)| &\leq C\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{u}}_h^n\|\|\nabla \hat{\mathbf{e}}_h^n\| \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M)\|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{e}}_h^n\|. \qquad (5.3.74)
\end{aligned}
$$

Using Young's inequality in (5.3.74), we obtain

$$
2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h^2(\hat{\mathbf{e}}_h^n)| \leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_H^n\|^2 + \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.75)
$$

To bound the third term in the right hand side of (5.3.69), we write it as

$$\begin{aligned}|\Lambda_h^3(\boldsymbol{\phi}_h)| &= |b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &= |b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_H^n - \mathbf{U}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &= |b(\mathbf{U}_H^n, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{e}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)|. \end{aligned} \tag{5.3.76}$$

With the help of (3.2.11), Theorem 5.7 and Lemma 5.2, we find that

$$\begin{aligned} e^{\alpha t_n}|\Lambda_h^3(\hat{\mathbf{e}}_h^n)| &= e^{-\alpha t_n}|b(\hat{\mathbf{U}}_H^n, \hat{\mathbf{e}}^n, \hat{\mathbf{e}}_h^n) - b(\hat{\mathbf{e}}_H^n, \hat{\mathbf{u}}_h^{*n}, \hat{\mathbf{e}}_h^n)| \\ &\le C\big(\|\nabla \hat{\mathbf{U}}_H^n\|\|\nabla \hat{\mathbf{e}}^n\|\|\nabla \hat{\mathbf{e}}_h^n\| + \|\nabla \hat{\mathbf{e}}_H^n\|\|\nabla \hat{\mathbf{u}}_h^{*n}\|\|\nabla \hat{\mathbf{e}}_h^n\|\big) \\ &\le C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla \hat{\mathbf{e}}^n\| + \|\nabla \hat{\mathbf{e}}_H^n\|\big)\|\nabla \hat{\mathbf{e}}_h^n\|. \end{aligned} \tag{5.3.77}$$

Applying Young's inequality in (5.3.77), we arrive at

$$\begin{aligned} 2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h^3(\hat{\mathbf{e}}_h^n)| &\le C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\big(\|\nabla \hat{\mathbf{e}}^n\|^2 + \|\nabla \hat{\mathbf{e}}_H^n\|^2\big) \\ &\quad + \epsilon k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}_h^n\|^2. \end{aligned} \tag{5.3.78}$$

For the estimation of the fourth term on the right hand side of (5.3.69), we first rewrite it as

$$\begin{aligned} |\Lambda_h^4(\boldsymbol{\phi}_h)| &= |b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &= |b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) \\ &+ b(\mathbf{U}^n, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &= |b(\mathbf{U}^n, \mathbf{e}_H^n - \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{e}^n, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &= |b(\mathbf{U}^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{e}^n, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)|. \end{aligned} \tag{5.3.79}$$

Apply (3.2.11) and estimate in Lemmas 5.1, 5.2 and 5.11 to obtain

$$
\begin{aligned}
e^{\alpha t_n}|\Lambda_h^4(\hat{\mathbf{e}}_h^n)| =& e^{-\alpha t_n}|b(\hat{\mathbf{U}}^n, \hat{\mathbf{e}}_H^n, \hat{\mathbf{e}}_h^n) - b(\hat{\mathbf{U}}^n, \hat{\mathbf{e}}^n, \hat{\mathbf{e}}_h^n) + b(\hat{\mathbf{e}}^n, \hat{\mathbf{u}}_H^n - \hat{\mathbf{u}}_h^{*n}, \hat{\mathbf{e}}_h^n)| \\
&\leq C\big(\|\nabla\hat{\mathbf{U}}^n\|\|\nabla\hat{\mathbf{e}}_H^n\| + \|\nabla\hat{\mathbf{U}}^n\|\|\nabla\hat{\mathbf{e}}^n\| + \|\nabla(\hat{\mathbf{u}}_H^n - \hat{\mathbf{u}}_h^{*n})\|\|\nabla\hat{\mathbf{e}}^n\|\big)\|\nabla\hat{\mathbf{e}}_h^n\| \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla\hat{\mathbf{e}}^n\| + \|\nabla\hat{\mathbf{e}}_H^n\|\big)\|\nabla\hat{\mathbf{e}}_h^n\|. \qquad (5.3.80)
\end{aligned}
$$

A use of Young's inequality in (5.3.80) now yields

$$
\begin{aligned}
2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h^4(\hat{\mathbf{e}}_h^n)| \leq& C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\big(\|\nabla\hat{\mathbf{e}}^n\|^2 + \|\nabla\hat{\mathbf{e}}_H^n\|^2\big) \\
&+ \epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.81)
\end{aligned}
$$

A combination of (5.3.72), (5.3.75), (5.3.78) and (5.3.81) leads to

$$
\begin{aligned}
|I_3^N| = 2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}_h^n)| \leq& C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\big(\|\nabla\hat{\mathbf{e}}_H^n\|^2 + \|\nabla\hat{\mathbf{e}}^n\|^2\big) \\
&+ C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2 + 4\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.82)
\end{aligned}
$$

Using estimates obtained from Theorems 5.8 and 5.9 in (5.3.82), we find that

$$
\begin{aligned}
|I_3^N| \leq& C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)k^2 + C(\epsilon)ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2 \\
&+ 4\epsilon ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2. \qquad (5.3.83)
\end{aligned}
$$

An application of (5.3.68) and (5.3.83) in (5.3.67) with $\epsilon = \frac{\nu}{6}$ yields

$$
\begin{aligned}
\|\hat{\mathbf{e}}_h^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2 \leq& C(\kappa, \nu, \alpha, \lambda_1, M)k^2 \\
&+ C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2 + C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^N\|^2\|\hat{\mathbf{e}}_h^N\|^2. \qquad (5.3.84)
\end{aligned}
$$

Using Theorem 5.7 in (5.3.84), we obtain

$$\|\hat{\mathbf{e}}_h^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2$$
$$+ C(\nu)ke^{-\alpha k}\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2 + C(\nu,\kappa,M)ke^{-\alpha k}\big(\|\hat{\mathbf{e}}_h^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2\big). \quad (5.3.85)$$

Again, using the fact that $(1 - C(\nu,\kappa,M)ke^{-\alpha k})$ can be made positive for $0 < k \le k_0$ for some $k_0 > 0$, we then use discrete Gronwall's lemma in (5.3.85) to arrive at

$$\|\hat{\mathbf{e}}_h^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2 + \beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\exp(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2). \quad (5.3.86)$$

Now, with the help of Theorem 5.7, we bound

$$\exp(k\sum_{n=1}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2) \le C(\kappa,\nu,\alpha,\lambda_1,M). \quad (5.3.87)$$

A use of (5.3.87) in (5.3.86) would complete the proof. □

We recall below (5.3.7), in order to establish bound for pressure in **Step 3**, as

$$(\bar{\partial}_t\mathbf{e}_h^n,\boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{e}_h^n,\boldsymbol{\phi}_h) = -\nu a(\mathbf{e}_h^n,\boldsymbol{\phi}_h) + (\sigma_h^n,\boldsymbol{\phi}_h) + \kappa a(\sigma_h^n,\boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h)\ \forall\boldsymbol{\phi}_h \in \mathbf{J}_h, \quad (5.3.88)$$

where $\sigma_h^n = \mathbf{u}_{ht}^n - \bar{\partial}_t\mathbf{u}_h^n$ and

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) + b(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}_H^n,\mathbf{U}_h^n,\boldsymbol{\phi}_h) + b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &- b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h) + b(\mathbf{U}^n,\mathbf{U}_H^n - \mathbf{U}^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n},\mathbf{u}_H^n - \mathbf{u}_h^{*n},\boldsymbol{\phi}_h). \quad (5.3.89)\end{aligned}$$

Choose $\boldsymbol{\phi}_h = \bar{\partial}_t\mathbf{e}_h^n$ in (5.3.88) to obtain

$$\|\bar{\partial}_t\mathbf{e}_h^n\|^2 + \kappa\|\nabla\bar{\partial}_t\mathbf{e}_h^n\|^2 = -\nu a(\mathbf{e}_h^n,\bar{\partial}_t\mathbf{e}_h^n) + (\sigma_h^n,\bar{\partial}_t\mathbf{e}_h^n) + \kappa a(\sigma_h^n,\bar{\partial}_t\mathbf{e}_h^n) + \Lambda_h(\bar{\partial}_t\mathbf{e}_h^n). \quad (5.3.90)$$

Using (5.3.69), we observe that

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= \big(b(\mathbf{u}_h^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h)\big) + \big(b(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h)\big) + \big(b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ &\quad - b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)\big) + \big(b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)\big) \\ &= \Lambda_h^1(\boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h) + \Lambda_h^3(\boldsymbol{\phi}_h) + \Lambda_h^4(\boldsymbol{\phi}_h). \end{aligned} \tag{5.3.91}$$

With the help of (5.3.70), (3.2.11), Lemma 5.6 and Theorem 5.7, we arrive at

$$\begin{aligned}|\Lambda_h^1(\boldsymbol{\phi}_h)| &= |-b(\mathbf{u}_h^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{e}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h)| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)(\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}_h^n\|)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.92}$$

Similarly, Using (5.3.73), (3.2.11), Lemma 5.6 and Theorem 5.7, we bound $\Lambda_h^2(\boldsymbol{\phi}_h)$ as

$$\begin{aligned}|\Lambda_h^2(\boldsymbol{\phi}_h)| &= |-b(\mathbf{e}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{e}_H^n\|\|\nabla \mathbf{u}_h^n\| + \|\nabla \mathbf{U}_H^n\|\|\nabla \mathbf{e}_h^n\|\big)\|\nabla \boldsymbol{\phi}_h\| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}_h^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.93}$$

From (5.3.76) (with $\hat{\mathbf{e}}_h^n$ replaced by $\boldsymbol{\phi}_h$), Theorem 5.7 and Lemma 5.2, we obtain

$$\begin{aligned}|\Lambda_h^3(\boldsymbol{\phi}_h)| &= |b(\mathbf{U}_H^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{U}_H^n\|\|\nabla \mathbf{e}^n\|\|\nabla \boldsymbol{\phi}_h\| + \|\nabla \mathbf{e}_H^n\|\|\nabla \mathbf{u}_h^{*n}\|\|\nabla \boldsymbol{\phi}_h\|\big) \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}_H^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.94}$$

Recalling (5.3.79), (3.2.11), Lemmas 5.1, 5.2 and 5.11, we note that

$$\begin{aligned}|\Lambda_h^4(\boldsymbol{\phi}_h)| &= |b(\mathbf{U}^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{e}^n, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)| \\ &\leq C\big(\|\nabla \mathbf{U}^n\|\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{U}^n\|\|\nabla \mathbf{e}^n\| + \|\nabla(\mathbf{u}_H^n - \mathbf{u}_h^{*n})\|\|\nabla \mathbf{e}^n\|\big)\|\nabla \boldsymbol{\phi}_h\| \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}_H^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \tag{5.3.95}$$

A use of (5.3.92)-(5.3.95) in (5.3.91) yields

$$|\Lambda_h(\boldsymbol{\phi}_h)| \leq C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}_h^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \tag{5.3.96}$$

With the help of (5.3.96), Cauchy-Schwarz's inequality and Young's inequality in (5.3.90), we observe that

$$\begin{aligned}\|\bar{\partial}_t \mathbf{e}_h^n\|^2 + \kappa\|\nabla\bar{\partial}_t \mathbf{e}_h^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\sigma_h^n\|^2 + \|\nabla\sigma_h^n\|^2 \\ + \|\nabla \mathbf{e}_H^n\|^2 + \|\nabla \mathbf{e}^n\|^2 + \|\nabla \mathbf{e}_h^n\|^2). \end{aligned} \tag{5.3.97}$$

Proceeding along the similar lines as in (5.3.58) (with $\mathbf{u}_h^*$ replaced by $\mathbf{u}_h$) and using Lemma 5.6, we arrive at

$$\begin{aligned}\|e^{\alpha t_n}\sigma_h^n\|^2 &\leq \frac{1}{k^2}\bigg(\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2 ds\bigg)\bigg(\int_{t_{n-1}}^{t_n}(t_n-s)^2 ds\bigg) \\ &= \frac{k}{3}\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2\, ds \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)ke^{2\alpha t_n}\int_{t_{n-1}}^{t_n} e^{-2\alpha s} ds \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 e^{2\alpha k^*}, \end{aligned} \tag{5.3.98}$$

for $k^* \in (0,k)$.
Similarly, we obtain

$$\|e^{\alpha t_n}\nabla\sigma_h^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 e^{2\alpha k^*}. \tag{5.3.99}$$

A use of (5.3.98)-(5.3.99), Theorems 5.8, 5.9 and 5.11 in (5.3.97) leads to

$$\|\bar{\partial}_t \mathbf{e}_h^n\|^2 + \kappa\|\nabla\bar{\partial}_t \mathbf{e}_h^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2\ e^{-2\alpha t_n}. \tag{5.3.100}$$

Next, we use the following equivalent form of (5.3.3) as: find a sequence of functions

$\{\mathbf{U}_h^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P_h^n\}_{n\geq 1} \in L_h$ such that

$$
\begin{aligned}
(\bar{\partial}_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \qquad (5.3.101)\\
= b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) + (P_h^n, \nabla \cdot \boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,\\
(\nabla \cdot \mathbf{U}_h^n, \chi_h) = 0 \quad \forall \chi_h \in L_h.
\end{aligned}
$$

After considering (5.2.90) at $t = t_n$, we subtract it from (5.3.101) to obtain

$$
\begin{aligned}
(\boldsymbol{\rho}_h^n, \nabla \cdot \boldsymbol{\phi}_h) =& (\bar{\partial}_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_h^n, \boldsymbol{\phi}_h)\\
&- (\sigma_h^n, \boldsymbol{\phi}_h) - \kappa a(\sigma_h^n, \boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h), \qquad (5.3.102)
\end{aligned}
$$

where $\boldsymbol{\rho}_h^n = P_h^n - p_h(t_n)$ and $\Lambda_h(\boldsymbol{\phi}_h)$ is defined in (5.3.91).
Applying Cauchy-Schwarz's inequality, (2.2.3), (5.3.96) in (5.3.102) to write

$$
(\boldsymbol{\rho}_h^n, \nabla \cdot \boldsymbol{\phi}_h) \leq C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\bar{\partial}_t \nabla \mathbf{e}_h^n\| + \|\nabla \mathbf{e}_h^n\| + \|\nabla \sigma_h^n\| + \|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\|\big). \qquad (5.3.103)
$$

Then, we use (5.3.99)-(5.3.100), Theorems 5.8, 5.9 and 5.11 in (5.3.103) to arrive at

$$
\|\boldsymbol{\rho}_h^n\| \leq C(\kappa, \nu, \alpha, \lambda_1, M) k \; e^{-\alpha t_n}. \qquad (5.3.104)
$$

Now a use of (5.3.104), Theorems 5.4, 5.6 and 5.11 completes the proof of the following Theorem.

Theorem 5.12. *Under the assumptions of Theorems 5.4 and 5.11, the following hold true:*

$$
\|\mathbf{u}(t_n) - \mathbf{U}_h^n\| \leq C\Bigg(h^2 + hH^{(2-\delta)} + H^{(5-2\delta)} + k\Bigg)e^{-\alpha t_n}
$$

and

$$
\|p(t_n) - P_h^n\| \leq C(h + H^{5-2\delta} + k)e^{-\alpha t_n}.
$$

Remark 5.3.1. *In case, we choose* $h = H^{2-\delta}$ *with* $\delta > 0$ *arbitrary small for* $\mathbb{R}^2$ *and* $\delta = 1/2$ *for* $\mathbb{R}^3$*, we arrive at the following optimal estimates for velocity and pressure*

$$\|\mathbf{u}(t_n) - \mathbf{U}_h^n\| \leq C(h^2 + k)e^{-\alpha t_n}$$

and

$$\|p(t_n) - P_h^n\| \leq C(h + k)e^{-\alpha t_n}.$$

5.4 Backward Difference Scheme

In this section, we apply backward difference scheme to two-grid semidiscrete approximations to obtain second order accuracy in time. The second order Backward differencing scheme applied to (5.2.5)-(5.2.7) is as follows:

Algorithm:

Step 1: Solve nonlinear system on coarse mesh $\mathcal{T}_H$: for $\boldsymbol{\phi}_H \in \mathbf{J}_H$, we seek $\{\mathbf{U}_H^n\}_{n\geq 1} \in \mathbf{J}_H$ such that

$$(D_t^{(2)}\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \kappa a(D_t^{(2)}\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu a(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) = 0 \;\; n \geq 2, \tag{5.4.1}$$
$$(\bar{\partial}_t \mathbf{U}_H^1, \boldsymbol{\phi}_H) + \kappa a(\bar{\partial}_t \mathbf{U}_H^1, \boldsymbol{\phi}_H) + \nu a(\mathbf{U}_H^1, \boldsymbol{\phi}_H) + b(\mathbf{U}_H^1, \mathbf{U}_H^1, \boldsymbol{\phi}_H) = 0,$$

where $D_t^{(2)}\mathbf{U}_H^n = \frac{1}{2k}(3\mathbf{U}_H^n - 4\mathbf{U}_H^{n-1} + \mathbf{U}_H^{n-2})$.

Step 2: Update on fine mesh $\mathcal{T}_h$ with one Newton iteration: find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$\begin{aligned}(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h \;\; n \geq 2, \\ (\bar{\partial}_t \mathbf{U}^1, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}_H^1, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}_H^1, \mathbf{U}^1, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^1, \mathbf{U}_H^1, \boldsymbol{\phi}_h).\end{aligned} \tag{5.4.2}$$

Step 3: Correct on fine mesh $\mathcal{T}_h$: find $\mathbf{U}_h^n \in \mathbf{J}_h$ such that

$$
\begin{aligned}
(D_t^{(2)}\mathbf{U}_h^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \quad (5.4.3)\\
+ b(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h \quad n \geq 2,\\
(\bar{\partial}_t \mathbf{U}_h^1, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}_h^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^1, \boldsymbol{\phi}_h) + b(\mathbf{U}_h^1, \mathbf{U}_H^1, \boldsymbol{\phi}_h)\\
+ b(\mathbf{U}_H^1, \mathbf{U}_h^1, \boldsymbol{\phi}_h) = b(\mathbf{U}_H^1, \mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}_H^1 - \mathbf{U}^1, \boldsymbol{\phi}_h).
\end{aligned}
$$

Similar to the Section **4.4**, the results of this section are derived with the help of identity (4.4.4). Hence, we recall it as follows:

$$
\begin{aligned}
2e^{2\alpha t_n}(a^n, 3a^n - 4a^{n-1} + a^{n-2}) = \|\hat{a}^n\|^2 - \|\hat{a}^{n-1}\|^2 + (1 - e^{2\alpha k})(\|\hat{a}^n\|^2 + \|\hat{a}^{n-1}\|^2) \quad (5.4.4)\\
+ \|\delta^2 \hat{a}^{n-1}\|^2 + \|2\hat{a}^n - e^{\alpha k}\hat{a}^{n-1}\|^2 - \|2\hat{a}^{n-1} - e^{\alpha k}\hat{a}^{n-2}\|^2,
\end{aligned}
$$

where

$$
\delta^2 \hat{a}^{n-1} = e^{\alpha k}\hat{a}^n - 2\hat{a}^{n-1} + e^{\alpha k}\hat{a}^{n-2}.
$$

We set $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H(t_n)$, $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h^*(t_n)$ and $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h(t_n)$.
Considering (5.2.5)-(5.2.7) at $t = t_n$ and subtracting the resulting equations from (5.4.1)-(5.4.3), respectively, to write
Equation for **Step 1**:

$$
\begin{aligned}
(D_t^{(2)}\mathbf{e}_H^n, \boldsymbol{\phi}_H) + \kappa a(D_t^{(2)}\mathbf{e}_H^n, \boldsymbol{\phi}_H) + \nu a(\mathbf{e}_H^n, \boldsymbol{\phi}_H) = (\sigma_{2H}^n, \boldsymbol{\phi}_H)\\
+ \kappa a(\sigma_{2H}^n, \boldsymbol{\phi}_H) + \Lambda_H(\boldsymbol{\phi}_H) \ \forall \boldsymbol{\phi}_H \in \mathbf{J}_H \quad n \geq 2 \quad (5.4.5)
\end{aligned}
$$

and for $n = 1$,

$$
\begin{aligned}
(\bar{\partial}_t \mathbf{e}_H^1, \boldsymbol{\phi}_H) + \kappa a(\bar{\partial}_t \mathbf{e}_H^1, \boldsymbol{\phi}_H) + \nu a(\mathbf{e}_H^1, \boldsymbol{\phi}_H) = (\sigma_H^1, \boldsymbol{\phi}_H)\\
+ \kappa a(\sigma_H^1, \boldsymbol{\phi}_H) + \Lambda_H^1(\boldsymbol{\phi}_H) \ \forall \boldsymbol{\phi}_H \in \mathbf{J}_H, \quad (5.4.6)
\end{aligned}
$$

where $\sigma_{2H}^n = \mathbf{u}_{Ht}^n - D_t^{(2)}\mathbf{u}_H^n$, $\Lambda_H(\boldsymbol{\phi}_H) = b(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_H) - b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H)$, $\sigma_H^1 = \mathbf{u}_{Ht}^1 - \bar{\partial}_t \mathbf{u}_H^1$ and $\Lambda_H^1(\boldsymbol{\phi}_H) = b(\mathbf{u}_H^1, \mathbf{u}_H^1, \boldsymbol{\phi}_H) - b(\mathbf{U}_H^1, \mathbf{U}_H^1, \boldsymbol{\phi}_H)$.

Equation for **Step 2**:

$$\begin{aligned}
&(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) \\
&\qquad = (\sigma_2^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_2^n, \boldsymbol{\phi}_h) + \Lambda^*(\boldsymbol{\phi}_h)\ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h \quad n \geq 2
\end{aligned} \tag{5.4.7}$$

and for $n = 1$,

$$\begin{aligned}
&(\bar{\partial}_t \mathbf{e}^1, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^1, \boldsymbol{\phi}_h) \\
&\qquad = (\sigma^1, \boldsymbol{\phi}_h) + \kappa a(\sigma^1, \boldsymbol{\phi}_h) + \Lambda^{*1}(\boldsymbol{\phi}_h)\ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,
\end{aligned} \tag{5.4.8}$$

where $\sigma_2^n = \mathbf{u}_{ht}^{*n} - D_t^{(2)}\mathbf{u}_h^{*n}$, $\sigma^1 = \mathbf{u}_{ht}^{*1} - \bar{\partial}_t \mathbf{u}_h^{*1}$,

$$\begin{aligned}
\Lambda^*(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) \\
&\quad - b(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) - b(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_H)
\end{aligned} \tag{5.4.9}$$

and

$$\begin{aligned}
\Lambda^{*1}(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^{*1}, \mathbf{u}_H^1, \boldsymbol{\phi}_h) - b(\mathbf{U}^1, \mathbf{U}_H^1, \boldsymbol{\phi}_h) + b(\mathbf{u}_H^1, \mathbf{u}_h^{*1}, \boldsymbol{\phi}_h) \\
&\quad - b(\mathbf{U}_H^1, \mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^1, \mathbf{U}_H^1, \boldsymbol{\phi}_H) - b(\mathbf{u}_H^1, \mathbf{u}_H^1, \boldsymbol{\phi}_H).
\end{aligned} \tag{5.4.10}$$

Similarly, equation in **Step 3** satisfies

$$\begin{aligned}
&(D_t^{(2)}\mathbf{e}_h^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_h^n, \boldsymbol{\phi}_h) \\
&\qquad = (\sigma_{2h}^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_{2h}^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h)\ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h \quad n \geq 2
\end{aligned} \tag{5.4.11}$$

and for $n = 1$,

$$\begin{aligned}
&(\bar{\partial}_t \mathbf{e}_h^1, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}_h^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_h^1, \boldsymbol{\phi}_h) \\
&\qquad = (\sigma_h^1, \boldsymbol{\phi}_h) + \kappa a(\sigma_h^1, \boldsymbol{\phi}_h) + \Lambda_h^1(\boldsymbol{\phi}_h)\ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,
\end{aligned} \tag{5.4.12}$$

where $\sigma_{2h}^n = \mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n$, $\sigma_h^1 = \mathbf{u}_{ht}^1 - \bar{\partial}_t\mathbf{u}_h^1$,

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) \;=\;& b(\mathbf{u}_h^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h) - b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) + b(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}_H^n,\mathbf{U}_h^n,\boldsymbol{\phi}_h) + b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ & -b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h) + b(\mathbf{U}^n,\mathbf{U}_H^n - \mathbf{U}^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*n},\mathbf{u}_H^n - \mathbf{u}_h^{*n},\boldsymbol{\phi}_h) \qquad (5.4.13)\end{aligned}$$

and

$$\begin{aligned}\Lambda_h^1(\boldsymbol{\phi}_h) \;=\;& b(\mathbf{u}_h^1,\mathbf{u}_H^1,\boldsymbol{\phi}_h) - b(\mathbf{U}_h^1,\mathbf{U}_H^1,\boldsymbol{\phi}_h) + b(\mathbf{u}_H^1,\mathbf{u}_h^1,\boldsymbol{\phi}_h) - b(\mathbf{U}_H^1,\mathbf{U}_h^1,\boldsymbol{\phi}_h) + b(\mathbf{U}_H^1,\mathbf{U}^1,\boldsymbol{\phi}_h)\\ & -b(\mathbf{u}_H^1,\mathbf{u}_h^{*1},\boldsymbol{\phi}_h) + b(\mathbf{U}^1,\mathbf{U}_H^1 - \mathbf{U}^1,\boldsymbol{\phi}_h) - b(\mathbf{u}_h^{*1},\mathbf{u}_H^1 - \mathbf{u}_h^{*1},\boldsymbol{\phi}_h). \qquad (5.4.14)\end{aligned}$$

Remark 5.4.1. *Note that, in all the three error equations results for case $n = 1$ is obtained by using backward Euler method.*

Next, we recall a *priori* bounds for discrete solution $\mathbf{U}_H^n$ of (5.4.1) and error estimates of **Step 1** which are already worked out in Section **4.4** (Lemma 4.2 and Theorem 4.4).

Theorem 5.13. *With $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, choose k_0 small so that for $0 < k \le k_0$*

$$\frac{\nu k\lambda_1}{\kappa\lambda_1 + 1} + 1 > e^{2\alpha k}. \qquad (5.4.15)$$

Then, the discrete solution $\mathbf{U}_H^n$, $n \ge 1$ of (5.4.1) satisfies the following a priori bound:

$$(\|\mathbf{U}_H^n\|^2 + \kappa\|\nabla\mathbf{U}_H^n\|^2) + e^{-2\alpha t_N}\; k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{U}_H^i\|^2 \le C(\kappa,\nu,\alpha,\lambda_1)e^{-2\alpha t_n}(\|\mathbf{U}_H^0\|^2 + \kappa\|\nabla\mathbf{U}_H^0\|^2).$$

□

Theorem 5.14. *Assume that $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ and choose $k_0 \ge 0$ such that for $0 < k \le k_0$, (5.4.15) is satisfied. Let $u_H(t)$ be a solution of (5.2.5) and $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H(t_n)$, for $n = 1, 2, \cdots, N$. Then, for some positive constant $C = C(\kappa,\nu,\alpha,\lambda_1,M)$, there holds*

$$\|\mathbf{e}_H^n\|^2 + \kappa\|\nabla\mathbf{e}_H^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_H^i\|^2 \le Ck^4 e^{-2\alpha t_n}. \qquad (5.4.16)$$

□

To arrive at the error estimates, we require the following exponential decay property of the discrete solution $\mathbf{U}^N$ of **Step 2**.

Lemma 5.12. *With* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, *choose* k_0 *small such that for* $0 < k \le k_0$ *(5.4.15) is satisfied. Then, the discrete solution* $\mathbf{U}^N$, $n \ge 1$ *of (5.4.2) satisfies the following a priori bound:*

$$(\|\mathbf{U}^n\|^2 + \kappa\|\nabla\mathbf{U}^n\|^2) + e^{-2\alpha t_n}\ k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 \le C(\kappa,\nu,\alpha,\lambda_1)e^{-2\alpha t_n}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2).$$

Proof. Multiply (5.4.2) by $e^{\alpha t_n}$ and substitute $\boldsymbol{\phi}_h = \hat{\mathbf{U}}^n$. Then, using identity (5.4.4) and (3.2.1), we obtain

$$\begin{aligned}
&\frac{1}{4}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2) + \nu\|\nabla\hat{\mathbf{U}}^n\|^2 + \Big(\frac{1-e^{2\alpha k}}{4k}\Big)\Big(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2\Big) \\
&\quad + \Big(\frac{1-e^{2\alpha k}}{4k}\Big)\Big(\|\hat{\mathbf{U}}^{n-1}\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^{n-1}\|^2\Big) + \frac{1}{4k}\|\delta^2\hat{\mathbf{U}}^{n-1}\|^2 + \frac{1}{4k}\kappa\|\delta^2\nabla\hat{\mathbf{U}}^{n-1}\|^2 \\
&\quad + \frac{1}{4k}\Big((2\hat{\mathbf{U}}^n - e^{\alpha k}\hat{\mathbf{U}}^{n-1})^2 - (2\hat{\mathbf{U}}^{n-1} - e^{\alpha k}\hat{\mathbf{U}}^{n-2})^2\Big) + \frac{\kappa}{4k}\big((2\nabla\hat{\mathbf{U}}^n - e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-1})^2 \\
&\quad - (2\nabla\hat{\mathbf{U}}^{n-1} - e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-2})^2\big) \\
&= -e^{-\alpha t_n}b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n) + e^{-\alpha t_n}b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n). \qquad (5.4.17)
\end{aligned}$$

Using the non-negativity property of the fifth and sixth terms on the left hand side of (5.4.17), we drop these terms. Multiply (5.4.17) by $4ke^{-2\alpha k}$, sum over $n = 2$ to N and use (2.2.3) and (4.4.7) to yield

$$\begin{aligned}
&\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + k\Big(4\nu e^{-2\alpha k} - 2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \\
&\quad + \|2e^{-\alpha k}\hat{\mathbf{U}}^N - \hat{\mathbf{U}}^{N-1}\|^2 + \kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^N - \nabla\hat{\mathbf{U}}^{N-1}\|^2 \\
&\le 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n) - 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{-\alpha t_n}b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n_H, \hat{\mathbf{U}}^n) + \big(\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^1\|^2\big) \\
&\quad + (2e^{-\alpha k}\hat{\mathbf{U}}^1 - \mathbf{U}^0)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1 - \nabla\mathbf{U}^0)^2 \\
&= I_1^N + I_2^N + \big(\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^1\|^2\big) + (2e^{-\alpha k}\hat{\mathbf{U}}^1 - \mathbf{U}^0)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1 - \nabla\mathbf{U}^0)^2. \quad (5.4.18)
\end{aligned}$$

With the help of (3.2.11), Young's inequality and Theorem 5.13, we arrive at

$$
\begin{aligned}
|I_1^N| &= |4ke^{-2\alpha k}\sum_{n=2}^{N} e^{-\alpha t_n} b(\hat{\mathbf{U}}_H^n, \hat{\mathbf{U}}_H^n, \hat{\mathbf{U}}^n)| \\
&\leq Cke^{-2\alpha k}\sum_{n=2}^{N} e^{-\alpha t_n}\|\nabla \hat{\mathbf{U}}_H^n\|^2 \|\nabla \hat{\mathbf{U}}^n\| \\
&\leq C(\kappa, M, \epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla \hat{\mathbf{U}}_H^n\|^2 + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla \hat{\mathbf{U}}^n\|^2 \\
&\leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)(\|\mathbf{U}_H^0\|^2 + \kappa\|\nabla \mathbf{U}_H^0\|^2) + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla \hat{\mathbf{U}}^n\|^2. \qquad (5.4.19)
\end{aligned}
$$

An application of (3.2.10) with Young's inequality yields

$$
\begin{aligned}
|I_2^N| &= |4ke^{-2\alpha k}\sum_{n=2}^{N} e^{-\alpha t_n} b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}_H^n, \hat{\mathbf{U}}^n)| \\
&\leq Cke^{-2\alpha k}\sum_{n=2}^{N} e^{-\alpha t_n}\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{U}}_H^n\|\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla \hat{\mathbf{U}}^n\|^{\frac{1}{2}} \\
&\leq Cke^{-2\alpha k}\sum_{n=2}^{N} e^{-\alpha t_n}\|\hat{\mathbf{U}}^n\|\|\nabla \hat{\mathbf{U}}^n\|\|\nabla \hat{\mathbf{U}}_H^n\| \\
&\leq C(\epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla \hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{U}}^n\|^2 + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla \hat{\mathbf{U}}^n\|^2. \qquad (5.4.20)
\end{aligned}
$$

For the last three terms on the right hand side of (5.4.18), we write (5.3.15) for $n = 1$ and obtain

$$
\begin{aligned}
&\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla \hat{\mathbf{U}}^1\|^2) + \Big(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla \hat{\mathbf{U}}^1\|^2 \\
&\quad \leq -e^{-\alpha k}e^{-\alpha k} b(\hat{\mathbf{U}}^1, \hat{\mathbf{U}}_H^1, \hat{\mathbf{U}}^1) + e^{-\alpha k}e^{-\alpha k} b(\hat{\mathbf{U}}_H^1, \hat{\mathbf{U}}_H^1, \hat{\mathbf{U}}^1). \qquad (5.4.21)
\end{aligned}
$$

Multiply (5.4.21) by $2k$ to arrive at

$$\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^1\|^2 + 2k\bigg(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{U}}^1\|^2$$
$$\leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 + 2ke^{-2\alpha k}\big(b(\hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1) - b(\hat{\mathbf{U}}^1, \hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1)\big)$$
$$\leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 + I_1^1 + I_2^1. \tag{5.4.22}$$

We use (3.2.11), Theorem 5.7 and observe that

$$\begin{aligned}|I_1^1| &= |2ke^{-2\alpha k} b(\hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1)| \\ &\leq Cke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1_H\|^2\|\nabla\hat{\mathbf{U}}^1\| \\ &\leq C(\kappa, M, \epsilon)ke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1_H\|^2 + \epsilon ke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2 \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)(\|\mathbf{U}^0_H\|^2 + \kappa\|\nabla\mathbf{U}^0_H\|^2) + \epsilon ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2.\end{aligned} \tag{5.4.23}$$

Applying (3.2.10), we obtain

$$\begin{aligned}|b(\hat{\mathbf{U}}^1, \hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1)| &\leq C\|\hat{\mathbf{U}}^1\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^1\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^1_H\|\|\hat{\mathbf{U}}^1\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^1\|^{\frac{1}{2}} \\ &\leq C\|\hat{\mathbf{U}}^1\|\|\nabla\hat{\mathbf{U}}^1\|\|\nabla\hat{\mathbf{U}}^1_H\|.\end{aligned} \tag{5.4.24}$$

With the help of (5.4.24) and Young's inequality, we observe that

$$\begin{aligned}|I_2^1| &= |2ke^{-2\alpha k} b(\hat{\mathbf{U}}^1, \hat{\mathbf{U}}^1_H, \hat{\mathbf{U}}^1)| \\ &\leq C(\epsilon)ke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1_H\|^2\|\hat{\mathbf{U}}^1\|^2 + \epsilon ke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2 \\ &\leq C(\epsilon)ke^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1_H\|^2\|\hat{\mathbf{U}}^1\|^2 + \epsilon ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2.\end{aligned} \tag{5.4.25}$$

Using (5.4.23), (5.4.25) in (5.4.22) with $\epsilon = \frac{\nu}{2}$ and Theorem 5.7, we obtain

$$\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^1\|^2 + k\bigg(e^{-\alpha k}\nu - 2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg) \leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 \tag{5.4.26}$$
$$+ C(\kappa,\nu,\alpha,\lambda_1,M)(\|\mathbf{U}^0_H\|^2 + \kappa\|\nabla\mathbf{U}^0_H\|^2) + C(\nu,\kappa,M)ke^{-\alpha k}\big(\|\hat{\mathbf{U}}^1\|^2 + \kappa\nabla\hat{\mathbf{U}}^1\|^2\big).$$

Choose k small enough to obtain

$$\|\hat{\mathbf{U}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^1\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 + \|\mathbf{U}_H^0\|^2 + \kappa\|\nabla\mathbf{U}_H^0\|^2). \quad (5.4.27)$$

Applying Cauchy-Schwarz's inequality, Young's inequality and (5.4.27), we write:

$$(2e^{-\alpha k}\hat{\mathbf{U}}^1 - \mathbf{U}^0)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1 - \nabla\mathbf{U}^0)^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)(\|\mathbf{U}^0\|^2 + \|\nabla\mathbf{U}^0\|^2). \quad (5.4.28)$$

(5.4.19)-(5.4.20), (5.4.27) and (5.4.28) in (5.4.18) would lead us to the desired result. □

Theorem 5.15. *Assume that* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and choose* $k_0 \geq 0$ *such that for* $0 < k \leq k_0$*, (5.4.15) is satisfied. Let* $\mathbf{u}_h^*(t)$ *be a solution of (5.2.6) and* $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h^*(t_n)$*, for* $n = 1, 2, \cdots, N$*. Then, for some positive constant* $C = C(\kappa,\nu,\alpha,\lambda_1,M)$*, there holds,*

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2 \leq Ck^4 e^{-2\alpha t_n}.$$

Proof. Applying the similar sets of operations to (5.4.7) as in Theorem 4.4 leading to (4.4.15), we arrive at

$$\begin{aligned} & k\bar{\partial}_t(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + \|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2 + 4k\nu\|\nabla\hat{\mathbf{e}}^n\|^2 \\ & +(1-e^{2\alpha k})(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^{n-1}\|^2) \\ & +(2\hat{\mathbf{e}}^n - e^{\alpha k}\hat{\mathbf{e}}^{n-1})^2 - (2\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\hat{\mathbf{e}}^{n-2})^2 + \kappa(2\nabla\hat{\mathbf{e}}^n - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-1})^2 \\ & -\kappa(2\nabla\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-2})^2 \\ & = 4k(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa\,a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,e^{\alpha t_n}\Lambda^*(\hat{\mathbf{e}}^n). \end{aligned} \quad (5.4.29)$$

Taking a sum of (5.4.29) over $n = 2$ to N, using (4.4.7) along with $\mathbf{e}^o = 0$ and dividing by

$e^{2\alpha k}$, we arrive at

$$\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2) + (2e^{-\alpha k}\hat{\mathbf{e}}^N - \hat{\mathbf{e}}^{N-1})^2 \\
&+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N - \nabla\hat{\mathbf{e}}^{N-1})^2 + k\Big(4\nu e^{-2\alpha k} - 2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\
&\leq \|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 + (2e^{-\alpha k}\hat{\mathbf{e}}^1 - \mathbf{e}^o)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^1 - \nabla\mathbf{e}^o)^2 \\
&+4ke^{-2\alpha k}\sum_{n=2}^{N}(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa e^{-2\alpha k}\sum_{n=2}^{N}a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}\Lambda^*(\hat{\mathbf{e}}^n) \\
&\leq C(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2) + I_1^* + I_2^* + I_3^*,\ \ say. \qquad (5.4.30)
\end{aligned}$$

The derivation uses proof techniques of Theorem 4.4. A use of Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality to bound $|I_1^*|$ leads to

$$\begin{aligned}
|I_1^*| &\leq 4ke^{-2\alpha k}(\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2)^{\frac{1}{2}}(\sum_{n=2}^{N}\|\hat{\mathbf{e}}^n\|^2)^{\frac{1}{2}} \\
&\leq C(\epsilon,\lambda_1)ke^{-2\alpha k}\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.4.31)
\end{aligned}$$

Similar to (4.4.18), we observe that

$$\|e^{\alpha t_n}\sigma_2^n\|^2 \leq \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}^*(t)\|^2\,dt. \qquad (5.4.32)$$

An application of (5.4.32) yields

$$\begin{aligned}
k\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 &\leq \frac{k^4}{2}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}^*(t)\|^2 dt \\
&= \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_{n-2}}\|\mathbf{u}_{httt}^*(t)\|^2 dt \\
&\leq \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t}\|\mathbf{u}_{httt}^*(t)\|^2 dt. \qquad (5.4.33)
\end{aligned}$$

A *priori* bounds for $\mathbf{u}_h^*$ from Lemma 5.5 and (5.4.33) yield

$$k\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 \leq k^4e^{4\alpha k}\int_0^{t_N} e^{2\alpha t}\|\mathbf{u}_{httt}^*(t)\|^2dt$$
$$\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4e^{4\alpha k}e^{-2\alpha t_N}. \tag{5.4.34}$$

Using (5.4.34) in (5.4.31), we observe that

$$|I_1^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.4.35}$$

Similarly, for I_2^* we arrive at

$$|I_2^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.4.36}$$

Using (5.3.37) and (5.4.9), we note that

$$\Lambda^*(\boldsymbol{\phi}_h) = \Lambda_1(\boldsymbol{\phi}_h) + \Lambda_2(\boldsymbol{\phi}_h) + \Lambda_3(\boldsymbol{\phi}_h). \tag{5.4.37}$$

With the help of (5.4.37), (5.3.41), (5.3.43), (5.3.47) and Theorem 5.14, it follows that

$$|I_3^*| = 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}|\Lambda^*(\hat{\mathbf{e}}^n)|$$
$$= 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}\big(|\Lambda_1(\hat{\mathbf{e}}^n) + \Lambda_2(\hat{\mathbf{e}}^n) + \Lambda_3(\hat{\mathbf{e}}^n)|\big)$$
$$\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}_H^n\|^2 + C(\epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2$$
$$+ 3\epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \tag{5.4.38}$$
$$\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + C(\epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2 + 3\epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.$$

For the purpose of bounding the first term on the right hand side of (5.4.30), we choose $n = 1$ in (5.3.27) and obtain

$$\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2\big) + \bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^1\|^2 \tag{5.4.39}$$
$$= e^{-\alpha k}(e^{\alpha k}\sigma^1, \hat{\mathbf{e}}^1) + e^{-\alpha k}\kappa a(e^{\alpha k}\sigma^1, \hat{\mathbf{e}}^1) + e^{-\alpha k}e^{\alpha k}\Lambda(\hat{\mathbf{e}}^1).$$

Multiply (5.4.39) by $2k$, use $\mathbf{e}^0 = 0$, Cauchy-Schwarz's inequality, (2.2.3) and Young's inequality with the estimates (5.3.47) (for $n = 1$ and $\epsilon = \frac{\nu}{3}$) to observe that

$$\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 + 2k\bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^1\|^2 \tag{5.4.40}$$
$$\begin{aligned}
&\le 2ke^{-\alpha k}(e^{\alpha k}\sigma^1, \hat{\mathbf{e}}^1) + 2ke^{-\alpha k}\kappa a(e^{\alpha k}\sigma^1, \hat{\mathbf{e}}^1) + 2ke^{-\alpha k}e^{\alpha k}\Lambda(\hat{\mathbf{e}}^1)\\
&\le Ck^2e^{-2\alpha k}\big(\|e^{\alpha k}\sigma^1\|^2 + \kappa\|e^{\alpha k}\nabla\sigma^1\|^2\big) + \frac{1}{2}(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2)\\
&\quad + ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}_H^1\|^2 + C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}^1\|^2 + \nu ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}^1\|^2.
\end{aligned}$$

The estimates $\|\nabla\mathbf{e}_H^1\|^2$ for **Step 1** , is already derived in Section **4.4** (see (4.4.30)) as

$$\|\mathbf{e}_H^1\|^2 + \kappa\|\nabla\mathbf{e}_H^1\|^2 + k\|\nabla\mathbf{e}_H^1\|^2 \le C(\kappa, \nu, \alpha, \lambda_1, M)k^4e^{-2\alpha k}. \tag{5.4.41}$$

From (5.3.31) for $n = 1$, we note that

$$\begin{aligned}
\|e^{\alpha k}\sigma^1\|^2 &\le \frac{1}{k^2}\bigg(\int_0^k e^{2\alpha k}\|\mathbf{u}_{htt}^*(s)\|^2ds\bigg)\bigg(\int_0^k (k-s)^2ds\bigg)\\
&= \frac{k}{3}\int_0^k e^{2\alpha k}\|\mathbf{u}_{htt}^*(s)\|^2\,ds\\
&\le C(\kappa, \nu, \alpha, \lambda_1, M)ke^{2\alpha k}\int_0^k e^{-2\alpha s}ds\\
&\le C(\kappa, \nu, \alpha, \lambda_1, M)k^2e^{2\alpha k^*},
\end{aligned} \tag{5.4.42}$$

where $k^* \in (0, k)$.

Similarly, we observe that

$$\|e^{\alpha k}\nabla\sigma^1\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{2\alpha k^*}. \tag{5.4.43}$$

An application of (5.4.41)-(5.4.43) in (5.4.40) yields

$$\begin{aligned}\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2+k\Big(\nu e^{-\alpha k}-2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{e}}^1\|^2 \\ \le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4+C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}^1\|^2.\end{aligned} \tag{5.4.44}$$

Using (5.4.35)-(5.4.36), (5.4.38) with $\epsilon=\frac{2\nu}{5}$, (5.4.44), $\mathbf{e}^0=0$ and bounds from Theorem 5.13 in (5.4.30), we obtain

$$\begin{aligned}&\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2)+(2e^{-\alpha k}\hat{\mathbf{e}}^N-\hat{\mathbf{e}}^{N-1})^2\\ &\quad+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N-\nabla\hat{\mathbf{e}}^{N-1})^2+2k\Big(\nu e^{-2\alpha k}-\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2+C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}^1\|^2\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)ke^{-\alpha k}\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2+C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^N\|^2\|\hat{\mathbf{e}}^N\|^2\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)ke^{-\alpha k}\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}^n\|^2\\ &\quad+Cke^{-\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2).\end{aligned} \tag{5.4.45}$$

Choose k_0, so that (5.4.15) is satisfied and $(1-Cke^{-\alpha k})>0$ for $0<k\le k_0$. An appeal to Gronwall's lemma leads to

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4\exp(k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2). \tag{5.4.46}$$

The bounds obtained from Theorem 5.13 yield

$$k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}_H^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)$$

and this would lead us to the desired result. □

The above derivation holds true for $n \ge 2$. For the case $n = 1$, we use (5.4.44) and Theorem 5.13. Then, with a choice of k such that $(1 - Cke^{-\alpha k}) > 0$, we arrive at the following estimate:

$$\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 + k\|\nabla\hat{\mathbf{e}}^1\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4e^{-2\alpha k}. \tag{5.4.47}$$

To derive bounds for pressure, we require the estimates for $\|D_t^2\mathbf{e}^n\|$.
For that purpose, substitute $\boldsymbol{\phi}_h = D_t^2\mathbf{e}^n$ in (5.4.7) and arrive at

$$\|D_t^{(2)}\mathbf{e}^n\|^2 + \|\nabla D_t^{(2)}\mathbf{e}^n\|^2 = -\nu a(\mathbf{e}^n, D_t^2\mathbf{e}^n) + (\sigma_2^n, D_t^2\mathbf{e}^n) + \kappa a(\sigma_2^n, D_t^2\mathbf{e}^n) + \Lambda^*(D_t^2\mathbf{e}^n). \tag{5.4.48}$$

Using (5.3.57), we observe that

$$|\Lambda^*(\boldsymbol{\phi}_h)| \le C(\kappa,\nu,\alpha,\lambda_1,M)(\|\nabla\mathbf{e}_H^n\| + \|\nabla\mathbf{e}^n\|)\|\nabla\boldsymbol{\phi}_h\|. \tag{5.4.49}$$

A use of Cauchy-Schwarz's inequality, Young's inequality, (2.2.3) and (5.4.49) (with $\boldsymbol{\phi}_h$ replaced by $D_t^2\mathbf{e}^n$) in (5.4.48) leads to

$$\|D_t^{(2)}\mathbf{e}^n\|^2 + \|\nabla D_t^{(2)}\mathbf{e}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)(\|\nabla\mathbf{e}^n\|^2 + \|\nabla\sigma_2^n\|^2 + \|\nabla\mathbf{e}_H^n\|^2). \tag{5.4.50}$$

To estimate the second term on the right hand side of (5.4.50), we use (5.4.32), estimates from Lemma 5.5 and obtain

$$
\begin{aligned}
\|e^{\alpha t_n}\sigma_2^n\|^2 &\le \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}^*(t)\|^2\,dt \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^3\,e^{2\alpha t_n}\int_{t_{n-2}}^{t_n} e^{-2\alpha t}dt \\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k^*}, \qquad (5.4.51)
\end{aligned}
$$

where $k^* \in (0,k)$.

An application of Theorems 5.14, 5.15 and (5.4.51) in (5.4.50) yields

$$
\|D_t^{(2)}\mathbf{e}^n\|^2 + \|\nabla D_t^{(2)}\mathbf{e}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{-2\alpha t_n}. \qquad (5.4.52)
$$

Next, to arrive at the error estimates for the pressure, let us consider equivalent form of (5.4.2) as follows: find a sequence of functions $\{\mathbf{U}^n\}_{n\ge1} \in \mathbf{H}_h$ and $\{P^n\}_{n\ge1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$
\begin{aligned}
&(D_t^{(2)}\mathbf{U}^n,\boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n,\boldsymbol{\phi}_h) + b(\mathbf{U}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) \qquad (5.4.53)\\
&\qquad + b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h) = b(\mathbf{U}_H^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) + (P^n,\nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h \quad n\ge 2,\\
&(\bar{\partial}_t\mathbf{U}^1,\boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^1,\boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1,\boldsymbol{\phi}_h) + b(\mathbf{U}^1,\mathbf{U}_H^1,\boldsymbol{\phi}_h)\\
&\qquad + b(\mathbf{U}_H^1,\mathbf{U}^1,\boldsymbol{\phi}_h) = b(\mathbf{U}_H^1,\mathbf{U}_H^1,\boldsymbol{\phi}_h) + (P^1,\nabla\cdot\boldsymbol{\phi}_h),\\
&(\nabla\cdot\mathbf{U}^n,\chi_h) = 0 \qquad \forall\chi_h \in L_h.
\end{aligned}
$$

Consider (5.2.75) at $t = t_n$ and subtract it from (5.4.53) to arrive at

$$
\begin{aligned}
(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) &= (D_t^{(2)}\mathbf{e}^n,\boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{e}^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n,\boldsymbol{\phi}_h) \qquad (5.4.54)\\
&\quad - (\sigma_2^n,\boldsymbol{\phi}_h) - \kappa a(\sigma_2^n,\boldsymbol{\phi}_h) - \Lambda^*(\boldsymbol{\phi}_h),
\end{aligned}
$$

where $\boldsymbol{\rho}^n = P^n - p_h^*(t_n)$.

With the help of Cauchy-Schwarz's inequality, (2.2.3) and (5.4.49), we write

$$|(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h)| \leq C(\kappa,\nu,\lambda_1)\big(\|\nabla D_t^{(2)}\mathbf{e}^n\| + \|\nabla\mathbf{e}^n\| + \|\nabla\sigma_2^n\| + \|\nabla\mathbf{e}_H^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \quad (5.4.55)$$

A use of Theorems 5.14, 5.15, (5.4.51) and (5.4.52) in (5.4.55) yields the desired pressure error estimates, that is,

$$\|\boldsymbol{\rho}^n\| \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{-\alpha t_n}, \;\; n\geq 2. \quad (5.4.56)$$

To deal with the case $n=1$, we use the estimates derived for backward Euler method. Substitute $n=1$ in (5.3.64) and obtain

$$|(\boldsymbol{\rho}^1, \nabla\cdot\boldsymbol{\phi}_h)| \leq C(\kappa,\nu,\alpha,\lambda_1,M)(\big(\|\bar{\partial}_t\nabla\mathbf{e}^1\| + \|\nabla\mathbf{e}^1\| + \|\nabla\mathbf{e}_H^1\| + \|\nabla\sigma^1\|\big)\|\nabla\boldsymbol{\phi}_h\|. \quad (5.4.57)$$

Applying (5.3.59), (5.3.61) and (5.4.47) along with the estimates from Theorem 5.8 in (5.4.57) would provide the desired estimates, that is,

$$\|\boldsymbol{\rho}^1\| \leq C(\kappa,\nu,\alpha,\lambda_1,M)ke^{-\alpha k}. \quad (5.4.58)$$

As a result of (5.4.47), (5.4.56), (5.4.58), Theorems 5.3, 5.5 and 5.15, we have the following theorem:

Theorem 5.16. *Under the assumption of Theorems 5.3, 5.9 and 5.15, the following hold true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq C(h^{2-j} + H^{3-\delta} + k^2)e^{-\alpha t_n} \;\; j = 0,\, 1,$$

and,

$$\|p(t_n) - P^n\| \leq Ce^{-\alpha t_n}(h + H^{3-\delta} + k^{2-\gamma}),$$

where

$$\gamma = \begin{cases} 0 & \textit{if } n \geq 2; \\ 1 & \textit{if } n = 1. \end{cases}$$

Next, we present the derivation of error estimates for velocity in **Step 3**.

Theorem 5.17. *Assume that* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and choose* $k_0 \geq 0$ *such that for* $0 < k \leq k_0$*, (5.4.15) is satisfied. Let* $u_h(t)$ *be a solution of (5.2.7) and* $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h(t_n)$*, for* $n = 1, 2, \cdots, N$*. Then, for some positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$*, there holds,*

$$\|\mathbf{e}_h^n\|^2 + \kappa\|\nabla\mathbf{e}_h^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_h^i\|^2 \leq Ck^4 e^{-2\alpha t_n}.$$

Proof. We apply a sequence of operations to (5.4.11) for obtaining an expression similar to (5.4.30), that is,

$$\begin{aligned}
&\|\hat{\mathbf{e}}_h^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^N\|^2 + e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}_h^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}_h^{n-1}\|^2) + (2e^{-\alpha k}\hat{\mathbf{e}}_h^N - \hat{\mathbf{e}}_h^{N-1})^2 \\
&\quad + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}_h^N - \nabla\hat{\mathbf{e}}_h^{N-1})^2 + k\left(4\nu e^{-2\alpha k} - 2\Big(\frac{1-e^{-2\alpha k}}{k}\Big)\Big(\kappa + \frac{1}{\lambda_1}\Big)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2 \\
&\leq \|\hat{\mathbf{e}}_h^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^1\|^2 + (2e^{-\alpha k}\hat{\mathbf{e}}_h^1 - \mathbf{e}_h^o)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}_h^1 - \nabla\mathbf{e}_h^o)^2 \\
&\quad + 4ke^{-2\alpha k}\sum_{n=2}^{N}(e^{\alpha t_n}\sigma_{2h}^n, \hat{\mathbf{e}}_h^n) + 4k\,\kappa e^{-2\alpha k}\sum_{n=2}^{N}a(e^{\alpha t_n}\sigma_{2h}^n, \hat{\mathbf{e}}_h^n) + 4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}_h^n) \\
&\leq C(\|\hat{\mathbf{e}}_h^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^1\|^2) + I_1^{Nh} + I_2^{Nh} + I_3^{Nh}, \ \textit{say}.
\end{aligned} \tag{5.4.59}$$

As in the proof of Theorem 5.15 (see (5.4.35)-(5.4.36)), we bound I_1^{Nh} and I_2^{Nh} as follows:

$$|I_1^{Nh}| + |I_2^{Nh}| \leq C(\kappa, \nu, \alpha, \lambda_1, M, \epsilon)k^4 + 2\epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2. \tag{5.4.60}$$

For I_3^{Nh}, we use techniques applied to two-grid backward Euler method for handling the nonlinear terms as these are same in both methods for all three steps and arrive at

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= \big(b(\mathbf{u}_h^n,\mathbf{u}_H^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)-b(\mathbf{U}_H^n,\mathbf{U}_h^n,\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &\quad -b(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\big)+\big(b(\mathbf{U}^n,\mathbf{U}_H^n-\mathbf{U}^n,\boldsymbol{\phi}_h)-b(\mathbf{u}_h^{*n},\mathbf{u}_H^n-\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\big)\\ &= \Lambda_h^1(\boldsymbol{\phi}_h)+\Lambda_h^2(\boldsymbol{\phi}_h)+\Lambda_h^3(\boldsymbol{\phi}_h)+\Lambda_h^4(\boldsymbol{\phi}_h).\end{aligned} \tag{5.4.61}$$

A use of (5.3.72), (5.3.75), (5.3.78), (5.3.81) and estimates from Theorems 5.14 and 5.15 yields

$$\begin{aligned}|I_3^{Nh}| = 4ke^{-2\alpha k}\sum_{n=2}^{N} e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}_h^n)| &\le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + C(\epsilon)ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}_H^n\|^2\|\hat{\mathbf{e}}_h^n\|^2\\ &\quad + 4\epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}_h^n\|^2.\end{aligned} \tag{5.4.62}$$

Substituting $n=1$ in (5.3.66), we observe that

$$\begin{aligned}&\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}_h^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}_h^1\|^2\big)+\bigg(\nu e^{-\alpha k}-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}_h^1\|^2\\ &\quad = e^{-\alpha k}(e^{\alpha k}\sigma_h^1,\hat{\mathbf{e}}_h^1)+e^{-\alpha k}\kappa a(e^{\alpha k}\sigma_h^1,\hat{\mathbf{e}}_h^1)+e^{-\alpha k}e^{\alpha k}\Lambda(\hat{\mathbf{e}}_h^1).\end{aligned} \tag{5.4.63}$$

With the help of similar kind of analysis as in (5.4.42) and a use of Lemma 5.6, we observe that

$$\begin{aligned}\|e^{\alpha k}\sigma_h^1\|^2 &\le \frac{1}{k^2}\bigg(\int_0^k e^{2\alpha k}\|\mathbf{u}_{htt}(s)\|^2 ds\bigg)\bigg(\int_0^k (k-s)^2 ds\bigg)\\ &= \frac{k}{3}\int_0^k e^{2\alpha k}\|\mathbf{u}_{htt}(s)\|^2\,ds\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)ke^{2\alpha k}\int_0^k e^{-2\alpha s}ds\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{2\alpha k^*},\end{aligned} \tag{5.4.64}$$

where $k^*\in(0,k)$.

Also, using (5.3.82) for $n = 1$ along with proper choice of ϵ and Theorems 5.14, 5.15, we write nonlinear term as

$$
\begin{aligned}
2ke^{-\alpha k}e^{\alpha k}|\Lambda_h(\hat{\mathbf{e}}_h^1)| \leq & C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)ke^{-\alpha k}\big(\|\nabla\hat{\mathbf{e}}_H^1\|^2+\|\nabla\hat{\mathbf{e}}^1\|^2\big) \\
& + C(\epsilon)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}_h^1\|^2 + 4\epsilon ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}_h^1\|^2 \\
\leq & C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}_h^1\|^2 \\
& + \nu ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}_h^1\|^2.
\end{aligned} \tag{5.4.65}
$$

After multiplying (5.4.63) by $2k$, use (5.4.64) and (5.4.65) in the resulting equation and obtain

$$
\begin{aligned}
\|\hat{\mathbf{e}}_h^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^1\|^2 + k\bigg(\nu e^{-\alpha k} - 2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}_h^1\|^2 \\
\leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + C(\nu)ke^{-\alpha k}\|\nabla\hat{\mathbf{U}}_H^1\|^2\|\hat{\mathbf{e}}_h^1\|^2.
\end{aligned} \tag{5.4.66}
$$

A use of (5.4.60), (5.4.62) and (5.4.66) in (5.4.59) and Gronwall's lemma completes the proof. □

Note that, the above result holds true for $n \geq 2$. We require to deal case $n = 1$ separately as it corresponds to backward Euler method. Here, we use (5.4.66) and Theorem 5.13, with a choice of k such that $(1 - Cke^{-\alpha k}) > 0$ to obtain the following results:

$$
\|\hat{\mathbf{e}}_h^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}_h^1\|^2 + k\|\nabla\hat{\mathbf{e}}_h^1\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4e^{-2\alpha k}. \tag{5.4.67}
$$

In order to derive pressure estimates, firstly we derive bounds for $\|D_t^2\mathbf{e}_h^n\|$. Substitute $\boldsymbol{\phi}_h = D_t^2\mathbf{e}_h^n$ in (5.4.11) to obtain

$$
\|D_t^{(2)}\mathbf{e}_h^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}_h^n\|^2 = -\nu a(\mathbf{e}_h^n, D_t^2\mathbf{e}_h^n) + (\sigma_{2h}^n, D_t^2\mathbf{e}_h^n) + \kappa a(\sigma_{2h}^n, D_t^2\mathbf{e}_h^n) + \Lambda_h(D_t^2\mathbf{e}_h^n). \tag{5.4.68}
$$

Using (5.3.96), we write

$$|\Lambda_h(D_t^{(2)}\mathbf{e}_h^n)| \le C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\nabla\mathbf{e}_H^n\| + \|\nabla\mathbf{e}^n\| + \|\nabla\mathbf{e}_h^n\|\big)\|\nabla D_t^{(2)}\mathbf{e}_h^n\|. \tag{5.4.69}$$

Applying Cauchy-Schwarz's inequality, Young's inequality, (2.2.3) and (5.4.69) to arrive at

$$\|D_t^{(2)}\mathbf{e}_h^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}_h^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)\big(\|\nabla\mathbf{e}_h^n\|^2 + \|\nabla\sigma_{2h}^n\|^2 + \|\nabla\mathbf{e}_H^n\|^2 + \|\nabla\mathbf{e}^n\|^2\big). \tag{5.4.70}$$

To bound $\|\nabla\sigma_{2h}^n\|$, we recall (5.4.51) (with $\mathbf{u}_h^*$ replaced by $\mathbf{u}_h$) and Lemma 5.6 to obtain

$$\begin{aligned}\|e^{\alpha t_n}\sigma_{2h}^n\|^2 &\le \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|^2\,dt\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^3\,e^{2\alpha t_n}\int_{t_{n-2}}^{t_n} e^{-2\alpha t}dt\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k^*},\end{aligned} \tag{5.4.71}$$

where $k^* \in (0,k)$.
Similarly, we note that

$$\|e^{\alpha t_n}\nabla\sigma_{2h}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k^*}. \tag{5.4.72}$$

With the help of Theorems 5.14, 5.15, 5.17 and (5.4.72), we arrive at

$$\|D_t^{(2)}\mathbf{e}_h^n\|^2 + \|\nabla D_t^{(2)}\mathbf{e}_h^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{-2\alpha t_n}. \tag{5.4.73}$$

For pressure error estimates, consider equivalent form of (5.4.3), that is, find a sequence of functions $\{\mathbf{U}_h^n\}_{n\ge1} \in \mathbf{H}_h$ and $\{P_h^n\}_{n\ge1} \in L_h$ as solutions of the following equations:

$$\begin{aligned}(D_t^{(2)}\mathbf{U}_h^n,\boldsymbol{\phi}_h) &+ \kappa a(D_t^{(2)}\mathbf{U}_h^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^n,\boldsymbol{\phi}_h) + b(\mathbf{U}_h^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h) + b(\mathbf{U}_H^n,\mathbf{U}_h^n,\boldsymbol{\phi}_h)\\ &= b(\mathbf{U}_H^n,\mathbf{U}^n,\boldsymbol{\phi}_h) + b(\mathbf{U}^n,\mathbf{U}_H^n-\mathbf{U}^n,\boldsymbol{\phi}_h) + (P_h^n,\nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h\in\mathbf{H}_h \quad n\ge 2,\end{aligned} \tag{5.4.74}$$

$$
\begin{aligned}
&(\bar{\partial}_t \mathbf{U}_h^1, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}_h^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}_h^1, \boldsymbol{\phi}_h) + b(\mathbf{U}_h^1, \mathbf{U}_H^1, \boldsymbol{\phi}_h) + b(\mathbf{U}_H^1, \mathbf{U}_h^1, \boldsymbol{\phi}_h) \\
&\quad = b(\mathbf{U}_H^1, \mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}_H^1 - \mathbf{U}^1, \boldsymbol{\phi}_h) + (P_h^1, \nabla \cdot \boldsymbol{\phi}_h), \\
&\qquad (\nabla \cdot \mathbf{U}_h^n, \chi_h) = 0 \quad \forall \chi_h \in L_h.
\end{aligned}
$$

Consider (5.2.90) at $t = t_n$ and subtract it from (5.4.74) to arrive at

$$
\begin{aligned}
(\boldsymbol{\rho}_h^n, \nabla \cdot \boldsymbol{\phi}_h) &= (D_t^{(2)} \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)} \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_h^n, \boldsymbol{\phi}_h) \\
&\quad -(\sigma_{2h}^n, \boldsymbol{\phi}_h) - \kappa a(\sigma_{2h}^n, \boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h).
\end{aligned} \tag{5.4.75}
$$

Using Cauchy-Schwarz's inequality, (2.2.3) and (5.3.96), we obtain

$$
|(\boldsymbol{\rho}_h^n, \nabla \cdot \boldsymbol{\phi}_h)| \le C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\nabla D_t^{(2)} \mathbf{e}_h^n\| + \|\nabla \mathbf{e}_h^n\| + \|\nabla \sigma_{2h}^n\| + \|\nabla \mathbf{e}_H^n\| + \|\nabla \mathbf{e}^n\|\big)\|\nabla \boldsymbol{\phi}_h\|. \tag{5.4.76}
$$

A use of Theorems 5.14, 5.15, 5.17, (5.4.72) and (5.4.73) in (5.4.76) leads to the desired result, that is,

$$
\|\boldsymbol{\rho}_h^n\| \le C(\kappa, \nu, \alpha, \lambda_1, M) k^2 e^{-\alpha t_n}, \;\; n \ge 2. \tag{5.4.77}
$$

To consider case $n = 1$, we substitute $n = 1$ in (5.3.103) as this corresponds to backward Euler method and observe that

$$
|(\boldsymbol{\rho}_h^1, \nabla \cdot \boldsymbol{\phi}_h)| \le C(\kappa, \nu, \alpha, \lambda_1, M)\big(\|\bar{\partial}_t \nabla \mathbf{e}_h^1\| + \|\nabla \mathbf{e}_h^1\| + \|\nabla \sigma_h^1\| + \|\nabla \mathbf{e}_H^1\| + \|\nabla \mathbf{e}^1\|\big). \tag{5.4.78}
$$

We apply (5.4.64), (5.4.67) and (5.3.100) along with Theorems 5.8, 5.9 (for $n = 1$) to arrive at

$$
\|\boldsymbol{\rho}_h^1\| \le C(\kappa, \nu, \alpha, \lambda_1, M) k e^{-\alpha k}. \tag{5.4.79}
$$

Using results obtained in (5.4.67), (5.4.77), (5.4.79), Theorems 5.4, 5.6 and 5.17, we have the following theorem.

Theorem 5.18. *Under the assumption of Theorems 5.4, 5.11 and 5.17, the following hold true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}_h^n\|_j \leq C(h^{2-j} + H^{5-2\delta} + k^2)e^{-\alpha t_n} \quad j = 0, 1,$$

and,

$$\|p(t_n) - P_h^n\| \leq Ce^{-\alpha t_n}(h + H^{5-2\delta} + k^{2-\gamma}),$$

where

$$\gamma = \begin{cases} 0 & \text{if } n \geq 2; \\ 1 & \text{if } n = 1. \end{cases}$$

and $\delta > 0$, arbitrary small for two dimensions and $\delta = \frac{1}{2}$ for three dimensions.

5.5 Numerical Experiments

In this section, we present numerical results which support the theoretical estimates obtained in Theorems 5.12 and 5.18, by employing two fully discrete schemes applied to (5.2.5)-(5.2.7). As in Chapter **4**, for space discretization, we use P_2-P_0 mixed finite element space. Thus, we consider the finite dimensional subspaces $\mathbf{V}_\mu$ and W_μ of $\mathbf{H}_0^1$ and L^2 respectively, which satisfy the approximation properties in (**B1**) and (**B2**), as:

$$\mathbf{V}_\mu = \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \left(C(\bar{\Omega})\right)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \mathcal{T}_\mu\},$$
$$W_\mu = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \mathcal{T}_\mu\},$$

where μ is the index for mesh sizes H and h for coarse and fine grid, respectively, and τ_μ denotes the triangulation of the domain $\bar{\Omega}$. Now, we discuss the fully discrete two-grid finite element formulations of (5.2.5)-(5.2.7) using backward Euler method and second order backward difference scheme.

Let k be the time step and $\mathbf{U}_\mu^n$ be the approximation of $\mathbf{u}(t)$ in $\mathbf{V}_\mu$ at $t = t_n = nk$.

The backward Euler approximation to (5.2.5)-(5.2.7) can be stated as follows:

Algorithm:

Step 1: Solve nonlinear system on coarse mesh $\mathcal{T}_H$: given $\mathbf{U}_H^{n-1}$, find the pair $(\mathbf{U}_H^n, P_H^n)$ satisfying:

$$
\begin{aligned}
&(\mathbf{U}_H^n, \mathbf{v}_H) + (\kappa + \nu k)\ a(\mathbf{U}_H^n, \mathbf{v}_H) + k\ b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_H) - k\ (P_H^n, \nabla \cdot \mathbf{v}_H) \\
&\quad = (\mathbf{U}_H^{n-1}, \mathbf{v}_H) + \kappa a(\mathbf{U}_H^{n-1}, \mathbf{v}_h) + k\ (\mathbf{f}(t_n), \mathbf{v}_H) \qquad \forall \mathbf{v}_H \in \mathbf{V}_H, \\
&\qquad\qquad (\nabla \cdot \mathbf{U}_H^n, w_H) = 0 \qquad \forall w_H \in W_H.
\end{aligned} \tag{5.5.1}
$$

Step 2: Update on fine mesh $\mathcal{T}_h$ with one Newton iteration: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$
\begin{aligned}
&(\mathbf{U}^n, \mathbf{v}_h) + (\kappa + \nu k)\ a(\mathbf{U}^n, \mathbf{v}_h) + k\ b(\mathbf{U}^n, \mathbf{U}_H^n, \mathbf{v}_h) + k\ b(\mathbf{U}_H^n, \mathbf{U}^n, \mathbf{v}_h) - k\ (P^n, \nabla \cdot \mathbf{v}_h) \\
&\quad = k\ b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_h) + (\mathbf{U}^{n-1}, \mathbf{v}_h) + \kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) + k\ (\mathbf{f}(t_n), \mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&\qquad\qquad (\nabla \cdot \mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.
\end{aligned} \tag{5.5.2}
$$

Step 3: Correct on fine mesh $\mathcal{T}_h$: given $\mathbf{U}_h^{n-1}$, find the pair $(\mathbf{U}_h^n, P_h^n)$ satisfying:

$$
\begin{aligned}
&(\mathbf{U}_h^n, \mathbf{v}_h) + (\kappa + \nu k)\ a(\mathbf{U}_h^n, \mathbf{v}_h) + k\ b(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{v}_h) + k\ b(\mathbf{U}_H^n, \mathbf{U}_h^n, \mathbf{v}_h) \\
&\qquad - k\ (P_h^n, \nabla \cdot \mathbf{v}_h) = k\ b(\mathbf{U}_H^n, \mathbf{U}^n, \mathbf{v}_h) + kb(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \mathbf{v}_h) \\
&\qquad + (\mathbf{U}_h^{n-1}, \mathbf{v}_h) + \kappa a(\mathbf{U}_h^{n-1}, \mathbf{v}_h) + k\ (\mathbf{f}(t_n), \mathbf{v}_h), \\
&\qquad (\nabla \cdot \mathbf{U}_h^n, w_h) = 0 \qquad \forall w_h \in W_h.
\end{aligned} \tag{5.5.3}
$$

Similarly, two grid second order backward difference approximation to (5.2.5)-(5.2.7) is as follows:

Algorithm:

Step 1: Solve nonlinear system on coarse mesh $\mathcal{T}_H$: given $\mathbf{U}_H^{n-1}$, find the pair $(\mathbf{U}_H^n, P_H^n)$ satisfying:

$$\begin{aligned}(3\mathbf{U}_H^n, \mathbf{v}_H) + (\kappa + 2\nu k)\ a(\mathbf{U}_H^n, \mathbf{v}_H) + 2\Delta t\ b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_H) - 2k\ (P_H^n, \nabla \cdot \mathbf{v}_H^n)\\ = 4(\mathbf{U}_H^{n-1}, \mathbf{v}_H) + 4\ \kappa a(\mathbf{U}_H^{n-1}, \mathbf{v}_H) - (\mathbf{U}_H^{n-2}, \mathbf{v}_H) - \kappa\ a(\mathbf{U}_H^{n-2}, \mathbf{v}_H)\\ + k\ (\mathbf{f}(t_n), \mathbf{v}_H) \quad \forall \mathbf{v}_H \in \mathbf{V}_H, \qquad (5.5.4)\\ (\nabla \cdot \mathbf{U}_H^n, w_H) = 0 \qquad \forall w_H \in W_H.\end{aligned}$$

Step 2: Update on fine mesh $\mathcal{T}_h$ with one Newton iteration: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned}(3\mathbf{U}^n, \mathbf{v}_h) + (\kappa + 2\nu k)\ a(\mathbf{U}^n, \mathbf{v}_h) + 2kt\ b(\mathbf{U}^n, \mathbf{U}_H^n, \mathbf{v}_h) + 2k\ b(\mathbf{U}^n, \mathbf{U}_H^n, \mathbf{v}_h)\\ -2k\ (P^n, \nabla \cdot \mathbf{v}_h^n) = 4(\mathbf{U}^{n-1}, \mathbf{v}_h) + 4\ \kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) - (\mathbf{U}^{n-2}, \mathbf{v}_H) - \kappa\ a(\mathbf{U}^{n-2}, \mathbf{v}_H)\\ +2k\ b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_h) + k\ (\mathbf{f}(t_n), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \qquad (5.5.5)\\ (\nabla \cdot \mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned}$$

Step 3: Correct on fine mesh $\mathcal{T}_h$: given $\mathbf{U}_h^{n-1}$, find the pair $(\mathbf{U}_h^n, P_h^n)$ satisfying:

$$\begin{aligned}(3\mathbf{U}_h^n, \mathbf{v}_h) + (\kappa + 2\nu k)\ a(\mathbf{U}_h^n, \mathbf{v}_h) + 2k\ b(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{v}_h) + 2k\ b(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{v}_h)\\ -2k\ (P_h^n, \nabla \cdot \mathbf{v}_h^n) = 4(\mathbf{U}_h^{n-1}, \mathbf{v}_h) + 4\ \kappa a(\mathbf{U}_h^{n-1}, \mathbf{v}_h) - (\mathbf{U}_h^{n-2}, \mathbf{v}_H) - \kappa\ a(\mathbf{U}_h^{n-2}, \mathbf{v}_H)\\ +2k\ b(\mathbf{U}_H^n, \mathbf{U}^n, \mathbf{v}_h) + 2kb(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \mathbf{v}_h) + 2k\ (\mathbf{f}(t_n), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \qquad (5.5.6)\\ (\nabla \cdot \mathbf{U}_h^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned}$$

We solve (5.5.1)-(5.5.3) (or (5.5.4)-(5.5.6)) following the similar technique as mentioned in Section **5** of Chapter **4**.

We choose $\nu = 1$, $\kappa = 10^{-2}$ with $\Omega = (0,1) \times (0,1)$ and time $t = [0,1]$. Here, $\bar{\Omega}$ is subdivided into two triangulations, composed of closed triangles: one coarse triangulation with mesh size H and one fine triangulation with mesh size h such that $h = \mathcal{O}(H^{2-\delta})$ with $\delta = 1/2$, arbitrary small. The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in $\mathbf{L^2}$-norm, of $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm for velocity and of $\mathcal{O}(h)$ in L^2-norm for pressure with a choice of $k = \mathcal{O}(h^2)$ for backward Euler method and $k = \mathcal{O}(h)$ for second order backward difference scheme.

Example 5.5.1. *In this example, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is*
$u_1 = 10e^{-t}x^2(x-1)^2y(y-1)(2y-1), \;\; u_2 = -10e^{-t}y^2(y-1)^2x(x-1)(2x-1), \;\; p = ye^{-t}.$

Here, Table 5.1 gives the numerical errors and convergence rates obtained on successively refined meshes for the backward Euler method with $k = \mathcal{O}(h^2)$ applied to two grid system (5.2.5)-(5.2.7) and Table 5.2 presents the errors and convergence rates for second order backward difference scheme with $k = \mathcal{O}(h)$. Figure 5.1 graphically depicts the order of convergence for velocity in $\mathbf{L^2}$ and $\mathbf{H}^1$-norms. In Figure 5.2, we have shown the graphs of order of convergence for velocity in $\mathbf{L^2}$ and $\mathbf{H}^1$-norms for backward difference scheme. In Figure 5.3, we depict the convergence plots of pressure for both backward Euler method and backward difference scheme, respectively. These results support the optimal theoretical convergence rates obtained in Theorems 5.12 and 5.18.

Example 5.5.2. *In the second example, the right hand side function f is constructed in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is*

$$u_1 = te^{-t^2} sin^2(3\pi x)\; sin(6\pi y),\; u_2 = -te^{-t^2} sin^2(3\pi y)\; sin(6\pi x),$$
$$p = te^{-t}\; sin(2\pi x)\; sin(2\pi y).$$

The results obtained are shown in Table 5.3 and Table 5.4. In Table 5.3, we present the numerical results showing the error estimates for velocity in $\mathbf{L^2}$ and $\mathbf{H}^1$-norms and for pressure in L^2-norm for backward Euler method. These results are in agreement with the theoretical bounds in Theorem 5.12 and in Table 5.4, we exhibit the error for velocity (pressure) in $\mathbf{L^2}$ and $\mathbf{H}^1$-norms (L^2-norm) for second order backward difference scheme. These results satisfy the optimal theoretical results in Theorem 5.18.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/8	0.365017×10^{-3}		0.010998		0.037912	
1/16	0.097081×10^{-3}	1.910691	0.005371	1.033832	0.019200	0.981523
1/32	0.025053×10^{-3}	1.954179	0.002687	0.999380	0.009660	0.991044
1/64	0.006360×10^{-3}	1.977711	0.001348	0.994756	0.004846	0.994975

Table 5.1: Errors and convergence rates for backward Euler method with $k = \mathbb{O}(h^2)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/8	0.369685×10^{-3}		0.011052		0.037549	
1/16	0.097491×10^{-3}	1.922940	0.005371	1.040915	0.019204	0.967363
1/32	0.025108×10^{-3}	1.957108	0.002687	0.999372	0.009660	0.991316
1/64	0.006372×10^{-3}	1.978301	0.001348	0.994757	0.004846	0.994979

Table 5.2: Errors and convergence rates for backward difference scheme with $k = \mathbb{O}(h)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L^2}}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/8	0.028118		1.537607		0.186313	
1/16	0.003710	2.921958	0.463199	1.730983	0.063223	1.559194
1/32	0.000472	2.971908	0.124017	1.901084	0.016051	1.977744
1/64	0.000063	2.894693	0.032022	1.953411	0.006437	1.318130

Table 5.3: Errors and convergence rates for backward Euler method with $k = \mathbb{O}(h^2)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/8	0.028032		1.537645		0.186275	
1/16	0.003663	2.935964	0.463208	1.730989	0.063019	1.563564
1/32	0.000466	2.973856	0.124020	1.901081	0.016014	1.976394
1/64	0.000063	2.868781	0.032023	1.953379	0.006431	1.316084

Table 5.4: Errors and convergence rates for backward difference scheme with $k = \mathbb{O}(h)$.

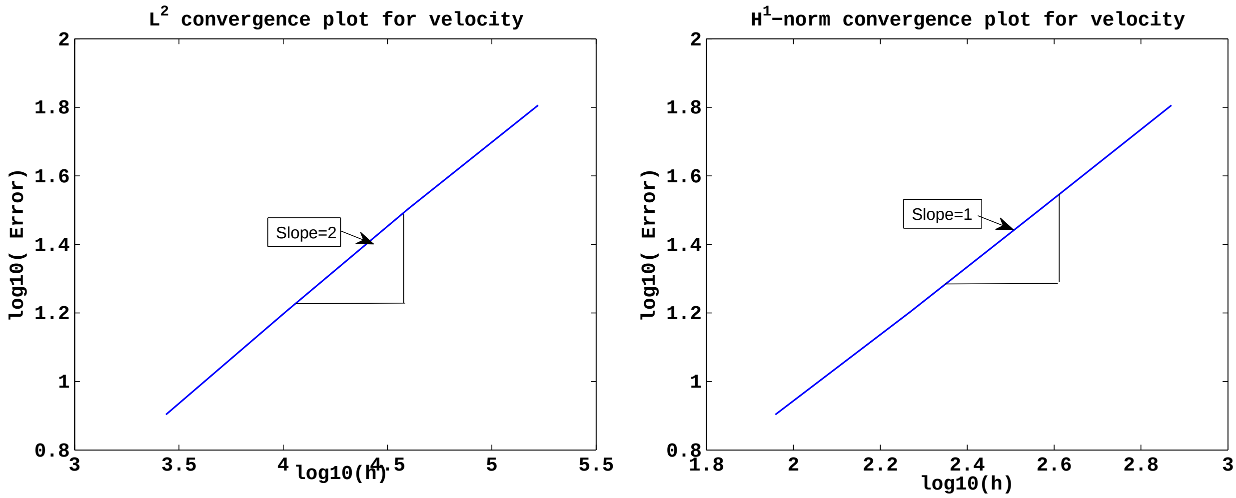

Figure 5.1: Convergence plots of velocity for backward Euler method.

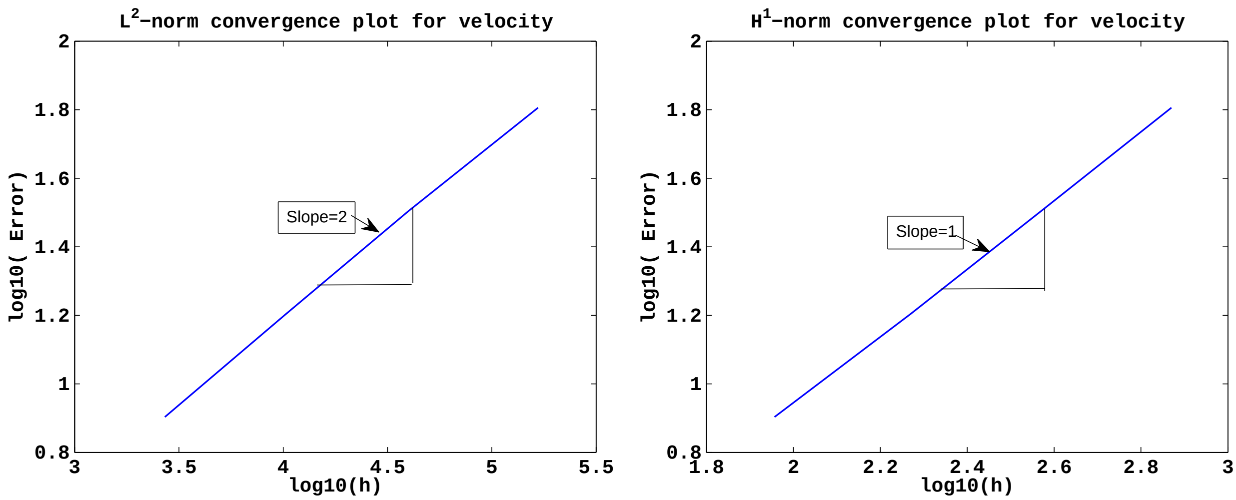

Figure 5.2: Convergence plots of velocity for backward difference scheme.

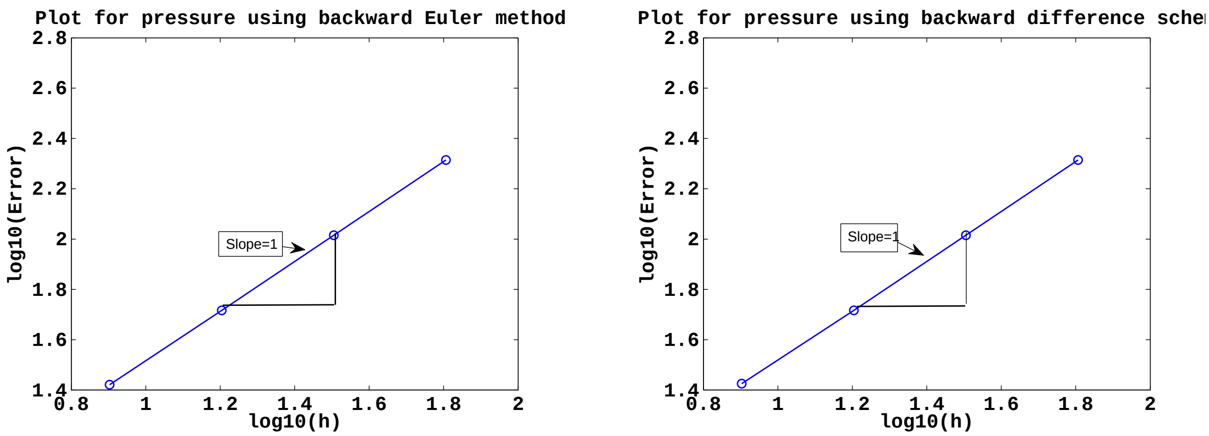

Figure 5.3: L^2-norm convergence of pressure for backward Euler method and backward difference scheme respectively.

Chapter 6

Summary and Future Plans

The main objective of our work is to study the finite element Galerkin approximations to the equations of motion arising in the Kelvin-Voigt model, which appears in the class of linear viscoelastic fluids. In this chapter, we summarize briefly our results and discuss about future plans.

6.1 Summary

In this section, we summarize our results.

In Chapter **2**, we have proved the global existence of a unique weak solution to the equations of motion arising in the Kelvin-Voigt viscoelastic model when the forcing function is zero using Faedo Galerkin method and standard compactness arguments. Further, we have derived a *priori* bounds based on energy arguments which provide new regularity results for the solution. Moreover, these results exhibit exponential decay property in time. For decay property, an exponential weight plays a crucial role. Special care is taken to avoid algebraic growth in time.

In Chapter **3**, finite element Galerkin method is applied to discretize the problem in spatial variable, while keeping the time variable continuous. Thus, we obtain a semidiscrete scheme. For the semidiscrete scheme, we derive optimal error estimates in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}^1)$-norm for the velocity and $L^\infty(L^2)$-norm for the pressure, which reflect the exponential decay property in time. Again, as in Chapter **2**, exponential weights play a crucial role in establishing the decay property in time, while standard energy arguments yield optimal error estimates in $L^\infty(\mathbf{H}^1)$-norm for velocity. For optimal error estimates in $L^\infty(\mathbf{L^2})$-norm for velocity, we follow the following proof techniques. We first split the error by introducing a Galerkin approximation to a linearized Kelvin-Voigt model. Essentially, we decompose the error into two parts: one due to linearization and the other to take care of the effect of nonlinearity. In order to obtain optimal error estimates in $L^\infty(\mathbf{L}^2)$-norm, we introduce a new auxiliary projection through a modification of the Stokes operator, named as Sobolev-Stokes projection. This plays a role similar to the role played by auxiliary elliptic projection in the context of parabolic equations in [103]. For the error due to the nonlinearity, we apply energy arguments with a suitable use of exponential weight and establish the optimal error bound. Then making use of estimates derived for the auxiliary projection, the error estimates due to the linearized model and due to nonlinearity, we have recovered the optimality of $L^\infty(\mathbf{L}^2)$ error estimates for the velocity. Finally, with the help of uniform inf-sup condition and error estimates for the velocity, we have established optimal error estimates for the pressure. Special care has been taken to preserve the exponential decay property in time even for the error estimates. In Chapter **4**, an attempt is made to discretize the semidiscrete problem discussed in Chapter **3** by replacing the time derivatives by suitable finite difference quotients. Thus, we obtain complete discrete schemes. In the first part of this chapter, we have discussed a backward Euler method which is an implicit scheme and established the existence of a unique solution to the fully discrete scheme at each time level by using a variant of Brouwer fixed point argument and a standard uniqueness technique. Optimal error estimates in $\ell^\infty(\mathbf{L^2})$ and $\ell^\infty(\mathbf{H}^1)$-norms for the velocity and $\ell^\infty(L^2)$-norm for the pressure are derived, which preserve exponential decay property in time. For achieving the decay property, we need to introduce exponential weight function at each time level and special care is taken to tackle the additional terms which appear

as a by product of the discrete problem. Since at each time level, we obtain a nonlinear system of algebraic equations, which is computationally more expensive, we have then used a linearized backward Euler method which preserves the optimal order of convergence. All these two Euler schemes are first order in time, therefore, in the later part of this chapter, we deal with a second order backward difference scheme. After obtaining a *priori* bounds of the discrete solution, we have given a remark on wellposedness of the discrete problem. Then, we have derived optimal error bounds which again exhibit exponential decay in time. Unlike backward Euler method, the introduction of the exponential weight function gives rise to special type of difficulties, which demands a more careful analysis. Finally, several numerical experiments have been conducted to confirm our theoretical findings.
In Chapter **5**, we have employed a two level method based on Newton's iteration for resolving the nonlinearity present in our problem. Essentially, we solve the nonlinear system on a coarse grid of size H and with two updates of nonlinear term, we solve a linear system on a finer grid of size h. We have derived a *priori* estimates for semidiscrete solutions and have established optimal velocity error estimates with the help of Sobolev-Stokes projection, that is, we have recovered an error of the order h^2 in $L^\infty(\mathbf{L}^2)$-norm and h in $L^\infty(\mathbf{H}^1)$-norm provided $h = \mathcal{O}(H^{2-\delta})$, $\delta > 0$ arbitrary small for two dimensions and $\delta = \frac{1}{2}$ for three dimensions. Then, we have applied a first order accurate backward Euler method and a second order backward difference scheme for the time discretization of two level algorithm. We have obtained a *priori* bounds for the discrete solution. Armed with all these a *priori* estimates, we have derived fully discrete optimal error estimates for complete discrete scheme which exhibit exponential decay property in time. Then, we have worked out some numerical examples to support the theoretical estimates.

6.2 Future Plans

- In this dissertation, we have obtained global solvability and optimal error estimates for both semidiscrete and fully discrete schemes for the 2D and 3D Kelvin-Voigt model when the forcing function $f = 0$. For non zero f, that is, $f \neq 0$, it may be worth while to extend the theoretical results of this thesis. Note that, we may not obtain exponential decay property in time. Even for optimal error analysis, special care must be taken, when $f \neq 0$.

- In Chapter **2** and in subsequent chapters, we obtain constants in various bounds which depend on $\frac{1}{\kappa}$ or $\exp(\frac{1}{\kappa})$ simultaneously, say for example in Chapter **2**, $\|\nabla \mathbf{u}(t)\| \leq \frac{C}{\kappa}$, which plays a crucial role in subsequent regularity results. As $\kappa \to 0$, it is expected that the solution of Kelvin-Voigt model should converge to the solution of the Navier-Stokes system. Therefore, it is desirable to obtain bounds which do not blow up as $\kappa \to 0$. More pertinent issue now: *How do the true solutions as well as discrete solutions behave as* $\kappa \to 0$? This may form a part of future work.

- The analysis in this dissertation involves the coupling of $\mathbf{u}$ and p by the incompressibility condition *div* $\mathbf{u}$=0. It is difficult to construct a finite element space having divergence free property. To overcome this problem, an extensive amount of literature is available for Navier Stokes equations, for example the work of J. Shen [93] and literature, therein. The author has established optimal error estimates for penalized Navier-Stokes equations, where the divergence free condition is penalized with penalty parameter ϵ. These estimates are of order $\mathcal{O}(\epsilon)$, where ϵ is penalty parameter. He has applied backward Euler method to the penalized system and obtained optimal error estimates. In future, we would like to work with a penalized Kelvin-Voigt model, where we will penalize the divergence free condition with penalty parameter ϵ, such that as $\epsilon \to 0$, the solutions of the penalized Kelvin-Voigt model converge to the solution of Kelvin-Voigt model. We would like to apply the finite element Galerkin method to the penalized Kelvin-Voigt model and work out the error analysis.

- The error estimates in finite element analysis depend on the estimates of exact solution, which in general is rarely known. For the Kelvin-Voigt model, in future, we would like to establish a posteriori error estimates in terms of computable quantities, depending on the data and discrete solution. These quantities are used to achieve a solution having specified accuracy in an optimal manner and to make computational method more effective.

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1. Bajpai S., Nataraj N., Pani A. K., Damazio P. and Yuan J. Y., *Semidiscrete Galerkin method for equations of motion arising in Kelvin-Voigt model of viscoelastic fluid flow*, NMPDE, DOI: 10.1002/num.21735.

2. Bajpai S., Nataraj N. and Pani A. K., *On fully discrete finite element schemes for equations of motion of Kelvin-Voigt fluids*, Int. J. Numer. Anal. Mod. **10** (2) (2013), 481-507.