FINITE VOLUME ELEMENT METHODS FOR INCOMPRESSIBLE MISCIBLE DISPLACEMENT PROBLEMS IN POROUS MEDIA

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CERTIFICATE OF COURSE WORK

This is to certify that Mr. Sarvesh Kumar was admitted to the candidacy of the Ph.D. Degree on 18 July, 2002, after successfully completing all the courses required for the Ph.D. Degree programme. The details of the course work done are given below.

Sr. No.	Course No.	Course Name	Credits
1.	MA 417	Ordinary Differential Equations	8.00
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3.	MA 827	Analysis	6.00
4.	MA 559	Functional Analysis	AU
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6.	MA 828	Functional Analysis	6.00
7.	MA 838	Special Topics in Mathematics II	6.00
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Abstract

The main objective of this dissertation is to study finite volume element methods (FVEMs) for incompressible miscible displacement problems in porous media. The mathematical model describing such a displacement in a reservoir gives rise to a system of coupled nonlinear partial differential equations consisting of the pressure-velocity equation or just the pressure equation which is of elliptic type and the concentration equation which is of parabolic type.

Mixed finite volume element procedures have been applied for the pressure equation to obtain an accurate approximation to the Darcy velocity which, in turn, yields a better approximation of the concentration. Since FVEMs are conservative, we have applied a standard FVEM for approximation of the concentration equation. Discontinuous Galerkin finite element methods are also element wise conservative and are easy to implement compared to other conforming and nonconforming finite elements methods. Therefore, an attempt has also been made to apply a discontinuous Galerkin FVEM for approximating the concentration equation. Then existence of a unique discrete solution is proved. Using backward-Euler difference method, we have discussed a fully discrete scheme and *a priori* error estimates in $L^\infty(L^2)$ norm are derived for velocity, pressure and concentration for both the schemes under appropriate smoothness on the exact solutions. Since the concentration equation is convection dominated diffusion type, the standard numerical schemes fail to provide a physically relevant solution because these methods suffer from grid orientation effects. One way to minimize the grid orientation effect is to use modified methods of characteristics (MMOC). We apply MMOC combined with standard FVEM for approximating the concentration equation. Moreover, *a priori* error estimates are derived for the velocity and concentration in the $L^\infty(L^2)$ norm. Further, some numerical experiments are conducted at the end of Chapters 2 through 4 to support our theoretical findings. Finally, the thesis deals with informal observations regarding the possible extension of the present work.

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Chapter 1

Introduction

The main objective of this dissertation is to study finite volume element methods (FVEM) for a coupled system of nonlinear elliptic and parabolic equations arising in incompressible miscible displacement problems in porous media.

1.1 Motivation

An oil reservoir is a porous medium, whose pores contain some hydrocarbon components, collectively called as "Oil". There are mainly three stages of oil recovery.
Primary Recovery. In this stage, the oil or gas is produced by simple natural decomposition. This stage ends rapidly when the pressure equilibrium between the oil field and the atmosphere is attained. This way upto 10 to 15 percent of the total amount of oil and gas can be recovered.
To produce more oil from the field, one may think of pumping out oil through the wells and, thereby, driving the remaining oil towards these wells. But this process has the following main disadvantages:

- The pressure around the wells may fall below the bubble pressure (see [18]) of the oil. Hence, mostly gases will be produced and the heavier components will remain trapped in the field.
- If the pressure in the fluid phase is diminished, this may lead to the collapse of the

rocks which, in turn, results in a low permeability field and, hence, it will be difficult to recover oil subsequently.

Secondary recovery. To overcome the above mentioned difficulties, one may divide the wells into two sets: **injection** and **production wells**. In order to push the remaining oil towards the production well, an inexpensive fluid (e.g. water) is injected through the injection wells into the porous medium. This helps to maintain a high pressure and flow rate in the reservoir. In this stage, we have the following two possibilities:

(a) If the pressure is maintained above the bubble pressure of the oil, then the flow in the reservoir is two-phase immiscible type (say, water and oil) with no mass transfer between the two phases.

(b) If the pressure goes below the bubble pressure at some points, then the oil may get split into two phases (liquid and gaseous). Then the flow in the reservoir is of three-phase type, water phase, which does not exchange mass with the other phases and two hydrocarbon phases (liquid which is called black oil and gas) which exchange mass when the pressure and temperature change.

Even in the best case scenario, this stage may produce only $25\% - 35\%$ of the oil contained in the field. The main reasons for this low recovery are:

(i) There are some regions which are never flooded by water and, hence, the residual oil in that part of the reservoir is not recovered.

(ii) Even in the completely flooded regions, a non negligible part of the oil like $20 - 30\%$ remains trapped in the pores due to the capillary forces. In literature, it is called residual oil.

(iii) In comparison to the oil which is heavy and viscous, the water is extremely mobile. Therefore, instead of pushing the oil, the water finds its own way very quickly to the production wells. This is called the *fingering effect*. Thereafter, only water will come out through the production wells.

Tertiary or enhanced recovery. To recover more oil which is left behind after the first two recovery stages, the miscibility of the fluids must be improved, see [18]. The miscibility is sought by increasing the field temperature, or by introduction of other components (usually expensive) like certain polymers or carbon dioxide flooding. Since the demand of oil is increasing day by day and the prices are going up, these alternatives are now seriously considered as a viable option to produce more oil. The mathematical model which describes polymer flooding gives rise to a system of strongly coupled partial differential equations consisting of an elliptic equation in pressure and a convection dominated diffusion equation in concentration and this process is known as incompressible miscible displacement in a porous medium (see [67]). However, as we have mentioned above, another alternative is to use carbon dioxide flooding and in this case, the displacement process is known as compressible miscible displacement.

In the present thesis, we study finite volume element methods for the approximation of incompressible miscible displacement problems in porous media. In general, the displacement problem arises from the natural law of conservation. The standard Galerkin finite element methods may fail to satisfy the conservation law, but the FVEM are conservative in nature. Therefore, these methods are more suitable for the approximation of the displacement problems in porous media. Most of the commercial packages like ECLIPSE are also based on finite volume methods. Moreover, these methods are also widely used in the approximation of conservation laws, computational fluid mechanics, etc, see, [13, 45, 54, 61]. For more applications and details of FVEM, we refer to [64]. Since the discontinuous Galerkin (DG) methods are also element wise conservative, an attempt has been made in this dissertation to apply discontinuous Galerkin finite volume element methods to approximate the concentration equation.

1.2 Notations and Preliminaries

In this section, we introduce some standard notations which will be used throughout the thesis.

Let Ω be a bounded domain in $\mathbb{R}^d$, that is, the d-dimensional Euclidean space and $\partial\Omega$

denote its boundary. Let $L^p(\Omega)$ denote the linear space of equivalence classes of measurable functions ϕ, defined on Ω, with

$$\int_\Omega |\phi(x)|^p \, dx < \infty.$$

The space $L^p(\Omega)$ equipped with the norm

$$\|\phi\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(x)|^p \, dx \right)^{1/p}, \qquad 1 \le p < \infty,$$

is a Banach space. For $p = \infty$, let $L^\infty(\Omega)$ be the linear space consisting of all functions ϕ that are essentially bounded on Ω, which is equipped with the norm

$$\|\phi\|_{L^\infty} = ess \sup_{x \in \Omega} |\phi(x)|.$$

For $p = 2$, we denote the inner product and norm on $L^2(\Omega)$ as

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x) dx \quad \text{and} \quad \|\phi\| = \left(\int_\Omega |\phi(x)|^2 \, dx \right)^{1/2},$$

respectively. It is well known that $L^2(\Omega)$ is a Hilbert space with respect to the inner product $(\cdot, \cdot)$.

A multi index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ is a d-tuple with non-negative integers $\alpha_i \ge 0$ and its order is denoted by $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Set the α^{th} order partial derivative as

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

For non-negative integer s and $1 \le p \le \infty$, the Sobolev space of order (s, p) over Ω, denoted by $W^{s,p}(\Omega)$ is defined as the set of functions in $L^p(\Omega)$ whose generalized derivatives up to order s are also in $L^p(\Omega)$, i.e.,

$$W^{s,p}(\Omega) = \{\phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega), \quad |\alpha| \le s\}.$$

This is also a Banach space with the norm

$$\|\phi\|_{s,p,\Omega} = \|\phi\|_{s,p} = \left(\sum_{|\alpha| \le s} \|D^\alpha \phi\|_{L^p}^p \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and for $p = \infty$,

$$\|\phi\|_{s,\infty,\Omega} = \|\phi\|_{s,\infty} = \sup_{|\alpha|\le s} \|D^\alpha \phi\|_{L^\infty(\Omega)}.$$

We also introduce seminorms denoted by $|\cdot|_{s,p}$ which are defined as

$$|\phi|_{s,p,\Omega} = |\phi|_{s,p} = \left(\sum_{|\alpha|=s} \|D^\alpha \phi\|_{L^p}^p \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and for $p = \infty$,

$$|\phi|_{s,\infty,\Omega} = |u|_{s,\infty} = \sup_{|\alpha|=s} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

When $p = 2$, we denote $W^{s,2}(\Omega)$ by simply $H^s(\Omega)$. Note that $H^s(\Omega)$ is a Hilbert space with the natural inner product defined by

$$(\phi, \psi) = \sum_{|\alpha|\le s} \int_\Omega D^\alpha \phi D^\alpha \psi dx \qquad\qquad \forall \phi,\ \psi \in H^s(\Omega),$$

and induced norm

$$\|\phi\|_s = \left(\sum_{|\alpha|\le s} \|D^\alpha \phi\|_{L^2}^2 \right)^{1/2}.$$

For our notational convenience, we write $H^s(\Omega)$ simply by H^s.

The dual space of $H^s(\Omega)$ is denoted by $H^{-s}(\Omega)$ and is equipped with the norm

$$\|\phi\|_{-s} = \sup_{\psi\in H^s(\Omega)/\{0\}} \frac{|(\phi,\psi)|}{\|\psi\|_s}.$$

We denote by $L^q(a, b; W^{s,p}(\Omega))$, $1 \le q,\ p \le \infty,\ s \ge 0$, the space of functions $\psi : [a, b] \longrightarrow W^{s,p}(\Omega)$ such that $\|\psi(t)\|_{s,p,\Omega} \in L^q(a, b)$, see [43, pp.285].

The norm on $L^q(a, b; W^{s,p}(\Omega))$ is defined as

$$\|\phi\|_{L^q(a,b;W^{s,p}(\Omega))} = \left(\int_a^b \|\phi(t)\|_{s,p}^q ds \right)^{1/q} \qquad 1 \le q < \infty,$$

and for $q = \infty$

$$\|\phi\|_{L^\infty(a,b;W^{s,p}(\Omega))} = ess \sup_{t\in(a,b)} \|\phi(t)\|_{s,p}.$$

We would also use the following matrix notations.

For a matrix $A = (a_{ij}(x))_{1\leq i,j\leq 2}$, with $x \in \Omega$, we define the following norms:

$$|A|_1 = \max_{1\leq j\leq 2} \sum_{i=1}^{2} |a_{ij}(x)|, \qquad |A|_2 = \left(\sum_{i,j=1}^{2} |a_{ij}(x)|^2 \right)^{1/2}, \tag{1.2.1}$$

$$\|A\|_{(L^2(\Omega))^{2\times 2}} = \left(\sum_{i,j=1}^{2} \int_\Omega |a_{ij}(x)|^2 dx \right)^{1/2}. \tag{1.2.2}$$

Also, we have

$$\frac{1}{\sqrt{2}}|A|_1 \leq |A|_2 \leq \sqrt{2}|A|_1. \tag{1.2.3}$$

We frequently use the following standard inequalities.

Young's Inequality. For $a,\ b \geq 0$ and $\epsilon > 0$, the following inequality holds:

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}. \tag{1.2.4}$$

LEMMA 1.2.1 **(Hölder's inequality, [52])** *Let* $1 \leq p, q < \infty$ *be such that* $1/p + 1/q = 1$ *and* $\Omega \subset \mathbb{R}^d$*. Further, let* $\phi \in L^p(\Omega)$ *and* $\psi \in L^q(\Omega)$*. Then*

$$\left| \int_\Omega \phi\psi \ dx \right| \leq \left(\int_\Omega |\phi|^p \ dx \right)^{1/p} \left(\int_\Omega |\psi|^q \ dx \right)^{1/q}.$$

LEMMA 1.2.2 **(Generalized Hölder's inequality, [52])** *Let* $1 \leq p,\ q,\ r < \infty$ *be such that* $1/p + 1/q + 1/r = 1$ *and* $\Omega \subset \mathbb{R}^d$*. Further, let* $\phi \in L^p(\Omega)$*,* $\psi \in L^q(\Omega)$ *and* $\chi \in L^r(\Omega)$*. Then*

$$\left| \int_\Omega \phi\psi\chi \ dx \right| \leq \left(\int_\Omega |\phi|^p \ dx \right)^{1/p} \left(\int_\Omega |\psi|^q \ dx \right)^{1/q} \left(\int_\Omega |\chi|^r \ dx \right)^{1/r}.$$

LEMMA 1.2.3 **(Cauchy-Schwarz inequality, [68])** *Let* $1 \leq p, q < \infty$ *be such that* $1/p + 1/q = 1$*. Suppose that* $\{a_i\}_{i=1}^N$ *and* $\{b_i\}_{i=1}^N$ *are positive real numbers. Then*

$$\left(\sum_{i=1}^{N} a_i b_i \right) \leq \left(\sum_{i=1}^{N} a_i^p \right)^{1/p} \left(\sum_{i=1}^{N} b_i^q \right)^{1/q}.$$

LEMMA 1.2.4 **(Generalized Cauchy-Schwarz inequality, [68])** *For* $1 \leq p, q, r < \infty$ *with* $1/p + 1/q + 1/r = 1$*, let* $\{a_i\}_{i=1}^N$*,* $\{b_i\}_{i=1}^N$ *and* $\{c_i\}_{i=1}^N$ *be positive real numbers. Then*

$$\left(\sum_{i=1}^{N} a_i b_i c_i \right) \leq \left(\sum_{i=1}^{N} a_i^p \right)^{1/p} \left(\sum_{i=1}^{N} b_i^q \right)^{1/q} \left(\sum_{i=1}^{N} c_i^r \right)^{1/r}.$$

LEMMA 1.2.5 **(Poincaré-Friedrich's Inequality, [8, pp. 102])** *Let Ω be open, bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Let $v \in H^1(\Omega)$ be such that*

$$\int_\Omega v \ dx = 0.$$

Then

$$\|v\|_{0,\Omega} \leq C\|\nabla v\|_{0,\Omega}, \tag{1.2.5}$$

where $C = C(\Omega)$ is positive constant.

LEMMA 1.2.6 **(Green's Formula [52])** *Let u and v be in $H^1(\Omega)$. Then for $1 \leq i \leq d$, the following integration by parts formula holds:*

$$\int_\Omega u\frac{\partial v}{\partial x_i}dx = -\int_\Omega v\frac{\partial u}{\partial x_i}dx + \int_{\partial\Omega} uvn_i ds,$$

where n_i is the i^{th} component of the outward normal to the boundary $\partial\Omega$.

LEMMA 1.2.7 **(Gronwall's Lemma [69])** *Let $g(t)$ be a continuous function and let $h(t)$ be a nonnegative continuous function on the interval $t_0 \leq t \leq t_0 + a$. If a continuous function $\phi(t)$ has the following property*

$$\phi(t) \leq g(t) + \int_{t_0}^t \phi(s)h(s)ds, \qquad \textit{for } t \in [t_0, t_0 + a],$$

then

$$\phi(t) \leq g(t) + \int_{t_0}^t g(s)h(s)exp\left[\int_s^t h(\tau)d\tau\right]ds, \qquad \textit{for } t \in [t_0, t_0 + a].$$

In particular, when $g(t) = K$ is a nonnegative constant, then we have

$$\phi(t) \leq K exp\left[\int_{t_0}^t h(s)ds\right], \qquad \textit{for } t \in [t_0, t_0 + a].$$

We note that for a nondecreasing nonnegative function g we obtain the above result with K replaced by $g(t)$. We also use the following discrete form of the Gronwall's Lemma, proof of which can be found in Pani *et al.* [65].

LEMMA 1.2.8 **(Discrete Gronwall's Lemma)** *Let $\{\xi^n\}$ be a sequence of nonnegative numbers satisfying*

$$\xi^n \leq \alpha^n + \sum_{j=0}^{n-1} \beta^j \xi^j, \qquad \text{for } n \geq 0,$$

where α^n is a nondecreasing sequence and β^j's are nonnegative. Then

$$\xi^n \leq \alpha^n exp\left(\sum_{j=0}^{n-1} \beta^j\right), \qquad \text{for } n \geq 0.$$

Throughout the thesis, we use the notations $C,\ C_i$ for $i = 1, 2, 3 \cdots$ to denote generic positive constants.

1.3 The Mathematical Model

In this section, we derive a mathematical model which describes the miscible displacement of one incompressible fluid by another in a porous medium. We study the flow of one incompressible fluid flooding from the injection well into a petroleum reservoir that mixes with the originally resident fluid to reduce the surface tension with an intention to push the oil towards production wells. The invading and displaced fluid are referred to as the solvent and resident fluid, respectively. We further assume that the solvent and resident fluid mix in all proportions forming a single phase and we neglect the influence of gravity. Let $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ be a rectangular reservoir with unit thickness. The assumption of unit thickness on the reservoir Ω is quite reasonable because the height (in metres) is very small compared to the length and breadth (in kilometres in both directions) of the reservoir.

Let c denote the concentration of the solvent/invading fluid in the fluid mixture. The miscibility of the components imply that the Darcy velocity $\mathbf{u}$ of the fluid satisfies

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall (x,t) \in \Omega \times J = (0, T], \tag{1.3.1}$$

and the incompressibility implies that

$$\nabla \cdot \mathbf{u} = q \qquad \forall (x,t) \in \Omega \times J, \tag{1.3.2}$$

where $x = (x_1, x_2) \in \Omega$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ and $p(x,t)$ are, respectively, the Darcy velocity and the pressure of the fluid mixture, $\mu(c)$ is the concentration dependent viscosity of the mixture, $\kappa(x)$ is the 2×2 permeability tensor of the medium, $q(x,t)$ represents the fluid flow rates at injection and production wells. We assume that there is no change in the volume due to the mechanical mixing. Here, the diffusion-dispersion tensor $D(\mathbf{u})$ consisting of molecular-diffusion and mechanical dispersion (due to mechanical mixing, see Peacemen [66]) is given by

$$D(\mathbf{u}) = \phi(x)\Big[d_m I + |\mathbf{u}|\Big(d_l E(\mathbf{u}) + d_t(I - E(\mathbf{u}))\Big)\Big], \tag{1.3.3}$$

where d_m is the molecular diffusion, d_l and d_t are, respectively, the longitudinal and transverse dispersion coefficients, $E(\mathbf{u})$ is the tensor that projects onto $\mathbf{u}$ direction, whose ij^{th} component is given by

$$(E(\mathbf{u}))_{ij} = u_i u_j / |\mathbf{u}|^2; \;\; 1 \leq i, j \leq 2, \;\; |\mathbf{u}|^2 = u_1^2 + u_2^2,$$

I being the identity matrix of order 2 and $\phi(x)$ denotes the porosity of the medium. We note that in realistic situations mechanical dispersion is more important compared to molecular diffusion and also $d_l > d_t$. The conservation of mass in the mixture satisfies the following equation

$$\phi(x)\rho\frac{\partial c}{\partial t} + \rho\nabla\cdot(c\mathbf{u}) - \rho\nabla\cdot(D(\mathbf{u})\nabla c) = \tilde{c}\rho q \qquad \forall (x,t) \in \Omega \times J, \tag{1.3.4}$$

where ρ is the density of the fluid mixture and $\tilde{c}$ is the concentration of the injected fluid at the injection well. Using (1.3.2), the equation (1.3.4) can be rewritten in the following form

$$\phi(x)\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D(\mathbf{u})\nabla c) = (\tilde{c} - c)q \qquad \forall (x,t) \in \Omega \times J. \tag{1.3.5}$$

The above equation (1.3.5) is in non-divergence form. Hence, the system of equations describing the incompressible miscible displacement of one fluid by another in a porous medium is given by

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall (x,t) \in \Omega \times J, \tag{1.3.6}$$

$$\nabla\cdot\mathbf{u} = q \qquad \forall (x,t) \in \Omega \times J, \tag{1.3.7}$$

$$\phi(x)\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D(\mathbf{u})\nabla c) = g(x,t,c) \qquad \forall (x,t) \in \Omega \times J. \tag{1.3.8}$$

Assume that no flow occurs across the boundary $\partial\Omega$, i.e.,

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{1.3.9}$$

$$D(\mathbf{u})\nabla c\cdot\mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{1.3.10}$$

and the initial condition

$$c(x,0) = c_0(x) \qquad \forall x \in \Omega, \tag{1.3.11}$$

where

$$g(x,t,c) = g(c) = (\tilde{c} - c)q, \tag{1.3.12}$$

and $c_0(x)$ represents the initial concentration and $\mathbf{n}$ denotes the unit exterior normal to $\partial\Omega$. For physically relevant situations, c_0 must satisfy $0 \leq c_0(x) \leq 1$. For well-posedness, the following compatibility condition is imposed on the data

$$\int_\Omega q(x,t)dx = 0 \qquad \forall t \in J. \tag{1.3.13}$$

This can be easily derived from (1.3.6)-(1.3.7) and (1.3.9). Here, the equation (1.3.13) indicates that for an incompressible flow with an impermeable boundary, the amount of injected fluid and the amount of fluid produced are equal. In general, equations (1.3.6)-(1.3.7) and (1.3.8) are referred as the pressure-velocity equations or just the pressure equation and the concentration equation, respectively. Since the equations (1.3.6)-(1.3.11) are strongly coupled and nonlinear, to find an analytic solution of this system would be a very difficult task. Therefore, one resorts to numerical methods for solving the above system of equations approximately. In the next two sections, we discuss the theoretical and computational issues related to the system (1.3.6)-(1.3.11).

1.4 Theoretical Issues

When one looks for the theoretical analysis of this model, we come across the following two main difficulties:

(i) In general, the viscosity depends on the concentration in the following manner:

$$\mu(c) = \mu(0)\left[1 + (M^{1/4} - 1)c\right]^{-4}, \qquad c \in [0,1], \tag{1.4.1}$$

where $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio. With (1.4.1), the pressure equation (1.3.6)-(1.3.7) becomes potentially degenerate. There are two reasons for occurrence of this degeneracy. The degeneracy occurs when either $c < 0$ and $M > 1$ (non-physical case) or $c > 1$ and $M \leq 1$ (physical case).

(ii) In the concentration equation the diffusion and convection terms may have unbounded coefficients due to the potentially unbounded velocity. For example, it can be seen easily that for $\mathbf{u} \in (C(\bar{\Omega}))^2$ the following inequality holds true:

$$\phi(d_m + d_t|\mathbf{u}|)|\xi|^2 \leq D(\mathbf{u})\xi \cdot \xi \leq \phi(d_m + d_l|\mathbf{u}|)|\xi|^2 \quad \forall \xi \in \mathbb{R}^2. \tag{1.4.2}$$

Now it is clear from (1.4.2) that when the velocity $\mathbf{u}$ is unbounded, then $D(\mathbf{u})$ is also unbounded.

In the past, efforts have been made to show existence and uniqueness of solution to the system (1.3.6)-(1.3.11) under some reasonable regularity assumptions on the data. Sammon [74] in 1986, has proved existence of a unique strong solution with the assumption that the matrix D is independent of velocity $\mathbf{u}$, i.e., $D(x) = \phi(x)d_m I$. It is also assumed in [74] that the mobility ratio $M = 1$, i.e., $\mu(c) =$ constant. To overcome the first difficulty, that is, the degeneracy, Feng [49] in 1995, instead of defining $\mu(c)$ for all real numbers c by using (1.4.1), has extended $\xi(c) = \mu(c)^{-1}$ to $\mathbb{R}$ in a reasonable way so that $\xi \in W^{2,\infty}(\mathbb{R})$ and there exists a positive constant ξ_0 such that

$$0 < \xi_0^{-1} \leq \xi(c) \leq \xi_0 < \infty \qquad \forall c \in \mathbb{R}. \tag{1.4.3}$$

Based on the method of regularization the original problem is approximated by a family of regularized problems. In [49] the author has considered the following regularized problem corresponding to the system (1.3.6)-(1.3.11): Given $\epsilon > 0$,

$$\mathbf{u}_\epsilon = -\frac{\kappa(x)}{\mu(c_\epsilon)}\nabla p_\epsilon \qquad \forall (x,t) \in \Omega \times J, \tag{1.4.4}$$

$$\nabla \cdot \mathbf{u}_\epsilon = q_\epsilon \qquad \forall (x,t) \in \Omega \times J, \tag{1.4.5}$$

$$\phi(x)\frac{\partial c_\epsilon}{\partial t} + \mathbf{u}_\epsilon \cdot \nabla c_\epsilon - \nabla \cdot (D(\mathbf{v}_\epsilon)\nabla c_\epsilon) = (\tilde{c} - c_\epsilon)q_\epsilon \quad \forall (x,t) \in \Omega \times J, \tag{1.4.6}$$

$$\mathbf{u}_\epsilon \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{1.4.7}$$

$$D(\mathbf{v}_\epsilon)\nabla c_\epsilon \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{1.4.8}$$

$$c_\epsilon(x,0) = c_0^\epsilon(x) \qquad \forall x \in \Omega, \tag{1.4.9}$$

$$\int_\Omega p_\epsilon dx = 0 \qquad \forall t \in J, \tag{1.4.10}$$

where $\mathbf{v}_\epsilon = \dfrac{1}{1+\epsilon|\mathbf{u}_\epsilon|}$. It is shown that each regularized problem possesses one and only one semi-classical solution. Using some uniform estimates for this family of regularized approximate solutions and applying compactness arguments, a weak solution to the original boundary value problem, i.e., (1.3.6)-(1.3.11) is proved.

Subsequently, Chen and Ewing [20] in 1999 have also studied the mathematical analysis of (1.3.6)-(1.3.11). They have shown that the system (1.3.6)-(1.3.8) with various boundary conditions possesses a weak solution under physically reasonable hypothesis on the data. However, it is difficult to prove the uniqueness in their setting. In stead of regularization, they have discretized the system in temporal direction to obtain a system of elliptic PDEs at each time level. Then using Rothe method and existence results for elliptic PDEs, a sequence of approximations is derived on the whole time interval. Finally, a limiting procedure is used to prove existence of a weak solution. More recently, Choquet [21] has studied the analysis for compressible miscible displacement problem in porous media. More attention has been paid to take care of the difficulty occurring through the strong coupling. To show the existence of relevant weak solutions, the author has used non-classical estimates and renormalization tools.

1.5 Computational Issues

In the last few decades, many numerical methods have been proposed in literature for obtaining good approximations of the miscible displacement problems in porous media. In this section, we discuss the computational difficulties associated with the simulation of the incompressible miscible displacement problems described in (1.3.6)-(1.3.11) that too in tertiary recovery process.

It is well known that in realistic situations the matrix $D(\mathbf{u})$ in the concentration equation (1.3.8) is very small in comparison to the convective or transport term, and hence, the concentration equation is strongly convection dominated. Unfortunately, most of the standard numerical methods exhibit *grid orientation* [1] which really affects the numerical solution and may not be accepted as a physically relevant solution. Todd [79] has noted that whether the spatial discretization is taken either parallel to the direction of the streamlines connecting the injection and production wells (parallel grid), or diagonally to the direction of the streamlines, the solution obtained from these two grids are different. Therefore, it is not easy to decide a priori which grid should be taken for the approximation. Different numerical schemes have been proposed, in literature, to minimize the grid orientation effect. Finite difference methods are very popular in petroleum simulation more because of their computational simplicity. But these methods suffer from grid orientation effects. Some of the earlier results with special attention to grid orientation effect can be found in [62, 71, 79]. Finite element methods are also successful in eliminating the grid orientation effect, provided an effective numerical diffusion term is added to these schemes. For an extensive reference on Galerkin methods for incompressible displacement problems in porous media, we refer to Wheeler [73] Ewing [37] and Douglas *et al.* [40] and references, therein. Below, we discuss some finite element methods applied to pressure and concentration equations.

1.5.1 Pressure equation

Since the concentration equation (1.3.8) depends explicitly on the velocity, it is desirable to find a good approximation of the velocity. The standard finite element, finite volume and finite difference methods for approximating pressure equation (1.3.6)-(1.3.7) first determine an approximation, say p_h to the pressure p and then, in order to compute the velocity $\mathbf{u_h}$ from p_h, one has to differentiate or take the difference quotient of p_h and multiply by a rough function κ/μ, where $\mathbf{u_h}$ is an approximation to the velocity $\mathbf{u}$. This process may not yield an accurate approximation for $\mathbf{u_h}$. Therefore, for a more accurate approximation $\mathbf{u_h}$

[1] A numerical discretization procedure is said to exhibit grid orientation effect if the discrete solution is sensitive to the spatial orientation of the grids.

of the velocity $\mathbf{u}$, it is natural to consider both p and $\mathbf{u}$ as primary variables. To achieve this, we split the pressure equation into a couple of first order equations (1.3.6)-(1.3.7) and then apply mixed methods. In the past, mixed finite elements methods are proposed in the literature, see, Douglas *et al.* [38, 37], Darlow *et al.*[31], Duran [42], Ewing *et al.* [44] and Dawson *et al.* [34] for approximating the pressure equation in incompressible miscible displacement problems. There is hardly any result on mixed finite volume element methods for the approximation of the pressure equation. *Therefore, in Chapter* 2*, an attempt has been made to introduce and analyze mixed FVEM for approximating the pressure equation.*

1.5.2 Concentration equation

Since the concentration equation (1.3.8) is a convection dominated diffusion equation, the solution of (1.3.8) varies rapidly from one point to other in the domain. Therefore, standard numerical methods fail to provide an accurate solution of the concentration. To overcome this difficulty, different numerical methods which provide appropriate numerical diffusion have been proposed in the past for the approximation of the concentration equation.

One such method is the modified method of characteristics (MMOC) which has been proposed in literature to deal with the grid orientation effect. The basic idea behind using MMOC is to combine the time derivative and the convective term as a directional derivative and apply time-stepping along the characteristics. Since the magnitude of the derivative is small compared to the magnitude in the direction of time, this procedure allows us to use larger and accurate time-stepping in the direction of time. In [2, 34, 41, 42], a modified method of characteristics combined with finite element method has been studied for the approximation of the concentration equation. *Based on this analysis, in Chapter* 4*, MMOC combined with the standard finite volume element method for the approximation of the concentration equation has been applied and a priori error estimates in* $L^\infty(L^2)$ *norm for the concentration as well as velocity has been derived.*

Douglas and Dupont [35] have introduced and analyzed a C^0- interior penalty method which uses interior penalties across the interior edges of the triangular elements of the finite element mesh in the direction of the normal derivatives enforcing the approximate solution to lie between C^0 and C^1- finite element spaces. The grid orientation effect is then reduced by introducing numerical diffusion through penalties. Wheeler and Darlow [83] have extended this procedure to the convection dominated diffusion equation for the incompressible miscible displacement in porous media, with the assumption that the matrix $D(\mathbf{u})$ is independent of velocity $\mathbf{u}$. Later, Das and Pani [32] applied the same technique to slightly compressible miscible displacement problem with the same assumption that $D(\mathbf{u})$ is independent of $\mathbf{u}$, i.e., only molecular diffusion is considered and the effect of tensor dispersion is neglected. But in physical problems the mechanical dispersion is more important than the molecular diffusion. Subsequently, in the thesis of Ali [1], the result has been extended to slightly compressible miscible displacement problems when the dispersion matrix D depends on $\mathbf{u}$. *In Chapter* 3*, we have applied a discontinuous Galerkin finite volume element method for the approximation of the concentration equation when the matrix* $D(\mathbf{u})$ *depends on* $\mathbf{u}$ *and have also derived the error estimates in* $L^\infty(L^2)$ *norm for the velocity as well as for the concentration.*

Sun *et al.* [76] applied the mixed FEM for pressure-velocity equation and discontinuous Galerkin FEM for approximating the concentration equation. Further, Sun and Wheeler [77] applied symmetric and non symmetric discontinuous Galerkin methods for the approximation of the concentration equation by assuming that the velocity is known and is time independent. The Eulerian-Lagrangian localized adjoint method (ELLAM) has been used to approximate the concentration equation in [80]. Daoqi Yang [84], considered the mixed methods with dynamic finite element spaces, i.e., different number of element and different basis functions were adopted at different time levels.

Efficient time-stepping procedure. Since the mathematical model which describes the miscible displacement is a coupled system of nonlinear partial differential equation (1.3.6)-(1.3.11), the fully discrete schemes give rise to a very large system of linear algebraic equation at each time level. Moreover, at each time step the matrices may change with time, so that a use of direct methods may be expensive, especially for the concentration equation. Dougals *et al.* [36] have observed that the computational cost can be minimized for quasi-linear parabolic equations by using an iterative time-stepping method. The basic idea is to factorize only once instead of factorizing a different large matrix at each time level, and then update after a fixed number of time steps. Further, a preconditioner is used in an iterative procedure and to stabilize the process, a few iterations are performed at each time level. This saves a substantial amount of computational cost. The conjugate gradient methods are one of those iteration procedures which can be used for this purpose. Ewing and Russell [47] have applied preconditioned conjugate gradient method without reducing the order of convergence for the approximation of incompressible miscible displacement problems in porous media. They have also derived *a priori* error estimates. Subsequently, Russell [72], has also studied time-stepping procedure combined with method of characteristics by extending the analysis [47]. It is further observed that the pressure and velocity are more smooth in time than the concentration and therefore large time steps can be used in computing the pressure and velocity than the concentration. Such analysis without loosing the order of convergence has been discussed by Ewing *et al.* [44] and Russell *et al.* [72].

1.6 Literature review on finite volume element methods

The finite volume element method, like finite element method and finite difference method is a numerical technique for approximating the solutions of partial differential equations. The basic idea of the FVEM is to apply Gauss divergence theorem for the elliptic operator on each computational cells, which converts the volume integral to a boundary integral. The idea is old and the resulting method comes under a variety of names, e.g., the generalized difference methods [60], box method [7] and the covolume methods [25, 27].

1.6.1 The standard finite volume element methods

The standard finite volume element method can be considered as a Petrov-Galerkin finite element method in which the trial space is chosen as C^0- piecewise linear polynomials on the finite element partition of the domain and the test space, as piecewise constants over the control volumes to be defined in Chapter 2. Since the test space is piecewise constants, computationally, the FVEM are less intensive compared to the standard FEM.

In case of nonstructured triangular meshes, Bank and Rose [7] have analyzed a finite volume method, which is called as box method, for the Poisson as well as more general elliptic problems. They have considered a nonuniform triangulation of a polygonal domain in $\mathbb{R}^2$, which satisfies the minimum angle condition, i.e., there exists a constant, say $\theta_0 > 0$ such that all the angles of the triangle are bounded below by θ_0. In order to construct the dual partition of the domain, a point z_T is chosen inside each triangle T and is connected with midpoint of each side of triangle T. They have also shown that the derived error estimates are comparable with those obtained from the standard Galerkin finite element methods using piecewise linear polynomials. A similar technique has been used by Cai [10] for the approximation of a self-adjoint elliptic problem in a two dimensional domain. But the choice of the interior point z_T, which was important in the analysis is taken as either circumcenter, orthocenter, incenter or centroid of the triangle T. Optimal error estimate has been derived only in the H^1- norm in [7, 10].

Jiangou *et al.* [51] have also analyzed FVEM for a general self adjoint elliptic problem with mixed boundary conditions and derived optimal error estimates in energy norm without putting any restriction on the mesh. Further, a counter example has been provided to show that an expected L^2- error estimate may not exist in the usual sense. It is conjectured that the FVE solution cannot have optimal order of convergence if the exact solution is in H^2 and the source term f in L^2.

For second order linear elliptic problems, Li *et al.* [60] have obtained the following L^2 error estimate:

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{W^{3,p}(\Omega)}, \qquad p > 1,$$

where u is the exact solution and u_h is the FV approximation of u. Note that the regularity on the exact solution seems to be too high compared to the finite element methods. In [25, 27], optimal H^1, $W^{1,\infty}$- estimates and superconvergence results in H^1 and $W^{1,\infty}$- norms have been derived by extending the analysis of [60]. In addition, the following maximum norm estimate

$$\|u - u_h\|_\infty \leq Ch^2 \left(\|u\|_{2,\infty} + \|u\|_3\right)$$

is also proved in [25, 27]. However, in all these papers, H^3- regularity of the exact solution is assumed. Chatzipantelidis [15] has also studied FVEM with nonconforming Crouzeix-Raviart linear element and has derived optimal error estimate in L^2- norm, but he has failed to mention that the H^1- regularity on the source term is essential for deriving optimal error estimates in L^2- norm. Recently, Ewing *et al.* [46] have presented the L^2 and L^∞- error estimates for the following elliptic problem: Given f, find u such that

$$\begin{aligned} \nabla \cdot (A\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded convex polygon in $\mathbb{R}^2$ with boundary $\partial\Omega$ and A is a 2×2 symmetric, positive definite matrix in Ω. In this paper, they have derived the following L^2 and L^∞- error estimates

$$\|u - u_h\|_0 \le C\left(h^2\|u\|_2 + h^{1+\beta}\|f\|_\beta\right),$$

and

$$\|u - u_h\|_\infty \le Ch^2|ln\frac{1}{h}|\left(\|u\|_{2,\infty} + h^{1+\beta}\|f\|_\beta\right).$$

The above results leads to the optimal convergence rate of the FVEM if $f \in H^\beta$ with $\beta \ge 1$. Li [59] and Chatzipantelidis *et al.* [16] have studied the finite volume method for nonlinear elliptic problems and derived *a priori* error estimates.

Chatzipantelidis *et al.* [17] have discussed the piecewise linear standard FVEM for the approximation of the parabolic problems in a convex polygonal domain. They have obtained optimal H^1 and L^2- error estimates by assuming suitable regularity conditions on the initial data. The authors [25, 27, 60] also have studied the FVEM for the parabolic problem and have derived optimal L^2- error estimates with higher order regularity assumption on the exact solution compared to the regularity results used for the standard finite element methods. Ewing *et al.* [45, 75] have discussed *a priori* error estimates for the parabolic integro-differential equations. More recently, Kumar *et al.* in [56] have studied a standard FVEM with and without numerical quadrature for the second order hyperbolic problems and have derived optimal error estimates in L^2 and H^1- norms and quasi-optimal estimates in L^∞ -norm.

1.6.2 Mixed finite volume or covolume methods

In a covolume method, one uses two different kind of grids: a primal grid and a dual grid. Mixed covolume methods can also be thought of as a Petrov-Galerkin method. The analysis of these methods is based on the tools borrowed from the mixed finite element methods. Using a transfer operator which maps the trial space to the test space, the mixed covolume methods can be put in the framework of mixed finite element methods. This transfer operator plays a vital role in deriving the optimal error estimates. Earlier, Chou *et al.* [23, 26] have discussed and analyzed mixed covolume or finite volume element method for the second order linear elliptic problems in two dimensional domains. In [26], the velocity $\mathbf{u}$ and pressure p have been approximated by the lowest order Raviart-Thomas element space on triangles, while in [23] rectangular elements have been used to approximate the solutions. In these two papers, *a priori* error bounds for the velocity in L^2 and $H(div)$- norms have been derived. For the nonstaggered quadrilateral grids, Chou *et al.* in [24] have constructed a mixed finite volume method for elliptic problems with Dirichlet boundary condition. Like in [23, 26], they have also used the Raviart-Thomas spaces for approximating velocity and pressure and derived the following error estimates:

$$\|p - p_h\|_0 + h\|p - p_h\|_1 \leq Ch^2\|f\|_1,$$

and

$$\|\mathbf{u} - \mathbf{u_h}\|_0 + \|\mathrm{div}\mathbf{u} - \mathrm{div}\mathbf{u_h}\|_0 \leq Ch(\|\mathbf{u}\|_1 + \|f\|_1).$$

Chou in [22] has discussed the convergence of the mixed covolume method for the Stokes equation. To approximate the velocity, instead of using lowest-order Raviart-Thomas element, nonconforming linear polynomials have been used whereas to approximate the pressure, piecewise constant polynomials are used. *A priori* error estimates are derived in L^2-norm for the velocity as well as for the pressure.

Based on the analysis of Milner [63], Kwak *et al.* [57] have extended the results of [23, 26] to quasi-linear elliptic problems. The author in [53] has also discussed mixed finite volume methods for the approximation of a nonlinear elliptic problem. More recently, Tongke [81] has discussed a mixed finite volume method on rectangular mesh for the biharmonic equation and compared the analysis with [26]. In this dissertation, a mixed finite volume element method is applied to approximate both pressure and velocity in the pressure equation.

1.6.3 Discontinuous Galerkin finite volume element methods

Keeping in mind the advantages of the FVEM and the discontinuous Galerkin methods, it is natural to think of discontinuous Galerkin finite volume element methods (DGFVEM) for the numerical approximation of the second order partial differential equations. In these methods, the support of the control volumes are small compared to the standard FVM [60] and mixed FVM [26]. Also the control volumes have support inside the triangle in which they belong and there is no contribution from the adjacent triangles. This property of the control volumes makes the DGFVEM more suitable for parallel computing.

The DGFVEM for elliptic problems has been discussed by Ye [85] and Chou *et al.* [28]. Further in [85], optimal error estimates in broken H^1-norm and suboptimal estimates in L^2- norm have also been derived. More recently, Kumar *et al.* [55] have developed and analyzed a one parameter family of DGFVE methods for approximating the solution of the second order linear elliptic problems and derived optimal error estimates in broken H^1 and L^2-norms. They have also reported numerical experiments to support their theoretical results. In this thesis, an attempt has been made to apply the DGFVEM for approximating the concentration equation.

1.7 Layout of the Thesis

The organization of the thesis is as follows. While Chapter 1 is introductory in nature, in Chapter 2, we apply mixed FVEM for approximation of the pressure-velocity equation and a standard FVEM for the approximation of the concentration equation. *A priori* error estimates in $L^\infty(L^2)$ norm are derived for the pressure, velocity and concentration. Existence and uniqueness results for the discrete solution are also discussed in details. Some numerical experiments are conducted using the data from [80] to corroborate our theoretical findings.

Taking into account the advantage of discontinuous Galerkin method and FVEM, Chapter

3 is devoted to DGFVEM for approximating the concentration equation. We also apply mixed FVEM for the approximation of the pressure equation. Then existence of a unique discrete solution has been proved. *A priori* error estimates have been derived for the velocity and concentration in the $L^\infty(L^2)$ norm. The final section of this chapter is devoted to some numerical experiments.

Since the concentration equation is convection dominated, in Chapter 4, we have applied a modified method of characteristics combined with standard FVEM for the approximation of the concentration equation and a mixed FVEM for the approximation of the pressure equation. *A priori* error estimates have been derived for velocity and concentration in the $L^\infty(L^2)$ norm and numerical experiments are also reported to substantiate the theoretical findings.

Finally, Chapter 5 is devoted to the critical evaluation of the present work. Some of the main results of this thesis are highlighted. We also discuss the scope of other discontinuous Galerkin methods for the approximation of the concentration equation. We conclude this chapter with a possible extension of the present work.

Chapter 2

Finite Volume Element Approximations

In this chapter, we discuss finite volume element methods (FVEMs) for incompressible miscible displacement problems in porous media.

2.1 Introduction

We now recall from Chapter 1, a mathematical model describing the miscible displacement of one incompressible fluid by another in a reservoir Ω in $\mathbb{R}^2$ of unit thickness with boundary $\partial\Omega$ over a time period of $J = (0, T]$ given by

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall (x,t) \in \Omega \times J, \tag{2.1.1}$$

$$\nabla \cdot \mathbf{u} = q \qquad \forall (x,t) \in \Omega \times J, \tag{2.1.2}$$

$$\phi(x)\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D(\mathbf{u})\nabla c) = g(c) \qquad \forall (x,t) \in \Omega \times J, \tag{2.1.3}$$

with boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{2.1.4}$$

$$D(\mathbf{u})\nabla c\cdot\mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{2.1.5}$$

and initial condition

$$c(x,0) = c_0(x) \qquad \forall x \in \Omega. \tag{2.1.6}$$

For the approximation of the pressure-velocity equation, we use mixed FVEM and for the concentration equation, we apply the standard FVEM. *A priori* error estimates in $L^\infty(L^2)$ norm are derived for velocity, pressure and concentration for semidiscrete and fully discrete schemes.

We now make the following assumptions on the coefficients $D,\ \phi,\ \kappa,\ \mu$, the forcing functions g and q through out the thesis:

Assumptions

(A_1) : The matrix D is uniformly positive definite, i.e., there exists a positive constant α independent of x and $\mathbf{u}$ such that

$$\sum_{i,j=1}^{2} D_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^2.$$

(A_2) : The functions μ and g are Lipschitz continuous, i.e., there exist Lipschitz constants C_1 and C_2 such that for $(x,t) \in \overline{\Omega} \times \overline{J}$

$$|g(c_1) - g(c_2)| \leq C_1|c_1 - c_2|, \tag{2.1.7}$$

$$|\mu(c_1) - \mu(c_2)| \leq C_2|c_1 - c_2|. \tag{2.1.8}$$

(A_3) : The functions $\phi,\ \mu,\ \kappa$ and q are bounded, i.e., there exist positive constants $\phi_*,\ \phi^*,\ \mu_*,\ \mu^*,\ \kappa_*,\ \kappa^*,\ q^*,\ D_*,\ D^*$ such that

$$0 < \phi_* \leq \phi(x) \leq \phi^*, \tag{2.1.9}$$

$$0 < \mu_* \leq \mu(x,c) \leq \mu^*, \tag{2.1.10}$$

$$0 < \kappa_* \leq \kappa(x) \leq \kappa^*, \tag{2.1.11}$$

$$|q(x)| \leq q^*, \tag{2.1.12}$$

$$0 < D_* \leq D(x,\mathbf{u}) \leq D^*. \tag{2.1.13}$$

(A_4) : The diffusion-dispersion tensor $D(\mathbf{u})$ satisfies

$$D(\mathbf{u}) \in [W^{2,\infty}(\Omega)]^{2\times 2}. \tag{2.1.14}$$

(A_5) : The problem (2.1.1)-(2.1.6) has a unique smooth solution $\{p, c\}$ as demanded by the error analysis.

The authors in [20, 49, 74] have discussed existence of a unique weak solution of the above system (2.1.1)-(2.1.6) under suitable assumptions on the data. The pressure-velocity equation is elliptic type while the concentration equation is convection dominated diffusion type. Since in the concentration equation only velocity is present, one would like to find a good approximation of the velocity. Therefore, for approximating velocity, it is natural to think of some mixed methods, which provide more accurate solution for the velocity compared to the standard finite element methods.

Earlier, Douglas *et al.* [37, 38], Ewing *et al.* [48] and Darlow *et al.* [31] have discussed the mixed finite element method for approximating the velocity as well as pressure and a standard Galerkin method for the concentration equation. They have also derived optimal error estimates in $L^\infty(L^2)$ norm for the velocity and concentration. Moreover, in [37] authors have proposed a modification of mixed methods when the flow is located at injection and production wells. Yang [84] has considered mixed methods with dynamic finite element spaces, i.e., different number of elements and different basis functions are adopted at different time levels. Compared to the conforming finite element methods (FEM), the finite volume methods are conservative in nature and hence, they preserve the physical conservative properties.

In this chapter, we discuss a mixed FVEM for approximating the pressure-velocity equations (2.1.1)-(2.1.2) and a standard FVEM for the approximation of the concentration equation (2.1.3). Moreover, we present some numerical experiments to support our theoretical results.

This chapter is organized as follows. Section 2.1 is introductory in nature. In Section 2.2, the weak formulation for the incompressible miscible displacement problems in porous media is described. In Section 2.3, we state and prove some auxiliary results to be used in our subsequent analysis. The existence and uniqueness results for the discrete problem is also discussed. *A priori* error estimates of velocity, pressure and concentration for the semidiscrete scheme are presented in Section 2.4. In Section 2.5, we discuss the fully discrete scheme and derive *a priori* error bounds. Finally in Section 2.6, the numerical procedure is discussed and some numerical experiments are conducted to substantiate the theoretical results obtained in this chapter.

2.2 Weak formulation

Let $H(\text{div};\Omega) = \{\mathbf{v} = (v_1, v_2) \in (L^2(\Omega))^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$ be associated with the norm

$$\|\mathbf{v}\|^2_{H(\text{div};\Omega)} = \|\mathbf{v}\|^2_{(L^2(\Omega))^2} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}, \tag{2.2.1}$$

where $\|\mathbf{v}\|^2_{(L^2(\Omega))^2} = \|v_1\|^2 + \|v_2\|^2$. Further, let

$$U = \{\mathbf{v} \in H(\text{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

The pressure-velocity equations (2.1.1)-(2.1.2) with the Neumann boundary condition (2.1.4) has a unique solution for the pressure upto an additive constant. This non-uniqueness may be avoided by considering the following quotient space:

$$W = L^2(\Omega)/\mathbb{R}.$$

Multiply (2.1.1) and (2.1.2) by $\mathbf{v} \in U$ and $w \in W$, respectively, and integrate over Ω. Further, use of Green's formula and $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, yields the following weak formulation: Find $(\mathbf{u}, p) : \overline{J} \longrightarrow U \times W$ satisfying

$$(\kappa^{-1}\mu(c)\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = 0 \quad \forall\mathbf{v} \in U, \tag{2.2.2}$$

$$(\nabla\cdot\mathbf{u}, w) = (q, w) \qquad \forall w \in W. \tag{2.2.3}$$

Similarly, multiplying (2.1.3) by $z \in H^1(\Omega)$ and integrating over Ω, we obtain using (2.1.5) a weak formulation for the concentration equation (2.1.3) as follows:
Find a map $c : \overline{J} \longrightarrow H^1(\Omega)$ such that for $t \in (0, T]$,

$$\begin{aligned} (\phi\frac{\partial c}{\partial t}, z) + (\mathbf{u}\cdot\nabla c, z) + a(\mathbf{u}; c, z) &= (g(c), z) \quad \forall z \in H^1(\Omega), \\ c(x, 0) &= c_0(x) \quad \forall x \in \Omega, \end{aligned} \tag{2.2.4}$$

where $(\cdot,\cdot)$ denotes the standard L^2- inner product and $a(\mathbf{u};\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ is a bilinear form defined by

$$a(\mathbf{u}; \phi, \psi) = \int_\Omega D(\mathbf{u})\nabla\phi\cdot\nabla\psi dx \quad \forall\phi,\ \psi \in H^1(\Omega).$$

In fact, in order that (2.2.4) makes sense, it is necessary that $\mathbf{u} \cdot \nabla c \in L^2(\Omega)$.
Since D is positive definite, the bilinear form $a(\mathbf{u}; \cdot, \cdot)$ satisfies the following condition

$$a(\mathbf{u}; \phi, \phi) \geq \alpha |\phi|_1^2 \quad \forall \phi \in H^1(\Omega), \tag{2.2.5}$$

where $|\cdot|_1$ denotes the usual semi-norm on $H^1(\Omega)$.

2.3 Finite volume element approximation

We use a mixed finite volume element method for the simultaneous approximation of velocity and pressure in (2.1.1)-(2.1.2) and a standard finite volume element method for the approximation of the concentration in (2.1.3). For this purpose, we introduce three kinds of grids: one primal grid and two dual grids.
Let $\mathcal{T}_h = \{T\}$ be a regular, quasi-uniform partition of the domain $\bar{\Omega}$ into closed triangles T, that is, $\overline{\Omega} = \cup_{T \in \mathcal{T}_h} \overline{T}$. Let $h_T = diam(T)$ and $h = \max_{T \in \mathcal{T}_h} h_T$. Let $P_1, P_2 \cdots, P_{N_h}$ and $M_1, M_2 \cdots, M_{N_m}$ denote respectively the vertices and midpoints of the edges of the triangles in the triangulation $\mathcal{T}_h$, where N_h and N_m are the total number of vertices and total number of midpoints of the sides of the triangles of $\mathcal{T}_h$.
Let the trial function spaces U_h and W_h associated with the approximation of velocity and pressure respectively be the lowest order Raviart-Thomas space for triangles defined by

$$U_h = \{\mathbf{v_h} \in U : \mathbf{v_h}|_T = (a + bx, c + by) \ \ \forall T \in \mathcal{T}_h\}, \tag{2.3.1}$$

and

$$W_h = \{w_h \in W : w_h|_T \text{ is a constant } \forall T \in \mathcal{T}_h\}. \tag{2.3.2}$$

Next, we construct the dual partition for the pressure-velocity equation and the related test spaces. The dual grid $\mathcal{T}_h^*$ consists of interior quadrilaterals and boundary triangles which are constructed as follows. For an interior mid-side node, the associated dual element is a quadrilateral. This is the union of two triangles formed by joining the end points of the side of the triangle on which the mid-side node lies with the barycenter of the triangles which share the mid-side node. For a mid-side node of a triangle which lies on the boundary $\partial\Omega$,

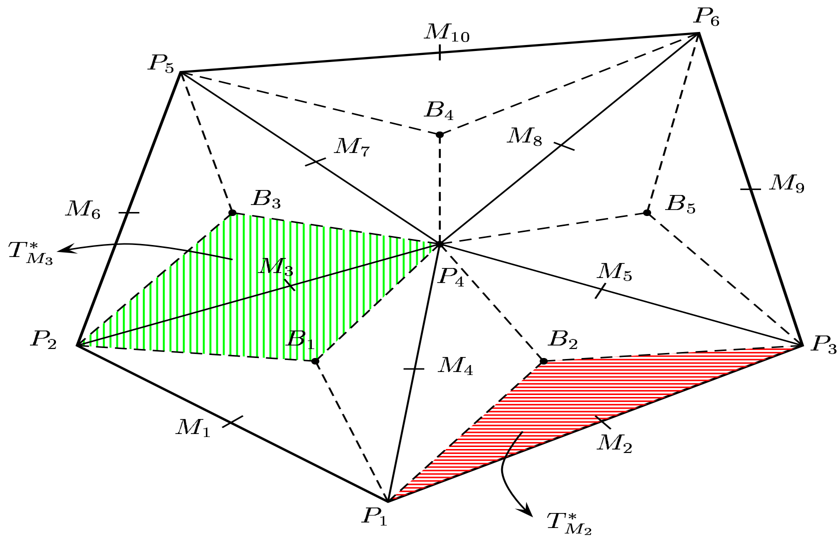

Figure 2.1: Primal grid $\mathcal{T}_h$ and dual grid $\mathcal{T}_h^*$

the dual element is the border triangle obtained by joining the end points of the edges of the triangle in $\mathcal{T}_h$ on which the mid side node lies with the barycenter of the triangle. For example, in Figure 2.1, the interior mid side node M_3 belongs to $\triangle P_1P_4P_2$ and $\triangle P_2P_4P_5$. The dual element associated with M_3 is the quadrilateral $P_2B_1P_4B_3P_2$ (say $T^*_{M_3}$), where B_1 and B_3 are the barycenters of the triangles $\triangle P_1P_4P_2$ and $\triangle P_2P_4P_5$, respectively. Similarly, for the boundary mid-side node M_2, the associated dual element is $\triangle P_3P_1B_2$ (say $T^*_{M_2}$). In general, let T^*_M denote the dual element corresponding to the mid-side node M. The union of all the dual elements/control volume elements form a partition $\mathcal{T}_h^*$ of $\bar{\Omega}$. The test space V_h is defined by

$$V_h = \left\{ \mathbf{v_h} \in (L^2(\Omega))^2 : \mathbf{v_h}|_{T^*_M} \text{ is a constant vector } \forall T^*_M \in \mathcal{T}_h^* \text{ and } \ \mathbf{v_h}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

For connecting our trial and test spaces, we define a transfer operator $\gamma_h : U_h \longrightarrow V_h$ by

$$\gamma_h \mathbf{v_h}(x) = \sum_{i=1}^{N_m} \mathbf{v_h}(M_i)\chi_i^*(x) \qquad \forall x \in \Omega, \tag{2.3.3}$$

where χ_i^*'s are the scalar characteristic functions corresponding to the control volume $T^*_{M_j}$ defined by

$$\chi_i^*(x) = \begin{cases} 1, & \text{if } x \in T^*_{M_i} \\ 0, & \text{elsewhere.} \end{cases}$$

Multiplying (2.1.1) by $\gamma_h \mathbf{v_h} \in V_h$, integrating over the control volumes $T_M^* \in \mathcal{T}_h^*$, applying the Gauss's divergence theorem and summing up over all the control volumes, we obtain

$$(\kappa^{-1}\mu(c)\mathbf{u}, \gamma_h \mathbf{v_h}) - \sum_{i=1}^{N_m} \mathbf{v_h}(M_i) \cdot \int_{T_{M_i}^*} p \; \mathbf{n}_{T_{M_i}^*} ds = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{2.3.4}$$

where $\mathbf{n}_{T_{M_i}^*}$ denotes the outward normal vector to the boundary of $T_{M_i}^*$. Set

$$b(\gamma_h \mathbf{v_h}, w_h) = -\sum_{i=1}^{N_m} \mathbf{v_h}(M_i) \cdot \int_{\partial T_{M_i}^*} w_h \; \mathbf{n}_{T_{M_i}^*} ds \quad \forall \mathbf{v_h} \in U_h, \; \forall w_h \in W_h. \tag{2.3.5}$$

Then, the mixed FVE approximation corresponding to (2.1.1)-(2.1.2) can be written as: find $(\mathbf{u_h}, p_h) : \overline{J} \longrightarrow U_h \times W_h$ such that for $t \in (0, T]$

$$(\kappa^{-1}\mu(c_h)\mathbf{u_h}, \gamma_h \mathbf{v_h}) + b(\gamma_h \mathbf{v_h}, p_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{2.3.6}$$

$$(\nabla \cdot \mathbf{u_h}, w_h) = (q, w_h) \qquad \forall w_h \in W_h, \tag{2.3.7}$$

where c_h is an approximation to c obtained from (2.3.9).

Now, we introduce a dual mesh $\mathcal{V}_h^*$ based on $\mathcal{T}_h$ which will be used for the approximation of the concentration equation. For an interior vertex of $\mathcal{T}_h$, identify the barycenters of the triangles in which this vertex lies and also the midpoints of the edges connecting this vertex with the adjacent vertices. The dual element associated with the vertex is obtained by joining successively these midpoints and the barycenters of the triangles which these mid-side points belong to. For example, in Figure 2.2, for the interior vertex P_4, the associated dual element is $M_4B_2M_5B_5M_8B_4M_7B_3M_3B_1M_4$ (say $V_{P_4}^*$). Similarly, for the vertex on the boundary $\partial\Omega$, say P_1, the associated dual element is $P_1M_2B_2M_4B_1M_1P_1$ (say $V_{P_1}^*$). In general, let V_P^* denote the dual element associated with the vertex P. The union of all these dual elements also form a partition $\mathcal{V}_h^*$ of $\bar{\Omega}$, corresponding to the primal partition $\mathcal{T}_h$. For applying the standard finite volume element method to approximate the concentration, we define the trial space M_h on $\mathcal{T}_h$ and the test space L_h on $\mathcal{V}_h^*$ as follows:

$$M_h = \left\{ z_h \in C^0(\bar{\Omega}) \; : \; z_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h \right\},$$

and

$$L_h = \left\{ w_h \in L^2(\Omega) \; : \; w_h|_{V_P^*} \text{ is a constant } \quad \forall V_P^* \in \mathcal{V}_h^* \right\}.$$

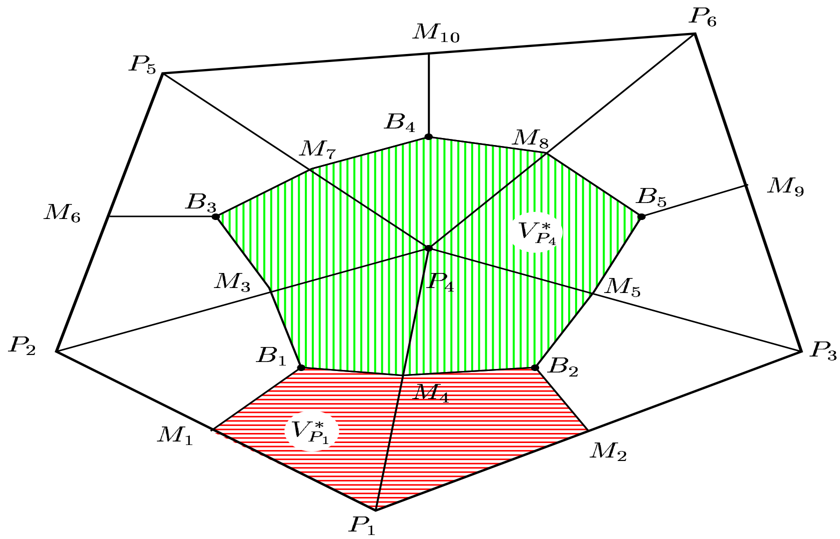

Figure 2.2: Primal grid $\mathcal{T}_h$ and dual grid $\mathcal{V}^*_h$

Again, we define a transfer function $\Pi^*_h : M_h \longrightarrow L_h$ by

$$\Pi^*_h z_h(x) = \sum_{j=1}^{N_h} z_h(P_j)\chi_j(x) \qquad \forall x \in \Omega, \tag{2.3.8}$$

where χ_j's are the characteristic functions corresponding to the control volume $V^*_{P_j}$ given by

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in V^*_{P_j} \\ 0, & \text{elsewhere.} \end{cases}$$

The FVE approximation c_h of c is to seek $c_h : \overline{J} \longrightarrow M_h$ such that for $t \in (0, T]$,

$$\begin{aligned} \big(\phi\frac{\partial c_h}{\partial t}, \Pi^*_h z_h\big) + (\mathbf{u_h}\cdot\nabla c_h, \Pi^*_h z_h) + a_h(\mathbf{u_h}; c_h, z_h) &= (g(c_h), \Pi^*_h z_h) \quad \forall z_h \in M_h \quad (2.3.9)\\ c_h(0) &= c_{0,h}, \end{aligned}$$

where $c_{0,h}$ is an approximation to c_0 to be defined later and the bilinear form $a_h(\mathbf{v};\cdot,\cdot)$ is defined by

$$a_h(\mathbf{v}; \chi, \psi_h) = -\sum_{j=1}^{N_h} \int_{\partial V^*_{P_j}} \Big(D(\mathbf{v})\nabla\chi \cdot \mathbf{n}_{P_j}\Big)\Pi^*_h \psi_h \; ds, \tag{2.3.10}$$

$\mathbf{n}_{P_j}$ being the unit outward normal to the boundary of $V^*_{P_j}$ with $\mathbf{v} \in U, \chi \in H^1(\Omega)$ and $\psi_h \in M_h$.

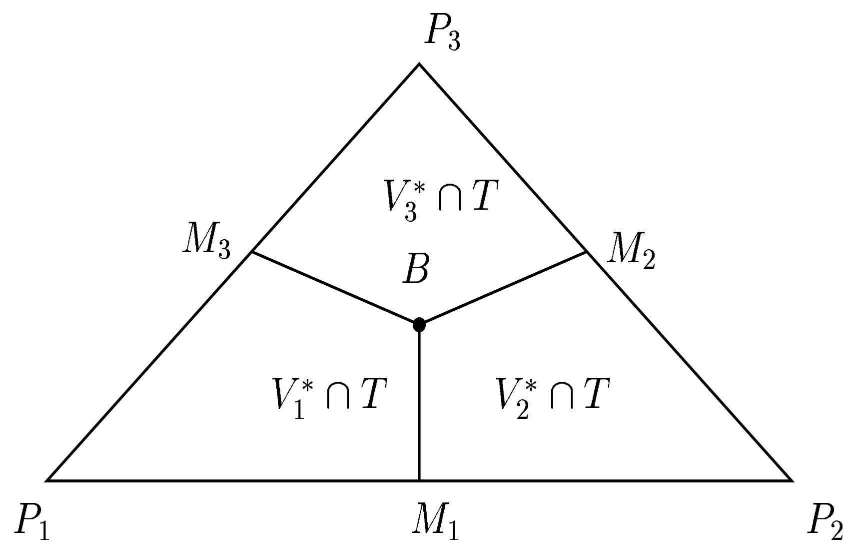

Figure 2.3: A triangular partition

REMARK 2.3.1 *Three grids are introduced one each for the pressure, velocity and concentration variables. This is to balance the number of unknowns and the equations in the coupled system (2.3.6)-(2.3.7) and (2.3.9).*

Next, we discuss the existence and uniqueness of solution for the discrete system (2.3.6)-(2.3.7) and (2.3.9). For this purpose, we now recall some results from [26] and [60].

2.3.1 Some Auxiliary Results

We define the following numerical quadrature formulae on a triangle $T \in \mathcal{T}_h$ which is exact for polynomials of degree one and two, respectively,:

$$\int_T \chi_h dx = \frac{|T|}{3}\Big(\chi_h(P_1) + \chi_h(P_2) + \chi_h(P_3)\Big), \tag{2.3.11}$$

and

$$\int_T \chi_h dx = \frac{|T|}{3}\Big(\chi_h(M_1) + \chi_h(M_2) + \chi_h(M_3)\Big), \tag{2.3.12}$$

where P_1, P_2, P_3 are the vertices of triangle T and $M_1, M_2,\ M_3$ denote the midpoints of the sides $P_1P_2,\ P_2P_3$ and P_1P_3, respectively, (see Figure 2.3). Here, $|T|$ denotes the area of the triangle T.

We also use frequently the following trace inequality [14, pp. 417]: for $w \in H^1(T)$,

$$\|w\|_{\partial T}^2 \leq C\left(h_T^{-1}\|w\|_T^2 + h_T|w|_{1,T}^2\right), \tag{2.3.13}$$

where $\|w\|_{\partial T}^2 = \displaystyle\int_{\partial T} w^2 ds$ and ∂T denoting the boundary of the triangle T. Further, we need the following inverse inequalities (see [29, pp. 141]):

$$\|\chi\|_{1,\infty} \leq Ch^{-1}\|\chi\|_1 \quad \forall \chi \in M_h, \tag{2.3.14}$$

and

$$\|\chi\|_1 \leq Ch^{-1}\|\chi\| \quad \forall \chi \in M_h. \tag{2.3.15}$$

By the usual interpolation theory, the operator Π_h^* has the following approximation property [27, pp. 466]:

$$\|\chi - \Pi_h^*\chi\|_{0,k} \leq Ch^\beta|\chi|_{s,k}, \quad 0 \leq \beta \leq s \leq 1, \ \ 1 \leq k \leq \infty. \tag{2.3.16}$$

For our future use, let us introduce the following notations. For $T \in \mathcal{T}_h$ with vertices P_1, P_2 and P_3, set

$$|\phi_h|_{0,h,T} = \left\{\frac{|T|}{3}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)\right\}^{1/2}, \tag{2.3.17}$$

and

$$|\phi_h|_{1,h,T} = \left\{(|\frac{\partial \phi_h}{\partial x}|^2 + |\frac{\partial \phi_h}{\partial y}|^2)|T|\right\}^{1/2}, \tag{2.3.18}$$

where $|T|$ is the area of triangle T and $\phi_j = \phi_h(P_j), \ 1 \leq j \leq 3$.

Define the discrete norms for $\phi_h \in M_h$ as

$$\|\phi_h\|_{0,h} = \left(\sum_{T\in\mathcal{T}_h} |\phi_h|_{0,h,T}^2\right)^{1/2}, \ |\phi_h|_{1,h} = \left(\sum_{T\in\mathcal{T}_h} |\phi_h|_{1,h,T}^2\right)^{1/2},$$

and

$$\|\phi_h\|_{1,h} = \left(\|\phi_h\|_{0,h}^2 + |\phi_h|_{1,h}^2\right)^{1/2}.$$

We also use the notation $\|\phi_h\|_T$ to denote $\|\phi_h\|_{0,T} = \left(\displaystyle\int_T \phi_h^2 dx\right)^{1/2}$.

The following lemma establishes a relation between the discrete norms and the continuous norms on the Sobolev spaces.

LEMMA 2.3.1 *[60, pp. 124] For $\phi_h \in M_h$, $|\cdot|_{1,h}$ and $|\cdot|_1$ are identical. Further, $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent to $\|\cdot\|$ and $\|\cdot\|_1$, respectively, that is, there exist positive constants $C_3, \cdots, C_6 > 0$, independent of h, such that*

$$C_3\|\phi_h\|_{0,h} \leq \|\phi_h\| \leq C_4\|\phi_h\|_{0,h} \quad \forall \phi_h \in M_h, \tag{2.3.19}$$

and

$$C_5||\phi_h||_{1,h} \leq ||\phi_h||_1 \leq C_6||\phi_h||_{1,h} \quad \forall \phi_h \in M_h. \tag{2.3.20}$$

Proof. Since $\dfrac{\partial \phi_h}{\partial x}$ and $\dfrac{\partial \phi_h}{\partial y}$ are constants on a triangle T, the norms $|\cdot|_1$ and $|\cdot|_{1,h}$ are identical. Now using the quadrature formula (2.3.12), we obtain

$$\begin{aligned}
\|\phi_h\|_T^2 &= \int_T |\phi_h|^2 dx = \frac{|T|}{3}\left(\phi_h(M_1)^2 + \phi_h(M_2)^2 + \phi_h(M_3)^2\right) \\
&= \frac{|T|}{3}\left[\left(\frac{\phi_1+\phi_2}{2}\right)^2 + \left(\frac{\phi_2+\phi_3}{2}\right)^2 + \left(\frac{\phi_1+\phi_3}{2}\right)^2\right] \\
&= \frac{|T|}{12}\left[\phi_1^2 + \phi_2^2 + \phi_3^2 + (\phi_1+\phi_2+\phi_3)^2\right]. \qquad (2.3.21)
\end{aligned}$$

Using Young's inequality (1.2.4) with $\epsilon = 1$, (2.3.21) can be written as

$$\begin{aligned}
\|\phi_h\|_T^2 &= \frac{|T|}{12}\left[2(\phi_1^2 + \phi_2^2 + \phi_3^2) + 2\phi_1\phi_2 + 2\phi_2\phi_3 + 2\phi_1\phi_3\right] \\
&\leq \frac{|T|}{4}\left[\phi_1^2 + \phi_2^2 + \phi_3^2\right]. \qquad (2.3.22)
\end{aligned}$$

A use of (2.3.17) yields

$$\|\phi_h\|^2 \leq \|\phi_h\|_{0,h}^2. \tag{2.3.23}$$

From (2.3.23) and (2.3.21), we find that

$$\frac{1}{4}\|\phi_h\|_{0,h}^2 \leq \|\phi_h\|^2 \leq \|\phi_h\|_{0,h}^2. \tag{2.3.24}$$

Now the estimate (2.3.20) follows from (2.3.24) and the fact that $|\cdot|_1$ and $|\cdot|_{1,h}$ are identical. This completes the proof. ■

LEMMA 2.3.2 *The following results hold true for* $\forall \phi_h \in M_h$,

$$\int_T (\phi_h - \Pi_h^* \phi_h) \; dx = 0 \qquad \forall T \in \mathcal{T}_h, \tag{2.3.25}$$

and

$$\int_{\partial T} (\phi_h - \Pi_h^* \phi_h) \; ds = 0 \tag{2.3.26}$$

Proof. Since ϕ_h is linear on each triangle T, from (2.3.11), we obtain

$$\begin{aligned} \int_T (\phi_h - \Pi_h^* \phi_h) dx &= \int_T \phi_h dx - \sum_{i=1}^3 \int_{V_i^* \cap T} \Pi_h^* \phi_h dx \\ &= \int_T \phi_h dx - \sum_{i=1}^3 \phi_i |V_i^* \cap T|, \end{aligned}$$

where $|V_i^* \cap T|$ denotes the area of the control volume $V_i^* \cap T$.
Since $|V_i^* \cap T| = \frac{|T|}{3}$, $i = 1, 2, 3$, we find that

$$\begin{aligned} \int_T (\phi_h - \Pi_h^* \phi_h) dx &= \frac{|T|}{3} (\phi_1 + \phi_2 + \phi_3) - \sum_{i=1}^3 \phi_i \frac{|T|}{3} \\ &= 0. \end{aligned}$$

This proves (2.3.25). Now (ii) follows directly from the definition of Π_h^* and this completes the rest of the proof. ■

Now introduce the following function

$$\epsilon_h(\psi, \chi_h) = (\psi, \chi_h) - (\psi, \Pi_h^* \chi_h) \quad \forall \chi_h \in M_h. \tag{2.3.27}$$

LEMMA 2.3.3 *[68, pp. 40] Let* $z \in P_1(T)$ *and* z_T *be the average of* z *on* T*, i.e.,* $z_T = \frac{1}{|T|} \int_T z \; dx$. *Then*

$$\|z - z_T\|_{0,T} \le C h_T |\nabla z|_{0,T}. \tag{2.3.28}$$

Proof. Let $v = z - z_T$, then

$$\int_T v \; dx = \int_T (z - z_T) dx = \int_T z \; dx - \int_T z_T \; dx = 0.$$

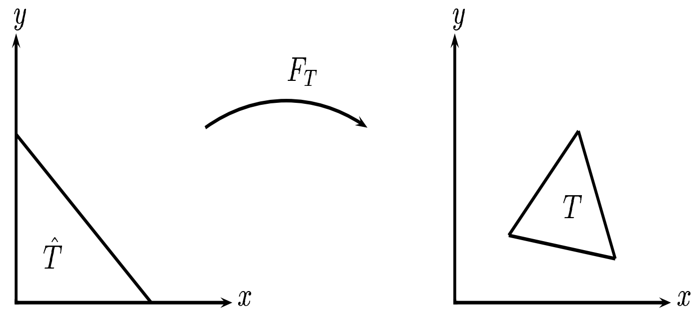

Figure 2.4: Reference element $\hat{T}$ and mapping F_T from $\hat{T}$ to the element T

Now by using a scaling argument and Lemma 1.2.5, we have

$$\|v\|_{0,T} \leq h_T \|\hat{v}\|_{0,\hat{T}} \leq C(\hat{T}) h_T \|\hat{\nabla}\hat{v}\|_{0,\hat{T}} \leq h_T \|\nabla v\|_{0,T}, \tag{2.3.29}$$

where $\hat{T}$ is the reference triangle corresponding to the triangle T, see Figure 2.4. Since $v = z - z_T$ and z_T is constant, it is follows from (2.3.29) that

$$\|z - z_T\|_{0,T} \leq C h_T |\nabla z|_{0,T}. \tag{2.3.30}$$

This completes the proof. ■

Lemma 2.3.4 *For $\chi_h \in M_h$ and $\psi \in H^1(\Omega)$, there exists a positive constant C independent of h such that*

$$|\epsilon_h(\psi, \chi_h)| \leq C h^2 |\psi|_1 |\chi_h|_1.$$

Proof. Using (2.3.25), (2.3.16), (2.3.28), we obtain

$$\begin{aligned} \int_T \psi(\chi_h - \Pi_h^* \chi) dx &= \int_T (\psi - \psi_T)(\chi_h - \Pi_h^* \chi) dx \\ &\leq \|\psi - \psi_T\|_T \|\chi_h - \Pi_h^* \chi_h\|_T \leq C h^2 |\psi|_{1,T} |\chi_h|_{1,T}. \end{aligned} \tag{2.3.31}$$

Sum up over all triangles $T \in \mathcal{T}_h$ to complete the rest of the proof. ■

Remark 2.3.2 *In general, we can say that ϵ_h has the following property (see, [15, pp. 317]): for $\chi \in M_h$ and $\psi \in W^{i,p}(\Omega)$ with $i, j = 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$|\epsilon_h(\psi, \chi)| \leq C h^{i+j} |\psi|_{W^{i,p}} |\chi|_{W^{j,q}}. \tag{2.3.32}$$

LEMMA 2.3.5 *[76, pp. 332] The matrix $D(\mathbf{u})$ defined in (1.3.3) is uniformly Lipschitz continuous, i.e., there exists a constant C such that for $\mathbf{u}$ and $\mathbf{v} \in (L^2(\Omega))^2$,*

$$\|D(\mathbf{u}) - D(\mathbf{v})\|_{(L^2(\Omega))^{2\times 2}} \leq C\|\mathbf{u} - \mathbf{v}\|_{(L^2(\Omega))^2}. \tag{2.3.33}$$

Proof. Using (1.2.1) and (1.3.3), we obtain

$$\begin{aligned} |D(\mathbf{u}) - D(\mathbf{v})|_1 &= \sum_{i=1}^{2} \max_{j=1,2} |D(\mathbf{u})_{i,j} - D(\mathbf{v})_{i,j}| \\ &\leq \sum_{i=1}^{2} \max_{j=1,2} |\phi(x)| \left|(d_l - d_t)\left(\frac{u_i u_j}{|\mathbf{u}|} - \frac{v_i v_j}{|\mathbf{v}|}\right) + d_t \delta_{ij}(|\mathbf{u}| - |\mathbf{v}|)\right|. \end{aligned}$$

Using (2.1.9), we find that

$$|D(\mathbf{u}) - D(\mathbf{v})|_1 \leq \phi^* \left(\sum_{i=1}^{2} |d_l - d_t| \max_{j=1,2}\left|\frac{u_i u_j}{|\mathbf{u}|} - \frac{v_i v_j}{|\mathbf{v}|}\right| + 2d_t \left||\mathbf{u}| - |\mathbf{v}|\right|\right).$$

Note that

$$\begin{aligned} \frac{u_i u_j}{|\mathbf{u}|} - \frac{v_i v_j}{|\mathbf{v}|} &= \frac{u_i u_j}{|\mathbf{u}|} - \frac{u_i v_j}{|\mathbf{u}|} + \frac{u_i v_j}{|\mathbf{u}|} - \frac{u_i v_j}{|\mathbf{v}|} + \frac{u_i v_j}{|\mathbf{v}|} - \frac{v_i v_j}{|\mathbf{v}|} \\ &= \frac{u_i(u_j - v_j)}{|\mathbf{u}|} + \frac{u_i v_j(|\mathbf{v}| - |\mathbf{u}|)}{|\mathbf{u}||\mathbf{v}|} + \frac{v_j(u_i - v_i)}{|\mathbf{v}|} \\ &\leq 2|\mathbf{u} - \mathbf{v}| + (|\mathbf{v}| - |\mathbf{u}|) \\ &\leq 3|\mathbf{u} - \mathbf{v}|. \end{aligned}$$

Hence,

$$|D(\mathbf{u}) - D(\mathbf{v})|_1 \leq 2K_1(d_t + 3|d_l - d_t|)|\mathbf{u} - \mathbf{v}|. \tag{2.3.34}$$

Using (1.2.3) and (2.3.34), we obtain

$$|D(\mathbf{u}) - D(\mathbf{v})|_2 \leq 2^{1/2}|D(\mathbf{u}) - D(\mathbf{v})|_1 \leq 2^{3/2}K_1(d_l + 3|d_l - d_t|)|\mathbf{u} - \mathbf{v}|. \tag{2.3.35}$$

Now integrate over Ω to complete the rest of the proof. ■

The following lemma yields a relation between the bilinear forms $a(\mathbf{u};\cdot,\cdot)$ and $a_h(\mathbf{u};\cdot,\cdot)$, the proof of which is based on the ideas of a similar result in [46, pp. 1871].

LEMMA 2.3.6 *Assume that* $\chi_h, \psi_h \in M_h$. *Then*

$$\begin{aligned} a_h(\mathbf{u};\chi_h,\psi_h) &= a(\mathbf{u};\chi_h,\psi_h) + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla\chi_h\cdot\mathbf{n})(\Pi_h^*\psi_h-\psi_h)ds \\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(D(\mathbf{u})\nabla\chi_h)(\psi_h-\Pi_h^*\psi_h)dx. \end{aligned} \tag{2.3.36}$$

Moreover, the following inequality holds:

$$a_h(\mathbf{u};\chi_h,\psi_h) \;\geq\; a(\mathbf{u};\chi_h,\psi_h) - Ch|\psi_h|_1|\phi_h|_1. \tag{2.3.37}$$

Proof. A use of Gauss's divergence theorem on each of $V_j^*\cap T, (j=1,2,3)$, (see Figure 2.3) yields

$$a_h(\mathbf{u};\chi_h,\psi_h) = -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\Pi_h^*\psi_h\int_{\partial V_j^*\cap T}(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds, \tag{2.3.38}$$

with $\mathbf{n}$ denoting the unit outward normal to $\partial V_j^*\cap T$. Now (2.3.38) can be rewritten as:

$$\begin{aligned} a_h(\mathbf{u};\chi_h,\psi_h) &= \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\Pi_h^*\psi_h\int_{P_jP_{j+1}}(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds \\ &\quad - \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*\cap T}\Pi_h^*\psi_h\nabla\cdot(D(\mathbf{u})\nabla\chi_h)\;dx \\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\psi_h(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds \\ &\quad - \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*\cap T}\Pi_h^*\psi_h\nabla\cdot(D(\mathbf{u})\nabla\chi_h)\;dx. \end{aligned}$$

Applying Green's formula on triangle T for the second term, we obtain

$$\begin{aligned} a_h(\mathbf{u};\chi_h,\psi_h) &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds + \sum_{T\in\mathcal{T}_h}\int_T D(\mathbf{u})\nabla\chi_h\cdot\nabla\psi_h\;dx \\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T\nabla\cdot(D(\mathbf{u})\nabla\chi_h)\psi_h\;dx - \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*\cap T}\Pi_h^*\psi_h\nabla\cdot(D(\mathbf{u})\nabla\chi_h)\;dx \\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\;ds + \sum_{T\in\mathcal{T}_h}\int_T D(\mathbf{u})\nabla\chi_h\cdot\nabla\psi_h\;dx \\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T\nabla\cdot(D(\mathbf{u})\nabla\chi_h)(\psi_h-\Pi_h^*\psi_h)\;dx. \end{aligned}$$

This proves (2.3.36). To prove (2.3.37), we first use (2.3.26) to obtain

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}ds=\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D-\overline{D}_T)\cdot\nabla\chi_h.\mathbf{n}\ ds,$$

where $\overline{D}_T=D(x_c),\quad x_c\in\partial T$. Since $|D-\overline{D}_T|_\infty\le Ch\|D\|_{1,\infty}$ (see [46, pp. 1873]), we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\ ds\le Ch\|D\|_{1,\infty}\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)\nabla\chi_h.\mathbf{n}\ ds. \tag{2.3.39}$$

Using the Cauchy-Schwarz inequality, the trace inequality (2.3.13) and (2.3.16) in (2.3.39), we arrive at

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\Pi_h^*\psi_h-\psi_h)(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\ ds\le Ch\|D\|_{1,\infty}\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\Pi_h^*\psi_h-\psi_h|^2ds\right)^{1/2}\\
&\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\nabla\chi_h\cdot\mathbf{n}|^2ds\right)^{1/2}\\
\le\ & Ch\left(\sum_{T\in\mathcal{T}_h}h^{-1}\|\Pi_h^*\psi_h-\psi_h\|_T^2+h|\Pi_h^*\psi_h-\psi_h|_{1,T}^2)\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h^{-1}|\nabla\chi_h|_{1,T}^2+h|\chi_h|_{2,T}^2\right)^{1/2}\\
\le\ & Ch\left(\sum_{T\in\mathcal{T}_h}|\psi_h|_{1,T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}|\chi_h|_{1,T}^2\right)^{1/2}\\
\le\ & Ch|\psi_h|_1|\chi_h|_1.
\end{aligned}\tag{2.3.40}$$

In the last inequality, we have used the fact that χ_h is linear on triangle T, i.e., $|\chi_h|_{2,T}=0$. Again use $|\chi_h|_{2,T}=0$, the Cauchy-Schwarz inequality and (2.3.16) to obtain

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}\int_T\nabla\cdot(D(\mathbf{u})\nabla\chi_h)(\psi_h-\Pi_h^*\psi_h)dx\ &\le\ \|D\|_{1,\infty}\left(\sum_{T\in\mathcal{T}_h}\int_T|\nabla\chi_h|^2dx\right)^{1/2}\\
&\left(\sum_{T\in\mathcal{T}_h}\int_T|\psi_h-\Pi_h^*\psi_h|^2dx\right)^{1/2}\\
&\le\ Ch|\chi_h|_1|\psi_h|_1.
\end{aligned}\tag{2.3.41}$$

Now (2.3.37) follows from (2.3.40) and (2.3.41). This completes the proof. ■

Remark 2.3.3 : *Note that (2.3.36) also holds true for* $\chi \in H^1(\Omega)$.

Lemma 2.3.7 *Under the assumption that the matrix* D *is positive definite, there exists a positive constant* α_0 *independent of* h *such that*

$$a_h(\mathbf{u_h}; \chi_h, \chi_h) \geq \alpha_0 |\chi_h|_1^2 \qquad \forall \chi_h \in M_h. \tag{2.3.42}$$

Proof. Since the matrix D is uniformly positive, we find that

$$a(\mathbf{u_h}; \chi_\mathbf{h}, \chi_\mathbf{h}) \geq \alpha |\chi_h|_1^2. \tag{2.3.43}$$

Use (2.3.37) and (2.3.43) to obtain

$$a_h(\mathbf{u_h}; \chi_h, \chi_h) \geq \alpha |\chi_h|_1^2 - Ch|\chi_h|_1^2. \tag{2.3.44}$$

Choose h small so that $\alpha - Ch = \alpha_0 > 0$ and this completes the rest of the proof. ■

Lemma 2.3.8 *[60, pp. 240] The operator* Π_h^* *has the following properties.*

(i) For $\Pi_h^* : M_h \longrightarrow L_h$ *defined in (2.3.8)*

$$(\phi_h, \Pi_h^* \psi_h) = (\psi_h, \Pi_h^* \phi_h) \qquad \forall \phi_h,\ \psi_h \in M_h. \tag{2.3.45}$$

(ii) With $|||\phi_h||| = (\phi_h, \Pi_h^* \phi_h)^{1/2}$*, the norms* $|||\cdot|||$ *and* $\|\cdot\|$ *are equivalent on* U_h*, that is, there exist positive constants* C_7 *and* C_8*, independent of* h*, such that*

$$C_7||\phi_h|| \leq |||\phi_h||| \leq C_8||\phi_h|| \qquad \forall \phi_h \in M_h. \tag{2.3.46}$$

(iii) Π_h^* *is stable with respect to the* L^2 *norm, i.e., there exists a positive constant* C *independent of* h *such that*

$$\|\Pi_h^* \chi_h\| \leq C\|\chi_h\| \qquad \forall \chi_h \in M_h. \tag{2.3.47}$$

Proof. To prove (i), we note that

$$(\phi_h, \Pi_h^* \psi_h) = \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{V_j^* \cap T} \phi_h\ \Pi_h^* \psi_h\ dx.$$

Using the definition of Π_h^* and the quadrature formula (2.3.11), we obtain with $\phi_j = \phi_h(P_j)$

$$\begin{aligned}\sum_{j=1}^{3}\int_{V_j\cap T}\phi_h\Pi_h^*\psi_h dx &= \psi_1\int_{V_1^*\cap T}\phi_h dx + \psi_2\int_{V_2^*\cap T}\phi_h dx + \psi_3\int_{V_3^*\cap T}\phi_h dx \\ &= \psi_1\left(2\phi_1 + \phi_h(M_1) + \phi_h(M_3) + 2\phi_h(B)\right)\frac{|V_1^*\cap T|}{6} \\ &\quad +\psi_2\left(2\phi_2 + \phi_h(M_1) + \phi_h(M_2) + 2\phi_h(B)\right)\frac{|V_2^*\cap T|}{6} \\ &\quad +\psi_3\left(2\phi_3 + \phi_h(M_2) + \phi_h(M_3) + 2\phi_h(B)\right)\frac{|V_3^*\cap T|}{6}.\end{aligned}$$

Using $\phi_h(B) = \dfrac{\phi_1+\phi_2+\phi_3}{3}$ and $\phi_h(M_i) = \dfrac{\phi_{i+1}+\phi_i}{2}$, $i = 1,2,3$, $\phi_4 = \phi_1$, we find that

$$\begin{aligned}\sum_{j=1}^{3}\int_{V_j\cap T}\phi_h\Pi_h^*\psi_h dx &= \frac{|T|}{108}\Big[\psi_1(22\phi_1 + 7\phi_2 + 7\phi_3) + \psi_2\left(22\phi_2 + 7\phi_1 + 7\phi_3\right) \\ &\quad +\psi_3\left(22\phi_3 + 7\phi_1 + 7\phi_2\right)\Big] \\ &= \frac{|T|}{54}[\psi_1,\psi_2,\psi_3]\begin{pmatrix}22 & 7 & 7\\ 7 & 22 & 7\\ 7 & 7 & 22\end{pmatrix}[\phi_1,\phi_2,\phi_3]^T,\end{aligned}$$

where we have used the fact that $|V_j^*\cap T| = \dfrac{|T|}{3}, j = 1,2,3$, see Figure 2.3. This proves that the inner product $(\cdot,\gamma\cdot)$ is symmetric.

For (ii) that is for the equivalence of the norms, we now rewrite

$$\begin{aligned}\sum_{j=1}^{3}\gamma\phi_h|_{V_j^*}\int_{V_j^*\cap T}\phi_h dx &= \frac{|T|}{108}\left[22(\phi_1^2+\phi_2^2+\phi_3^2) + 14(\phi_1\phi_2 + \phi_2\phi_3 + \phi_1\phi_3)\right] \\ &= \frac{|T|}{54}\left[15(\phi_1^2+\phi_2^2+\phi_3^2) + 7(\phi_1+\phi_2+\phi_3)^2\right]. \qquad (2.3.48)\end{aligned}$$

The equivalence of the norms follow from (2.3.48) and (2.3.21). This completes the proof of (ii).

In order to prove (iii), we note that

$$\|\Pi_h^*\chi_h\|^2 = \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*\cap T}|\Pi_h^*\chi_h|^2 dx.$$

Now using the definition of Π_h^*, we obtain

$$\sum_{j=1}^{3}\int_{V_j^*\cap T}|\Pi_h^*\chi_h|^2dx = \sum_{j=1}^{3}\chi_j^2\ |V_j^*\cap T| = \sum_{j=1}^{3}\chi_j^2\ \frac{|T|}{3}. \tag{2.3.49}$$

Now (iii) follows from (2.3.21) and (2.3.49). This completes the rest of the proof. ■

LEMMA 2.3.9 *[26, pp. 1854] The operator γ_h defined in (2.3.3) has the following properties:*

$$(a)\quad \|\gamma_h\mathbf{v_h}\|_{(L^2(\Omega))^2} \le \|\mathbf{v_h}\|_{(L^2(\Omega))^2} \quad \forall \mathbf{v_h}\in U_h, \tag{2.3.50}$$

$$(b)\quad \|\mathbf{v_h}-\gamma_h\mathbf{v_h}\|_{(L^2(\Omega))^2} \le Ch\|\mathbf{v_h}\|_{H(\mathrm{div};\Omega)}. \tag{2.3.51}$$

$$(c)\quad b(\gamma_h\mathbf{v_h},w_h) = -(\nabla\cdot\mathbf{v_h},w_h) \qquad \forall \mathbf{v_h}\in U_h,\ \forall w_h\in W_h. \tag{2.3.52}$$

(d) There exists a positive constant C which depends on the bounds of κ^{-1} and μ and is independent of h such that

$$\Big(\kappa^{-1}\mu(c_h)\mathbf{v_h},\gamma_h\mathbf{v_h}\Big) \ge C\|\mathbf{v_h}\|^2_{H(\mathrm{div};\Omega)} \quad \forall \mathbf{v_h}\in U_h, \tag{2.3.53}$$

with $\nabla\cdot\mathbf{v_h}$=0.

Proof. Since $\mathbf{v_h}$ is linear on triangle T, (a) can be proved in the same way as we have proved (iii) of Lemma 2.3.8. To prove (b), we proceed as follows. Note that

$$\|\mathbf{v_h}-\gamma_h\mathbf{v_h}\|^2_{(L^2(\Omega))^2} = \sum_{T\in\mathcal{T}_h}\int_T|\mathbf{v_h}-\gamma_h\mathbf{v_h}|^2dx. \tag{2.3.54}$$

Using the definition of γ_h and referring to Figure 2.5, for $\mathbf{v_h}=(v_h^1,v_h^2)$, we obtain

$$\begin{aligned}\int_T|\mathbf{v_h}-\gamma_h\mathbf{v_h}|^2dx &= \sum_{j=1}^{3}\int_{\triangle P_{j+1}BP_j}|\mathbf{v_h}(x)-\gamma_h\mathbf{v_h}(x)|^2dx\\ &= \sum_{j=1}^{3}\int_{\triangle P_{j+1}BP_j}|\mathbf{v_h}(x)-\mathbf{v_h}(M_j)|^2dx\\ &= \sum_{j=1}^{3}\int_{\triangle P_{j+1}BP_j}\sum_{k=1}^{2}(|v_h^k(x)-v_h^k(M_j)|^2dx.\end{aligned}$$

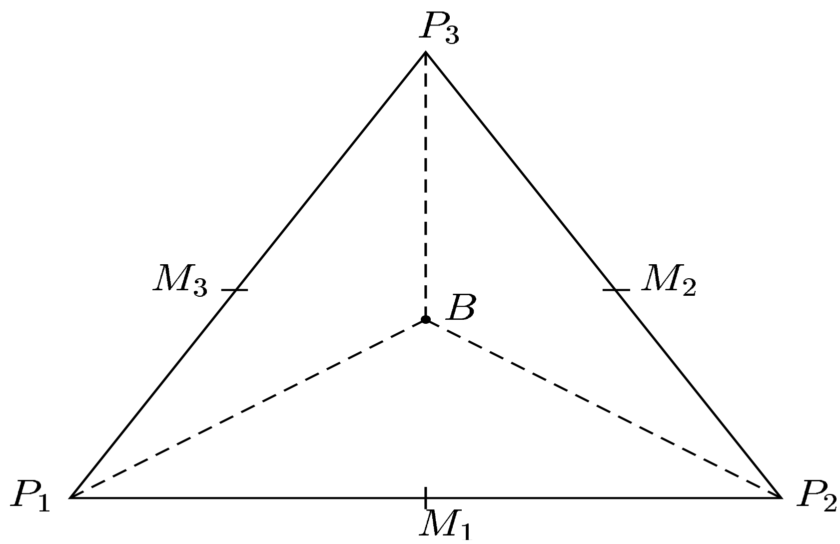

Figure 2.5: Triangle T

Using Taylor series expansion, we find that

$$\begin{aligned}\int_T |\mathbf{v_h} - \gamma_h \mathbf{v_h}|^2 dx &= \sum_{j=1}^{3} \int_{\triangle P_{j+1}BP_j} \left(|x - M_j|^2 \sum_{k=1}^{2} |\nabla v_h^k|^2 \right) dx \\ &\leq h^2 \sum_{j=1}^{3} \int_{\triangle P_{j+1}BP_j} |\nabla \cdot \mathbf{v_h}|^2 dx = h^2 \int_T |\nabla \cdot \mathbf{v_h}|^2 dx. \quad (2.3.55)\end{aligned}$$

Take the summation over all triangles $T \in \mathcal{T}_h$ to complete the rest of the proof for (b).

The bilinear form $b(\gamma_h \cdot, \cdot)$ can be written as

$$\begin{aligned}b(\gamma_h \mathbf{v_h}, w_h) &= -\sum_{i=1}^{N_m} \mathbf{v_h}(M_i) \cdot \int_{T^*_{M_i}} w_h \mathbf{n}_{T^*_{M_i}} \\ &= -\sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{P_{j+1}BP_j} \mathbf{v_h}(M_j) w_h \cdot \mathbf{n}. \quad (2.3.56)\end{aligned}$$

A use of Gauss's divergence theorem on $\triangle P_{j+1}BP_j$, yields

$$\begin{aligned}\sum_{j=1}^{3} \int_{P_{j+1}BP_j} \mathbf{v_h}(M_j) w_h \cdot \mathbf{n} ds &= -\sum_{j=1}^{3} \int_{P_{j+1}P_j} \mathbf{v_h}(M_j) w_h \cdot \mathbf{n} \; ds \\ &+ \sum_{j=1}^{3} \int_{\triangle P_{j+1}BP_j} \nabla \cdot (\mathbf{v_h}(M_j) w_h) \; dx.\end{aligned}$$

Using the fact that $\mathbf{v_h}$ is linear on each triangle and M_j is the mid point of P_jP_{j+1}, we find that

$$\begin{aligned}
\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{v_h}(M_j)w_h\cdot\mathbf{n} &= -\sum_{j=1}^{3}\int_{P_{j+1}P_j}\mathbf{v_h}(M_j)w_h\cdot\mathbf{n}\;ds = -\sum_{j=1}^{3}w_h(\mathbf{v_h}(M_j)\cdot\mathbf{n})\;|P_jP_{j+1}| \\
&= -\sum_{j=1}^{3}w_h\frac{\mathbf{v_h}(P_j)+\mathbf{v_h}(P_{j+1})}{2}\cdot\mathbf{n}\;|P_jP_{j+1}| \\
&= -\sum_{j=1}^{3}\int_{P_jP_{j+1}}w_h\mathbf{v_h}\cdot\mathbf{n}\;ds \\
&= -\sum_{\partial T}w_h\mathbf{v_h}\cdot\mathbf{n}\;ds = -w_h\int_T\nabla\cdot\mathbf{v_h}\;dx. \qquad (2.3.57)
\end{aligned}$$

This completes the proof for (c).

Since $\nabla\cdot\mathbf{v_h}=0$, to prove (2.3.53), it is enough to show that

$$(\mathbf{v_h},\gamma_h\mathbf{v_h})\geq C\|\mathbf{v_h}\|_{(L^2(\Omega))^2}. \qquad (2.3.58)$$

This can be proved using the same arguments as in the proof of (i) and (ii) in Lemma 2.3.8. This completes the proof. ∎

LEMMA 2.3.10 *[19, pp. 130] There exists a positive constant β independent of h such that the following inf-sup condition holds true:*

$$\sup_{\mathbf{0}\neq\mathbf{v_h}\in\mathbf{U_h}}\frac{(\nabla\cdot\mathbf{v_h},w_h)}{\|\mathbf{v_h}\|_{H(\mathrm{div};\Omega)}}\geq\beta\|w_h\| \quad \forall w_h\in W_h. \qquad (2.3.59)$$

2.3.2 Existence and Uniqueness of Discrete Solution

Using (2.3.52), the problem (2.3.6)-(2.3.7) yields a system of linear algebraic equations for a given c_h. To show the existence of a solution, it is enough to prove the uniqueness of the solution of the corresponding homogeneous system

$$(\kappa^{-1}\mu(c_h)\mathbf{u_h},\gamma_h\mathbf{v_h})-(\nabla\cdot\mathbf{v_h},p_h)=0 \quad \forall\mathbf{v_h}\in U_h, \qquad (2.3.60)$$

$$(\nabla\cdot\mathbf{u_h},w_h)=0 \qquad \forall w_h\in W_h. \qquad (2.3.61)$$

For uniqueness, it is sufficient to show that $\mathbf{u_h}=0$, $p_h=0$. Substitute $w_h=\nabla\cdot\mathbf{u}_h$ in (2.3.61) to obtain $\nabla\cdot\mathbf{u_h}=0$. Put $w_h=p_h$ in (2.3.61) and $\mathbf{v_h}=\mathbf{u_h}$ in (2.3.60) and use

(2.3.53) to obtain

$$\|\mathbf{u_h}\|_{H(\mathrm{div};\Omega)} = 0.$$

This implies that $\mathbf{u_h} = 0$. Use $\mathbf{u_h} = 0$ in (2.3.60) and the inf-sup condition (2.3.59), to find that $p_h = 0$. Hence, for a given c_h, there exists a unique solution $\Big(\mathbf{u_h}(c_h), \mathbf{p_h}(c_h)\Big)$ satisfying (2.3.6)-(2.3.7). On substituting in (2.3.9), we obtain a system of nonlinear ordinary differential equations in c_h. An appeal to Picard's Theorem yields the existence of a unique solution in $(0, t_h)$ for some $0 < t_h \leq T$. To continue the solution for all $t \in J$, we need an *a priori* bound for c_h. Now the quasi-uniformity of the mesh implies that

$$\|\mathbf{u_h}\|_{(L^\infty(\Omega))^2} \leq Ch^{-1}\|\mathbf{u_h}\|_{(L^2(\Omega))^2}. \tag{2.3.62}$$

For *a priori* bound, choose $z_h = c_h$ in (2.3.9) and use (2.3.62) to bound $\mathbf{u_h}$. Then a use of Lemma 2.3.8 yields

$$\frac{1}{2}\frac{d}{dt}(\phi c_h, \Pi_h^* c_h) + a_h(\mathbf{u_h}; c_h, c_h) \leq S(h)\|c_h\|^2, \tag{2.3.63}$$

where $S(h) = O(h^{-1})$. For a given $\mathbf{u_h}$, the positive-definiteness of $a_h(\mathbf{u_h}; c_h, c_h)$ given in (2.3.42) with (2.1.9) yields *a priori* estimates in L^2 and H^1- norms for c_h. Now the *a priori* bound $\|c_h\|_{L^\infty(L^2)}$ can be used to show the existence of an uniqueness solution c_h of the concentration equation for all $t \in J$ and for a fixed h. This completes the part of unique solvability of (2.3.6)-(2.3.7) and (2.3.9).

2.4 Error estimates

In this section, we discuss the error estimates for the semidiscrete method. First of all, we derive the estimates for the velocity and pressure in terms of the concentration using the Raviart- Thomas projection and L^2 projection. Then for finding the estimates for the concentration, we split $c - c_h = (c - R_h c) + (R_h c - c_h)$, where R_h is the Ritz projection to be defined in (2.4.27) . In Lemma 2.4.1 and Lemma 2.4.3, we derive, respectively, H^1 and L^2- error estimates for R_h. Based on these estimates, we finally obtain *a priori* error estimates for the concentration in $L^\infty(L^2)$- norm.

2.4.1 Estimates for the velocity

Let Π_h be the usual Raviart-Thomas projection $\Pi_h : U \longrightarrow U_h$ defined by

$$(\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}), w_h) = 0 \quad \forall w_h \in W_h, \tag{2.4.1}$$

which has the following approximation properties (see [19, pp. 163]):

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{(L^2(\Omega))^2} \leq Ch\|\mathbf{u}\|_{(H^1(\Omega))^2}, \tag{2.4.2}$$

$$\|\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})\| \leq Ch\|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)}, \tag{2.4.3}$$

and (see [58, pp. 48])

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{(L^\infty(\Omega))^2} \leq Ch\|\mathbf{u}\|_{(W^{1,\infty}(\Omega))^2}. \tag{2.4.4}$$

Let P_h be the L^2- projection of W onto W_h defined by

$$(p - P_h p, w_h) = 0 \qquad \forall w_h \in W_h. \tag{2.4.5}$$

The operator P_h satisfies the following approximation property (see [19, pp. 163]):

$$\|p - P_h p\| \leq Ch\|p\|_{H^1(\Omega)}. \tag{2.4.6}$$

Further, the following inverse property holds:

$$\|\mathbf{v_h}\|_{(L^\infty(\Omega))^2} \leq Ch^{-1}\|\mathbf{v_h}\|_{(L^2(\Omega))^2} \qquad \forall v_h \in U_h. \tag{2.4.7}$$

Now, we introduce the following auxiliary functions $(\tilde{\mathbf{u}}_h, \tilde{p}_h) : [0, T] \longrightarrow U_h \times W_h$ satisfying

$$(\kappa^{-1}\mu(c)\tilde{\mathbf{u}}_h, \mathbf{v_h}) - (\nabla \cdot \mathbf{v_h}, \tilde{p}_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{2.4.8}$$

$$(\nabla \cdot \tilde{\mathbf{u}}_h, w_h) = (q, w_h) \qquad \forall w_h \in W_h. \tag{2.4.9}$$

For a proof of the the existence and uniqueness of the solution of (2.4.8)-(2.4.9), we refer to [9, pp. 52]. The following error estimates for $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ can be obtained by using the properties of Raviart-Thomas projection Π_h and L^2 projection P_h defined in (2.4.1) and (2.4.5), respectively:

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2} + \|p - \tilde{p_h}\| \leq Ch\left(\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1\right), \tag{2.4.10}$$

and

$$\|\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h)\| \leq Ch\|\nabla \cdot \mathbf{u}\|_1, \tag{2.4.11}$$

where the constant C is independent of h, but may depend on the bounds of μ and κ^{-1} given in (2.1.10) and (2.1.11). For a proof, see [19, pp. 166].

The bound for $\tilde{\mathbf{u}}_h$ in L^∞- norm can be found by using (2.4.2), (2.4.4), (2.4.7) and (2.4.10) as follows:

$$\begin{aligned}
\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2} &\leq \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2} + \|\mathbf{u}\|_{(L^\infty(\Omega))^2} \\
&\leq \|\mathbf{u} - \Pi_h\mathbf{u}\|_{(L^\infty(\Omega))^2} + \|\Pi_h\mathbf{u} - \tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2} + \|\mathbf{u}\|_{(L^\infty(\Omega))^2} \\
&\leq C\left(h\|\mathbf{u}\|_{(W^{1,\infty}(\Omega))^2} + h^{-1}\|\Pi_h\mathbf{u} - \tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2} + \|\mathbf{u}\|_{(L^\infty(\Omega))^2}\right) \\
&\leq C\Big(h\|\mathbf{u}\|_{(W^{1,\infty}(\Omega))^2} + h^{-1}\|\Pi_h\mathbf{u} - \mathbf{u}\|_{(L^2(\Omega))^2} \\
&\quad + h^{-1}\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2} + \|\mathbf{u}\|_{(L^\infty(\Omega))^2}\Big) \\
&\leq C\Big(h\|\mathbf{u}\|_{(W^{1,\infty}(\Omega))^2} + \|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1 + \|\mathbf{u}\|_{(L^\infty(\Omega))^2}\Big).
\end{aligned} \tag{2.4.12}$$

Theorem 2.4.1 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$, respectively, be the solutions of (2.2.2)-(2.2.3) and (2.3.6)-(2.3.7). Then, there exists a positive constant C, independent of h, but dependent on the bounds of κ^{-1} and μ such that*

$$\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + \|p - p_h\| \leq C\left[\|c - c_h\| + h(\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1)\right], \tag{2.4.13}$$

and

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u_h})\| \leq Ch\|\nabla \cdot \mathbf{u}\|_1, \tag{2.4.14}$$

provided $\mathbf{u}(t) \in (H^1(\Omega))^2,\ \nabla \cdot \mathbf{u}(t) \in H^1(\Omega)$ *and* $p(t) \in H^1(\Omega)$.

Proof. Write $\mathbf{u} - \mathbf{u_h} = (\mathbf{u} - \tilde{\mathbf{u}}_h) + (\tilde{\mathbf{u}}_h - \mathbf{u_h})$ and $p - p_h = (p - \tilde{p}_h) + (\tilde{p}_h - p_h)$. Since the estimates of $\mathbf{u} - \tilde{\mathbf{u}}_h$ and $p - \tilde{p}_h$ are known from (2.4.10), it is sufficient to estimate $\tilde{\mathbf{u}}_h - \mathbf{u_h}$ and $\tilde{p}_h - p_h$. Let $\tilde{\mathbf{e_1}}_h = \tilde{\mathbf{u}}_h - \mathbf{u_h}$ and $\tilde{e_2}_h = \tilde{p}_h - p_h$. Using (2.3.52) in (2.3.6)-(2.3.7), we obtain

$$(\kappa^{-1}\mu(c_h)\mathbf{u_h}, \gamma_h\mathbf{v_h}) - (\nabla \cdot \mathbf{v_h}, p_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{2.4.15}$$

$$(\nabla \cdot \mathbf{u_h}, w_h) = (q, w_h) \qquad \forall w_h \in W_h. \tag{2.4.16}$$

Subtracting (2.4.15) from (2.4.8) and (2.4.16) from (2.4.9), we find that

$$\begin{aligned}(\kappa^{-1}\mu(c_h)\tilde{\mathbf{e}}_{\mathbf{1}h},\gamma_h\mathbf{v_h}) \quad - \quad & (\nabla\cdot\mathbf{v_h},\tilde{e}_{2h}) = -(\kappa^{-1}\mu(c)\tilde{\mathbf{u}}_h,\mathbf{v_h}-\gamma_h\mathbf{v_h}) \\ & -(\kappa^{-1}(\mu(c)-\mu(c_h))\tilde{\mathbf{u}}_h,\gamma_h\mathbf{v_h}) \qquad \forall \mathbf{v_h}\in U_h, \qquad (2.4.17)\\ (\nabla\cdot\tilde{\mathbf{e}}_{\mathbf{1}h},w_h) \quad = \quad & 0 \qquad\qquad \forall w_h\in W_h. \qquad (2.4.18)\end{aligned}$$

Since $\nabla\cdot U_h\subset W_h$, take $w_h=\nabla\cdot\tilde{\mathbf{e}}_{\mathbf{1}h}$ in (2.4.18) to arrive at

$$\|\nabla\cdot\tilde{\mathbf{e}}_{\mathbf{1}h}\|=0, \tag{2.4.19}$$

and hence, using (2.2.1), we obtain

$$\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{H(\mathrm{div};\Omega)}=\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}. \tag{2.4.20}$$

Choosing $\mathbf{v_h}=\tilde{\mathbf{e}}_{\mathbf{1}h}$ in (2.4.17) and $w_h=\tilde{e}_{2h}$ in (2.4.18), we arrive at

$$(\kappa^{-1}\mu(c_h)\tilde{\mathbf{e}}_{\mathbf{1}h},\gamma_h\tilde{\mathbf{e}}_{\mathbf{1}h})=-(\kappa^{-1}\mu(c)\tilde{\mathbf{u}}_h,\tilde{\mathbf{e}}_{\mathbf{1}h}-\gamma_h\tilde{\mathbf{e}}_{\mathbf{1}h})-(\kappa^{-1}(\mu(c)-\mu(c_h))\tilde{\mathbf{u}}_h,\gamma_h\tilde{\mathbf{e}}_{\mathbf{1}h}).$$

Using (2.3.51), (2.3.53) with (2.1.10)-(2.1.11) and (2.1.8), we obtain

$$\begin{aligned}\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|^2_{H(\mathrm{div};\Omega)} \quad \leq \quad & C\Big(\|\tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2}\|\tilde{\mathbf{e}}_{\mathbf{1}h}-\gamma_h\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}+\|c-c_h\|\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}\Big)\\ \leq \quad & C\Big(h\|\tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2}\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{H(\mathrm{div};\Omega)}+\|c-c_h\|\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}\Big).\end{aligned} \tag{2.4.21}$$

Substitute (2.4.21) in (2.4.20) to find that

$$\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}\leq C\Big(h\|\tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2}+\|c-c_h\|\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}\Big). \tag{2.4.22}$$

For estimating $\tilde{e}_{2h}$, choose $\mathbf{v_h}=\tilde{\mathbf{e}}_{\mathbf{1}h}$ in (2.4.17), use (2.1.10)-(2.1.11), (2.1.8) and (2.4.20) to obtain

$$\begin{aligned}(\nabla\cdot\tilde{\mathbf{e}}_{\mathbf{1}h},\tilde{e}_{2h}) \quad \leq \quad & C\Big(h\|\tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2}+\|c-c_h\|\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}\\ & +\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}\Big)\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}.\end{aligned} \tag{2.4.23}$$

A use of the inf-sup condition (2.3.59) on the left hand side of (2.4.23) yields

$$\|\tilde{e}_{2h}\| \quad \leq \quad C\Big(h\|\tilde{\mathbf{u}}_h\|_{(L^2(\Omega))^2}+\|c-c_h\|\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}+\|\tilde{\mathbf{e}}_{\mathbf{1}h}\|_{(L^2(\Omega))^2}\Big). \tag{2.4.24}$$

Using (2.4.10), (2.4.12), (2.4.22) and (2.4.24) can be written as

$$\|\tilde{\mathbf{e_1}}_h\|_{(L^2(\Omega))^2} \leq C\Big(h\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|c - c_h\|\Big), \tag{2.4.25}$$

and

$$\|\tilde{e_2}_h\| \leq C\Big(h\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|c - c_h\|\Big), \tag{2.4.26}$$

where the constant C depends on $\|\tilde{\mathbf{u}}_h\|_{(L^\infty(\Omega))^2}$ derived in (2.4.12). An application of the triangle inequality completes the proof of (2.4.13). Now the estimate for (2.4.14) directly follows from (2.4.19) and (2.4.11). This completes the rest of the proof. ■

2.4.2 Estimates for the concentration

Let $R_h : H^1(\Omega) \longrightarrow M_h$ be the projection of c defined by

$$A(\mathbf{u}; c - R_h c, \chi) = 0 \qquad \forall \chi \in M_h, \tag{2.4.27}$$

where

$$A(\mathbf{u}; \psi, \chi) = a_h(\mathbf{u}; \psi, \chi) + (\mathbf{u} \cdot \nabla\psi, \chi) + (\lambda\psi, \chi) \quad \forall \chi \in M_h. \tag{2.4.28}$$

The function λ is chosen in such a way that $A(\cdot; \cdot, \cdot)$ is coercive.
Since

$$\begin{aligned}(\mathbf{u} \cdot \nabla\psi, \psi) &= \int_\Omega \psi\mathbf{u} \cdot \nabla\psi dx = -\int_\Omega \nabla \cdot (\psi\mathbf{u})\psi dx + \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}\psi^2 dx \\ &= -\int_\Omega \nabla \cdot (\psi\mathbf{u})\psi dx = -\int_\Omega \psi\mathbf{u} \cdot \nabla\psi dx - \int_\Omega \nabla \cdot \mathbf{u}\psi^2 dx,\end{aligned}$$

we obtain

$$(\mathbf{u} \cdot \nabla\psi, \psi) = -\frac{1}{2}(\nabla \cdot \mathbf{u}, \psi^2) = -\frac{1}{2}(q, \psi^2). \tag{2.4.29}$$

If we choose $\lambda = 1 + \frac{1}{2}q$, then $(\mathbf{u} \cdot \nabla\chi, \chi) + (\lambda\chi, \chi) = (\chi, \chi)$ for $\chi \in M_h$.
Now we derive the error bound in H^1 and L^2 norms for $c - R_h c$. Let I_h be the continuous interpolant onto M_h satisfying the following approximation properties. For $\phi \in H^{k+1}(\Omega)$ with $k \geq 1$, we have [29]:

$$\|\phi - I_h\phi\|_j \leq h^{k+1-j}\|\phi\|_{k+1} \quad j = 0, 1. \tag{2.4.30}$$

Moreover, if $\phi \in W^{2,\infty}(\Omega)$, then

$$\|\phi - I_h\phi\|_{1,\infty} \leq Ch\|\phi\|_{2,\infty}. \tag{2.4.31}$$

LEMMA 2.4.1 *There exists a positive constant C independent of h such that*

$$\|c - R_h c\|_1 \leq Ch\|c\|_2, \tag{2.4.32}$$

provided $c \in H^2(\Omega)$, for $t \in (0,T]$ a.e.

Proof. The coercivity and boundedness of bilinear form $A(\mathbf{u};\cdot,\cdot)$ with (2.4.27) yield

$$\begin{aligned}\|I_hc - R_hc\|_1^2 &\leq CA(\mathbf{u}; I_hc - R_hc, I_hc - R_hc)\\ &\leq CA(\mathbf{u}; I_hc - c, I_hc - R_hc)\\ &\leq C\|c - I_hc\|_1\|I_hc - R_hc\|_1,\end{aligned}$$

and hence,

$$\|I_hc - R_hc\|_1 \leq C\|c - I_hc\|_1, \tag{2.4.33}$$

where C depends on the bound of $D(\mathbf{u})$ given in (2.1.13). Combine the estimates (2.4.33) and (2.4.30) and use the triangle inequality to complete the proof. ■

For deriving the L^2- error bounds for $c - R_hc$, we need the following Lemma.

LEMMA 2.4.2 *There exists a positive constant C such that for $\psi \in H^1(\Omega)$ and $\chi_h \in M_h$*

$$|\epsilon_a(\mathbf{u}; c - R_hc, \psi_h)| \leq Ch^2\left(|g|_1 + |\mathbf{u}\cdot\nabla\mathbf{c}|_1 + |\phi\frac{\partial c}{\partial t}|_1 + \|c\|_2\right)|\psi_h|_1 \quad \forall\psi_h \in M_h, \tag{2.4.34}$$

where $\epsilon_a(\mathbf{u};\psi,\chi_h) = a(\mathbf{u};\psi,\chi_h) - a_h(\mathbf{u};\psi,\chi_h)$.

Proof. Using (2.3.36) (see Remark 2.3.3 also), we find that

$$\begin{aligned}|\epsilon_a(\mathbf{u}; c - R_hc, \psi_h)| &\leq \left|\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(D(\mathbf{u})\nabla(c - R_hc))(\psi_h - \Pi_h^*\psi_h)dx\right|\\ &\quad + \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla(c - R_hc)\cdot\mathbf{n})(\psi_h - \Pi_h^*\psi_h)ds\right|\\ &= J_1 + J_2, \quad \text{say.}\end{aligned} \tag{2.4.35}$$

To bound J_1, first we use the fact that $R_h c$ is linear on each triangle T to obtain

$$\begin{aligned} J_1 &= \left| \sum_{T\in\mathcal{T}_h} \int_T \nabla\cdot(D(\mathbf{u})\nabla(c-R_hc))\,(\psi_h-\Pi_h^*\psi_h)dx \right| \\ &= \left| \sum_{T\in\mathcal{T}_h} \int_T \Big(\nabla\cdot(D(\mathbf{u})\nabla c) - (\nabla\cdot D(\mathbf{u}))\cdot\nabla R_hc\Big)(\psi_h-\Pi_h^*\psi_h)dx \right|. \end{aligned}$$

Now use (2.1.3), (2.3.25) (2.3.32) to obtain

$$\begin{aligned} J_1 &\le \left| \sum_{T\in\mathcal{T}_h} \int_T \left(-g+\mathbf{u}\cdot\nabla c+\phi\frac{\partial c}{\partial t}\right)(\psi_h-\Pi_h^*\psi_h)dx \right| \\ &\quad + \left| \sum_{T\in\mathcal{T}_h} \int_T [(\nabla\cdot D(\mathbf{u}) - (\nabla\cdot D(\mathbf{u}))_T)\cdot\nabla R_hc]\,(\psi_h-\Pi_h^*\psi_h)dx \right| \\ &\le Ch^2\left(|g|_1+|\mathbf{u}\cdot\nabla\mathbf{c}|_1+|\phi\frac{\partial c}{\partial t}|_1+\|c\|_2\right)|\psi_h|_1, \end{aligned}$$

where $(\nabla\cdot D(\mathbf{u}))_T$ denotes the average value of $\nabla\cdot D(\mathbf{u})$ on triangle T.

Based on the analysis in [46, pp. 1873], we estimate J_2 as follows. Note that an appeal to the continuity of $\nabla c\cdot\mathbf{n}$ with (2.3.26) yields

$$J_2 = \left| \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big((D-\overline{D}_T)\nabla(c-R_hc)\cdot\mathbf{n}\Big)(\psi_h-\Pi_h^*\psi_h)ds \right|,$$

where $D = D(\mathbf{u})$ and $\overline{D}_T$ is a function such that for any edge of a triangle $T\in\mathcal{T}_h$,

$$\overline{D}_T(x) = D(x_c), \qquad\qquad x\in E,$$

and x_c is the mid point of E. Since $|D(x)-\overline{D}_T|\le Ch_T\|D\|_{1,\infty}$, we use trace inequality (2.3.13) and (2.4.32) to arrive at

$$\begin{aligned} J_2 &\le Ch|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla(c-R_hc)\cdot\mathbf{n})(\psi_h-\Pi_h^*\psi_h)ds| \\ &\le Ch\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|(\nabla(c-R_hc)\cdot\mathbf{n})|^2\right)^{1/2}\times\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\psi_h-\Pi_h^*\psi_h|^2\right)^{1/2} \\ &\le Ch\left(h_T^{-1/2}\|c-R_hc\|_1+h_T^{1/2}\|c\|_2\right)\times\left(h_T^{-1/2}\|\psi_h-\Pi_h^*\psi_h\|+h_T^{1/2}|\psi_h|_1\right) \\ &\le Ch^2\|c\|_2|\psi_h|_1. \end{aligned}$$

Substitute the estimates of J_1 and J_2 in (2.4.35) to complete the rest of the proof. ■

LEMMA 2.4.3 *: There exists a positive constant C independent of h such that*

$$\|c - R_h c\| \leq Ch^2 \left(\|c\|_2 + |g|_1 + |\mathbf{u} \cdot \nabla c|_1 + |\phi \frac{\partial c}{\partial t}|_1 \right), \tag{2.4.36}$$

provided $c \in H^2(\Omega)$, $\mathbf{u} \cdot \nabla c \in H^1(\Omega)$ and $\frac{\partial c}{\partial t} \in H^1(\Omega)$ for $t \in (0, T]$ a.e.

Proof. To obtain optimal L^2 error estimates for $c - R_h c$, we now appeal to Aubin-Nitsche duality argument. Let $\psi \in H^2(\Omega)$ be a solution of the following adjoint problem

$$\begin{aligned} -\nabla \cdot (D(\mathbf{u})\nabla\psi + \mathbf{u}\psi) + \lambda\psi &= c - R_h c \qquad \text{in } \Omega, \\ (D(\mathbf{u})\nabla\psi + \mathbf{u}\psi) \cdot \mathbf{n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.4.37}$$

which satisfies the elliptic regularity condition:

$$\|\psi\|_2 \leq C\|c - R_h c\|. \tag{2.4.38}$$

Multiply the above equation by $c - R_h c$ and integrate over Ω. An integration by parts and a use of (2.4.27) yield

$$\begin{aligned} \|c - R_h c\|^2 &= a(\mathbf{u}; , \psi, c - R_h c) - (\mathbf{u} \cdot \nabla\psi, c - R_h c) - (\nabla \cdot \mathbf{u}\psi, c - R_h c) + \lambda(\psi, c - R_h c) \\ &= \Big[a(\mathbf{u}; c - R_h c, \psi - \psi_h) + (\mathbf{u} \cdot \nabla(c - R_h c), \psi - \psi_h) \\ &\quad + \lambda(c - R_h c, \psi - \psi_h) \Big] + \epsilon_a(\mathbf{u}; c - R_h c, \psi_h) \qquad \forall \psi_h \in M_h \\ &= I_1 + I_2, \quad \text{say.} \end{aligned} \tag{2.4.39}$$

For I_1, use (2.4.32) to find that

$$\begin{aligned} |I_1| &= |a(\mathbf{u}; c - R_h c, \psi - \psi_h) + (\mathbf{u} \cdot \nabla(c - R_h c), \psi - \psi_h) + \lambda(c - R_h c, \psi - \psi_h)| \\ &\leq C\|c - R_h c\|_1 \|\psi - \psi_h\|_1 \\ &\leq Ch\|c\|_2 \|\psi - \psi_h\|_1. \end{aligned} \tag{2.4.40}$$

The bound for I_2 follows from Lemma 2.4.2 and hence,

$$|I_2| \leq Ch^2 \left(|g|_1 + |\mathbf{u} \cdot \nabla \mathbf{c}|_1 + |\phi \frac{\partial c}{\partial t}|_1 + \|c\|_2 \right) |\psi_h|_1. \tag{2.4.41}$$

Substitute (2.4.40) and (2.4.41) in (2.4.39) to find that

$$\|c - R_h c\|^2 \leq C \left[h\|c\|_2 \|\psi - \psi_h\|_1 + h^2 \left(|g|_1 + |\mathbf{u} \cdot \nabla c|_1 + |\phi \frac{\partial c}{\partial t}|_1 + \|c\|_2 \right) |\psi_h|_1 \right]. \tag{2.4.42}$$

Now choose $\psi_h = I_h\psi$ in (2.4.42). Then use elliptic regularity condition (2.4.38) with (2.4.30) to obtain

$$\|c - R_h c\| \leq Ch^2 \left(\|c\|_2 + |g|_1 + |\mathbf{u}\cdot\nabla c|_1 + |\phi\frac{\partial c}{\partial t}|_1) \right),$$

and this completes the proof. ■

For $\|R_h c\|_{1,\infty}$, we use inverse inequality (2.3.14), (2.4.31) (2.4.30) and (2.4.33) to obtain

$$\begin{aligned}
\|R_h c\|_{1,\infty} &\leq \|c - R_h c\|_{1,\infty} + \|c\|_{1,\infty} \\
&\leq \|c - I_h c\|_{1,\infty} + \|I_h c - R_h c\|_{1,\infty} + \|c\|_{1,\infty} \\
&\leq C\left(\|c - I_h c\|_{1,\infty} + h^{-1}\|I_h c - R_h c\|_1 + \|c\|_{1,\infty}\right) \\
&\leq C\|c\|_{2,\infty}. \qquad (2.4.43)
\end{aligned}$$

LEMMA 2.4.4 *There exists a positive constant C such that $\forall\theta \in M_h$,*

$$|a_h(\mathbf{u}; R_h c, \theta) - a_h(\mathbf{u_h}; R_h c, \theta)| \leq C\left(\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u} - \mathbf{u_h})\|\right)|\theta|_1. \quad (2.4.44)$$

Proof. Note that

$$\begin{aligned}
|a_h(\mathbf{u}; R_h c, \theta) - a_h(\mathbf{u_h}; R_h c, \theta)| &= \left|\sum_{i=1}^{N_h}\int_{\partial V_i^*}(D(\mathbf{u}) - D(\mathbf{u_h}))\nabla R_h c\cdot\mathbf{n}_i\ \Pi_h^*\theta\ ds\right| \\
&= \left|\sum_{T\in\mathcal{T}_h} K_T\right|,
\end{aligned}$$

where $K_T = \sum_{l=1}^{3}\int_{\partial V_l^*\cap T}(D(\mathbf{u}) - D(\mathbf{u_h}))\nabla R_h c\cdot\mathbf{n}_l\theta_l\ ds$ and $\theta_l = \theta(P_l)$, see Figure 2.3. For each triangle T, K_T can be written as

$$K_T = \sum_{l=1}^{3}\int_{M_l B}(D(\mathbf{u}) - D(\mathbf{u_h}))\nabla R_h c\cdot\mathbf{n}_l(\theta_{l+1} - \theta_l)\ ds \qquad (\theta_4 = \theta_1).$$

Using the Cauchy-Schwarz inequality and (2.4.43), we obtain

$$\begin{aligned}
K_T &\leq \sum_{l=1}^{3}|\theta_{l+1} - \theta_l|\int_{M_l B}|(D(\mathbf{u}) - D(\mathbf{u_h}))\nabla R_h c\cdot\mathbf{n}_l|\ ds \\
&\leq C\sum_{l=1}^{3}|\theta_{l+1} - \theta_l|\ \|D(\mathbf{u}) - D(\mathbf{u_h})\|_{(L^2(M_l B))^{2\times 2}}\left(\text{meas}(M_l B)\right)^{1/2}.
\end{aligned}$$

A use of the trace inequality (2.3.13) and (2.3.33) yields

$$\begin{aligned} K_T &\leq C\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\ \|\mathbf{u}-\mathbf{u_h}\|_{(L^2(M_lB))^2}h_T^{1/2} \\ &\leq Ch_T^{1/2}\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\left[h_T^{-1/2}\|\mathbf{u}-\mathbf{u_h}\|_T+h_T^{1/2}\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|_T\right]. \qquad (2.4.45)\end{aligned}$$

Now using Taylor series expansion and (2.3.18), we find that

$$\begin{aligned} |\theta_{l+1}-\theta_l| &\leq h_T\left[|\frac{\partial\theta}{\partial x}|+|\frac{\partial\theta}{\partial y}|\right]\leq\left[\left(|\frac{\partial\theta}{\partial x}|^2+|\frac{\partial\theta}{\partial y}|^2\right)h_T^2\right]^{1/2} \\ &\leq C|\theta|_{1,h,T}, \qquad\qquad l=1,2,3. \qquad (2.4.46)\end{aligned}$$

Substitute (2.4.46) in (2.4.45) to arrive at

$$K_T\leq C|\theta|_{1,h,T}\left(\|\mathbf{u}-\mathbf{u_h}\|_T+h_T\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|_T\right).$$

With the estimates for K_T and Lemma 2.3.1, we obtain

$$|a_h(\mathbf{u};R_hc,\theta)-a_h(\mathbf{u_h};R_hc,\theta)|\leq C\left(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2}+h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\right)|\theta|_1,$$

and this completes the rest of the proof. ■

Theorem 2.4.2 *Let c and c_h be the solutions of (2.1.3) and (2.3.9) respectively, and let $c_h(0)=c_{0,h}=R_hc(0)$. Then, for sufficiently small h, there exists a positive constant $C(T)$ independent of h, but dependent on the bounds of κ^{-1} and μ such that*

$$\begin{aligned} \|c-c_h\|^2_{L^\infty(J;L^2)} &\leq C(T)\left[\int_0^T\left(h^4(\|c\|_2^2+\|g\|_1^2+\|\mathbf{u}\cdot\nabla c\|_1^2+\|\phi\frac{\partial c}{\partial t}\|_1^2\right.\right. \\ &\quad+\|c_t\|_2^2+\|g_t\|_1^2+\|(\mathbf{u}\cdot\nabla c)_t\|_1^2+\|\phi\frac{\partial^2c}{\partial t^2}\|_1^2 \\ &\quad+\|\mathbf{u}\|^2_{(H^1(\Omega))^2}+\|p\|_1^2+\|\nabla\cdot\mathbf{u}\|_1^2) \\ &\quad\left.\left.+h^2(\|\mathbf{u}\|^2_{(H^1(\Omega))^2}+\|p\|_1^2)\right)ds\right], \qquad (2.4.47)\end{aligned}$$

provided, $c,\ c_t\in L^2(J;H^2(\Omega)),\ \mathbf{u}\in L^2(J;(H^1(\Omega))^2),$ and $\mathbf{u}\cdot\nabla c,\ (\mathbf{u}\cdot\nabla c)_t,\ g,\ g_t,\ \nabla\cdot\mathbf{u},\ p,\ \phi\frac{\partial c}{\partial t},\ \phi\frac{\partial^2c}{\partial t^2}\in L^2(J;H^1(\Omega))$.

Proof. Write $c - c_h = (c - R_h c) + (R_h c - c_h) = \rho + \theta$. Since the estimates of ρ are known, we need to find only the estimates of θ.

Multiply (2.1.3) by $\Pi_h^* z_h$, integrate over Ω and subtract the resulting equation from (2.3.9) to obtain

$$\left(\phi\frac{\partial\theta}{\partial t}, \Pi_h^* z_h\right) + (\mathbf{u}\cdot\nabla c, \Pi_h^* z_h) - (\mathbf{u_h}\cdot\nabla c_h, \Pi_h^* z_h) + a_h(\mathbf{u}; c, z_h)$$
$$-a_h(\mathbf{u_h}; c_h, z_h) = -\left(\phi\frac{\partial\rho}{\partial t}, \Pi_h^* z_h\right) + (g(c) - g(c_h), \Pi_h^* z_h) \quad \forall z_h \in M_h. \quad (2.4.48)$$

Using the definition of ϵ_h in (2.3.27), (2.4.48) can be rewritten as

$$\begin{aligned}\left(\phi\frac{\partial\theta}{\partial t}, \Pi_h^* z_h\right) &- (\mathbf{u_h}.\nabla c_h, z_h) + \epsilon_h(\mathbf{u_h}.\nabla c_h, z_h) + (\mathbf{u}\cdot\nabla c, z_h) - \epsilon_h(\mathbf{u}\cdot\nabla c, z_h)\\ &+ a_h(\mathbf{u}; c, z_h) - a_h(\mathbf{u_h}, c_h, z_h) = -\left(\phi\frac{\partial\rho}{\partial t}, \Pi_h^* z_h\right)\\ &+(g(c) - g(c_h), \Pi_h^* z_h) \quad \forall z_h \in M_h. \qquad (2.4.49)\end{aligned}$$

Put $z_h = \theta$ in (2.4.49) and use the definition of R_h to obtain

$$\begin{aligned}\left(\phi\frac{\partial\theta}{\partial t}, \Pi_h^*\theta\right) &+ (\mathbf{u_h}\cdot\nabla\theta, \theta) + a_h(\mathbf{u_h}; \theta, \theta) = -\left(\phi\frac{\partial\rho}{\partial t}, \Pi_h^*\theta\right) + (\lambda\rho, \theta) + ((\mathbf{u_h} - \mathbf{u})\cdot\nabla R_h c, \theta)\\ &+ [\epsilon_h(\mathbf{u}\cdot\nabla c, \theta) - \epsilon_h(\mathbf{u_h}\cdot\nabla c_h, \theta)] - [a_h(\mathbf{u}; R_h c, \theta) - a_h(\mathbf{u_h}; R_h c, \theta)]\\ &+ \left(g(c) - g(c_h), \Pi_h^*\theta\right) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \text{ say.} \qquad (2.4.50)\end{aligned}$$

To estimate I_1, we use the Cauchy-Schwarz inequality, boundedness of ϕ (see,(2.1.9)) and (2.3.47) to obtain

$$|I_1| = |\left(\phi\frac{\partial\rho}{\partial t}, \Pi_h^*\theta\right)| \le C\|\frac{\partial\rho}{\partial t}\|\|\theta\|. \qquad (2.4.51)$$

Similarly,

$$|I_2| = |(\lambda\rho, \theta)| \le C\|\rho\|\|\theta\|. \qquad (2.4.52)$$

Using (2.4.43), I_3 is bounded as follows:

$$\begin{aligned}|I_3| &= |((\mathbf{u_h} - \mathbf{u})\cdot\nabla R_h c, \theta)| \le \|\mathbf{u_h} - \mathbf{u}\|_{(L^2(\Omega))^2}\|\nabla R_h c\|_{L^\infty}\|\theta\|\\ &\le C\|\mathbf{u_h} - \mathbf{u}\|_{(L^2(\Omega))^2}\|\|\theta\|, \qquad (2.4.53)\end{aligned}$$

where the constant C depends on the L^∞ bound of R_h given in (2.4.43).
The bound for I_4 is a bit technical and now we proceed as follows:

$$\begin{aligned} |I_4| &\leq |\epsilon_h(\mathbf{u}\cdot\nabla c,\theta)| + |\epsilon_h(\mathbf{u_h}\cdot\nabla c_h,\theta)| \\ &\leq |\epsilon_h(\mathbf{u}\cdot\nabla c,\theta)| + |\epsilon_h((\mathbf{u_h}-\mathbf{u})\cdot\nabla\theta,\theta)| + |\epsilon_h\left((\mathbf{u_h}-\mathbf{u})\cdot\nabla R_h c,\theta\right)| \\ &\quad + |\epsilon_h(\mathbf{u}\cdot\nabla\theta,\theta)| + |\epsilon_h(\mathbf{u}\cdot\nabla R_h c,\theta)| \\ &= A_1 + A_2 + A_3 + A_4 + A_5, \ \text{ say.} \end{aligned} \tag{2.4.54}$$

To estimate $A_1 \cdots A_5$, we use the bound for ϵ_h in (2.3.32) and the inverse inequalities (2.3.14)-(2.3.15) and (2.4.32) to find that

$$A_1 = |\epsilon_h(\mathbf{u}\cdot\nabla c,\theta)| \leq Ch^2\|\mathbf{u}\cdot\nabla c\|_1|\theta|_1, \tag{2.4.55}$$

$$\begin{aligned} A_2 &= |\epsilon_h((\mathbf{u_h}-\mathbf{u})\cdot\nabla\theta,\theta)| \\ &\leq Ch^2\left(\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\nabla\theta\|_{L^1}\right)|\theta|_{1,\infty} \\ &\leq C\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\ \|\theta\|\ |\theta|_1, \end{aligned} \tag{2.4.56}$$

$$\begin{aligned} A_3 = |\epsilon_h\left((\mathbf{u_h}-\mathbf{u})\cdot\nabla R_h c,\theta\right)| &\leq C\|\mathbf{u}-\mathbf{u_h}\|\ \|\nabla R_h c\|\ h\ \|\theta\|_{1,\infty} \\ &\leq C\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2}|\theta|_1, \end{aligned} \tag{2.4.57}$$

and

$$A_4 = |\epsilon_h\left(\mathbf{u}\cdot\nabla\theta,\theta\right)| \leq C|\theta|_1\|\theta\|. \tag{2.4.58}$$

Finally for A_5, we use the definition of $\epsilon_h(\mathbf{u};\cdot,\cdot)$ in (2.3.27) and (2.4.43) to obtain

$$A_5 \leq Ch^2\|\mathbf{u}\|_{(H^1(\Omega))^2}|\theta|_1.$$

Substituting the bounds for A_1 to A_5 in (2.4.54), we obtain following bound for I_4:

$$\begin{aligned} |I_4| &\leq C|\theta|_1\Big[h^2\|\mathbf{u}\cdot\nabla c\|_1 + \|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} + \|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\ \|\theta\| \\ &\quad + h^2\|\mathbf{u}\|_{H^1(\Omega)} + \|\theta\|\Big]. \end{aligned} \tag{2.4.59}$$

The bound for I_5 follows from Lemma 2.4.4, and hence,

$$|I_5| \leq C\left(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\right)|\theta|_1. \tag{2.4.60}$$

Using (2.1.7) and (2.3.47), I_6 can be estimated as

$$|I_6| \leq |(g(c_h)-g(c), \Pi_h^*\theta)| \leq C\|c-c_h\|\ \|\theta\|. \tag{2.4.61}$$

Now, we need to bound from below the left-hand side of (2.4.50).
Note that

$$(\mathbf{u_h}\cdot\nabla\theta,\theta) = -\frac{1}{2}(\nabla\cdot\mathbf{u_h},\theta^2) = -\frac{1}{2}(q\theta,\theta) - \frac{1}{2}(\nabla\cdot(\mathbf{u_h}-\mathbf{u}),\theta^2). \tag{2.4.62}$$

To estimate the second term in (2.4.62), we use (2.4.14)

$$\begin{aligned}|(\nabla\cdot(\mathbf{u_h}-\mathbf{u}),\theta^2)| &\leq \|\nabla\cdot(\mathbf{u_h}-\mathbf{u})\|_{L^2}\|\theta\|\ \|\theta\|_{L^\infty}\\ &\leq Ch\|\nabla\cdot\mathbf{u}\|_{H^1}\|\theta\|\ \|\theta\|_{L^\infty}\\ &\leq C\|\theta\|^2. \end{aligned}\tag{2.4.63}$$

The boundedness of q implies that

$$(q\theta,\theta) \leq C\|\theta\|^2. \tag{2.4.64}$$

Substitute the estimates for $I_1,\cdots,I_6$ in (2.4.50) and use (2.4.63)-(2.4.64), (2.3.42), Young's inequality $ab \leq \frac{1}{2}\epsilon a^2 + \frac{1}{2\epsilon}b^2$, non singularity of the function ϕ with standard kick back argument to obtain

$$\begin{aligned}\frac{d}{dt}|||\theta|||^2 + (\alpha-\epsilon)|\theta|_1^2 &\leq C\Big[\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|^4 + h^4(\|\mathbf{u}\cdot\nabla c\|_1^2 + \|\mathbf{u}\|^2_{(H^1(\Omega))^2})\\ &+ h^2\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|^2 + \|\rho\|^2 + \|\frac{\partial\rho}{\partial t}\|^2 + (1+\|\theta\|^2)\|\theta\|^2\Big]. \end{aligned}\tag{2.4.65}$$

Now, from (2.4.13),

$$\|\mathbf{u}-\mathbf{u_h}\| \leq C\Big(\|\rho\| + \|\theta\| + h(\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1)\Big). \tag{2.4.66}$$

A use of (2.4.36), (2.4.66) and (2.4.14) in (2.4.65) gives us

$$\begin{aligned}\frac{d}{dt}|||\theta|||^2 + \alpha_0|\theta|_1^2 &\leq C\Big[h^4(\|c\|_2^2 + \|g\|_1^2 + \|\mathbf{u}\cdot\nabla c\|_1^2 + \|\phi\frac{\partial c}{\partial t}\|_1^2\\ &+\|c_t\|_2^2 + \|g_t\|_1^2 + \|(\mathbf{u}\cdot\nabla c)_t\|_1^2 + \|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2\\ &+\|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2 + \|\nabla\cdot\mathbf{u}\|_1^2)\\ &+h^2(\|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2) + (1+\|\theta\|^2)\|\theta\|^2\Big]. \end{aligned}\tag{2.4.67}$$

To estimate the last term on the right hand side of (2.4.67), we follow the arguments given in [37]. Let $t^* \leq T$ be the largest time such that the induction hypothesis

$$\|\theta\|_{L^\infty(J;L^2)} \leq 1, \tag{2.4.68}$$

holds true $\forall t \leq t^*$. The existence of $t^* > 0$ for which (2.4.68) holds true can be justified in the following way. Since $c_h(0) = R_h c(0)$, this implies that $\theta(0) = 0$. An application to Gronwall's inequality (1.2.7) with (2.3.46), (2.4.67) and (2.4.68) yields

$$\begin{aligned}\|\theta\|^2_{L^\infty(J;L^2)} \;\leq\; & C(T)\Bigg[\int_0^T \Big(h^4(\|c\|_2^2 + \|g\|_1^2 + \|\mathbf{u}\cdot\nabla c\|_1^2 + \|\phi\frac{\partial c}{\partial t}\|_1^2 \\ & +\|c_t\|_2^2 + \|g_t\|_1^2 + \|(\mathbf{u}\cdot\nabla c)_t\|_1^2 + \|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2 \\ & +\|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2 + \|\nabla\cdot\mathbf{u}\|_1^2) \\ & +h^2(\|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2)\Big)ds\Bigg] \qquad \forall t \in (0,t^*] \text{ with } t^* \leq T. \end{aligned} \tag{2.4.69}$$

Choose $h_0 > 0$, small enough so that for $h \in (0,h_0]$, $\forall t \in (0,t^*]$ with $t^* \leq T$, we have from (2.4.69) that $\|\theta\|_{L^\infty(J;L^2)} \leq 1$. If $t^* < T$, by the continuity of the mapping $t \longrightarrow \|\theta\|_{L^\infty(J;L^2)}$, either $\|\theta\|_{L^\infty(J;L^2)} \leq 1$ $\forall 0 \leq t \leq T$, or there exists some t^{**} such that $t* < t^{**} < T$ and $\|\theta\|_{L^\infty(J;L^2)} > 1$. In both the cases, we get a contradiction due to the fact that t^* is the largest interval in $(0,T]$ such that $\|\theta\|_{L^\infty(J;L^2)} \leq 1$ and hence, $t^* = T$. Combine the estimates for ρ in (2.4.36) and θ given in (2.4.69) to complete the rest of the proof. ■

Combining the estimates derived in (2.4.13) and (2.4.47), we obtain the following estimates for $\mathbf{u} - \mathbf{u_h}$ and $p - p_h$.

Theorem 2.4.3 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$ be, respectively, the solutions of (2.2.2)-(2.2.3) and (2.3.6)-(2.3.7) and let $c_h(0) = c_{0,h} = R_h c(0)$. Then for sufficiently small h, there exists a positive constant $C(T)$ which is independent of h but may depend on the bounds of κ^{-1} and μ such that*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u_h}\|^2_{L^\infty(J;(L^2(\Omega))^2)} \;+\; & \|p-p_h\|^2_{L^\infty(J;L^2(\Omega))} \leq C(T)\Bigg[\int_0^T \Big(h^4(\|c\|_2^2 + \|g\|_1^2 + \|\mathbf{u}\cdot\nabla c\|_1^2 \\ & +\|\phi\frac{\partial c}{\partial t}\|_1^2 + \|c_t\|_2^2 + \|g_t\|_1^2 + \|(\mathbf{u}\cdot\nabla c)_t\|_1^2 + \|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2 + \|\mathbf{u}\|^2_{(H^1(\Omega))^2} \\ & +\|p\|_1^2 + \|\nabla\cdot\mathbf{u}\|_1^2) + h^2(\|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2)\Big)ds\Bigg]. \end{aligned} \tag{2.4.70}$$

2.5 Completely Discrete Scheme

In Section 2.4, we have discussed a semidiscrete scheme, i.e., we have discretized only the spatial domain Ω and not the time interval $[0, T]$ and have derived *a priori* error estimates for concentration, velocity and pressure. In this section, we introduce a completely discrete scheme, i.e., we also discretize the time variable using finite difference methods.

Let $0 = t_0 < t_1 < \cdots t_N = T$ be a given partition of the time interval $[0, T]$ with time step size Δt. Set $f^n = f(t_n)$ for a generic function f in time. Then, at time level t_n, the fully discrete problem corresponding to pressure-velocity equation (2.3.6)-(2.3.7) is to find $(\mathbf{u_h^n}, p_h^n) \in U_h \times W_h$ such that

$$(\kappa^{-1}\mu(c_h^n)\mathbf{u_h^n}, \gamma_h \mathbf{v_h}) + (\nabla \cdot \mathbf{v_h}, p_h^n) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{2.5.1}$$

$$(\nabla \cdot \mathbf{u_h^n}, w_h) = (q^n, w_h) \qquad \forall w_h \in W_h. \tag{2.5.2}$$

For the approximation of concentration at time level $t = t_{n+1}$, we use the approximate velocity at the previous time level $(t = t_n)$ and for approximating the time derivative $\dfrac{\partial c_h}{\partial t}$, we use the backward Euler difference scheme:

$$\frac{\partial c_h}{\partial t}|_{t=t_{n+1}} \approx \frac{c_h^{n+1} - c_h^n}{\Delta t_n}. \tag{2.5.3}$$

For the sake of convenience, we choose $\Delta t_n = \Delta t, \ \forall n = 1, 2, \cdots N$.

Now, the discrete problem corresponding to the concentration equation (2.3.9) is to find $c_h^{n+1} \in M_h$ such that

$$\begin{aligned}\Big(\phi\frac{(c_h^{n+1} - c_h^n)}{\Delta t}, \Pi_h^* z_h\Big) + (\mathbf{u_h^n} \cdot \nabla c_h^{n+1}, \Pi_h^* z_h)& \\ + a_h(\mathbf{u_h^n}; c_h^{n+1}, z_h) = (g(c_h^{n+1}), \Pi_h^* z_h) \quad \forall z_h \in M_h.&\end{aligned} \tag{2.5.4}$$

2.5.1 Error Estimates

The following error bound for velocity and pressure at $t = t_n$ is given in Theorem 2.4.1.

$$\|\mathbf{u^n} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + \|p^n - p_h^n\| \le C\left[\|c^n - c_h^n\| + h\left(\|\mathbf{u^n}\|_{(H^1(\Omega))^2} + \|p^n\|_1\right)\right], \tag{2.5.5}$$

and

$$\|\nabla \cdot (\mathbf{u^n} - \mathbf{u_h^n})\| \le Ch\|\nabla \cdot \mathbf{u^n}\|_1. \tag{2.5.6}$$

THEOREM 2.5.1 *Let c^m and c_h^m be the solutions of (2.1.3) and (2.5.4) at $t = t^m$, respectively, and let $c_h(0) = c_{0,h} = R_h c(0)$. Further, assume that $\Delta t = O(h)$. Then, for sufficiently small h, there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such that*

$$\max_{0\le m\le N}\|c^m - c_h^m\|^2 \le C\Big[h^4\Big(\|c\|^2_{L^\infty(0,T;H^2)} + \|g\|^2_{L^\infty(0,T;H^1)} + \|\mathbf{u}\cdot\nabla c\|^2_{L^\infty(0,T;H^1)}$$
$$+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)} + \|\frac{\partial c}{\partial t}\|^2_{L^2(0,T;H^2)} + \|g_t\|^2_{L^2(0,T;H^1)} + \|(\mathbf{u}\cdot\nabla c)_t\|^2_{L^2(0,T;H^1)}$$
$$+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big) + (\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)}$$
$$+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big) + h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \qquad (2.5.7)$$

Proof. Write $c^n - c_h^n = (c^n - R_hc^n) + (R_hc^n - c_h^n) = \rho^n + \theta^n$. Since the estimates for ρ^n are known from Lemma 2.4.3 at $t = t_n$, it is enough to obtain the bound for θ^n.
Multiply the concentration equation (2.1.3) by $\Pi_h^* z_h$ and integrate over Ω. Then, at $t = t_{n+1}$, we have

$$\Big(\phi\frac{\partial c^{n+1}}{\partial t}, \Pi_h^* z_h\Big) + \big(\mathbf{u^{n+1}}\cdot\nabla c^{n+1}, \Pi_h^* z_h\big)$$
$$+a_h(\mathbf{u^{n+1}}; c^{n+1}, z_h) = (g(c^{n+1}), \Pi_h^* z_h) \quad \forall z_h \in M_h, \qquad (2.5.8)$$

where $a_h(\cdot;\cdot,\cdot)$ is defined in (2.3.10).
Subtracting (2.5.4) from (2.5.8) and using (2.3.27), we obtain

$$\Big(\phi\frac{\partial c^{n+1}}{\partial t} - \phi\frac{c_h^{n+1} - c_h^n}{\Delta t}, \Pi_h^* z_h\Big) - (\mathbf{u_h^n}.\nabla c_h^{n+1}, z_h) + \epsilon_h(\mathbf{u_h^n}.\nabla c_h^{n+1}, z_h)$$
$$+(\mathbf{u^{n+1}}\cdot\nabla c^{n+1}, z_h) - \epsilon_h(\mathbf{u^{n+1}}\cdot\nabla c^{n+1}, z_h) + a_h(\mathbf{u^{n+1}}; c^{n+1}, z_h)$$
$$-a_h(\mathbf{u_h^n}, c_h^{n+1}, z_h) = (g(c^{n+1}) - g(c_h^{n+1}), \Pi_h^* z_h) \quad \forall z_h \in M_h. \qquad (2.5.9)$$

Choosing $z_h = \theta^{n+1}$ in (2.5.9) and using the definition of R_h given in (2.4.27), we obtain the following error equation:

$$\Big(\phi\frac{\theta^{n+1} - \theta^n}{\Delta t}, \Pi_h^*\theta^{n+1}\Big) + (\mathbf{u_h^n}\cdot\nabla\theta^{n+1}, \theta^{n+1})$$
$$+a_h(\mathbf{u_h^n}; \theta^{n+1}, \theta^{n+1}) = \Big[\epsilon_h(\mathbf{u^{n+1}}\cdot\nabla c^{n+1}, \theta^{n+1}) - \epsilon_h(\mathbf{u_h^n}\cdot\nabla c_h^{n+1}, \theta^{n+1})\Big]$$
$$+\Big[a_h(\mathbf{u_h^n}; R_hc^{n+1}, \theta^{n+1}) - a_h(\mathbf{u^{n+1}}, R_hc^{n+1}, \theta^{n+1})\Big] + (\mathbf{u_h^n} - \mathbf{u^{n+1}}\cdot\nabla R_hc, \theta^{n+1})$$

$$
\begin{aligned}
&-\left(\phi\frac{\rho^{n+1}-\rho^n}{\Delta t},\Pi_h^*\theta^{n+1}\right)-\left(\phi\frac{\partial c^{n+1}}{\partial t}-\phi\frac{c^{n+1}-c^n}{\Delta t},\Pi_h^*\theta^{n+1}\right)\\
&+(g(c^{n+1})-g(c_h^{n+1}),\Pi_h^*\theta^{n+1})+\lambda(c^{n+1}-R_hc^{n+1},\Pi_h^*\theta^{n+1})\\
&=J_1+J_2+J_3+J_4+J_5+J_6+J_7,\quad \text{say.}
\end{aligned}
\tag{2.5.10}
$$

Now, we estimate $J_i, i=1,2,\cdots,7$ one by one.

Repeating the same arguments, which we have used to bound the term I_4 of Theorem 2.4.2, we obtain the following bound for J_1:

$$
\begin{aligned}
|J_1| &\le |\epsilon_h(\mathbf{u^{n+1}}\cdot\nabla c^{n+1},\theta^{n+1})-\epsilon_h(\mathbf{u_h^n}\cdot\nabla c_h^{n+1},\theta^{n+1})|\\
&\le C|\theta^{n+1}|_1\Big[h^2\|\mathbf{u^{n+1}}\cdot\nabla c^{n+1}\|_1+\|\mathbf{u^{n+1}}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}\\
&\quad+\|\nabla\cdot(\mathbf{u^{n+1}}-\mathbf{u_h^n})\|\|\theta^{n+1}\|+h^2\|\mathbf{u^{n+1}}\|_{(H^1(\Omega))^2}+\|\theta^{n+1}\|\Big],
\end{aligned}
$$

and hence,

$$
\begin{aligned}
|J_1| &\le C|\theta^{n+1}|_1\Big[h^2\|\mathbf{u^{n+1}}\cdot\nabla c^{n+1}\|_1+\|\mathbf{u^n}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}\\
&\quad+\|\nabla\cdot(\mathbf{u^n}-\mathbf{u_h^n})\|\|\theta^{n+1}\|+h^2\|\mathbf{u^{n+1}}\|_{(H^1(\Omega))^2}+\|\theta^{n+1}\|\\
&\quad+\|\mathbf{u^{n+1}}-\mathbf{u^n}\|_{(L^2(\Omega))^2}+\|\nabla\cdot(\mathbf{u^{n+1}}-\mathbf{u^n})\|\|\theta^{n+1}\|\Big].
\end{aligned}
\tag{2.5.11}
$$

Using Lemma 2.4.4, J_2 can be bounded as

$$
\begin{aligned}
|J_2| &\le |a_h(\mathbf{u_h^n};R_hc^{n+1},\theta^{n+1})-a_h(\mathbf{u^{n+1}},R_hc^{n+1},\theta^{n+1})|\\
&\le C\left[\|\mathbf{u^{n+1}}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}+h\|\nabla\cdot(\mathbf{u^{n+1}}-\mathbf{u_h^n})\|\right]|\theta^{n+1}|_1\\
&\le C\Big[\|\mathbf{u^n}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}+h\|\nabla\cdot(\mathbf{u^n}-\mathbf{u_h^n})\|\\
&\quad+\|\mathbf{u^n}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}+h\|\nabla\cdot(\mathbf{u^{n+1}}-\mathbf{u^n})\|\Big]|\theta^{n+1}|_1.
\end{aligned}
\tag{2.5.12}
$$

To bound J_3, we use (2.4.43):

$$
\begin{aligned}
|J_3| &\le |(\mathbf{u_h^n}-\mathbf{u^{n+1}}\cdot\nabla R_hc,\theta^{n+1})|\le C\|\mathbf{u^{n+1}}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}\|\theta^{n+1}\|\\
&\le C\left[\|\mathbf{u^{n+1}}-\mathbf{u_h^n}\|_{(L^2(\Omega))^2}+\|\mathbf{u^{n+1}}-\mathbf{u^n}\|_{(L^2(\Omega))^2}\right]\|\theta^{n+1}\|.
\end{aligned}
\tag{2.5.13}
$$

Using (2.1.9) and (2.3.47), we bound J_4 as

$$
|J_4|\le\left|\left(\phi\frac{\rho^{n+1}-\rho^n}{\Delta t},\Pi_h^*\theta^{n+1}\right)\right|\le C\left\|\frac{\rho^{n+1}-\rho^n}{\Delta t}\right\|\|\theta^{n+1}\|.
\tag{2.5.14}
$$

Using the Cauchy-Schwarz inequality, we have

$$\left|\frac{\rho^{n+1}-\rho^n}{\Delta t}\right| \leq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} |\rho_t| ds \leq (\Delta t)^{-1/2}\left(\int_{t_n}^{t_{n+1}} |\rho_t|^2 ds\right)^{1/2}. \tag{2.5.15}$$

Using (2.5.15), we obtain

$$\left\|\frac{\rho^{n+1}-\rho^n}{\Delta t}\right\| \leq C(\Delta t)^{-1/2}\|\frac{\partial \rho}{\partial t}\|_{L^2(t_n,t_{n+1};L^2)}. \tag{2.5.16}$$

This implies that

$$|J_4| \leq C(\Delta t)^{-1/2}\|\frac{\partial \rho}{\partial t}\|_{L^2(t_n,t_{n+1};L^2)}\|\theta^{n+1}\|. \tag{2.5.17}$$

An application of Taylor series expansion and (2.3.47) gives us

$$\begin{aligned}
|J_5| &\leq \left|\left(\phi\frac{\partial c^{n+1}}{\partial t} - \phi\frac{c^{n+1}-c^n}{\Delta t}, \Pi_h^*\theta^{n+1}\right)\right| \leq \phi^* \left\|\frac{\partial c^{n+1}}{\partial t} - \frac{c^{n+1}-c^n}{\Delta t}\right\| \|\theta^{n+1}\| \\
&\leq C\|\theta^{n+1}\|\int_{t_n}^{t_{n+1}} \|c_{tt}\| ds \leq \|\theta^{n+1}\|\left(\Delta t\int_{t_n}^{t_{n+1}} \|c_{tt}\|^2 ds\right)^{1/2}.
\end{aligned} \tag{2.5.18}$$

Since the function g is uniformly Lipschitz continuous (see (2.1.7)), J_6 can be bounded in the following way:

$$\begin{aligned}
|J_6| &\leq |(g(c^{n+1}) - g(c_h^{n+1}), \Pi_h^*\theta^{n+1})| \leq C\|c^{n+1} - c_h^{n+1}\|\|\theta^{n+1}\| \\
&\leq C(\|\rho^{n+1}\| + \|\theta^{n+1}\|)\|\theta^{n+1}\|.
\end{aligned} \tag{2.5.19}$$

Again using (2.3.47), we obtain

$$|J_7| \leq |\lambda(c^{n+1} - R_h c^{n+1}, \Pi_h^*\theta^{n+1})| \leq C\|\rho^{n+1}\|\|\theta^{n+1}\|. \tag{2.5.20}$$

Since

$$|\mathbf{u^{n+1}} - \mathbf{u^n}|^2 = \left|\int_{t_n}^{t_{n+1}} \mathbf{u}_t\ ds\right|^2 \leq \Delta t\int_{t_n}^{t_{n+1}} |\mathbf{u}_t|^2 ds, \tag{2.5.21}$$

Hence,

$$\|\mathbf{u^{n+1}} - \mathbf{u^n}\|^2_{(L^2(\Omega))^2} \leq \Delta t\ \|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}, \tag{2.5.22}$$

and similarly,

$$\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u^n})\|^2_{L^2(\Omega)} \leq \Delta t\ \|\nabla\cdot\mathbf{u_t}\|^2_{L^2(t_n,t_{n+1};L^2(\Omega))}. \tag{2.5.23}$$

Now, we need to bound the term $(\mathbf{u_h^n}\cdot\nabla\theta^{n+1},\theta^{n+1})$ in (2.5.10) from below. Note that

$$\begin{aligned}(\mathbf{u_h^n}\cdot\nabla\theta^{n+1},\theta^{n+1}) &= -\frac{1}{2}(\nabla\cdot\mathbf{u_h^n},(\theta^{n+1})^2)\\ &= -\frac{1}{2}(q^n\theta^{n+1},\theta^{n+1})-\frac{1}{2}(\nabla\cdot(\mathbf{u_h^n}-\mathbf{u^n}),(\theta^{n+1})^2). \quad (2.5.24)\end{aligned}$$

Using (2.5.6), we obtain

$$\begin{aligned}|(\nabla\cdot(\mathbf{u_h^n}-\mathbf{u^n}),(\theta^{n+1})^2)| &\le \|\nabla\cdot(\mathbf{u_h^n}-\mathbf{u^n})\|\ \|\theta^{n+1}\|\ \|\theta^{n+1}\|_{L^\infty}\\ &\le Ch\|\nabla\cdot\mathbf{u^n}\|_{H^1(\Omega)}\|\theta^{n+1}\|\ \|\theta^{n+1}\|_{L^\infty}\\ &\le C\|\theta^{n+1}\|^2. \quad (2.5.25)\end{aligned}$$

The boundedness of q^n implies that

$$|(q^n\theta^{n+1},\theta^{n+1})|\le C\|\theta^{n+1}\|^2, \quad (2.5.26)$$

and hence, from (2.5.24)

$$(\mathbf{u_h^n}\cdot\nabla\theta^{n+1},\theta^{n+1})\ge C\|\theta^{n+1}\|^2. \quad (2.5.27)$$

As in semidiscrete case, we now make the following hypothesis:

$$\max_{0\le m\le N}\|\theta^m\|\le 1. \quad (2.5.28)$$

Substituting all the estimates derived in (2.5.11)-(2.5.20) with (2.5.22),(2.5.23) and (2.5.27) in (2.5.10), with an application of Young's inequality, we obtain

$$\begin{aligned}\left(\phi\frac{\theta^{n+1}-\theta^n}{\Delta t},\Pi_h^*\theta^{n+1}\right) &+ a_h(\mathbf{u_h};\theta^{n+1},\theta^{n+1})\le C\Big[(\Delta t)^{-1}\|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)}+\|\theta^{n+1}\|^2\\ &+ \Delta t\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)}\Big)\\ &+ \Delta t\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(t_n,t_{n+1},L^2)}+\|\mathbf{u^n}-\mathbf{u_h^n}\|^2_{(L^2(\Omega))^2}\\ &\quad +h^2\|\nabla.(\mathbf{u^n}-\mathbf{u_h^n})\|^2+\|\theta^{n+1}\|_1^2+\|\rho^{n+1}\|^2\Big]. \quad (2.5.29)\end{aligned}$$

A use of (2.3.42), (2.5.5) and (2.5.6) in (2.5.29) with kick back arguments yields

$$\begin{aligned}|||\theta^{n+1}|||^2-|||\theta^n|||^2 &\le C\Big[\Delta t\left(\|\theta^{n+1}\|^2+\|\theta^n\|^2+\|\rho^n\|^2+\|\rho^{n+1}\|^2\right)\\ &\quad +(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)}\\ &\quad +\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big)+\|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)}\\ &\quad +h^2\Delta t\left(h^2\|\nabla\cdot\mathbf{u^n}\|_1^2+\|\mathbf{u^n}\|^2_{(H^1(\Omega))^2}+\|p^n\|_1^2\right)\Big]. \quad (2.5.30)\end{aligned}$$

Taking summation over $n = 0 \cdots m-1$, we obtain

$$|||\theta^m|||^2 - |||\theta^0|||^2 \leq C\Big[\sum_{n=0}^{m-1}\Big\{\Delta t\left(\|\theta^{n+1}\|^2 + \|\theta^n\|^2 + \|\rho^n\|^2 + \|\rho^{n+1}\|^2\right) + \|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)}$$
$$+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)} + \|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big)$$
$$+h^2\Delta t\left(h^2\|\nabla\cdot\mathbf{u^n}\|_1^2 + \|\mathbf{u^n}\|^2_{(H^1(\Omega))^2} + \|p^n\|_1^2\right)\Big\}\Big]. \tag{2.5.31}$$

Use Gronwall's Lemma (Lemma 1.2.8), equivalence of the norms $|||\cdot|||$ and $\|\cdot\|$ given in (2.3.46) and the estimates of ρ to obtain

$$\|\theta^m\|^2 \leq C\Big[\|\theta^0\|^2 + h^4\Big(\|c\|^2_{L^\infty(0,T;H^2)} + \|g\|^2_{L^\infty(0,T;H^1)} + \|\mathbf{u}\cdot\nabla c\|^2_{L^\infty(0,T;H^1)}$$
$$+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)} + \|c_t\|^2_{L^2(0,T;H^2)} + \|g_t\|^2_{L^2(0,T;H^1)} + \|(\mathbf{u}\cdot\nabla c)_t\|^2_{L^2(0,T;H^1)}$$
$$+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big) + (\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)}$$
$$+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big) + h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \tag{2.5.32}$$

Now since $c_h(0) = R_h c(0)$, i.e., $\theta^0 = 0$, (2.5.32) implies that

$$\max_{0\leq m\leq N}\|\theta^m\|^2 \leq C\Big[h^4\Big(\|c\|^2_{L^\infty(0,T;H^2)} + \|g\|^2_{L^\infty(0,T;H^1)} + \|\mathbf{u}\cdot\nabla c\|^2_{L^\infty(0,T;H^1)} + \|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)}$$
$$+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|c_t\|^2_{L^2(0,T;H^2)} + \|g_t\|^2_{L^2(0,T;H^1)} + \|(\mathbf{u}\cdot\nabla c)_t\|^2_{L^2(0,T;H^1)} + \|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big)$$
$$+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;L^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2(\Omega)^2)} + \|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big)$$
$$+h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \tag{2.5.33}$$

Using (2.5.33), the hypothesis (2.5.28) can be justified with the assumption that $\Delta t = O(h)$ in the similar way, as we have proved the hypothesis (2.4.68). Now combined the estimate of θ and ρ to completes the rest of the proof. ∎

Using (2.5.5) and (2.5.7), we obtain the following error estimates for velocity as well as pressure .

THEOREM 2.5.2 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$ be, respectively, the solutions of (2.1.1)-(2.1.2) and (2.5.1)-(2.5.2) and let $c_h(0) = c_{0,h} =$*

$R_h c(0)$. *Further, assume that* $\Delta t = O(h)$. *Then for sufficiently small* h, *there exists a positive constant* $C(T)$ *independent of* h *but dependent on the bounds of* κ^{-1} *and* μ *such that*

$$\max_{0\le m\le N} \|\mathbf{u^m}-\mathbf{u_h^m}\|^2_{(L^2(\Omega))^2}+\|p^m-p_h^m\|^2 \le C\Big[h^4\Big(\|c\|^2_{L^\infty(0,T;H^2)}+\|g\|^2_{L^\infty(0,T;H^1)}$$
$$+\|\mathbf{u}\cdot\nabla c\|^2_{L^\infty(0,T;H^1)}+\|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)}+\|c_t\|^2_{L^2(0,T;H^2)}+\|g_t\|^2_{L^2(0,T;H^1)}+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)}$$
$$+\|(\mathbf{u}\cdot\nabla c)_t\|^2_{L^2(0,T;H^1)}+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big)+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;L^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2(\Omega)^2)}$$
$$+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big)+h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)}+\|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big].$$

2.6 Numerical Procedure

In this section, we discuss numerical methods applied to system (2.1.1)-(2.1.6). For the pressure equations (2.1.1)-(2.1.2), we apply mixed finite volume element method and for the approximation of the concentration equation (2.1.3), we use the standard finite volume method. We consider two test problems, one when only the molecular diffusion is present and the effect of dispersion coefficients is negligible and the second when dispersion coefficients are present. For our numerical experiments, we consider the following set of equations:

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall(x,t)\in\Omega\times J, \tag{2.6.1}$$
$$\nabla\cdot\mathbf{u} = q^+ - q^- \qquad \forall(x,t)\in\Omega\times J, \tag{2.6.2}$$
$$\phi\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(D(\mathbf{u})\nabla c)+cq^- = \bar{c}q^+ \qquad \forall(x,t)\in\Omega\times J, \tag{2.6.3}$$

with boundary conditions

$$\mathbf{u}\cdot\mathbf{n}=0 \qquad \forall(x,t)\in\partial\Omega\times J, \tag{2.6.4}$$
$$D(\mathbf{u})\nabla\cdot\mathbf{n}=0 \qquad \forall(x,t)\in\partial\Omega\times J, \tag{2.6.5}$$

and initial condition

$$c(x,0)=c_0(x) \qquad \forall x\in\Omega. \tag{2.6.6}$$

Here, $\mu(c)$ is the viscosity of the fluid mixture which depends on the concentration and is given by

$$\mu(c) = \mu(0)\left[(1-c) + M^{\frac{1}{4}}c\right]^{-4}, \tag{2.6.7}$$

where M is the mobility ratio between the resident and injected fluids and $\mu(0)$ is the viscosity of the resident fluid, $\bar{c}$ is the injection concentration and q^+ and q^- are the production and injection rates, respectively. Let $\mathcal{T}_h$ be an admissible regular, uniform triangulation of $\overline{\Omega}$ into closed triangles.

Experimentally, it has been observed that the velocity is much smoother in time compared to the concentration. It was suggested in [47] that for a good approximation to the concentration, one should take larger time step for the pressure equation than the concentration equation. Let $0 = t_0 < t_1 < \cdots t_M = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t_m = t_{m+1} - t_m$ for the pressure equation and $0 = t^0 < t^1 < \cdots t^N = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t^n = t^{n+1} - t^n$ for the concentration equation. We denote $C^n \approx c_h(t^n)$, $C_m \approx c_h(t_m)$, $U_m \approx \mathbf{u_h}(t_m)$ and $P_m \approx p_h(t_m)$.

If concentration step t^n relates to pressure steps by $t_{m-1} < t^n \le t_m$, we require a velocity approximation at $t = t^n$, which will be used in the concentration equation, based on $\mathbf{U}_{m-1}$ and earlier values. We define a velocity approximation [44, pp. 81] at $t = t^n$ by

$$E\mathbf{U^n} = \left(1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\right)\mathbf{U}_{m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{U}_{m-2} \qquad \text{for} \quad m \ge 2, \tag{2.6.8}$$

$$E\mathbf{U^n} = \mathbf{U}_0 \qquad \text{for} \ \ m = 1. \tag{2.6.9}$$

The discrete problem corresponding to pressure-velocity equations (2.3.6)-(2.3.7) is to find $(\mathbf{U}, P) : \{t_0, t_1, \cdots t_M\} \longrightarrow U_h \times W_h$ such that

$$\begin{aligned} (\kappa^{-1}\mu(C_m)\mathbf{U}_m, \gamma_h \mathbf{v_h}) + (\nabla\cdot\mathbf{v_h}, P_m) = 0 \quad \forall \mathbf{v_h} \in U_h \\ (\nabla\cdot\mathbf{U}_m, w_h) = (q^+ - q^-, w_h) \qquad \forall w_h \in W_h, \quad m \ge 0. \end{aligned} \tag{2.6.10}$$

Set $\frac{\partial C}{\partial t}|_{t=t_{n+1}} \approx \frac{C^{n+1} - C^n}{\Delta t^n}$. Then, the discrete problem corresponding to concentration equation (2.3.9) is to find $C : \{t^0, t^1, \cdots t^N\} \longrightarrow M_h$ such that

$$\begin{aligned} &\left(\phi\frac{(C^{n+1} - C^n)}{\Delta t_n}, \Pi_h^* z_h\right) + (E\mathbf{U^{n+1}}\cdot\nabla C^{n+1}, \Pi_h^* z_h) \\ &\quad + a_h(E\mathbf{U^{n+1}}; C^{n+1}, z_h) + (q^- c^{n+1}, \Pi_h^* z_h) = (\bar{c}q^+, \Pi_h^* z_h) \quad \forall z_h \in M_h. \end{aligned} \tag{2.6.11}$$

Using $C^0 = C_0 = R_h c_0(x)$, we first find (U_0, P_0) from (2.6.10) and then using U_0, we find C^1 from (2.6.11) and so on.

To put equations (2.6.10) and (2.6.11) in matrix form, let N_m be the total number of edges of the triangulation $\mathcal{T}_h$, $(M_i)_{i=1}^{N_m}$ be the midpoint of edges of triangulation $\mathcal{T}_h$, N_h be the total number of vertices of the triangulation $\mathcal{T}_h$ and N_t be the total number of triangles. Let $(\mathbf{\Phi_i})_{i=1}^{N_m}$ be the edge oriented basis functions for the trial space U_h and $\{\chi_l^*\}_{l=1,2,\cdots,N_t}$ be the characteristic functions corresponding to the triangles which form basis functions for the test space W_h. Now we discuss the construction of the basis functions $\mathbf{\Phi_i}$, see [6].

Construction of local and global basis functions for the space U_h:

Let the vertices of a triangle be denoted by P_1, P_2, P_3 and the edges opposite to the vertices be denoted by E_1, E_2 and E_3, respectively. Let the midpoints of E_1, E_2, E_3 be M_1, M_2 and M_3, respectively. We denote the coordinate of the vertices P_1, P_2, P_3 as $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, respectively (see Figure 2.6). Let $\mathbf{\Phi}_{\boldsymbol{E_1}}$, $\mathbf{\Phi}_{\boldsymbol{E_2}}$ and $\mathbf{\Phi}_{\boldsymbol{E_3}}$ be the three local basis functions corresponding to the edges E_1, E_2, E_3, respectively for the triangle $T = \triangle P_1 P_2 P_3$ such that

$$(\mathbf{\Phi_{E_i}}.\mathbf{n}_j)(M_j) = \delta_{ij},\ i, j = 1, 2, 3. \tag{2.6.12}$$

Here, $\mathbf{n}_j$ is the outward normal vector to the edge E_j. If we define local basis functions for the triangle T as

$$\mathbf{\Phi_{E_i}} = a_1 \frac{|E_i|}{2|T|}(x - x_i, y - y_i) \quad (a_1 = +1, \text{ or } = -1),$$

then it can be easily checked that $\mathbf{\Phi_{E_i}}$ satisfies (2.6.12). Here, $|E_i|$ denotes the length of the edge E_i and $|T|$ denotes the area of the triangle T. Now we will construct the global basis functions with the help of local basis functions. Let $\{\mathbf{\Phi_i}\}_1^{N_m}$ be the global basis functions corresponding to the edges e_i. Now referring to Figures 2.7 and 2.8, the global basis functions for U_h can be defined in the following way

$$\mathbf{\Phi_1} = \begin{cases} \mathbf{\Phi_{E_3}}, & \text{on } T_1 \\ 0, & \text{otherwise.} \end{cases}, \quad \mathbf{\Phi_2} = \begin{cases} \mathbf{\Phi_{E_2}}, & \text{on } T_1 \\ 0, & \text{otherwise.} \end{cases}$$

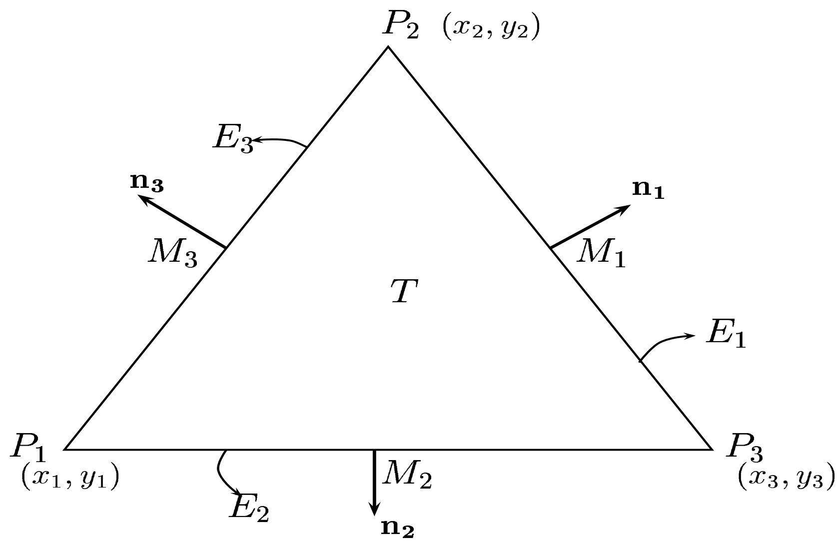

Figure 2.6: Normal vectors to the edges

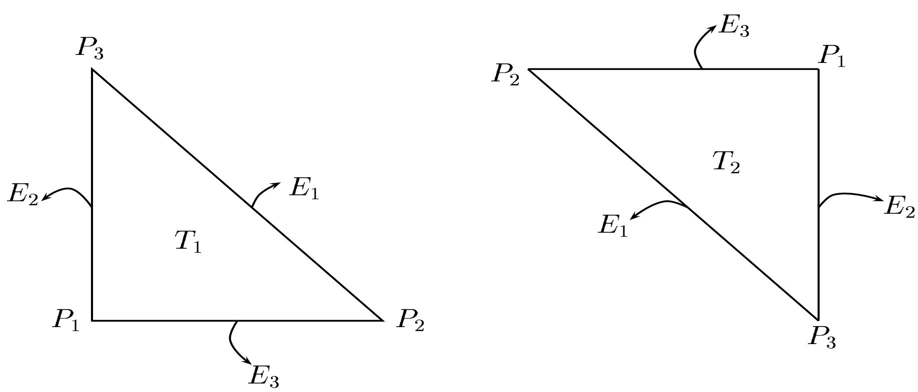

Figure 2.7: Local numbering (E_i) of edges

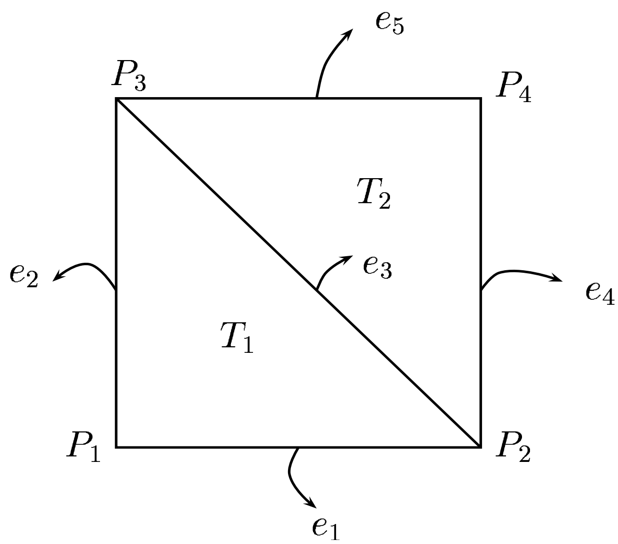

Figure 2.8: Global numbering (e_i) of edges

,

$$\mathbf{\Phi_3} = \begin{cases} \mathbf{\Phi_{E_1}}, & \text{on } T_1 \\ \mathbf{\Phi_{E_1}}, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{\Phi_4} = \begin{cases} \mathbf{\Phi_{E_2}}, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}, \quad \mathbf{\Phi_5} = \begin{cases} \mathbf{\Phi_{E_3}}, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}$$

So, $\mathbf{U}_m$ and P_m can be written as

$$\mathbf{U}_m(x) = \sum_{j=1}^{N_m} \alpha_j^m \mathbf{\Phi_j}(x), \quad P_m(x) = \sum_{l=1}^{N_t} \beta_l^m \chi_l^*(x) \tag{2.6.13}$$

with $\alpha_j = (\mathbf{u_h} \cdot \mathbf{n_j})(M_j)$, $\beta_l = p_h(B_l)$, $\mathbf{n}_j$ being the outward normal to the edge E_j and B_l being the barycenter corresponding to the triangle T_l. Use the definition of the transfer operator γ_h and (2.6.13), the equation (2.6.10) can be written in the matrix form

$$\begin{pmatrix} A_m & B_m \\ B_m^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^m \\ \boldsymbol{\beta}^m \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{F}^m \end{pmatrix}, \tag{2.6.14}$$

where, B_m^T is the transpose of B_m and the matrices are given by

$$\boldsymbol{\alpha}^m = (\alpha_j^m)_{j=1}^{N_m}, \quad \boldsymbol{\beta}^m = (\beta_l^m)_{l=1}^{N_t}$$

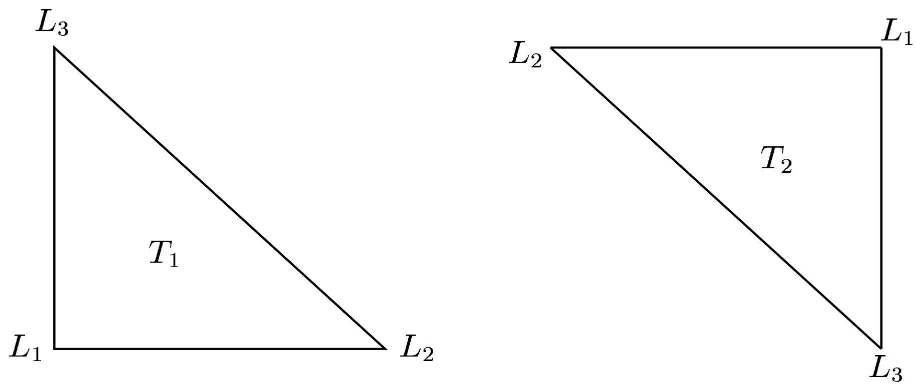

Figure 2.9: Local numbering (L_i) of vertices

$$A_m = (a_{ij})_{i,j=1}^{N_m} = \int_{K_{p_i}^*} \kappa^{-1}\mu(C_m(x))\mathbf{\Phi_j}\cdot\mathbf{\Phi_i}(M_i)dx,$$

$$B_m = (b_{lj})_{l=1,2\cdots N_t,\ j=1,2\cdots N_m} = \int_{T_l} \nabla\cdot\mathbf{\Phi_j}\ dx$$

and

$$\mathbf{F}_m = (f_l^m)_{l=1}^{N_t} = \int_{K_l} (q^+ - q^-)dx.$$

In the same say, referring to the Figures 2.9 and 2.10, we construct the global basis functions for the finite dimensional space M_h. Let λ_1, λ_2 and λ_3 be the barycentric coordinates corresponding to the any triangle $T \in \mathcal{T}_h$ and associated with the vertices P_1, P_2 and P_3, respectively (see [29, pp. 45]). Then the global basis functions $\{\Psi_i\}_{i=1}^{N_h}$ for M_h is defined in the following manner

$$\Psi_1 = \begin{cases} \lambda_1, & \text{on } T_1 \\ 0, & \text{otherwise.} \end{cases}, \quad \Psi_2 = \begin{cases} \lambda_2, & \text{on } T_1 \\ \lambda_3, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}$$

,

$$\Psi_3 = \begin{cases} \lambda_3, & \text{on } T_1 \\ \lambda_2, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}, \quad \Psi_4 = \begin{cases} \lambda_1, & \text{on } T_2 \\ 0, & \text{otherwise.} \end{cases}$$

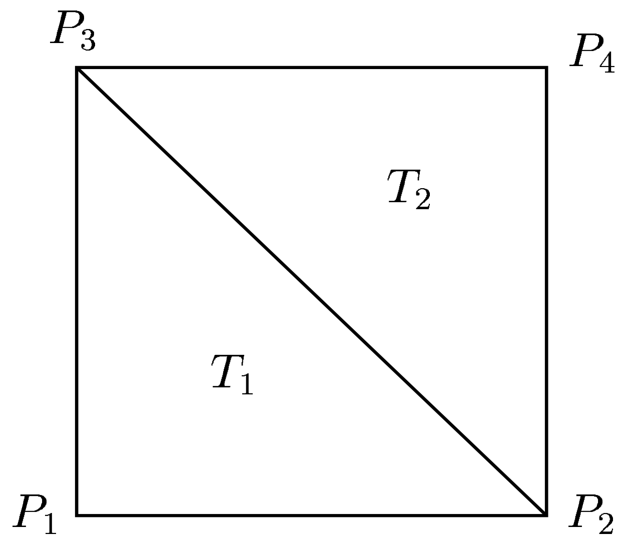

Figure 2.10: Global numbering (P_i) of vertices

If we set $C^n = \sum_{i=1}^{N_h} \gamma_i^n \Psi_i$, where Ψ_i's are the basis functions for the space M_h, then the concentration equation (2.6.11) can be written in the following matrix form:

$$[D^n + \Delta t^n (E^n + H^n + R^n)]\, \boldsymbol{\gamma}^{n+1} = D^n \boldsymbol{\gamma}^n + \Delta t^n \mathbf{G}^n, \tag{2.6.15}$$

where

$$\boldsymbol{\gamma}^n = (C^n(P_i))_{i=1}^{N_h}, \quad D^n = (d_{ij})_{i,j=1}^{N_h} = \int_{V_i} \psi_j dx,$$

$$E^n = (e_{ij})_{i,j=1}^{N_h} = \int_{V_i} E\mathbf{U^n} \cdot \nabla \Psi_j dx \quad H^n = (h_{ij})_{i,j=1}^{N_h} = -\int_{\partial V_i} D(E\mathbf{U^n}) \nabla \Psi_j \cdot n_j ds,$$

and

$$R^n = (r_{ij})_{i,j=1}^{N_h} = \int_{V_i} q^- \Psi_j dx, \qquad \mathbf{G}^n = (g_i^n)_{i=1}^{N_h} = \int_{V_i} \bar{c} q^+ dx.$$

2.6.1 Test Problems

For the test problems, we have taken the data from [80]. The spatial domain is $\Omega = (0, 1000) \times (0, 1000)$ ft^2 and the time period is $[0, 3600]$ days, viscosity of oil is $\mu(0) = 1.0$ cp. The injection well is located at the upper right corner $(1000, 1000)$ with the injection rate $q^+ = 30$ft^2/day and injection concentration $\bar{c} = 1.0$. The production well is located at

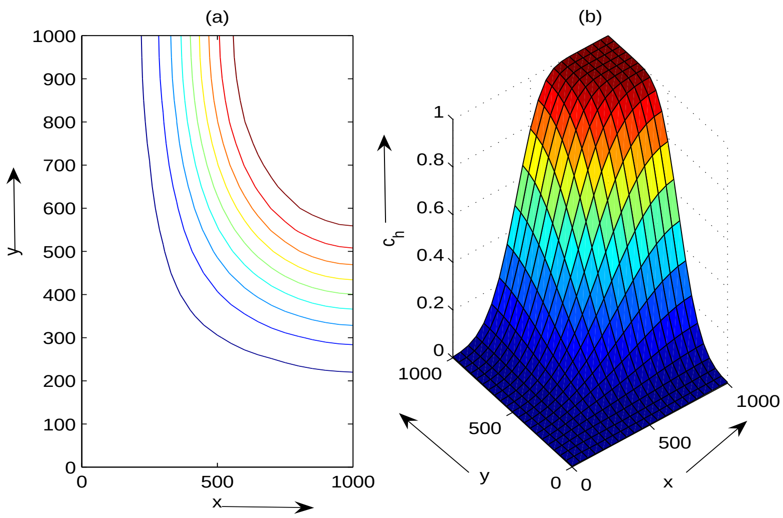

Figure 2.11: Contour (a) and surface plot (b) in Test 1 at $t = 3$ years.

the lower left corner with the production rate $q^- = 30\text{ft}^2/\text{day}$ and the initial concentration is $c(x, 0) = 0$. In the numerical simulation for spatial discretization we choose in 20 divisions on both x and y axes. For time discretization, we take $\Delta t_p = 360$ days and $\Delta t_c = 120$ days, i.e., we divide each pressure time interval into sub three intervals.

Test 1: We assume that the porous medium is homogeneous and isotropic. The permeability κ is 80. The porosity of the medium is $\phi = .1$ and the mobility ratio between the resident and injected fluid is $M = 1$. Further, we assume that the molecular diffusion is $d_m = 1$ and dispersion coefficients are zero.

The surface and contour plots for the concentration at $t = 3$ and $t = 10$ years are presented in Figure 2.11 and Figure 2.12, respectively. Since only molecular diffusion is present and viscosity is also independent of the velocity, Figure 2.11, shows that the velocity is radial and the contour plots for the concentration is almost circular until the invading fluid reaches the production well. Figure 2.12 shows that when these plots are reached at production well, the invading fluid continues to fill the whole domain until $c = 1$.

Test 2: In this test, the permeability tensor is same as in test 1. The adverse mobility ratio is $M = 41$. We assume that the physical diffusion and dispersion coefficients are given by $\phi d_m = 0.0\text{ft}^2/\text{day}$, $\phi d_t = 5.0\text{ft}$ and $\phi d_t = .5\text{ft}$. From (2.6.7), in test 1, $\mu(c)$ was

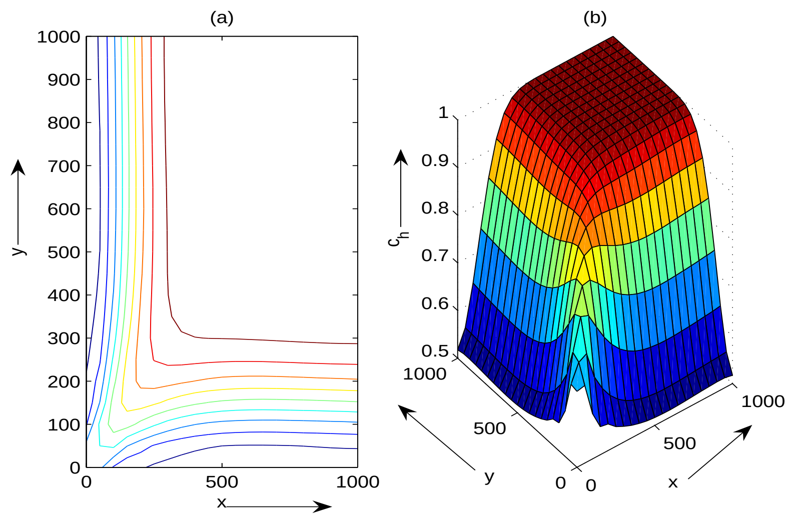

Figure 2.12: Contour (a) and surface plot (b) in Test 1 at $t = 10$ years.

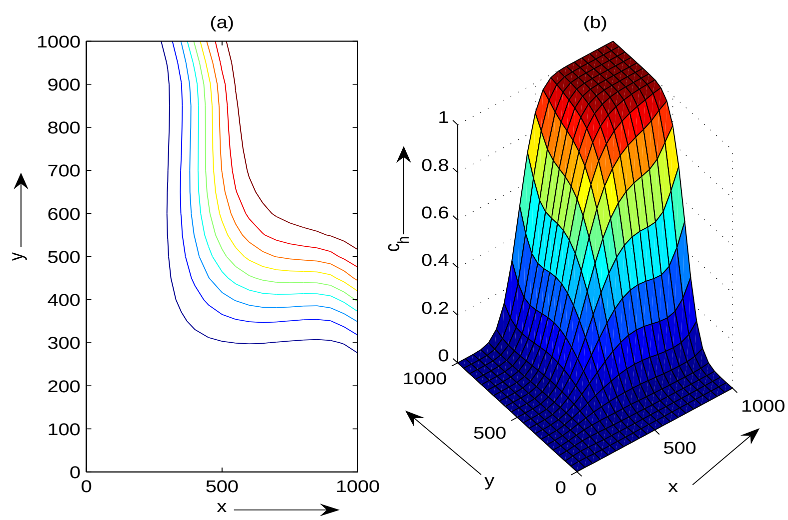

Figure 2.13: Contour (a) and surface plot (b) in Test 2 at $t = 3$ years.

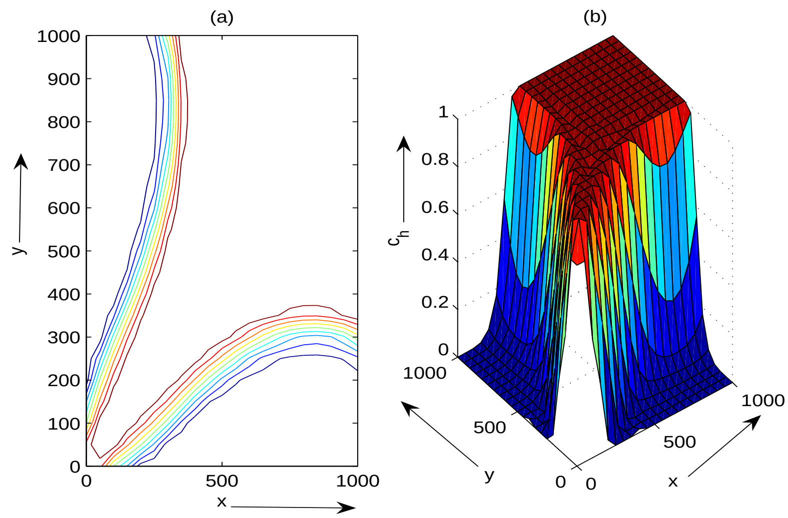

Figure 2.14: Contour (a) and surface plot (b) in Test 2 at $t = 10$ years.

independent of the concentration c but here $\mu(c)$ depends on c . The difference between the longitudinal and the transverse dispersity coefficients implies that the fluid flow is much faster along the diagonal direction see Figures 2.13 and 2.14.

Test 3**:** In this test we consider the numerical simulation of a miscible displacement problem with discontinuous permeability. Here, the data is same as given in Test 1 except the permeability of the medium $\kappa(x)$. We take $\kappa = 80$ on the sub domain $\Omega_L := (0, 1000) \times (0, 500)$ and $\kappa = 20$ on the sub domain $\Omega_U := (0, 1000) \times (500, 1000)$. The contour and surface plot at $t = 3$ and $t = 10$ years are given in Figure 2.15 and Figure 2.16 respectively.

Test 4**:**. In this test, we consider the miscible displacement problem with effect of numerical dispersion with discontinuous permeability. Here data is same as in given Test 2 except the permeability of the medium. We take $\kappa = 80$ on the sub domain $\Omega_L := (0, 1000) \times (0, 500)$ and $\kappa = 20$ on the sub domain $\Omega_U := (0, 1000) \times (500, 1000)$. The contour and surface plot at $T = 3$ and $T = 10$ years are given in Figure 2.17 and Figure 2.18 respectively. The lower half domain has a larger permeability than the upper half. Therefore, when the injecting fluid reaches the lower half domain, it starts moving much faster in the horizontal direction on this domain compared to the low permeability domain that is upper half domain. We observe that one should put the production well in a low

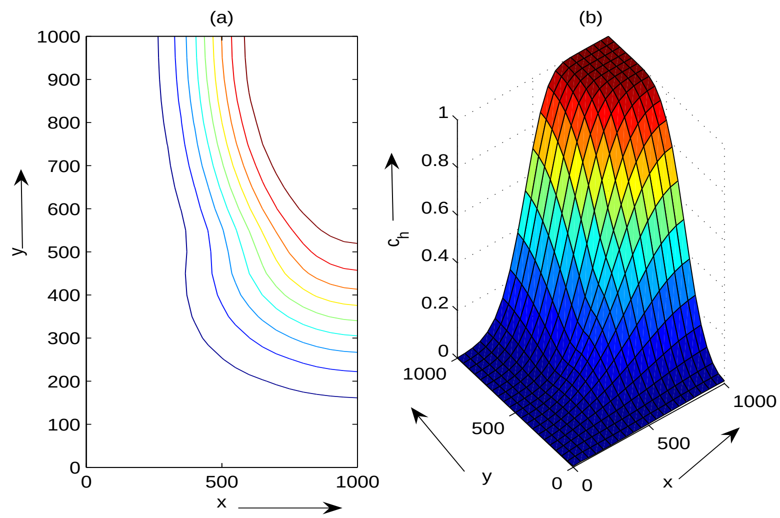

Figure 2.15: Contour (a) and surface plot (b) in Test 3 at $t = 3$ years.

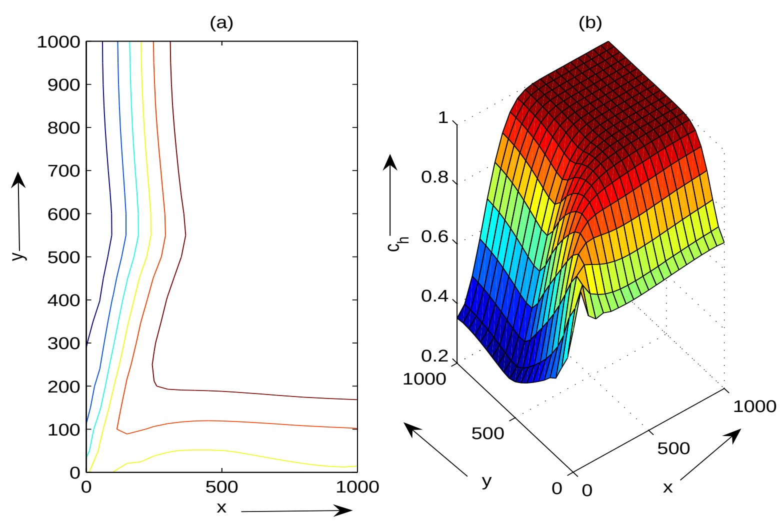

Figure 2.16: Contour (a) and surface plot (b) in Test 3 at $t = 10$ years.

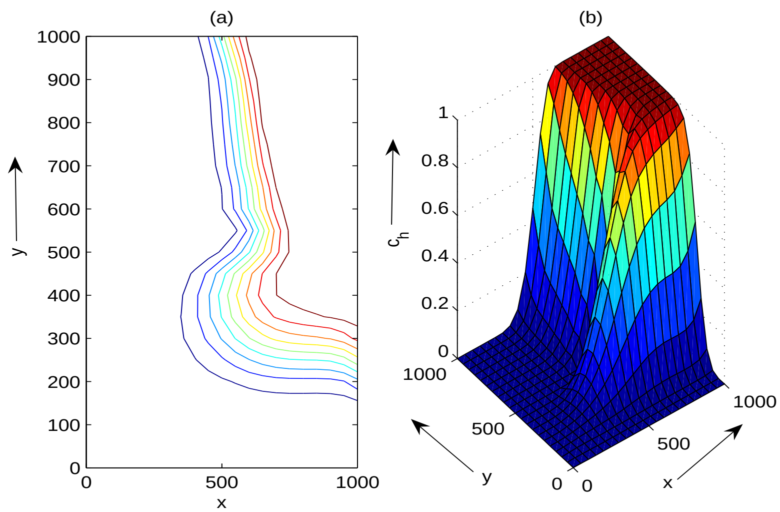

Figure 2.17: Contour (a) and surface plot (b) in Test 4 at $t = 3$ years.

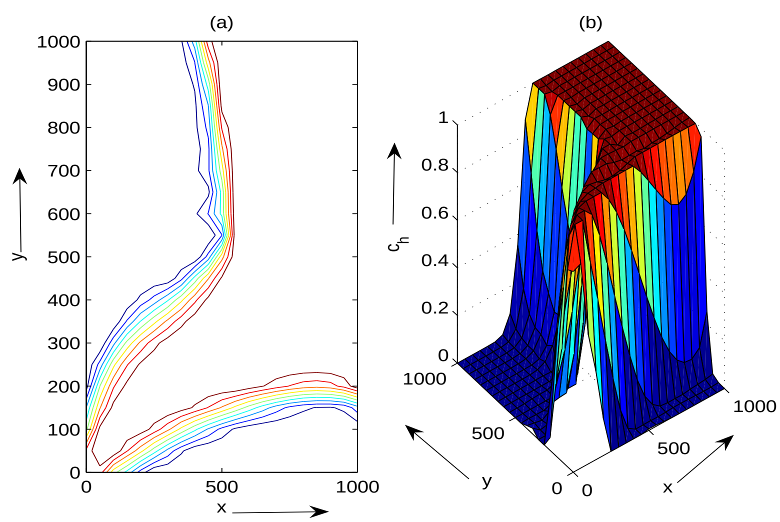

Figure 2.18: Contour (a) and surface plot (b) in Test 4 at $t = 10$ years

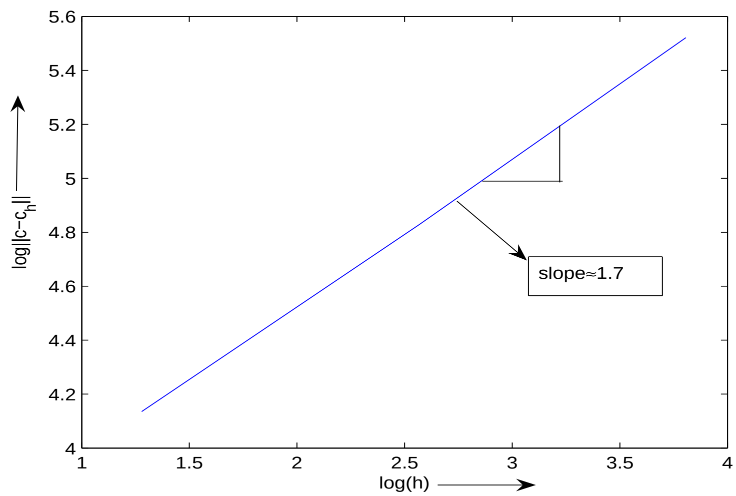

Figure 2.19: Order of convergence in L^2- norm for Test 1

permeability zone to increase the area swept by the injected fluid. It is also can be noted that the area occupied by the invading fluid at $t = 10$ year in Test 4 is larger compared to the area occupied by invading fluid in Test 2, where the permeability is constant. This tells us how the numerical simulations could help the decision making in the petroleum reservoir industry.

Order of Convergence: In order to verify our theoretical results we also compute the order of convergence for the concentration. We compute the order of convergence in L^2 norm. To discretize the time interval $[0, T]$, we take uniform time step $\Delta t = 360$ days for pressure and concentration equation. The computed order of convergence is given in Figure 2.19. Note that the computed order of convergence matches with the theoretical order of convergence derived in Theorem 2.5.1.

Chapter 3

Discontinuous Galerkin Finite Volume Element Approximations

3.1 Introduction

The main objective in this chapter is to study a discontinuous finite volume element method for the approximation of the concentration equation. As in Chapter 2, we have used a mixed finite volume element method for the approximation of the pressure-velocity equation. *A priori* error estimates in $L^\infty(L^2)$ norm are derived for velocity, pressure and concentration for the semidiscrete and the fully discrete schemes. Numerical results are presented to validate our theoretical results.

In recent years, there has been a renewed interest in Discontinuous Galerkin (DG) methods for the numerical approximation of partial differential equations. This is due to their flexibility in local mesh adaptivity and handling nonuniform degrees of approximation for solutions whose smoothness exhibit variation over the computational domain. DG methods have the advantage that they are element-wise conservative and are easy to implement with high degree of piecewise polynomials compared to other numerical methods such as conforming finite element methods, finite volume element methods, mixed finite element and finite volume methods. In conforming finite element methods, the approximating functions should satisfy some continuity criteria across the interelement boundaries, but in DG methods, we have the freedom to choose discontinuous functions in the finite element spaces.

The first work on DG methods for elliptic and parabolic problems trace back to the work by Douglas *et al.*[35] and Wheeler [82]. In 1973, Babuška [5] introduced a penalty method to impose the Dirichlet boundary condition weakly. Interior Penalty (IP) methods by Arnold [3] and Wheeler [82] arose from the observation that just as Dirichlet boundary conditions, interelement continuity of approximating functions could be imposed weakly instead of being built into the finite element space. This makes it possible to use spaces of discontinuous piecewise polynomials of higher degree. These IP methods are based on the Nitsche's symmetric formulation and hence are presently called as symmetric interior penalty Galerkin (SIPG) methods. The variational formulation of the SIPG methods is symmetric and adjoint consistent. This helps in developing optimal L^2- error estimates. However, for the coercivity of the associated bilinear form, we need to choose the penalty parameter large enough. Rivière *et al.* [70] and Houston *et al.* [50] have introduced and analyzed the non-symmetric interior penalty Galerkin (NIPG) methods. A significant property of NIPG method is that it is unconditionally stable with respect to the choice of the penalty parameter. Hence, this advantage has stimulated renewed interest in applying these methods to a large class of partial differential equations. It is noted that NIPG methods are not adjoint consistent. In this case, optimal L^2- error estimates can be derived by using super-penalty techniques, for more details, we refer to [4].

Keeping in mind the advantages of FVEM and DG methods, it is natural to think of discontinuous Galerkin finite volume element methods (DGFVEMs) for the numerical approximation of partial differential equations. In these methods, the support of the control volumes are small compared to the standard FVM [60], mixed FVM [26]. Also the control volumes have support inside the triangle in which they belong to and there is no contribution from adjacent triangles, see Figures 3.1 and 3.4. This property of the control volumes makes DGFVEM more suitable for parallel computing. DGFVEM for the elliptic problems has been discussed in [28, 55, 85] and for Stokes problem in [86].

As mentioned in Chapter 1, a mathematical model, which describes the miscible displacement of one incompressible fluid over another in a porous medium is given by the following set of partial differential equations.

For a given $T > 0$, the pressure p and the concentration c satisfy

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall (x,t) \in \Omega \times J, \tag{3.1.1}$$

$$\nabla \cdot \mathbf{u} = q \qquad \forall (x,t) \in \Omega \times J, \tag{3.1.2}$$

$$\phi(x)\frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u})\nabla c - \mathbf{u}c) = g(c) \qquad \forall (x,t) \in \Omega \times J, \tag{3.1.3}$$

with boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{3.1.4}$$

$$(D(\mathbf{u})\nabla c - \mathbf{u}c) \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{3.1.5}$$

and initial condition

$$c(x,0) = c_0(x) \qquad \forall x \in \Omega, \tag{3.1.6}$$

where $g(c) = \tilde{c}q$. Since the concentration equation (3.1.3) has the transport term $\mathbf{u} \cdot \nabla c$, which dominates the diffusion term, the solution of (3.1.3) varies rapidly from one part of the domain to the other. Standard Galerkin methods based on C^0- piecewise-polynomials for such problems show unacceptable oscillations in the approximation. A use of C^1-piecewise-polynomials smear the front excessively leading to very smooth approximations. To strike a balance between these two methods, Douglas and Dupont [35] have introduced and analyzed a new method which uses interior penalties across the interior edges of the triangles for convection dominated diffusion equation. Wheeler *et al.* [83] have extended this procedure for the approximation of convection dominated diffusion equation for incompressible miscible displacement problem in a porous media.

Sun *et al.* [76] applied the mixed FEM for pressure-velocity equation and discontinuous Galerkin FEM for approximating the concentration. Further, Sun and Wheeler [77] applied symmetric and nonsymmetric discontinuous Galerkin methods for approximation of the concentration equation by assuming that the velocity is known and time independent. In this chapter, we apply mixed FVEM for the approximation of pressure-velocity equations (3.1.1)-(3.1.2) and a discontinuous Galerkin FVEM for approximating the concentration equation (3.1.3).

This chapter is organized as follows. In Section 3.2, the weak formulation for incompressible miscible displacement problems in a porous media is described. In Section 3.3, we discuss the DGFVEM formulation. The existence and uniqueness of solution to the discrete problem is also discussed. *A priori* error estimates for velocity, pressure and concentration are presented in Section 3.4. Finally in Section 3.5, the numerical procedure is discussed and the results of some numerical experiments are presented.

3.2 Weak formulation

Define

$$U = \{\mathbf{v} \in H(\text{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

and

$$W = L^2(\Omega)/\mathbb{R}.$$

Then, the weak form the pressure-velocity equations (2.1.1)-(2.1.2) is to seek
$(\mathbf{u}, p) : \overline{J} \longrightarrow U \times W$ satisfying

$$(\kappa^{-1}\mu(c)\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = 0 \quad \forall \mathbf{v} \in U, \tag{3.2.1}$$

$$(\nabla\cdot\mathbf{u}, w) = (q, w) \qquad \forall w \in W. \tag{3.2.2}$$

The concentration equation (3.1.3) can be put in the weak form as follows. Find a differentiable map $c : \overline{J} \longrightarrow H^1(\Omega)$ such that

$$\begin{aligned} (\phi\frac{\partial c}{\partial t}, z) + a(\mathbf{u}; c, z) &= (g(c), z) \quad \forall z \in H^1(\Omega), \\ c(x,0) &= c_0(x) \quad \forall x \in \Omega, \end{aligned} \tag{3.2.3}$$

where $a(\mathbf{v};\cdot,\cdot) : H^1(\Omega)\times H^1(\Omega) \longrightarrow \mathbb{R}$ is a continuous bilinear form defined by

$$a(\mathbf{v};\phi,\psi) = \int_\Omega (D(\mathbf{v})\nabla\phi - \mathbf{v}\phi)\cdot\nabla\psi \;\; dx \quad \forall \phi,\ \psi \in H^1(\Omega), \mathbf{v} \in \mathbb{R}^2.$$

3.3 Discontinuous Finite volume element approximation

A mixed finite volume element approximation for the pressure equation is defined as:
Find $(\mathbf{u_h}, p_h) \in U_h \times W_h$ such that

$$(\kappa^{-1}\mu(c_h)\mathbf{u_h}, \gamma_h \mathbf{v_h}) + b(\gamma_h \mathbf{v_h}, p_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{3.3.1}$$

$$(\nabla \cdot \mathbf{u_h}, w_h) = (q, w_h) \qquad \forall w_h \in W_h, \tag{3.3.2}$$

where the bilinear form $b(\cdot, \gamma \cdot)$ is defined in (2.3.5). The trial spaces U_h and W_h for velocity and pressure, respectively, and the transfer operator γ_h are already defined in Section 2.3 of Chapter 2 (see, (2.3.1), (2.3.2) and (2.3.3)).
Let $\mathcal{T}_h$ be a regular, quasi-uniform triangulation of $\bar{\Omega}$ into closed triangles T. With Γ denoting the union of all the interior edges of the triangles T of $\mathcal{T}_h$, we now introduce the the dual mesh $\mathcal{V}_h^*$ based on $\mathcal{T}_h$ which will be used in the approximation of concentration equation. The dual partition $\mathcal{V}_h^*$ corresponding to the primal partition $\mathcal{T}_h$ is constructed as follows: Divide each triangle $T \in \mathcal{T}_h$ into three triangles by joining the barycenter B and the vertices of T as shown is Figure 3.1. In general, let V^* denote the dual element/control volume in $\mathcal{V}_h^*$, see Figure 3.2. The union of these sub-triangles form the dual partition $\mathcal{V}_h^*$ of $\bar{\Omega}$.

We introduce the standard definitions of jumps and averages [4] for scalar and vector functions as follows. For an interior edge e shared by two elements T_1 and T_2, having normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ pointing exterior to T_1 and T_2 (see Figure 3.3) respectively, the average $\langle \cdot \rangle$ and jump $[\cdot]$ on e for a scalar q and a vector $\mathbf{r}$ are defined, respectively, as:

$$\langle q \rangle = \frac{1}{2}(q_1 + q_2), \quad [q] = q_1 \mathbf{n_1} + q_2 \mathbf{n_2},$$

$$\langle \mathbf{r} \rangle = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \quad [\mathbf{r}] = \mathbf{r_1} \cdot \mathbf{n_1} + \mathbf{r_2} \cdot \mathbf{n_2},$$

where $q_i = (q|_{T_i})|_e, \;\; \mathbf{r}_i = (\mathbf{r}|_{T_i})|_e, \; i = 1, 2.$
In case e is an edge on $\partial\Omega$, we define

$$\langle q \rangle = q, \quad [q] = q\mathbf{n},$$

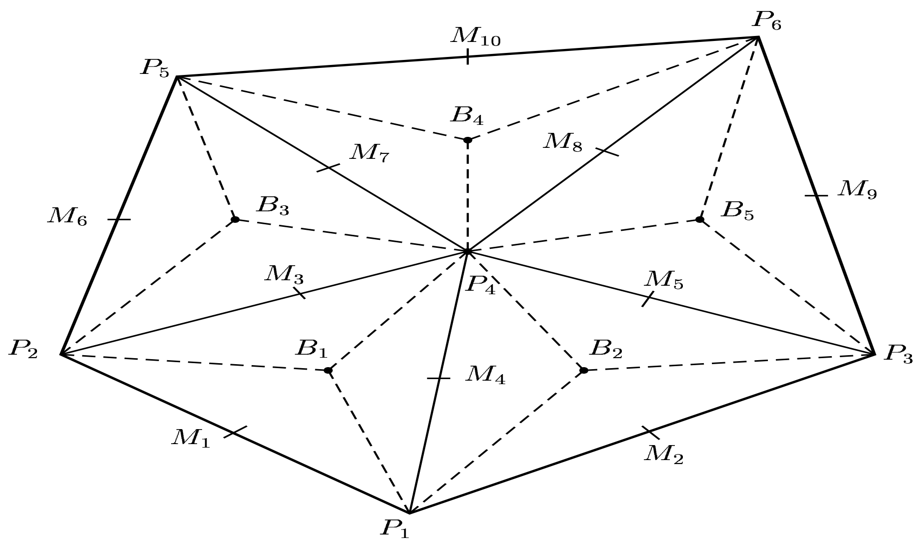

Figure 3.1: Triangular partition and dual elements.

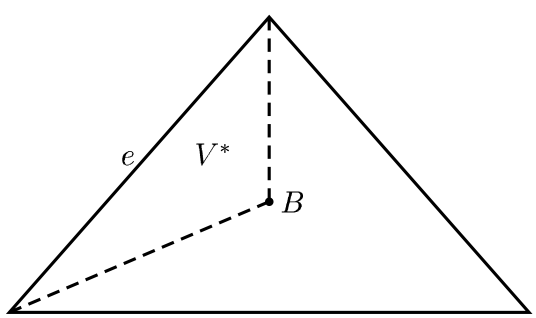

Figure 3.2: An element V^* in the dual partition.

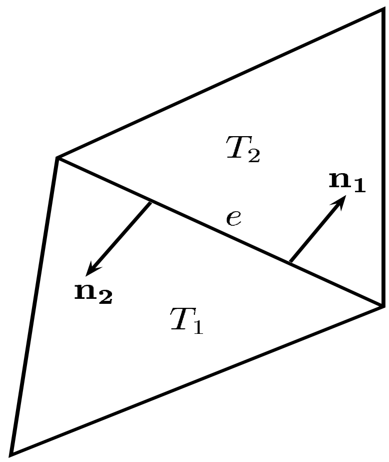

Figure 3.3: Outward normal vectors to the edge e.

$$\langle \mathbf{r} \rangle = \mathbf{r}, \quad [\mathbf{r}] = \mathbf{r} \cdot \mathbf{n},$$

$\mathbf{n}$ being the outward normal vector to the boundary $\partial\Omega$.

For applying DGFVEM to approximate the concentration equation, we define the finite dimensional trial and test spaces M_h and L_h on $\mathcal{T}_h$ and $\mathcal{V}_h^*$, respectively, as

$$M_h = \{v_h \in L^2(\Omega) : \quad v_h|_T \in P_1(T) \ \ \forall T \in \mathcal{T}_h\},$$

$$L_h = \{w_h \in L^2(\Omega) : \quad w_h|_{V^*} \in P_0(V^*) \ \ \forall V^* \in \mathcal{V}_h^*\},$$

where $P_m(T)$(resp. $P_m(V^*)$) denotes the polynomials of degree less than or equal to m defined on T(resp. V^*).

Let $M(h) = M_h + H^2(\Omega)$. Define

$$|||v|||^2 \;=\; |v|_{1,h}^2 + \sum_{e\in\Gamma} \frac{1}{h_e} \int_e [v]^2 ds, \tag{3.3.3}$$

and

$$|||v|||_1 \;=\; |||v||| + \|v\|, \tag{3.3.4}$$

where, $|v|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} |v|_{1,T}^2$.

For connecting the trial and test spaces, define the transfer operator $\gamma : M(h) \longrightarrow L_h$ as

$$\gamma v|_{V^*} = \frac{1}{h_e} \int_e v|_{V^*} ds, \qquad V^* \in \mathcal{V}_h^*, \tag{3.3.5}$$

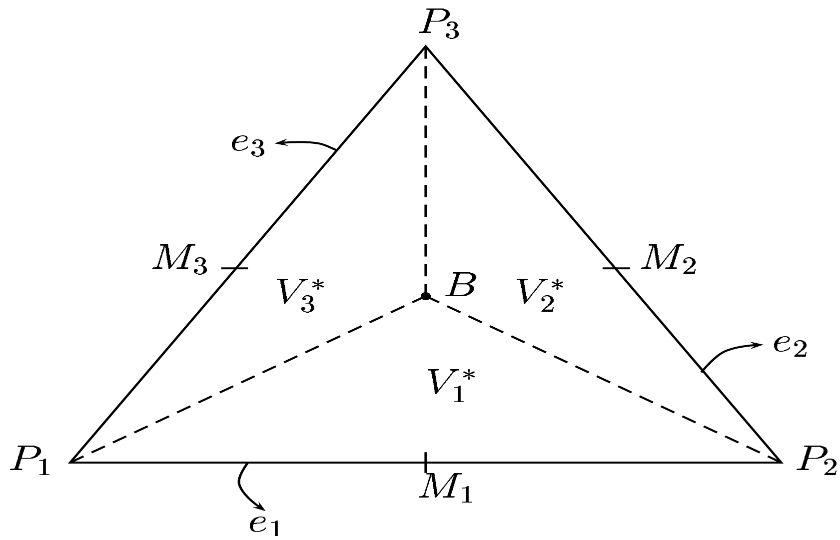

Figure 3.4: A triangular partition and its dual elements

where $'e'$ is an edge in T and V^* is the dual element in $\mathcal{V}_h^*$ containing e, h_e being the length of the edge e (see Figure 3.2). We also assume that h_e and h_T are equivalent, i.e., there exist positive constants C_1 and C_2 such that

$$C_1 h_e \leq h_T \leq C_2 h_e. \tag{3.3.6}$$

By usual interpolation theory, it is easy to see that the operator γ has the following approximation properties [29]: For $\chi \in H^1(T)$

$$\|\chi - \gamma\chi\|_T \leq Ch\|\nabla\chi\|_T. \tag{3.3.7}$$

We frequently use the following trace inequality [3, pp. 745].
For $w \in H^2(T)$ and for an edge e of triangle T, we have

$$\|w\|_{0,e}^2 \leq C\left(h_e^{-1}\|w\|_{0,T}^2 + h_e|w|_{1,T}^2\right), \tag{3.3.8}$$

where $\|w\|_{0,e}^2 = \displaystyle\int_e w^2 ds$.
Since γz_h is a constant over each control volume, multiplying (3.1.3) by $\gamma z_h \in L_h$, integrating, applying the Gauss's divergence theorem over the control volumes $V^* \in \mathcal{V}_h^*$ and summing up over all control volumes, we obtain

$$(\phi\frac{\partial c}{\partial t}, \gamma z_h) - \sum_{V^* \in \mathcal{V}_h^*}\int_{\partial V^*}(D(\mathbf{u})\nabla c - \mathbf{u}c).\mathbf{n}\gamma z_h = (g(c), \gamma z_h) \quad \forall z_h \in M_h, \tag{3.3.9}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary ∂V^* of V^*.

Let $V_j^* \in \mathcal{V}_h^*(j=1,2,3)$ be the three triangles in $T \in \mathcal{T}_h$, (see Figure 3.4). Then,

$$\sum_{V^*\in\mathcal{V}_h^*}\int_{\partial V^*}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds = \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{\partial V_j^*}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds$$
$$=\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds+\sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds, \quad (3.3.10)$$

where, $P_4 = P_1$, see Figure 3.4.

For any four real numbers a, b, c and d, we have

$$ac-bd=\frac{1}{2}(a+b)(c-d)+\frac{1}{2}(a-b)(c+d). \quad (3.3.11)$$

Since $[(D(\mathbf{u})\nabla c-\mathbf{u}c)]=0$ from (2.1.14), we have from (3.3.10) and (3.3.11)

$$\begin{aligned}\sum_{V^*\in\mathcal{V}_h^*}\int_{\partial V^*}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds &= \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla c-\mathbf{u}c)\cdot\mathbf{n}\gamma z_h ds\\ &+\sum_{e\in\Gamma}\int_e[\gamma z_h]\cdot\langle D(\mathbf{u})\nabla c-\mathbf{u}c\rangle ds. \quad (3.3.12)\end{aligned}$$

For a fixed positive real number M, and $\mathbf{v}\in\mathbb{R}^2$, define the following cut-off operator

$$\mathbf{v}^M=\mathcal{M}(\mathbf{v})(x)=\min\left(|\mathbf{v}(x)|,M\right)\frac{\mathbf{v}(x)}{|\mathbf{v}(x)|}, \quad (3.3.13)$$

Moreover, since later on we will assume that the Darcy velocity $\mathbf{u}\in(L^\infty(\Omega))^2$, then M can be chosen such that $\|\mathbf{u}\|_{(L^\infty(\Omega))^2}\le M$ and this implies that

$$\mathbf{u}^M=\mathcal{M}(\mathbf{u})=\mathbf{u}. \quad (3.3.14)$$

It is easy to check that the “cut-off” operator $\mathcal{M}$ is uniformly Lipschitz continuous in the following sense, (see [76, pp. 331]):

$$\|\mathcal{M}(\mathbf{u})-\mathcal{M}(\mathbf{v})\|_{(L^\infty(\Omega))^2}\le\|\mathbf{u}-\mathbf{v}\|_{(L^\infty(\Omega))^2}. \quad (3.3.15)$$

Proof of (3.3.15) is straightforward. Note that for a fixed $x\in\Omega$, we have

$$|\mathcal{M}(\mathbf{u})-\mathcal{M}(\mathbf{v})|(x)\le|\mathbf{u}-\mathbf{v}|(x), \quad (3.3.16)$$

which can be shown by taking three cases (i) $|\mathbf{u}| \leq M,\ |\mathbf{v}| \leq M$, (ii) $|\mathbf{u}| > M,\ |\mathbf{v}| \leq M$ and (iii) $|\mathbf{u}| > M,\ |\mathbf{v}| > M$ and using the definition of $\mathcal{M}$. Now take the essential supremum on both sides of (3.3.16) the proof is completed.
By definition of $\mathcal{M}$, it can be seen that

$$\|\mathbf{u_h}^M\|_{(L^\infty(\Omega))^2} \leq M. \tag{3.3.17}$$

Now we are in a position to define the DG finite volume scheme for the concentration equation. The discontinuous Galerkin finite volume element scheme corresponding to (3.2.3) is defined as:
Find $c_h(t) \in M_h$ such that

$$\begin{aligned} (\phi\frac{\partial c_h}{\partial t}, \gamma z_h) + A_h(\mathbf{u_h}^M; c_h, z_h) &= (g(c_h), \gamma z_h) \qquad \forall z_h \in M_h, \\ c_h(0) &= c_{0,h}. \end{aligned} \tag{3.3.18}$$

Here, $\mathbf{u_h}^M$ is the "cut-off" function of $\mathbf{u_h}$ defined in (3.3.13) and $c_{0,h}$ be an approximation to c_0 to be defined later, the bilinear form $A_h(\mathbf{v};\cdot,\cdot) : M(h) \times M(h) \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} A_h(\mathbf{v};\phi,\psi) &= A_1(\mathbf{v};\phi,\psi) - \sum_{e\in\Gamma}\int_e [\gamma\psi]\cdot\langle D(\mathbf{v})\nabla\phi - \mathbf{v}\phi\rangle ds \\ &\quad - \sum_{e\in\Gamma}\int_e [\gamma\phi]\cdot\langle D(\mathbf{v})\nabla\psi\rangle ds + \sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi]\cdot[\psi]ds \quad \forall \mathbf{v}\in\mathbb{R}^2, \end{aligned} \tag{3.3.19}$$

with $A_1(\mathbf{v};\phi,\psi) = -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\big(D(\mathbf{v})\nabla\phi - \mathbf{v}\phi\big)\cdot\mathbf{n}\,\gamma\psi\,ds$ and α is a penalty parameter to be defined later. Note that (3.3.18) is consistent with (3.3.9). Now, based on the analysis of [85], we prove the following two lemmas, which will be useful in proving the coercivity and boundedness of the bilinear form $A_h(\mathbf{u};\cdot,\cdot)$.

Lemma 3.3.1 *The following result holds true: For $\chi, \psi \in M(h)$, we have*

$$\begin{aligned} A_1(\mathbf{u};\chi,\psi) &= \sum_{T\in\mathcal{T}_h}\int_T (D(\mathbf{u})\nabla\chi - \mathbf{u}\chi)\cdot\nabla\psi \;\; dx + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla\chi - \mathbf{u}\chi)\cdot\mathbf{n})(\gamma\psi - \psi)ds \\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(D(\mathbf{u})\nabla\chi - \mathbf{u}\chi)(\psi - \gamma\psi)ds. \end{aligned} \tag{3.3.20}$$

Proof. A use of Gauss divergence theorem on each on the control volume $V_j^*, j = 1,2,3,$ (see Figure 3.4), yields

$$\begin{aligned}
A_1(\mathbf{u};\chi,\psi) &= -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\gamma\psi\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds \\
&= \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\gamma\psi\int_{P_jP_{j+1}}(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds-\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*}\gamma\psi\nabla\cdot(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)dx \\
&= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\gamma\psi-\psi)(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\psi(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds \\
&\quad -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*}\gamma\psi\nabla\cdot(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)dx
\end{aligned}$$

A use of Gauss's divergence theorem once more yields,

$$\begin{aligned}
A_1(\mathbf{u};\chi,\psi) &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\gamma\psi-\psi)(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds+\sum_{T\in\mathcal{T}_h}\int_{T}(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\nabla\psi\;dx \\
&\quad +\sum_{T\in\mathcal{T}_h}\int_{T}\nabla\cdot(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\psi\;dx-\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{V_j^*}\gamma\psi\nabla\cdot(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\;dx \\
&= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\gamma\psi-\psi)(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\mathbf{n}\;ds+\sum_{T\in\mathcal{T}_h}\int_{T}(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)\cdot\nabla\psi\;dx \\
&\quad +\sum_{T\in\mathcal{T}_h}\int_{T}\nabla\cdot(D(\mathbf{u})\nabla\chi-\mathbf{u}\chi)(\psi-\gamma\psi)dx.
\end{aligned}$$

This completes the proof. ■

Following the similar proof techniques, which we have used in Lemma 3.3.1. It is easy to check that for $\chi,\psi\in M(h)$, we have

$$\begin{aligned}
A_1(\mathbf{u};\chi,\psi) &= \sum_{T\in\mathcal{T}_h}\int_{T}(D(\mathbf{u})\nabla\chi)\cdot\nabla\psi\;\;dx+\sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla\chi)\cdot\mathbf{n})(\gamma\psi-\psi)ds \\
&\quad +\sum_{T\in\mathcal{T}_h}\int_{T}\nabla\cdot(D(\mathbf{u})\nabla\chi)(\psi-\gamma\psi)ds, \qquad (3.3.21)
\end{aligned}$$

where

$$A_1(\mathbf{u};\chi,\psi) = -\sum_{T}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla\chi)\cdot\mathbf{n}\gamma\psi\;ds$$

.

LEMMA 3.3.2 *If $\chi_h \in M_h$, then*

$$A_1(\mathbf{u};\chi_h,\chi_h) \geq \alpha_0 \sum_{T\in\mathcal{T}_h} \|\nabla\chi_h\|^2 - C_1 h|||\chi_h|||^2 - C_2\|\chi_h\|^2. \tag{3.3.22}$$

Proof. Rewrite $A_1(\mathbf{u},\chi_h,\psi_h)$ as

$$\begin{aligned} A_1(\mathbf{u},\chi_h,\psi_h) &= -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\ \gamma\psi_h\ ds \\ &\quad + \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{u}\cdot\mathbf{n}\chi_h\ \gamma\psi_h\ ds. \end{aligned} \tag{3.3.23}$$

Using the same argument as in the proof of (2.3.37), it can be easily proved that

$$-\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u})\nabla\chi_h)\cdot\mathbf{n}\gamma\chi_h\ ds \geq \alpha_0\sum_{T\in\mathcal{T}_h}\|\nabla\chi_h\|_T^2 - C_1 h|||\chi_h|||^2. \tag{3.3.24}$$

Since $\gamma\psi_h$ is constant on each control volume V^*, set $\gamma\psi_h|_{V_l^*} = \psi_l$. Using the Cauchy-Schwarz inequality and referring to Figure 3.4, we obtain

$$\begin{aligned} \sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{u}\cdot\mathbf{n}\chi_h\ \gamma\psi_h\ ds &= \sum_{l=1}^{3}\int_{P_lB}\mathbf{u}\cdot\mathbf{n_l}\chi_h(\psi_{l+1}-\psi_l)ds \qquad (\psi_4=\psi_1) \\ &\leq \sum_{l=1}^{3}|\psi_{l+1}-\psi_l|\int_{P_lB}\mathbf{u}\cdot\mathbf{n_l}\ \chi_h\ ds \leq C\sum_{l=1}^{3}|\psi_{l+1}-\psi_l|\int_{P_lB}\chi_h\ ds \end{aligned}$$

A use of the trace inequality (3.3.8) yields

$$\begin{aligned} \sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{u}\cdot\mathbf{n}\chi_h\gamma\psi_h\ ds &\leq C\sum_{l=1}^{3}|\psi_{l+1}-\psi_l|\ \|\chi_h\|_{L^2(P_lB)}(\mathrm{meas}(P_lB))^{1/2} \\ &\leq C\sum_{l=1}^{3}|\psi_{l+1}-\psi_l|\ \|\chi_h\|_{L^2(P_lB)}h_T^{1/2} \\ &\leq Ch_T^{1/2}\sum_{l=1}^{3}|\psi_{l+1}-\psi_l|\left[h_T^{-1/2}\|\chi_h\|_T + h_T^{1/2}\|\nabla\chi_h\|_T\right] \end{aligned} \tag{3.3.25}$$

Now using Taylor series expansion and (2.3.18), we find that

$$\begin{aligned} |\psi_{l+1}-\psi_l| &\leq h_T\left[|\frac{\partial\psi_h}{\partial x}| + |\frac{\partial\psi_h}{\partial y}|\right] \leq \left[\left(|\frac{\partial\psi_h}{\partial x}|^2 + |\frac{\partial\psi_h}{\partial y}|^2\right)h_T^2\right]^{1/2} \\ &\leq C|\psi_h|_{1,h,T}, \qquad\qquad l=1,2,3. \end{aligned} \tag{3.3.26}$$

Substituting (3.3.26) in (3.3.25), we arrive at

$$\sum_{j=1}^{3}\int_{P_{j+1}BP_j} \mathbf{u}\cdot\mathbf{n}\chi_h\ \gamma\psi_h\ ds \le C|\psi_h|_{1,h,T}\Big[\|\chi_h\|_T + h_T\|\nabla\chi_h\|_T\Big]. \tag{3.3.27}$$

Taking summation over all the triangles $T \in \mathcal{T}_h$, we obtain

$$\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j} \mathbf{u}\cdot\mathbf{n}\chi_h\ \gamma\psi_h\ ds \le C|||\psi_h|||\left(\|\chi_h\| + h|||\chi_h|||\right). \tag{3.3.28}$$

Substituting (3.3.24) and (3.3.28) in (3.3.23) and using Young's inequality, we complete the rest of the proof. ■

LEMMA 3.3.3 *[3, pp. 744] There exists a positive constant C independent of mesh size h such that*

$$\|\phi\| \le C|||\phi||| \qquad \forall\phi \in M(h). \tag{3.3.29}$$

LEMMA 3.3.4 *[60] The following results hold true: $\forall\phi_h \in M_h$,*

$$(i)\int_T(\phi_h - \gamma\phi_h)\ dx = 0 \qquad \forall T\in\mathcal{T}_h \qquad (ii)\int_e(\phi_h-\gamma\phi_h)\ ds = 0 \qquad \forall e\in\Gamma.$$

Proof. Using (3.3.5), we have

$$\begin{aligned}\int_T(\phi_h-\gamma\phi_h)dx &= \int_T\phi_h\ dx - \sum_{i=1}^{3}\int_{V_i^*}\gamma\phi_h\ dx\\ &= \int_T\phi_h\ dx - \sum_{i=1}^{3}\gamma\phi_h|_{V_i^*}|V_i^*|\\ &= \int_T\phi_h\ dx - \sum_{i=1}^{3}\frac{1}{h_{e_i}}\int_{e_i}\phi_h\ ds\ |V_i^*|\\ &= 0.\end{aligned}$$

Here $|V_i^*|$ denotes the area of the control volume V_i^*. In the last equality we have used $|V_i^*| = \dfrac{|T|}{3}$, $i = 1,2,3$ and the quadrature formula (2.3.11). Now (ii) follows directly from (3.3.5). This completes the proof. ■

Let f_T be the average value of f over the triangle T. The using Lemma 3.3.4, the Cauchy-Schwarz inequality and (3.3.7), we find that

$$\begin{aligned}\int_T f(\psi_h - \gamma\psi_h)dx &= \int_T (f - f_T)(\psi_h - \gamma\psi_h)dx \\ &\leq \|f - f_T\|_T\|\psi_h - \gamma\psi_h\|_T \leq Ch^2\|\nabla f\|_T\|\nabla\psi_h\|_T. \end{aligned} \quad (3.3.30)$$

Next, we show that the bilinear form $A_h(\mathbf{u};\cdot,\cdot)$ satisfies a Gärding type inequality. Using Cauchy-Schwarz inequality and the trace inequality (3.3.8), we arrive at

$$\begin{aligned}\sum_{e\in\Gamma}\int_e [\gamma\phi_h]\cdot\langle D(\mathbf{u})\nabla\psi_h\rangle ds &\leq \left(\sum_{e\in\Gamma} h_e^{-1}\int_e [\gamma\phi_h]^2 ds\right)^{1/2}\left(\sum_{e\in\Gamma} h_e\int_e \langle D(\mathbf{u})\nabla\psi_h\rangle^2 ds\right)^{1/2} \\ &\leq C\|D(\mathbf{u})\|_{0,\infty}\left(\sum_{e\in\Gamma}[\gamma\phi_h]_e^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}|\nabla\psi_h|_T^2\right)^{1/2} \\ &\leq C\left(\sum_{e\in\Gamma}[\gamma\phi_h]_e^2\right)^{1/2} |||\psi_h|||,\end{aligned}$$

where $[\gamma\phi_h]_e = \dfrac{1}{h_e}\displaystyle\int_e [\phi_h]ds$. Now using (3.3.5) and the Cauchy-Schwarz inequality, we obtain

$$[\gamma\phi_h]_e^2 = \left(\frac{1}{h_e}\int_e [\phi_h]ds\right)^2 \leq \left(\frac{1}{h_e}\right)^2\int_e [\phi_h]^2 ds\int_e ds = \int_e \frac{1}{h_e}[\phi_h]^2 ds. \quad (3.3.31)$$

This implies that

$$\sum_{e\in\Gamma}\int_e [\gamma\phi_h]\cdot\langle D(\mathbf{u})\nabla\psi_h\rangle ds \leq C\left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e [\phi_h]^2 ds\right)^{1/2} |||\psi_h|||, \quad (3.3.32)$$

and similarly,

$$\sum_{e\in\Gamma}\int_e [\gamma\phi_h]\cdot\langle(D(\mathbf{u})\nabla\psi_h - \mathbf{u}\psi_h\rangle ds \leq C\left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e [\phi_h]^2 ds\right)^{1/2} (|||\psi_h||| + \|\psi_h\|). \quad (3.3.33)$$

LEMMA 3.3.5 *There exist positive constants C and C_3 independent of h such that for α large enough, h small enough and $\mathbf{v} \in (L^\infty(\Omega))^2$,*

$$A_h(\mathbf{v};\phi_h,\phi_h) \geq C|||\phi_h|||^2 - C_3\|\phi_h\|^2 \qquad \forall\phi_h \in M_h. \quad (3.3.34)$$

Proof. Use (3.3.29), (3.3.32), (3.3.33) and (3.3.22) to obtain

$$\begin{aligned} A_h(\mathbf{v};\phi_h,\phi_h) &\geq \alpha_0 \sum_{T\in\mathcal{T}_h} \|\nabla\phi_h\|^2 - C_1 h|||\phi_h|||^2 - C_2\|\phi_h\|^2 - C|||\phi_h||| \left(\sum_{e\in\Gamma} \frac{1}{h_e}\int_e [\phi_h]^2 ds\right)^{1/2} \\ &\quad + \alpha \sum_{e\in\Gamma} \frac{1}{h_e}\int_e [\phi_h]^2 ds. \end{aligned}$$

A use of Young's inequality yields

$$\begin{aligned} A_h(\mathbf{v};\phi_h,\phi_h) &\geq \alpha_0 \sum_{T\in\mathcal{T}_h} \|\nabla\phi_h\|^2 - C_1 h|||\phi_h|||^2 - C_2\|\phi_h\|^2 - \frac{C^2}{2\alpha_0}\left(\sum_{e\in\Gamma} \frac{1}{h_e}\int_e [\phi_h]^2 ds\right) \\ &\quad - \frac{\alpha_0}{2}|||\phi_h|||^2 + \alpha \sum_{e\in\Gamma} \frac{1}{h_e}\int_e [\phi_h]^2 ds \\ &\geq \frac{\alpha_0}{2} \sum_{T\in\mathcal{T}_h} \|\nabla\phi_h\|^2 - C_1 h|||\phi_h|||^2 - C_2\|\phi_h\|^2 + \left(\alpha - \frac{C^2}{2\alpha_0} - \frac{\alpha_0}{2}\right) \sum_{e\in\Gamma} \frac{1}{h_e}\int_e [\phi_h]^2 ds \\ &\geq C(\alpha)|||\phi_h|||^2 - C_3\|\phi_h\|^2 - C_1 h|||\phi_h|||^2 \\ &\geq C|||\phi_h|||^2 - C_3\|\phi_h\|^2, \end{aligned}$$

where $C(\alpha) = \min\left(\frac{\alpha_0}{2}, \alpha - \frac{C^2}{2\alpha_0} - \frac{\alpha_0}{2}\right)$ and α_0 is the lower bound for matrix $D(\mathbf{v})$. Here we have to choose the parameter α such that the term $\left(\alpha - \frac{C^2}{2\alpha_0} - \frac{\alpha_0}{2}\right)$ is positive and h small enough such that $C = C(\alpha) - C_1 h > 0$. This completes the proof. ■

Lemma 3.3.6 *For $\phi,\psi \in M(h)$, we have*

$$\begin{aligned} |A_h(\mathbf{u};\phi,\psi)| &\leq C\Bigg[\left\{|||\phi||| + \left(\sum_{T\in\mathcal{T}_h} h_T^2|\phi|_{2,T}\right)^{1/2}\right\}|||\psi||| \\ &\quad + \left\{|||\psi||| + \left(\sum_{T\in\mathcal{T}_h} h_T^2|\psi|_{2,T}\right)^{1/2}\right\}|||\phi|||\Bigg]. \end{aligned} \tag{3.3.35}$$

Further if $\phi_h,\psi_h \in M_h$, then

$$|A_h(\mathbf{u};\phi_h,\psi_h)| \leq C|||\phi_h|||\ |||\psi_h|||. \tag{3.3.36}$$

Proof. Rewrite $A_h(\mathbf{u};\phi,\psi) = A_h^1(\mathbf{u};\phi,\psi) + A_h^2(\mathbf{u};\phi,\psi)$,

where

$$\begin{aligned} A_h^1(\mathbf{u};\phi,\psi) &= -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\big(D(\mathbf{u})\nabla\phi\big)\cdot\mathbf{n}\gamma\psi\ ds - \sum_{e\in\Gamma}\int_e[\gamma\psi].\langle D(\mathbf{u})\nabla\psi\rangle ds \\ &\quad -\sum_{e\in\Gamma}\int_e[\gamma\phi]\cdot\langle D(\mathbf{u})\nabla\psi\rangle ds + \sum_{e\in\Gamma}\int_e\frac{\alpha}{h_e}[\phi_h]\cdot[\psi]ds, \end{aligned} \tag{3.3.37}$$

and

$$A_h^2(\mathbf{u};\phi,\psi) = \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{u}\phi\cdot\mathbf{n}\gamma\psi + \sum_{e\in\Gamma}\int_e[\gamma\psi]\cdot\langle\mathbf{u}\phi\rangle ds. \tag{3.3.38}$$

To bound the first term of $A_h^1(\mathbf{u};\phi,\psi)$, we use (3.3.21) to obtain

$$\begin{aligned} &\Big|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\big(D(\mathbf{u})\nabla\phi\big)\cdot\mathbf{n}\gamma\psi\ ds\Big| \le \Big|\sum_{T\in\mathcal{T}_h}\int_T D(\mathbf{u})\nabla\phi\cdot\nabla\psi\ dx\Big| \\ &+\Big|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(D(\mathbf{u})\nabla\phi\cdot\mathbf{n})(\gamma\psi-\psi)ds\Big| + \Big|\sum_{T\in\mathcal{T}_h}\int_T\nabla\cdot(D(\mathbf{u})\nabla\phi)(\psi-\gamma\psi)ds\Big| \end{aligned}$$

A use of the Cauchy-Schwarz inequality gives us

$$\begin{aligned} &\Big|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\big(D(\mathbf{u})\nabla\phi\big)\cdot\mathbf{n}\gamma\psi\ ds\Big| \le \|D(\mathbf{u})\|_{0,\infty}\left(\sum_{T\in\mathcal{T}_h}|\nabla\phi|_T^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}|\nabla\psi|_T^2\right)^{1/2} \\ &+\|D(\mathbf{u})\|_{0,\infty}\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\frac{\partial\phi}{\partial\mathbf{n}}|^2ds\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\psi-\gamma\psi|^2ds\right)^{1/2} \\ &+\|D(\mathbf{u})\|_{1,\infty}\Big(\sum_{T\in\mathcal{T}_h}|\phi|_{1,T}\|\psi-\gamma\psi\|_T\Big) + \|D(\mathbf{u})\|_{0,\infty}\Big(\sum_{T\in\mathcal{T}_h}|\phi|_{2,T}\|\psi-\gamma\psi\|_T\Big). \end{aligned}$$

Using the trace inequality (3.3.8), we find that

$$\begin{aligned} \Big|\sum_{T\in\mathcal{T}_h}\sum_{j}\int_{P_{j+1}BP_j}\big(D(\mathbf{u})\nabla\phi\big)\cdot\mathbf{n}\gamma\psi\ ds\Big| &\le C\Bigg[|\phi|_{1,h}|\psi|_{1,h} + \left(\sum_{T\in\mathcal{T}_h}h_T^{-1}|\nabla\phi|_T^2 + h_T|\phi|_{2,T}^2\right)^{1/2} \\ &\quad\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}|\psi-\gamma\psi|_T^2 + h_T|\psi-\gamma\psi|_{1,T}^2\right)^{1/2} \\ &\quad+\sum_{T\in\mathcal{T}_h}h_T\left(|\phi|_{1,T}+|\phi|_{2,T}\right)|\psi|_{1,T}\Bigg]. \end{aligned}$$

Now using (3.3.7), we obtain

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\big(D(\mathbf{u})\nabla\phi\big)\cdot\mathbf{n}\gamma\psi\, ds\Big| &\leq C\Bigg[|\phi|_{1,h}|\psi|_{1,h}+\Bigg(\sum_{T\in\mathcal{T}_h}|\phi|_{1,T}^2+h_T^2|\phi|_{2,T}^2\Bigg)^{1/2}\\ &\qquad \Bigg(\sum_{T\in\mathcal{T}_h}|\psi|_{1,T}\Bigg)^{1/2}+\sum_{T\in\mathcal{T}_h}h_T\left(|\phi|_{1,T}+|\phi|_{2,T}\right)|\psi|_{1,T}\Bigg]\\ &\leq C\Bigg[|\phi|_{1,h}|\psi|_{1,h}+\Bigg(\sum_{T\in\mathcal{T}_h}h_T^2|\phi|_{2,T}^2\Bigg)^{1/2}|\psi|_{1,h}\Bigg] \qquad (3.3.39)\end{aligned}$$

Following the proof techniques of (3.3.32), we arrive at

$$|\sum_{e\in\Gamma}\int_e[\gamma\psi]\cdot\langle D(\mathbf{u})\nabla\phi\rangle ds|\leq C\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2ds\Bigg)^{1/2}\Bigg[\sum_{T\in\mathcal{T}_h}\left(|\phi|_{1,T}^2+h_T^2|\phi|_{2,T}^2\right)\Bigg]^{1/2}, \quad (3.3.40)$$

and

$$|\sum_{e\in\Gamma}\int_e[\gamma\phi]\cdot\langle D(\mathbf{u})\nabla\psi\rangle ds|\leq C\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2ds\Bigg)^{1/2}\Bigg[\sum_{T\in\mathcal{T}_h}\left(|\psi|_{1,T}^2+h_T^2|\psi|_{2,T}^2\right)\Bigg]^{1/2}. \quad (3.3.41)$$

Note that

$$\Bigg|\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]\cdot[\psi]ds\Bigg|\leq\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2ds\Bigg)^{1/2}\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2ds\Bigg)^{1/2}. \qquad (3.3.42)$$

Using (3.3.39)-(3.3.42) in (3.3.37), we obtain

$$\begin{aligned}|A_h^1(\mathbf{u};\phi,\psi)| \leq\; & C\Bigg[|\phi|_{1,h}|\psi|_{1,h}+\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2ds\Bigg)^{1/2}\Bigg(\sum_{T\in\mathcal{T}_h}\left(|\phi|_{1,T}^2+h_T^2|\phi|_{2,T}^2\right)\Bigg)^{1/2}\\ &+\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2ds\Bigg)^{1/2}\Bigg(\sum_{T\in\mathcal{T}_h}\left(|\psi|_{1,T}^2+h_T^2|\psi|_{2,T}^2\right)\Bigg)^{1/2}\\ &+\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2ds\Bigg)^{1/2}\Bigg(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2ds\Bigg)^{1/2}\Bigg].\end{aligned}$$

Using (3.3.3),

$$\begin{aligned}
|A_h^1(\mathbf{u};\phi,\psi)| &\leq C\Bigg[|\phi|_{1,h}|\psi|_{1,h} + \left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2 ds\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\left(|\phi|_{1,T}^2 + h_T^2|\phi|_{2,T}^2\right)\right)^{1/2} \\
&\quad +|\phi|_{1,h}|\psi|_{1,h} + \left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2 ds\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\left(|\psi|_{1,T}^2 + h_T^2|\psi|_{2,T}^2\right)\right)^{1/2} \\
&\quad + \left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\phi]^2 ds\right)^{1/2}\left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\psi]^2 ds\right)^{1/2}\Bigg] \\
&\leq C\Bigg[\left\{|||\phi||| + \left(\sum_{T\in\mathcal{T}_h} h_T^2|\phi|_{2,T}^2\right)^{1/2}\right\}|||\psi||| \\
&\quad +\left\{(|||\psi||| + \left(\sum_{T\in\mathcal{T}_h} h_T^2|\psi|_{2,T}^2\right)^{1/2}\right\}|||\phi|||\Bigg].
\end{aligned}$$

Now we proceed to estimate the first term in $A_h^2(\mathbf{u};\phi,\psi)$. From (3.3.28), we have

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\mathbf{u}\cdot\mathbf{n}\ \phi\ \gamma\psi\ ds\right| \leq C\ |||\psi|||\left(\|\phi\| + h|||\phi|||\right). \tag{3.3.43}$$

Using (3.3.31) it can be easily seen that

$$\left|\sum_{e\in\Gamma}\int_e[\gamma\psi]\cdot\langle\mathbf{u}\phi\rangle ds\right| \leq C|||\psi|||\ |||\phi|||. \tag{3.3.44}$$

Substitute (3.3.43) and (3.3.44) in (3.3.38), to obtain

$$|A_h^2(\mathbf{u};\phi,\psi)| \leq C\left(|||\phi|||\ |||\psi||| + |||\phi|||\ \|\psi\|\right). \tag{3.3.45}$$

Combining the estimates derived for $A_h^1(\mathbf{u};\phi,\psi)$ and $A_h^2(\mathbf{u};\phi,\psi)$ with (3.3.29), we complete the proof of (3.3.35).

In particular, if $\phi_h \in M_h$, then $|\phi_h|_{2,T} = 0$ and hence (3.3.36) directly follows from follows from (3.3.35). This completes the proof. ■

LEMMA 3.3.7 *The operator γ has the following properties. For $\phi_h, \psi_h \in M_h$,*

$$(\phi_h, \gamma\psi_h) = (\psi_h, \gamma\phi_h). \tag{3.3.46}$$

Moreover, with $|||\phi_h|||_h = (\phi_h, \gamma\phi_h)$ *norms* $|||\cdot|||_h$ *and* $\|\cdot\|$ *are equivalent, i.e., there exist positive constants* C_1 *and* C_2*, independent of* h *such that*

$$C_1\|\phi_h\| \leq |||\phi_h|||_h \leq C_2\|\phi_h\| \qquad \forall \phi_h \in M_h, \tag{3.3.47}$$

and

$$\|\gamma\phi_h\| \leq C\|\phi_h\| \qquad \forall \phi_h \in M_h. \tag{3.3.48}$$

Proof. Since

$$(\phi_h, \gamma\psi_h) = \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3} \gamma\psi_h|_{V_j^*} \int_{V_j^*\cap T} \phi_h \, dx,$$

using the definition of γ, we obtain

$$\begin{aligned}\sum_{j=1}^{3} \gamma\psi_h|_{V_j^*} \int_{V_j^*\cap T} \phi_h \, dx &= \gamma\psi_h|_{V_1^*} \int_{V_1^*\cap T} \phi_h \, dx + \gamma\psi_h|_{V_2^*} \int_{V_2^*\cap T} \phi_h \, dx + \gamma\psi_h|_{V_3^*} \int_{V_3^*\cap T} \phi_h \, dx \\ &= \frac{1}{h_{e_1}} \int_{e_1} \psi_h \, ds \int_{V_1^*\cap T} \phi_h \, dx + \frac{1}{h_{e_2}} \int_{e_2} \psi_h \, ds \int_{V_2^*\cap T} \phi_h \, dx \\ &\quad + \frac{1}{h_{e_3}} \int_{e_3} \psi_h \, ds \int_{V_3^*\cap T} \phi_h \, dx.\end{aligned}$$

Since ψ_h is linear on each triangle T, we use the following quadrature formula to compute the edge integral

$$\int_{e_i} \psi_h \, ds = \frac{\psi_i + \psi_{i+1}}{2} \qquad \text{for } i = 1, 2, 3. \tag{3.3.49}$$

Now a use of the quadrature formula (2.3.11) along with (3.3.49), yields

$$\begin{aligned}\sum_{j=1}^{3} \gamma\psi_h|_{V_j^*} \int_{V_j^*\cap T} \phi_h \, dx &= \left(\frac{\psi_1 + \psi_2}{2}\right)(\phi_1 + \phi_2 + \phi_B)\frac{|V_1^*|}{3} + \left(\frac{\psi_2 + \psi_3}{2}\right)(\phi_2 + \phi_3 + \phi_B)\frac{|V_2^*|}{3} \\ &\quad + \left(\frac{\psi_1 + \psi_3}{2}\right)(\phi_1 + \phi_3 + \phi_B)\frac{|V_3^*|}{3}.\end{aligned}$$

Since $\phi_B = \dfrac{\phi_1 + \phi_2 + \phi_3}{3}$, we have

$$\begin{aligned}
\sum_{j=1}^{3} \gamma \psi_h|_{V_j^*} \int_{V_j^* \cap T} \phi_h \, dx &= \frac{|T|}{54}\Big[(\psi_1 + \psi_2)(4\phi_1 + 4\phi_2 + \phi_3) + (\psi_2 + \psi_3)(4\phi_2 + 4\phi_3 + \phi_1) \\
&\quad + (\psi_3 + \psi_1)(4\phi_3 + 4\phi_1 + \phi_2)\Big] \\
&= \frac{|T|}{54}[\psi_1, \psi_2, \psi_3] \begin{pmatrix} 8 & 5 & 5 \\ 5 & 8 & 5 \\ 5 & 5 & 8 \end{pmatrix} [\phi_1, \phi_2, \phi_3]^T.
\end{aligned}$$

This prove that the inner product $(\cdot, \gamma \cdot)$ is symmetric. To prove the equivalence of the norms, rewrite

$$\begin{aligned}
\sum_{j=1}^{3} \gamma \phi_h|_{V_j^*} \int_{V_j^* \cap T} \phi_h \, dx &= \frac{|T|}{54}\left[8(\phi_1^2 + \phi_2^2 + \phi_3^2) + 10(\phi_1\phi_2 + \phi_2\phi_3 + \phi_1\phi_3)\right] \\
&= \frac{|T|}{54}\left[3(\phi_1^2 + \phi_2^2 + \phi_3^2) + 5(\phi_1 + \phi_2 + \phi_3)^2\right]. \qquad (3.3.50)
\end{aligned}$$

Since ϕ_h is linear on triangle T, we can use midpoint quadrature formula (2.3.12) to compute the integral

$$\begin{aligned}
\int_T |\phi_h|^2 \, dx &= \frac{|T|}{3}\left[\left(\frac{\phi_1 + \phi_2}{2}\right)^2 + \left(\frac{\phi_2 + \phi_3}{2}\right)^2 + \left(\frac{\phi_1 + \phi_3}{2}\right)^2\right] \\
&= \frac{|T|}{12}\left[\phi_1^2 + \phi_2^2 + \phi_3^2 + (\phi_1 + \phi_2 + \phi_3)^2\right]. \qquad (3.3.51)
\end{aligned}$$

Equivalence of norms follow from (3.3.50) and (3.3.51). This completes the proof for (3.3.47). To prove (3.3.48), note that

$$\|\gamma \phi_h\|^2 = \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{V_j^*} |\gamma \phi_h|^2 dx.$$

Since $\gamma \phi_h$ is constant over the control volumes, ($\gamma \phi_h|_{V_j^*} = \phi_j$, say), we have

$$\sum_{j=1}^{3} \int_{V_j^*} |\gamma \phi_h|^2 dx = \sum_{j=1}^{3} \phi_j^2 |V_j^*| = \sum_{j=1}^{3} \phi_j^2 \frac{|T|}{3}. \qquad (3.3.52)$$

Now (3.3.48) follows from (3.3.52) and (3.3.51) and this completes the proof. ■

For a given c_h, the existence and uniqueness of the discrete solution $\mathbf{u_h}$ and p_h can be shown in a similar way as in Chapter 2. Since $\mathbf{u_h^M}$ is the cut-off function of $\mathbf{u_h}$, the existence of $\mathbf{u_h}$ implies the existence of $\mathbf{u_h^M}$. To show the existence and uniqueness of the concentration in (3.3.18), we argue as follows. On substituting $(\mathbf{u_h^M}(c_h))$ in (3.3.18), we obtain a system of nonlinear ordinary differential equations in c_h. An appeal to Picard's theorem yields the existence of a unique solution in $(0, t_h)$ for some $0 < t_h \leq T$. To continue the solution for all $t \in J$, we need an *a priori* bound for c_h.

Choosing $z_h = c_h$ in (3.3.18) and using (3.3.46), we obtain

$$\frac{1}{2}\frac{d}{dt}(\phi c_h, \gamma c_h) + A_h(\mathbf{u_h^M}; c_h, c_h) \leq |(g(c_h), \gamma c_h)| \tag{3.3.53}$$

Using the Cauchy-Schwarz inequality and (3.3.48), we obtain

$$|(g(c_h), \gamma c_h)| \leq C\left(\|c_h\|^2 + \|\tilde{c}\|^2\right). \tag{3.3.54}$$

Substituting (3.3.34) and (3.3.54) in (3.3.53), we arrive at

$$\frac{1}{2}\frac{d}{dt}(\phi c_h, \gamma c_h) + C_2|||c_h|||^2 \leq C_3\left(\|c_h\|^2 + \|\tilde{c}\|^2\right). \tag{3.3.55}$$

Integrating from 0 to T and using (3.3.47), we obtain

$$\|c_h\|^2 + \int_0^T |||c_h|||^2 ds \leq C\left(\|c_h(0)\|^2 + \int_0^T \|c_h\|^2 ds + + \int_0^T \|\tilde{c}\|^2 ds\right) \tag{3.3.56}$$

A use of Gronwall's lemma in (3.3.56) gives an *a priori* estimate in L^2- norm for c_h. Now the *a priori* bound $\|c_h\|_{L^\infty(L^2)}$ can be used to show the existence of a unique solution c_h of the concentration equation for all $t \in J$. This completes the part of unique solvability of (3.3.1)-(3.3.2) and (3.3.18). ■

3.4 Error estimates

In this section, we discuss the error estimates for the semidiscrete scheme. The following theorem which gives the estimates for velocity and pressure has been proved in Chapter 2 (see Theorem 2.4.1).

THEOREM 3.4.1 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$, respectively, be the solutions of (3.2.1)-(3.2.2) and (3.3.1)-(3.3.2). Then, there exists a positive constant C independent of h, but dependent on the bounds of κ^{-1} and μ such that*

$$\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + \|p - p_h\| \leq C\left[\|c - c_h\| + h(\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1)\right], \tag{3.4.1}$$

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u_h})\| \leq Ch\|\nabla \cdot \mathbf{u}\|_1, \tag{3.4.2}$$

provided $\mathbf{u}(t) \in (H^1(\Omega))^2,\ \nabla \cdot \mathbf{u}(t) \in H^1(\Omega)$ *and* $p(t) \in H^1(\Omega)$.

Then for finding the estimates for concentration, we split $c - c_h = (c - R_h c) + (R_h c - c_h)$, where R_h is the Ritz projection to be defined below in (3.4.3). In Lemma 3.4.1 and Lemma 3.4.3, we derive respectively H^1 and L^2- error estimates for R_h. Finally using these estimates, we obtain a priori error estimates in L^2- norm for the concentration.

3.4.1 Elliptic projection

Let $R_h : H^1(\Omega) \longrightarrow M_h$ be the projection of c defined by

$$B(\mathbf{u}; c - R_h c, \chi_h) = 0 \qquad \forall \chi_h \in M_h, \tag{3.4.3}$$

where

$$B(\mathbf{u}; \psi, \chi_h) = A_h(\mathbf{u}; \psi, \chi_h) + (\lambda\psi, \chi_h) \quad \forall \chi_h \in M_h. \tag{3.4.4}$$

Using the boundedness of $A_h(\mathbf{u}; \cdot, \cdot)$, we have

$$|B(\mathbf{u}; \psi, \chi_\mathbf{h})| \left[\left\{ |||\psi|||_1 + \left(\sum_{T \in \mathcal{T}_h} h_T |\psi|_{2,T}^2\right)^{1/2} \right\} |||\chi_h|||_1 + |||\chi_h|||_1 |||\psi|||_1 \right]. \tag{3.4.5}$$

Since $A_h(\mathbf{u}; \chi_h, \chi_h) \geq C|||\chi_h|||^2 - C_2\|\chi_h\|^2$, if we choose λ such that $\lambda - C_2 \geq 0$. Then, $B(\mathbf{u}; \chi_h, \chi_h)$ will be coercive in the norm $|||\cdot|||_1$ defined in (3.3.4), i.e., there exists a positive constant C independent of h such that

$$B(\mathbf{u}; \chi_h, \chi_h) \geq C|||\chi_h|||_1^2 \qquad \forall \chi_h \in M_h. \tag{3.4.6}$$

Below we discuss L^2 and H^1- *a priori* error bounds for $c - R_h c$.

Let $I_h c \in M_h$ be an interpolant of c, which has the following approximation properties [29]:

$$|c - I_h c|_{s,T} \leq C h_T^{2-s} \|c\|_{2,T} \qquad \forall T \in \mathcal{T}_h, \quad s = 0, 1, 2. \tag{3.4.7}$$

Moreover, if $\phi \in W^{2,\infty}(\Omega)$, then

$$\|\phi - I_h \phi\|_{1,\infty} \leq C h \|\phi\|_{2,\infty}. \tag{3.4.8}$$

LEMMA 3.4.1 *There exists a positive constant C independent of h such that*

$$|||c - R_h c|||_1 \leq C h \|c\|_2, \tag{3.4.9}$$

provided $c \in H^2(\Omega)$.

Proof. Write $c - R_h c = (c - I_h c) + (I_h c - R_h c)$, first we derive the bound for $|||c - I_h c|||$. By the definition of $||| \cdot |||$, we obtain

$$|||c - I_h c|||^2 = |c - I_h c|_{1,h}^2 + \sum_{e \in \Gamma} \frac{1}{h_e} \int_e [c - I_h c]^2 ds \tag{3.4.10}$$

Using the trace inequality (3.3.8) and (3.4.7), we find that

$$\begin{aligned} \frac{1}{h_e} \int_e [c - I_h c]_e^2 ds &\leq C \left(h_e^{-2} \|c - I_h c\|_{0,T}^2 + |c - I_h c|_{1,T}^2 \right) \\ &\leq C h_T^2 \|c\|_{2,T}^2 \end{aligned} \tag{3.4.11}$$

Use (3.4.11), (3.4.10) and (3.4.7), to obtain

$$|||c - I_h c|||_1 \leq C h \|c\|_2. \tag{3.4.12}$$

The coercivity (3.4.6) and boundedness (3.4.5) of bilinear form $B(\mathbf{u}; \cdot, \cdot)$ with (3.4.3) yields

$$\begin{aligned} |||I_h c - R_h c|||_1^2 &\leq C B(\mathbf{u}; I_h c - R_h c, I_h c - R_h c) \\ &\leq C B(\mathbf{u}; I_h c - c, I_h c - R_h c) \\ &\leq C \Bigg[\left\{ |||c - I_h c|||_1 + \left(\sum_{T \in \mathcal{T}_h} h_T |c - I_h c|_{2,T}^2 \right)^{1/2} \right\} |||R_h c - I_h c|||_1 \\ &\quad + |||R_h c - I_h c|||_1 |||c - I_h c|||_1 \Bigg], \end{aligned} \tag{3.4.13}$$

and hence,

$$|||I_hc - R_hc|||_1 \leq C|||c - I_hc|||_1, \tag{3.4.14}$$

where C depends on bound of $D(\mathbf{u})$ given in (2.1.13). Combine the estimates of (3.4.12) and (3.4.14) and use the triangle inequality to complete the proof. ■

Before going to the L^2 error estimates, we introduce the following bilinear form $A(\mathbf{u};\cdot,\cdot) : M(h) \times M(h) \longrightarrow \mathbb{R}$ which will be helpful in deriving optimal error estimate in L^2 norm.

$$\begin{aligned} A(\mathbf{u};\phi,\psi) &= \sum_{T\in\mathcal{T}_h}\int_T (D(\mathbf{u})\nabla\phi - \mathbf{u}\phi)\cdot\nabla\psi\ dx - \sum_{e\in\Gamma}\int_e [\psi]\cdot\langle D(\mathbf{u})\nabla\phi - \mathbf{u}\phi\rangle ds \\ &\quad - \sum_{e\in\Gamma}\int_e [\phi]\cdot\langle D(\mathbf{u})\nabla\psi\rangle ds + \sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi]\cdot[\psi]ds. \end{aligned} \tag{3.4.15}$$

For our further use, let us introduce the following error function

$$\epsilon_a(\mathbf{u};\psi,\chi) = A(\mathbf{u};\psi,\chi) - A_h(\mathbf{u};\psi,\chi) \quad \forall\chi\in M_h. \tag{3.4.16}$$

Below, we will prove a lemma which plays an important role in the proof of *a priori* L^2 error estimates for the Ritz projection R_h.

Lemma 3.4.2 *There exists a positive constant C such that*

$$\begin{aligned} |\epsilon_a(\mathbf{u}, c - R_hc, \phi_h)| &\leq Ch^2\Big(\|g\|_1 + |\phi\frac{\partial c}{\partial t}|_1 + \|c\|_2 \\ &\quad + \|\nabla\cdot\mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2}\Big)\|\phi_h\|_1 \quad \forall\phi_h\in M_h. \end{aligned} \tag{3.4.17}$$

Proof. From (3.4.15), we obtain

$$\begin{aligned} A(\mathbf{u};u - R_hc,\phi_h) &= \sum_{T\in\mathcal{T}_h}\int_T (D(\mathbf{u})\nabla(c - R_hc) - \mathbf{u}(c - R_hc))\cdot\nabla\phi_h\ dx \\ &\quad - \sum_{e\in\Gamma}\int_e [\phi_h]\cdot\langle D(\mathbf{u})\nabla(c - R_hc) - \mathbf{u}(c - R_hc)\rangle ds \\ &\quad - \sum_{e\in\Gamma}\int_e [c - R_hc]\cdot\langle D(\mathbf{u})\nabla\phi_h\rangle ds \\ &\quad + \sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi_h]\cdot[c - R_hc]ds \\ &= T_1 + T_2 + T_3 + \sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi_h]\cdot[c - R_hc]ds, \end{aligned} \tag{3.4.18}$$

Similarly, we find that

$$\begin{aligned} A_h(\mathbf{u}; c - R_h c, \phi_h) &= -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}\left((D(\mathbf{u})\nabla(c-R_hc) - \mathbf{u}(c-R_hc))\cdot\mathbf{n}\right)\gamma\phi_h\, ds \\ &\quad -\sum_{e\in\Gamma}\int_e [\gamma\phi_h]\cdot\langle D(\mathbf{u})\nabla(c-R_hc) - \mathbf{u}(c-R_hc)\rangle ds \\ &\quad -\sum_{e\in\Gamma}\int_e [\gamma(c-R_hc)]\cdot\langle D(\mathbf{u})\nabla\phi_h\rangle ds \\ &\quad +\sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi_h]\cdot[c-R_hc]ds \\ &= T_{h_1} + T_{h_2} + T_{h_3} + \sum_{e\in\Gamma}\int_e \frac{\alpha}{h_e}[\phi_h]\cdot[c-R_hc]ds. \end{aligned} \tag{3.4.19}$$

From (3.4.18) and (3.4.19), we arrive at

$$|A(\mathbf{u}; c-R_hc, \phi_h) - A_h(\mathbf{u}; c-R_hc,\phi_h)| \le |T_1 - T_{h_1}| + |T_2 - T_{h_2}| + |T_3 - T_{h_3}|. \tag{3.4.20}$$

To estimate $|T_1 - T_{h_1}|$, we use Lemma 3.3.1 to obtain

$$\begin{aligned} |T_1 - T_{h_1}| &\le \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\phi_h - \gamma\phi_h)\big(D(\mathbf{u})\nabla(c-R_hc)) - \mathbf{u}(c-R_hc)\big)\cdot\mathbf{n}\, ds\right| \\ &\quad + \left|\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(D(\mathbf{u})\nabla(c-R_hc) - \mathbf{u}(c-R_hc))(\phi_h - \gamma\phi_h)dx\right| \\ &= |I_1| + |I_2|, \qquad \text{say.} \end{aligned} \tag{3.4.21}$$

To bound $|I_1|$ and $|I_2|$, we follow the proof techniques given in [46]. Since $\nabla c\cdot\mathbf{n}$ is continuous across the element boundaries, we find that

$$\begin{aligned} |I_1| &\le \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T} D(\mathbf{u})\nabla(c-R_hc)\cdot\mathbf{n}(\phi_h - \gamma\phi_h)ds\right| \\ &\quad + \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{u}(c-R_hc)\cdot\mathbf{n}(\phi_h-\gamma\phi_h)ds\right| \\ &\le \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\big((D-\bar{D}_T)\nabla(c-R_hc)\big)\cdot\mathbf{n}(\phi_h-\gamma\phi_h)ds\right| \\ &\quad + \|c-R_hc\|_\infty\left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{u}\cdot\mathbf{n}(\phi_h-\gamma\phi_h)ds\right|, \end{aligned} \tag{3.4.22}$$

where $D = D(\mathbf{u})$ and $\bar{D}_T$ is a function designed in a piecewise manner such that for any edge e of a triangle $T \in \mathcal{T}_h$,

$$\bar{D}_T(x) = D(x_c), \qquad \forall x \in e,$$

and x_c is the mid point of e. Since $|\bar{D}_T - D| \leq Ch\|D\|_{1,\infty}$, we obtain from the Cauchy-Schwarz inequality that

$$\begin{aligned} |I_1| &\leq Ch\|D\|_{1,\infty} \sum_{T\in\mathcal{T}_h} \int_{\partial T} |\phi_h - \gamma\phi_h|\, |\nabla(c - R_h c)\cdot \mathbf{n}| ds \\ &\quad + \|c\|_2 \left| \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\mathbf{u} - \bar{\mathbf{u}}_T)\cdot \mathbf{n}(\phi_h - \gamma\phi_h) ds \right|. \end{aligned}$$

Using (3.3.7) and (3.4.9), we obtain

$$\begin{aligned} |I_1| &\leq Ch\Big[\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\phi_h - \gamma\phi_h\|_T^2 + h_T|\phi_h - \gamma\phi_h|_{1,T}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}|c - R_h c|_{1,T}^2 + h_T|c - R_h c|_{2,T}^2 \right)^{1/2} \\ &\quad + \|c\|_2\, h_T^{1/2} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\phi_h - \gamma\phi_h\|_T^2 + h_T|\phi_h - \gamma\phi_h|_{1,T}^2 \right)^{1/2} \\ &\leq Ch^2\|\phi_h\|_1\|c\|_2. \end{aligned} \tag{3.4.23}$$

$|I_2|$ can be bounded as follows:

$$\begin{aligned} |I_2| &\leq \left| \sum_{T\in\mathcal{T}_h} \int_T \nabla\cdot(D(\mathbf{u})\nabla c - \mathbf{u}c) - \nabla\cdot(D(\mathbf{u})\nabla R_h c)(\phi - \gamma\phi_h) ds \right| \\ &\quad + \left| \sum_{T\in\mathcal{T}_h} \int_T \nabla\cdot(\mathbf{u}R_h c)(\phi - \gamma\phi_h) ds \right|. \end{aligned}$$

Using (3.1.3), we obtain

$$\begin{aligned} I_2 &\leq \left| \sum_{T\in\mathcal{T}_h} \int_T \left(g - \phi\frac{\partial c}{\partial t} - \nabla\cdot(D(\mathbf{u})\nabla R_h c) \right)(\phi - \gamma\phi_h) ds \right| \\ &\quad + \left| \sum_{T\in\mathcal{T}_h} \int_T (\nabla\cdot\mathbf{u}\; R_h c + \mathbf{u}\cdot\nabla R_h c)(\phi - \gamma\phi_h) ds \right| \end{aligned}$$

A use of (3.3.30) and (3.3.7) yields

$$|I_2| \;\leq\; Ch^2\left(\|g\|_1 + \|\phi\frac{\partial c}{\partial t}\|_1 + \|c\|_2 + \|\nabla\cdot\mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2}\right)\|\phi_h\|_1. \quad (3.4.24)$$

Combining (3.4.23), (3.4.24) and (3.4.21), we obtain the following bound for $|T_1 - T_{h_1}|$.

$$|T_1 - T_{h_1}| \leq Ch^2\left(\|g\|_1 + \|\phi\frac{\partial c}{\partial t}\|_1 + \|c\|_2 + \|\nabla\cdot\mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2}\right)\|\phi_h\|_1. \quad (3.4.25)$$

To obtain an estimate for $T_2 - T_{h_2}$, we note that

$$\begin{aligned} |T_2 - T_{h_2}| \;&\leq\; \left|\sum_{e\in\Gamma}\int_e [\phi_h - \gamma\phi_h]\cdot\langle D(\mathbf{u})\nabla(c - R_hc)\rangle ds\right| + \left|\sum_{e\in\Gamma}\int_e [\phi_h - \gamma\phi_h]\cdot\langle \mathbf{u}(c - R_hc)\rangle ds\right| \\ &= |J_1| + |J_2|, \quad \text{say.} \end{aligned} \quad (3.4.26)$$

Using the same argument as in I_1, we bound J_1 as

$$\begin{aligned} |J_1| \;&=\; \left|\sum_{e\in\Gamma}\int_e [\phi_h - \gamma\phi_h]\cdot\langle D(\mathbf{u})\nabla(c - R_hc)\rangle ds\right| \\ &=\; \left|\sum_{e\in\Gamma}\int_e [\phi_h - \gamma\phi_h]\cdot\langle (D - \bar{D}_T)\nabla(c - R_hc)\rangle ds\right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality and the trace inequality (3.3.8), (3.3.7) and (3.4.9), we find that

$$\begin{aligned} |J_1| \;&\leq\; Ch\|D\|_{1,\infty}\sum_{e\in\Gamma}\int_e |[\phi_h - \gamma\phi_h]|\;|\langle\nabla(c - R_hc)\rangle| ds \\ &\leq\; Ch\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\phi_h - \gamma\phi_h\|_T^2 + h_T|\phi_h - \gamma\phi_h|_{1,T}^2\right)^{1/2} \\ &\quad\times\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}|c - R_hc|_{1,T}^2 + h_T|c - R_hc|_{2,T}^2\right)^{1/2} \\ &\leq\; Ch\left(\sum_{T\in\mathcal{T}_h} h_T|\phi_h|_{1,T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} h_T|c|_{2,T}^2\right)^{1/2} \\ &\leq\; Ch^2\|\phi_h\|_1\|c\|_2. \end{aligned} \quad (3.4.27)$$

Similarly, we bound J_2 as

$$\begin{aligned}
|J_2| &= \left|\sum_{e\in\Gamma}\int_e[\phi_h-\gamma\phi_h]\cdot\langle\mathbf{u}(c-R_hc)\rangle ds\right| \\
&\leq \|c\|_2\left|\sum_{e\in\Gamma}\int_e[\phi_h-\gamma\phi_h]\cdot\langle\mathbf{u}-\bar{\mathbf{u}}_T\rangle ds\right| \\
&\leq Ch\|c\|_2\|\mathbf{u}\|_{(W_1^\infty(\Omega))^2}\sum_{e\in\Gamma}\int_e[\phi_h-\gamma\phi_h]ds \\
&\leq Ch^2\|c\|_2\|\phi_h\|_1. \qquad (3.4.28)
\end{aligned}$$

Substituting (3.4.27) and (3.4.28) in (3.4.26), we obtain

$$|T_2-T_{h_2}|\leq Ch^2\|\phi_h\|_1\|c\|_2. \qquad (3.4.29)$$

In order to estimate $T_3-T_{h_3}$, we rewrite it as

$$\begin{aligned}
|T_3-T_{h_3}| &= \left|\sum_{e\in\Gamma}\int_e[(c-R_hc)-\gamma(c-R_hc)]\cdot\langle D(\mathbf{u})\nabla\phi_h\rangle ds\right| \\
&= \left|\sum_{e\in\Gamma}\int_e[(c-R_hc)-\gamma(c-R_hc)]\cdot\langle (D-\bar{D}_T)\nabla\phi_h\rangle ds\right|,
\end{aligned}$$

and using the same arguments as in $|I_1|$, we arrive at

$$\begin{aligned}
|T_3-T_{h_3}| &\leq Ch\|D\|_{1,\infty}\sum_{e\in\Gamma}\int_e|[(c-R_hc)-\gamma(c-R_hc)]|\ |\langle\nabla\phi_h\rangle|ds \\
&\leq Ch\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|(c-R_hc)-\gamma(c-R_hc)\|_T^2+h_T|(c-R_hc)-\gamma(c-R_hc)|_{1,T}^2\right)^{1/2} \\
&\quad\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\phi_h\|_{1,T}^2\right)^{1/2} \\
&\leq Ch\left(\sum_{T\in\mathcal{T}_h}h_T|c-R_hc|_{1,T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\phi_h\|_{1,T}^2\right)^{1/2} \\
&\leq Ch\left(\sum_{T\in\mathcal{T}_h}h_T^2|c|_{2,T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\phi_h\|_{1,T}^2\right)^{1/2} \\
&\leq Ch^2\|c\|_2\|\phi_h\|_1. \qquad (3.4.30)
\end{aligned}$$

Now substitute (3.4.25), (3.4.29) and (3.4.30) in (3.4.20) to complete the rest of the proof. ■

LEMMA 3.4.3 *There exists a positive constant C independent of h such that*

$$\|c - R_h c\| \leq Ch^2 \left(\|c\|_2 + \|g\|_1 + \|\phi \frac{\partial c}{\partial t}\|_1 + \|\nabla \cdot \mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2} \right). \tag{3.4.31}$$

Proof. To obtain an optimal L^2 estimate for $c - R_h c$, we now appeal to Aubin-Nitsche duality argument. Let $\psi \in H^2(\Omega)$ be a solution of the following adjoint problem

$$\begin{aligned} -\nabla \cdot (D(\mathbf{u})\nabla\psi) - \mathbf{u} \cdot \nabla\psi + \lambda\psi &= c - R_h c \qquad \text{in } \Omega, \\ D(\mathbf{u})\nabla\psi \cdot \mathbf{n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.4.32}$$

satisfying the following elliptic regularity condition:

$$\|\psi\|_2 \leq C\|c - R_h c\|. \tag{3.4.33}$$

Multiply the first equation (3.4.32) by $c - R_h c$ and integrate over Ω. Then using (3.4.16) and (3.4.3), we arrive at

$$\begin{aligned} \|c - R_h c\|^2 &= A(\mathbf{u}; c - R_h c, \psi) + \lambda(\psi, c - R_h c) \\ &= \Big[A(\mathbf{u}; c - R_h c, \psi - \psi_h) + \lambda(c - R_h c, \psi - \psi_h) \Big] \\ &\quad + \epsilon_a(\mathbf{u}; c - R_h c, \psi_h) = I_1 + I_2, \text{ say.} \end{aligned} \tag{3.4.34}$$

For I_1, use (3.4.9) to obtain

$$\begin{aligned} |I_1| &= |A(\mathbf{u}; c - R_h c, \psi - \psi_h) + (\mathbf{u} \cdot \nabla(c - R_h c), \psi - \psi_h) + \lambda(c - R_h c, \psi - \psi_h)| \\ &\leq C \Bigg[\Bigg\{ |||c - R_h c|||_1 + \Bigg(\sum_{T \in \mathcal{T}_h} h_T |c - R_h c|_{2,T}^2 \Bigg)^{1/2} \Bigg\} |||\psi - \psi_h|||_1 \\ &\quad + |||\psi - \psi_h|||_1 |||c - R_h c|||_1 \Bigg] \\ &\leq Ch\|c\|_2 |||\psi - \psi_h|||_1. \end{aligned} \tag{3.4.35}$$

The following bound for I_2 follows from Lemma 3.4.2.

$$|I_2| \leq Ch^2 \left(\|g\|_1 + \|\phi \frac{\partial c}{\partial t}\|_1 + \|c\|_2 + \|\nabla \cdot \mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2} \right) \|\psi_h\|_1. \tag{3.4.36}$$

Substitute (3.4.35) and (3.4.36) in (3.4.34) to find that

$$\begin{aligned}\|c - R_h c\|^2 \;&\leq\; C\Bigg[h\|c\|_2\|\psi - \psi_h\|_1 + h^2\Big(\|g\|_1 + \|\phi\frac{\partial c}{\partial t}\|_1 + \|c\|_2 \\ &\quad + \|\nabla\cdot\mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2}\Big)\|\psi_h\|_1\Bigg]. \qquad (3.4.37)\end{aligned}$$

Now choose $\psi_h = I_h\psi$ in (3.4.37) and use (3.4.33) with (3.4.7) to obtain

$$\|c - R_h c\| \leq Ch^2\left(\|c\|_2 + \|g\|_1 + \|\phi\frac{\partial c}{\partial t}\|_1 + \|\nabla\cdot\mathbf{u}\|_1 + \|\mathbf{u}\|_{(H^1(\Omega))^2}\right),$$

and this completes the proof. ■

For finding a bound for $\|R_h c\|_{1,\infty}$, we use (2.3.14), (3.4.8) (3.4.9) and (3.4.14)

$$\begin{aligned}\|R_h c\|_{1,\infty} \;&=\; \|c - R_h c\|_{1,\infty} + \|c\|_{1,\infty} \\ &=\; \|c - I_h c\|_{1,\infty} + \|I_h c - R_h c\|_{1,\infty} + \|c\|_{1,\infty} \\ &\leq\; C\left(\|c - I_h c\|_{1,\infty} + h^{-1}\|I_h c - R_h c\|_1 + \|c\|_{1,\infty}\right) \\ &\leq\; C\|c\|_{2,\infty}. \qquad (3.4.38)\end{aligned}$$

LEMMA 3.4.4 *There exists a positive constant C such that*

$$\begin{aligned}|A_h(\mathbf{u}^M; R_h c, \theta) - A_h(\mathbf{u}_\mathbf{h}^M; R_h c, \theta)| \;&\leq\; C\Big(\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u} - \mathbf{u_h})\| \\ &\quad + |||\rho|||\Big)|||\theta||| \qquad \forall\theta \in M_h. \qquad (3.4.39)\end{aligned}$$

Proof: Using the definition of $A_h(\cdot;\cdot,\cdot)$, we obtain

$$\begin{aligned}|A_h(\mathbf{u}^M; R_h c, \theta) - A_h(\mathbf{u}_\mathbf{h}^M; R_h c, \theta)| \;&=\; \left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u}^M) - D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\cdot\mathbf{n}\gamma\theta\; ds\right| \\ &\quad + \left|\sum_{e\in\Gamma}\int_e[\gamma R_h c]\cdot\langle(D(\mathbf{u}^M) - D(\mathbf{u}_\mathbf{h}^M))\nabla\theta\rangle\; ds\right| \\ &\quad + \left|\sum_{e\in\Gamma}\int_e[\gamma\theta]\cdot\langle(D(\mathbf{u}^M) - D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\rangle\; ds\right| \\ &\quad + \left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(\mathbf{u}^M - \mathbf{u}_\mathbf{h}^M)R_h c\cdot\mathbf{n}\;\gamma\theta\; ds\right| \\ &\quad + \left|\sum_{e\in\Gamma}\int_e[\gamma\theta]\cdot\langle(\mathbf{u}^M - \mathbf{u}_\mathbf{h}^M)R_h c\rangle ds\right| \\ &=\; \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5, \qquad \text{say.}\end{aligned}$$

To estimate $\mathcal{A}_1$, we note that

$$\begin{aligned}\mathcal{A}_1 &= \left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u}^M)-D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\cdot\mathbf{n}\,\gamma\theta\;ds\right| \\ &= |\sum_{T\in\mathcal{T}_h}K_T|,\end{aligned}$$

where $K_T=\sum_{j=1}^{3}\int_{P_{j+1}BP_j}(D(\mathbf{u}^M)-D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\cdot\mathbf{n}\,\gamma\theta\;ds$. Since $\gamma\theta$ is constant over each control volume V^*, set $\gamma\theta|_{V_l^*}=\theta_l$. Referring to Figure 3.4, K_T can be written as follows:

$$K_T = \sum_{l=1}^{3}\int_{P_lB}(D(\mathbf{u}^M)-D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\cdot\mathbf{n}_l(\theta_{l+1}-\theta_l)ds \qquad (\theta_4=\theta_1)$$

Then using (3.4.38) and the Cauchy-Schwarz, we obtain

$$\begin{aligned}K_T &\le \sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\int_{P_lB}|(D(\mathbf{u}^M)-D(\mathbf{u}_\mathbf{h}^M))\nabla R_h c\cdot\mathbf{n}_l|ds \\ &\le C\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\;\|D(\mathbf{u}^M)-D(\mathbf{u}_\mathbf{h}^M)\|_{(L^2(P_lB))^{2\times2}}\left(\text{meas}(P_lB)\right)^{1/2}.\end{aligned}$$

Apply Lemma 2.3.5, trace inequality (3.3.8) and (3.3.15) to obtain

$$\begin{aligned}K_T &\le C\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\;\|\mathbf{u}^M-\mathbf{u}_\mathbf{h}^M\|_{(L^2(P_lB))^2}h_T^{1/2} \\ &\le C\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\;\|\mathbf{u}-\mathbf{u}_\mathbf{h}\|_{(L^2(P_lB))^2}h_T^{1/2} \\ &\le Ch_T^{1/2}\sum_{l=1}^{3}|\theta_{l+1}-\theta_l|\left[h_T^{-1/2}\|\mathbf{u}-\mathbf{u}_\mathbf{h}\|_T+h_T^{1/2}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_\mathbf{h})\|_T\right]. \qquad (3.4.40)\end{aligned}$$

Now using Taylor series expansion and (2.3.18), we obtain

$$\begin{aligned}|\theta_{l+1}-\theta_l| &\le h_T\left[|\frac{\partial\theta}{\partial x}|+|\frac{\partial\theta}{\partial y}|\right]\le\left[\left(|\frac{\partial\theta}{\partial x}|^2+|\frac{\partial\theta}{\partial y}|^2\right)h_T^2\right]^{1/2} \\ &\le C|\theta|_{1,h,T}, \qquad l=1,2,3. \qquad (3.4.41)\end{aligned}$$

Substitute (3.4.41) in (3.4.40) to arrive at

$$K_T\le C|\theta|_{1,h,T}\left(\|\mathbf{u}-\mathbf{u}_\mathbf{h}\|_T+h_T\|\nabla\cdot(\mathbf{u}-\mathbf{u}_\mathbf{h})\|_T\right).$$

With the estimates of K_T and Lemma 2.3.1, we obtain

$$\mathcal{A}_1 \le C\Big(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\Big)|||\theta|||. \tag{3.4.42}$$

Since $[\gamma c]=0$, we can write

$$\begin{aligned} \mathcal{A}_2 &= \left|\sum_{e\in\Gamma}\int_e [\gamma R_h c]\cdot\langle (D(\mathbf{u}^M)-D(\mathbf{u_h}^M))\nabla\theta\rangle\ ds\right| \\ &= \left|\sum_{e\in\Gamma}\int_e [\gamma R_h c-\gamma c]\cdot\langle (D(\mathbf{u}^M)-D(\mathbf{u_h}^M))\nabla\theta\rangle\ ds\right| \\ &\le \left|\sum_{e\in\Gamma}\int_e [\gamma\rho]\cdot\langle D(\mathbf{u}^M)\nabla\theta\rangle ds\right| + \left|\sum_{e\in\Gamma}\int_e [\gamma\rho]\cdot\langle D(\mathbf{u_h}^M)\nabla\theta\rangle ds\right|. \end{aligned} \tag{3.4.43}$$

Using the same argument as in deriving (3.3.32), we obtain

$$\left|\sum_{e\in\Gamma}\int_e [\gamma\rho]\cdot\langle D(\mathbf{u}^M)\nabla\theta\rangle ds\right| \le C|||\rho|||\ |||\theta|||, \tag{3.4.44}$$

and similarly,

$$\left|\sum_{e\in\Gamma}\int_e [\gamma\rho]\cdot\langle D(\mathbf{u_h}^M)\nabla\theta\rangle ds\right| \le C|||\rho|||\ |||\theta|||. \tag{3.4.45}$$

Substituting (3.4.44) and (3.4.45) in (3.4.43), we obtain

$$\mathcal{A}_2 \le C|||\rho|||\ |||\theta|||. \tag{3.4.46}$$

Now, we bound $\mathcal{A}_3$ as follows: Using (3.4.38) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mathcal{A}_3 &= \left|\sum_{e\in\Gamma}\int_e [\gamma\theta]\cdot\langle (D(\mathbf{u}^M)-D(\mathbf{u_h}^M))\nabla R_h c\rangle ds\right| \\ &\le \|\nabla R_h c\|_\infty \left|\sum_{e\in\Gamma}\int_e [\gamma\theta]\cdot\langle D(\mathbf{u}^M)-D(\mathbf{u_h}^M)\rangle ds\right| \\ &\le C\left(\sum_{e\in\Gamma}\int_e [\gamma\theta]^2 ds\right)^{1/2}\left(\sum_{e\in\Gamma}\|D(\mathbf{u}^M)-D(\mathbf{u_h}^M)\|^2_{(L^2(e))^{2\times 2}}\right)^{1/2}. \end{aligned}$$

Now using Lemma Lemma 2.3.5 and (3.3.15), we arrive at

$$\begin{aligned} \mathcal{A}_3 &\le C\left(\sum_{e\in\Gamma}\int_e [\gamma\theta]^2 ds\right)^{1/2}\left(\sum_{e\in\Gamma}\|\mathbf{u}^M-\mathbf{u_h}^M\|^2_{(L^2(e))^2}\right)^{1/2} \\ &\le C\left(\sum_{e\in\Gamma}\int_e [\gamma\theta]^2 ds\right)^{1/2}\left(\sum_{e\in\Gamma}\|\mathbf{u}-\mathbf{u_h}\|^2_{(L^2(e))^2}\right)^{1/2} \end{aligned}$$

Using the trace inequality (3.3.8), we obtain

$$\mathcal{A}_3 \leq C|||\theta|||\Big(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\Big). \tag{3.4.47}$$

$\mathcal{A}_4$ can be bound in a similar way as $\mathcal{A}_1$:

$$\mathcal{A}_4 \leq C|||\theta|||\Big(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\Big). \tag{3.4.48}$$

To bound $\mathcal{A}_5$, we use the same arguments used in $\mathcal{A}_3$ to obtain

$$\mathcal{A}_5 \leq C|||\theta|||\left(\|\mathbf{u}-\mathbf{u_h}\| + h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|\right) \tag{3.4.49}$$

Combine the estimates derived for $\mathcal{A}_1\cdots\mathcal{A}_5$ to complete the rest of the proof. ■

3.4.2 $L^\infty(L^2)$ estimates for concentration

In this subsection, we discuss an $L^\infty(L^2)$ norm error estimate for the concentration.

THEOREM 3.4.2 *Let c and c_h be the solutions of (3.2.3) and (3.3.18), respectively, and let $c_h(0) = c_{0,h} = R_h c(0)$. Then, for sufficiently small h, there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such that*

$$\begin{aligned}\|c-c_h\|^2_{L^\infty(J;L^2)} \leq C(T)\Bigg[\int_0^T \Big(& h^4(\|g\|_1^2 + \|\phi\frac{\partial c}{\partial t}\|_1^2 + \|\mathbf{u}\|_{(H^1(\Omega))^2} + \|\nabla\cdot\mathbf{u}\|_1 \\ &+\|c_t\|_2^2 + \|g_t\|_1^2 + \|\mathbf{u_t}\|_{(H^1(\Omega))^2} + \|\nabla\cdot\mathbf{u}_t\|_1 + \|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2 \\ &+h^2(\|c\|_2^2 + \|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2)\Big)ds\Bigg].\end{aligned} \tag{3.4.50}$$

Proof: Write $c - c_h = (c - R_h c) + (R_h c - c_h) = \rho + \theta$. Since the estimates of ρ are known, we need to find only the estimates of θ.
Multiply (3.1.3) by γz_h, integrate over Ω. Then subtract the resulting equation from (3.3.18) to obtain

$$\begin{aligned}\left(\phi\frac{\partial\theta}{\partial t},\gamma z_h\right) + A_h(\mathbf{u}^M;c,z_h) - A_h(\mathbf{u_h}^M;c_h,z_h) = -\left(\phi\frac{\partial\rho}{\partial t},\gamma z_h\right) \\ +(g(c)-g(c_h),\gamma z_h)\ \ \forall z_h\in M_h.\end{aligned} \tag{3.4.51}$$

Put $z_h = \theta$ and use the definition of R_h to obtain

$$\begin{aligned}\left(\phi\frac{\partial\theta}{\partial t},\gamma\theta\right)+A_h(\mathbf{u_h}^M;\theta,\theta) &= -\left(\phi\frac{\partial\rho}{\partial t},\gamma\theta\right)+(\lambda\rho,\theta)+\Big(g(c)-g(c_h),\gamma\theta\Big)\\ &\quad -\big[A_h(\mathbf{u}^M;R_hc,\theta)-A_h(\mathbf{u_h}^M;R_hc,\theta)\big]\\ &= I_1+I_2+I_3+I_4,\ \text{ say.}\end{aligned} \tag{3.4.52}$$

Now we estimate $I_j, j = 1, 2, 3, 4$ one by one. To estimate I_1, we use the Cauchy Schwartz inequality, boundedness of ϕ and (3.3.48) to obtain

$$|I_1| = |\left(\phi\frac{\partial\rho}{\partial t},\gamma\theta\right)| \leq C\|\frac{\partial\rho}{\partial t}\|\ \|\theta\|. \tag{3.4.53}$$

Similarly,

$$|I_2| = |(\lambda\rho,\theta)| \leq C\|\rho\|\ \|\theta\|. \tag{3.4.54}$$

Using (2.1.7) and (3.3.48), I_3 can be estimated as

$$|I_3| \leq |(g(c_h)-g(c),\gamma\theta)| \leq C\|c-c_h\|\ \|\theta\|. \tag{3.4.55}$$

The bound for I_4 follows from Lemma 3.4.4 and hence,

$$|I_4| \leq C\left(\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2}+h\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|+|||\rho|||\right)|||\theta|||. \tag{3.4.56}$$

Substitute the estimates of $I_1,\cdots,I_4$ in (3.4.52) and use (3.3.34), Young's inequality, non singularity of the function ϕ with standard kick back arguments to obtain

$$\begin{aligned}\frac{d}{dt}|||\theta|||_h^2+(\alpha_0-\epsilon)|||\theta|||^2 &\leq C\Big[h^2\|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|^2+|||\rho|||^2\|\mathbf{u}-\mathbf{u_h}\|^2_{(L^2(\Omega))^2}\\ &\quad +\|c-c_h\|^2+\|\rho\|^2+\|\frac{\partial\rho}{\partial t}\|^2+\|\theta\|^2\Big].\end{aligned} \tag{3.4.57}$$

Now, from (3.4.1),

$$\|\mathbf{u}-\mathbf{u_h}\|_{(L^2(\Omega))^2} \leq C\Big(\|\rho\|+\|\theta\|+h(\|\mathbf{u}\|_{(H^1(\Omega))^2}+\|p\|_1)\Big). \tag{3.4.58}$$

A use of (3.4.31), (3.4.58) and (3.4.2) in (3.4.57) yields

$$\begin{aligned}\frac{d}{dt}|||\theta|||_h^2+\alpha_1|||\theta|||^2 &\leq C\Big[h^4\Big(\|g\|_1^2+\|\mathbf{u}\|_{(H^1(\Omega))^2}+\|\nabla\cdot\mathbf{u}\|_1+\|\phi\frac{\partial c}{\partial t}\|_1^2\\ &\quad +\|c_t\|_2^2+\|g_t\|_1^2+\|\mathbf{u_t}\|_{(H^1(\Omega))^2}+\|\nabla\cdot\mathbf{u_t}\|_1+\|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2\Big)\\ &\quad +h^2(\|c\|_2^2+\|\mathbf{u}\|^2_{(H^1(\Omega))^2}+\|p\|_1^2)+\|\theta\|^2\Big].\end{aligned} \tag{3.4.59}$$

Since $c_h(0) = R_hc(0)$, this implies that $\theta(0) = 0$. An application of Gronwall's inequality with (3.3.47), (3.4.59) yields

$$\begin{aligned}\|\theta\|^2_{L^\infty(J;L^2)} \leq\ & C(T)\Big[\int_0^T \Big\{h^4\Big(\|g\|_1^2 + \|\mathbf{u}\|_{(H^1(\Omega))^2} + \|\nabla\cdot\mathbf{u}\|_1 \\ & +\|\phi\frac{\partial c}{\partial t}\|_1^2 + \|c_t\|_2^2 + \|g_t\|_1^2 + \|\mathbf{u_t}\|_{(H^1(\Omega))^2} + \|\nabla\cdot\mathbf{u_t}\|_1 \\ & +\|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2\Big) + h^2(\|c\|_2^2 + \|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2)\Big\}ds\Big]. \qquad (3.4.60)\end{aligned}$$

Use triangle inequality to complete the rest of the proof. ■

Combining the estimates derived in (3.4.18) and (3.4.50), we obtain the following estimates for the velocity and pressure.

THEOREM 3.4.3 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$, respectively, be the solutions of (3.2.1)-(3.2.2) and (3.3.1)-(3.3.2) and let $c_h(0) = c_{0,h} = R_hc(0)$. Then, for sufficiently small h, there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such that*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u_h}\|^2_{L^\infty(J;(L^2(\Omega))^2)} + \|p-p_h\|^2_{L^\infty(J;L^2(\Omega))} \leq\ & C(T)\Big[\int_0^T \Big\{h^4\Big(\|g\|_1^2 + \|\mathbf{u}\|_{(H^1(\Omega))^2} \\ & +\|\nabla\cdot\mathbf{u}\|_1 + \|\phi\frac{\partial c}{\partial t}\|_1^2 + \|c_t\|_2^2 + \|g_t\|_1^2 + \|\mathbf{u_t}\|_{(H^1(\Omega))^2} + \|\nabla\cdot\mathbf{u_t}\|_1 \\ & +\|\phi\frac{\partial^2 c}{\partial t^2}\|_1^2\Big) + h^2(\|c\|_2^2 + \|\mathbf{u}\|^2_{(H^1(\Omega))^2} + \|p\|_1^2)\Big\}ds\Big].\end{aligned}$$

3.5 Completely Discrete Scheme

In this section, we briefly discuss a fully discrete scheme. In order to approximate the time derivative, we use the Euler backward difference scheme. Let $0 = t_0 < t_1 < \cdots t_N = T$ be a given partition of the time interval $[0, T]$ with time step size $\Delta t_n = t_n - t_{n-1}$. For the sake of convenience we assume a uniform time step size Δt. Set $f^n = f(t_n)$ for a generic function f in time. Then, at time level t_n, the fully discrete problem corresponding to pressure-velocity equation (3.3.1)-(3.3.2) is to find $(\mathbf{u_h^n}, p_h^n) \in U_h \times W_h$ such that

$$(\kappa^{-1}\mu(c_h^n)\mathbf{u_h^n}, \gamma_h\mathbf{v_h}) + (\nabla\cdot\mathbf{v_h}, p_h^n) = 0 \quad \forall \mathbf{v_h} \in U_h, \qquad (3.5.1)$$

$$(\nabla\cdot\mathbf{u_h^n}, w_h) = (q^n, w_h) \qquad \forall w_h \in W_h. \qquad (3.5.2)$$

For the approximation of concentration at time level $t = t_{n+1}$, we use the approximate velocity at the previous time level $(t = t_n)$. Using backward Euler difference scheme, we have

$$\frac{\partial c_h}{\partial t}|_{t=t_{n+1}} \approx \frac{c_h^{n+1} - c_h^n}{\Delta t}. \tag{3.5.3}$$

Here also, we use the cut-off operator $\mathcal{M}(\mathbf{u_h})$ of the approximate velocity $\mathbf{u_h}$ at $t = t_n$ by

$$\mathcal{M}(\mathbf{u_h^n}) = \min\left(|\mathbf{u_h^n}|, M\right) \frac{\mathbf{u_h^n}}{|\mathbf{u_h^n}|}, \tag{3.5.4}$$

Since $\mathbf{u^n}$ is bounded, we have

$$\mathcal{M}(\mathbf{u^n}) = \mathbf{u^n}$$

Now, the discrete problem corresponding to the concentration equation (3.3.18) is to find $c_h^{n+1} \in M_h$ such that

$$\big(\phi\frac{(c_h^{n+1} - c_h^n)}{\Delta t}, \gamma z_h\big) + A_h(\mathcal{M}(\mathbf{u_h^n}); c_h^{n+1}, z_h) = (g(c_h^{n+1}), \gamma z_h) \quad \forall z_h \in M_h. \tag{3.5.5}$$

3.5.1 Error Estimates

In order to derive the error estimates for concentration, we need the following error bound for velocity and pressure at $t = t_n$, which is given in Theorem 2.4.1.

$$\|\mathbf{u^n} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + \|p^n - p_h^n\| \leq C\left[\|c^n - c_h^n\| + h\left(\|\mathbf{u^n}\|_{(H^1(\Omega))^2} + \|p^n\|_1\right)\right], \tag{3.5.6}$$

and

$$\|\nabla\cdot(\mathbf{u^n} - \mathbf{u_h^n})\| \leq Ch\|\nabla\cdot\mathbf{u^n}\|. \tag{3.5.7}$$

Now we prove our main theorem.

THEOREM 3.5.1 *Let c^m and c_h^m be the solutions of (3.2.3) and (3.5.5) respectively at $t = t_m$, $1 \leq m \leq N$, and let $c_h(0) = c_{0,h} = R_h c(0)$. Then for sufficiently small h, there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such* *that*

$$\max_{0\le m\le N}\|c^m-c_h^m\|^2 \le C\Big[h^4\Big(\|g\|^2_{L^\infty(0,T;H^1)}+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)}+\|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)}$$
$$+\|c_t\|^2_{L^2(0,T;H^2)}+\|g_t\|^2_{L^2(0,T;H^1)}+\|\mathbf{u}_t\|^2_{L^2(0,T;(H^1(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;H^1)}$$
$$+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big)+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)}$$
$$+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big)+h^2\Big(\|c\|^2_{L^\infty(0,T;H^2)}+\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)}+\|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big].\quad(3.5.8)$$

Proof. Write $c^n-c_h^n=(c^n-R_hc^n)+(R_hc^n-c_h^n)=\rho^n+\theta^n$. Since the estimates for ρ^n known from Lemmas 3.4.1 and 3.4.3 at $t=t_n$, it is enough to obtained the bound for θ^n. Multiply the concentration equation (3.1.3) by γz_h and integrate over Ω. Then, at $t=t_{n+1}$, we have

$$\left(\phi\frac{\partial c^{n+1}}{\partial t},\gamma z_h\right)+A_h(\mathbf{u^{n+1}};c^{n+1},z_h)=(g(c^{n+1}),\gamma z_h)\quad\forall z_h\in M_h\qquad(3.5.9)$$

Subtracting (3.5.5) from (3.5.9), we obtain

$$\left(\phi\frac{\partial c^{n+1}}{\partial t}-\phi\frac{c_h^{n+1}-c_h^n}{\Delta t},\gamma z_h\right)+A_h(\mathbf{u^{n+1}};c^{n+1},z_h)$$
$$-A_h(\mathcal{M}(\mathbf{u_h^n});c_h^{n+1},z_h)=(g(c^{n+1})-g(c_h^{n+1}),\gamma z_h)\quad\forall z_h\in M_h.\qquad(3.5.10)$$

Now using choosing $z_h=\theta^{n+1}$ and using (3.4.3), we obtain the following error equation:

$$\begin{aligned}\left(\phi\frac{\theta^{n+1}-\theta^n}{\Delta t},\gamma\theta^{n+1}\right) \;+\;& A_h(\mathcal{M}(\mathbf{u_h^n});\theta^{n+1},\theta^{n+1})=\Big[A_h(\mathcal{M}(\mathbf{u_h^n});R_hc^{n+1},\theta^{n+1})\\ &-A_h(\mathbf{u^{n+1}};R_hc^{n+1},\theta^{n+1}\Big]-\left(\phi\frac{\rho^{n+1}-\rho^n}{\Delta t},\gamma\theta^{n+1}\right)\\ &-\phi\left(\frac{\partial c^{n+1}}{\partial t}-\frac{c^{n+1}-c^n}{\Delta t},\gamma\theta^{n+1}\right)\\ &+(g(c^{n+1})-g(c_h^{n+1}),\gamma\theta^{n+1})+\lambda(\rho^{n+1},\gamma\theta^{n+1})\\ =\;& J_1+J_2+J_3+J_4+J_5,\quad\text{say.}\end{aligned}\qquad(3.5.11)$$

Using the same arguments as in the proof of Lemma 3.4.4, J_1 can be bounded in the following manner

$$\begin{aligned} |J_1| &\leq |A_h(\mathcal{M}(\mathbf{u_h^n}); R_h c^{n+1}, \theta^{n+1}) - A_h(\mathbf{u^{n+1}}, R_h c^{n+1}, \theta^{n+1})| \\ &\leq C\left[\|\mathbf{u^{n+1}} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u_h^n})\| + |||\rho^{n+1}|||\right] |||\theta^{n+1}||| \\ &\leq C\Big[\|\mathbf{u^{n}} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^{n}} - \mathbf{u_h^n})\| \\ &\quad + \|\mathbf{u^{n+1}} - \mathbf{u^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u^n})\| + |||\rho^{n+1}|||\Big] |||\theta^{n+1}|||. \end{aligned} \tag{3.5.12}$$

Using (3.3.48) and the Cauchy-Schwarz inequality, J_2 can be bounded easily as follows:

$$|J_2| \leq C(\Delta t)^{-1/2} \|\frac{\partial \rho}{\partial t}\|_{L^2(t_n,t_{n+1};L^2)} \|\theta^{n+1}\|. \tag{3.5.13}$$

An application of Taylor series expansion and (3.3.48) gives us

$$|J_3| \leq C\|\theta^{n+1}\| \left(\Delta t \int_{t_n}^{t_{n+1}} \|c_{tt}\|^2 ds\right)^{1/2}. \tag{3.5.14}$$

Since the function g is uniformly Lipschitz continuous (see (2.1.7)), J_4 can be bounded in the following way:

$$\begin{aligned} |J_4| &\leq |(g(c^{n+1}) - g(c_h^{n+1}), \gamma\theta^{n+1})| \leq C\|c^{n+1} - c_h^{n+1}\| \, \|\theta^{n+1}\| \\ &\leq (\|\rho^{n+1}\| + \|\theta^{n+1}\|)\|\theta^{n+1}\|. \end{aligned} \tag{3.5.15}$$

Again using (3.3.48), we obtain

$$J_5 \leq |\lambda(c^{n+1} - R_h c^{n+1}, \gamma\theta^{n+1})| \leq \|\rho^{n+1}\| \, \|\theta^{n+1}\|. \tag{3.5.16}$$

Since

$$|\mathbf{u^{n+1}} - \mathbf{u^n}|^2 = \left|\int_{t_n}^{t_{n+1}} \mathbf{u}_t \, ds\right|^2 \leq \Delta t \int_{t_n}^{t_{n+1}} |\mathbf{u}_t|^2 ds, \tag{3.5.17}$$

Hence,

$$\|\mathbf{u^{n+1}} - \mathbf{u^n}\|^2_{(L^2(\Omega))^2} \leq \Delta t \, \|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}, \tag{3.5.18}$$

and similarly,

$$\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u^n})\|^2_{L^2(\Omega)} \leq \Delta t \, \|\nabla\cdot\mathbf{u_t}\|^2_{L^2(t_n,t_{n+1};L^2)}. \tag{3.5.19}$$

Substituting (3.5.12)-(3.5.16) with (3.5.18), (3.5.19) in (3.5.11) and applying Young's inequality, we obtain the following equation

$$\begin{aligned}\Big(\phi\frac{\theta^{n+1}-\theta^{n}}{\Delta t},\gamma\theta^{n+1}\Big) &+ A_h(\mathcal{M}(\mathbf{u_h^n});\theta^{n+1},\theta^{n+1}) \leq C\Big[(\Delta t)^{-1}\|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)}+\|\theta^{n+1}\|^2\\
&+ \Delta t\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)}\Big)\\
&+ \Delta t\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(t_n,t_{n+1},L^2)}+\|\mathbf{u^n}-\mathbf{u_h^n}\|^2_{(L^2(\Omega))^2}+|||\rho^{n+1}|||^2\\
&+ h^2\|\nabla.(\mathbf{u^n}-\mathbf{u_h^n})|||^2+|||\theta^{n+1}|||^2+\|\rho^{n+1}\|^2\Big].\end{aligned}\tag{3.5.20}$$

Now, a use of (3.3.34), (3.5.6) and (3.5.7) in (3.5.20) with kick back arguments yields

$$\begin{aligned}|||\theta^{n+1}|||^2-|||\theta^{n}|||^2 \leq\ & C\Big[\Delta t\left(\|\theta^{n+1}\|^2+\|\theta^{n}\|^2+\|\rho^{n}\|^2+\|\rho^{n+1}\|^2+|||\rho^{n+1}|||^2\right)\\
&+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)}\\
&+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big)+\|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)}\\
&+h^2\Delta t\left(h^2\|\nabla\cdot\mathbf{u^n}\|^2_1+\|\mathbf{u^n}\|^2_{(H^1(\Omega))^2}+\|p^n\|^2_1\right)\Big].\end{aligned}\tag{3.5.21}$$

Taking summation over $n=0\cdots m-1$, we obtain

$$\begin{aligned}|||\theta^{m}|||^2_h-|||\theta^{0}|||^2_h \leq\ & C\Big[\sum_{n=0}^{m-1}\Big\{\Delta t\left(\|\theta^{n+1}\|^2+\|\theta^{n}\|^2+\|\rho^{n}\|^2+\|\rho^{n+1}\|^2+|||\rho^{n+1}|||\right)\\
&+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u})_t\|^2_{L^2(t_n,t_{n+1};L^2)}+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big)\\
&+\|\frac{\partial\rho}{\partial t}\|^2_{L^2(0,T,L^2)}+h^2\Delta t\Big(h^2\|\nabla\cdot\mathbf{u^n}\|^2_1\\
&+\|\mathbf{u^n}\|^2_{(H^1(\Omega))^2}+\|p^n\|^2_1\Big)\Big\}\Big].\end{aligned}\tag{3.5.22}$$

Use Gronwall's Lemma 1.2.8, equivalence of the norms $|||\cdot|||_h$ and $\|\cdot\|$ given in (3.3.47) and estimates of ρ to obtain

$$\begin{aligned}\|\theta^m\|^2 \leq\ & C\Big[\|\theta^0\|^2+h^4\Big(\|g\|^2_{L^\infty(0,T;H^1)}+\|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)}+\|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)}\\
&+\|c_t\|^2_{L^2(0,T;H^2)}+\|g_t\|^2_{L^2(0,T;H^1)}+\|\mathbf{u}_t\|^2_{L^2(0,T;(H^1(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;H^1)}\\
&+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big)+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)}+\|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)}\\
&+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big)+h^2\Big(\|c\|^2_{L^\infty(0,T;H^2)}+\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)}+\|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big].\end{aligned}\tag{3.5.23}$$

Now since $c_h(0) = R_h c(0)$, i.e., $\theta^0 = 0$, (3.5.23) implies that

$$\max_{0\leq m\leq N} \|\theta^m\|^2 \leq C\Big[h^4\Big(\|g\|^2_{L^\infty(0,T;H^1)} + \|\nabla\cdot\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|\phi\frac{\partial c}{\partial t}\|^2_{L^\infty(0,T;H^1)}$$
$$+\|c_t\|^2_{L^2(0,T;H^2)} + \|g_t\|^2_{L^2(0,T;H^1)} + \|\mathbf{u}_t\|^2_{L^2(0,T;(H^1(\Omega))^2)} + \|\nabla\cdot u_t\|^2_{L^2(0,T;H^1)}$$
$$+\|\phi\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T;H^1)}\Big) + (\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;L^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)}$$
$$+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(0,T,L^2)}\Big) + h^2\Big(\|c\|^2_{L^\infty(0,T;H^2)} + \|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \quad (3.5.24)$$

Now combined the estimate of θ and ρ to completes the rest of the proof. ■

Now using (3.5.6) and (3.5.8), we obtain the similar estimates as in Theorem 3.5.1 for velocity as well as pressure.

3.6 Numerical Procedure

In this section, we discuss the numerical method applied to pressure-velocity equation and concentration equation. We consider the following test problem, where only the molecular diffusion is present and the effect of dispersion coefficients are negligible. Find $c,\ p$, $\mathbf{u}$ such that

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall(x,t)\in\Omega\times J, \quad (3.6.1)$$

$$\nabla\cdot\mathbf{u} = q^+ - q^- \qquad \forall(x,t)\in\Omega\times J, \quad (3.6.2)$$

$$\phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D(\mathbf{u})\nabla c) + cq^- = \bar{c}q^+ \qquad \forall(x,t)\in\Omega\times J, \quad (3.6.3)$$

with boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \forall(x,t)\in\partial\Omega\times J, \quad (3.6.4)$$

$$D(\mathbf{u})\nabla\cdot\mathbf{n} = 0 \qquad \forall(x,t)\in\partial\Omega\times J, \quad (3.6.5)$$

and initial condition

$$c(x,0) = c_0(x) \qquad \forall x\in\Omega. \quad (3.6.6)$$

Here $\mu(c)$ is the viscosity of the fluid mixture which depends on the concentration as:

$$\mu(c) = \mu(0)\Big[(1-c) + M^{\frac{1}{4}}c\Big]^{-4}, \quad (3.6.7)$$

where M is the mobility ratio between the resident and injected fluids and $\mu(0)$ is the viscosity of the resident fluid, $\bar{c}$ is the injection concentration and q^+ and q^- are the production and injection rates, respectively. Let $\mathcal{T}_h$ be an admissible regular, uniform triangulation of $\overline{\Omega}$ into closed triangles

As we have mentioned in Chapter 2 that for a good approximation to the concentration, one has to take larger time step for the pressure equation than the concentration. Here also, we take different time step for pressure and concentration equation. Let $0 = t_0 < t_1 < \cdots t_M = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t_m = t_{m+1} - t_m$ for the pressure equation and $0 = t^0 < t^1 < \cdots t^N = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t^n = t^{n+1} - t^n$ for the concentration equation. We denote $C^n \approx c_h(t^n)$, $C_m \approx c_h(t_m)$, $U_m \approx \mathbf{u_h}(t_m)$ and $P_m \approx p_h(t_m)$.

If concentration step t^n relates to pressure steps by $t_{m-1} < t^n \leq t_m$, we require a velocity approximation at $t = t^n$, which will be used in the concentration equation, based on $\mathbf{U}_{m-1}$ and earlier values. We define a velocity approximation at $t = t^n$ by

$$E\mathbf{U^n} = \left(1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\right)\mathbf{U}_{m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{U}_{m-2} \qquad \text{for} \quad m \geq 2, \tag{3.6.8}$$

$$E\mathbf{U^n} = \mathbf{U}_0 \qquad \text{for} \ \ m = 1. \tag{3.6.9}$$

The discrete problem corresponding to pressure-velocity equation (3.3.1)-(3.3.2) is to find $(\mathbf{U}, P) : \{t_0, t_1, \cdots t_M\} \longrightarrow U_h \times W_h$ such that

$$(\kappa^{-1}\mu(C_m)\mathbf{U}_m, \gamma_h \mathbf{v_h}) + (\nabla \cdot \mathbf{v_h}, P_m) = 0 \quad \forall \mathbf{v_h} \in U_h \tag{3.6.10}$$

$$(\nabla \cdot \mathbf{U}_m, w_h) = (q^+ - q^-, w_h) \qquad \forall w_h \in W_h, \quad m \geq 0. \tag{3.6.11}$$

Set $\dfrac{\partial C}{\partial t}|_{t=t_{n+1}} \approx \dfrac{C^{n+1} - C^n}{\Delta t^n}$. Then, the discrete problem corresponding to concentration equation (3.3.18) is to find $C : \{t^0, t^1, \cdots t^N\} \longrightarrow M_h$ such that

$$\begin{aligned}&\big(\phi\frac{(C^{n+1} - C^n)}{\Delta t_n}, \Pi_h^* z_h\big) + (E\mathbf{U^{n+1}} \cdot \nabla C^{n+1}, \Pi_h^* z_h)\\ &\quad + A_h(E\mathbf{U^{n+1}}; C^{n+1}, z_h) + (q^- c^{n+1}, \Pi_h^* z_h) = (\bar{c}q^+, \Pi_h^* z_h) \quad \forall z_h \in M_h,\end{aligned} \tag{3.6.12}$$

where the bilinear form $A_h(\cdot; \cdot, \cdot)$ is defined in (3.3.19). To solve (3.6.12)-(3.6.11), we use mixed finite volume element method. Numerical procedure for solving (3.6.12)-(3.6.11), we have discussed in details in Chapter 2. Here we will discuss the numerical procedure for solving (3.6.12).

Now we write (3.6.12) in the matrix form. Let λ_1, λ_2 and λ_3 be the barycentric coordinates of the triangle $\triangle P_1P_2P_3$, associated with nodes P_1, P_2 and P_3, respectively. Since the finite dimensional space M_h is discontinuous, we take 1, λ_2 and λ_3 as local basis functions for a triangle $T \in \mathcal{T}_h$. Then, we construct the global basis functions Ψ_i's for M_h with the help of these local basis functions (for details see numerical procedure part of Chapter 2). Set $C^n = \sum_{i=1}^{N_h} \delta_i^n \Psi_i$, then the concentration equation (3.6.12) can be written in matrix form as

$$[D^n + \Delta t_n(E^n + H^n + R^n)]\, \boldsymbol{\delta}^{n+1} = D^n \boldsymbol{\delta}^n + \Delta t_n \mathbf{G}^n, \tag{3.6.13}$$

where

$$\boldsymbol{\delta}^n = (C^n(P_i))_{i=1}^{N_h}, \quad D^n = (d_{ij})_{i,j=1}^{N_h} = \int_{V_i^*} \phi\Psi_i\, \gamma\Psi_j\, dx,$$

$$E^n = (e_{ij})_{i,j=1}^{N_h} = \int_{V_i^*} (\mathbf{U}^n \cdot \nabla\Psi_i)\, \gamma\Psi_j\, dx$$

$$H^n = T_1^n + T_2^n + T_3^n + T_4^n,$$

with

$$[T_1^n(ij)] = -\sum_{T\in\mathcal{T}_h} \sum_{k=1}^{3} \int_{P_{k+1}BP_k} (D(\mathbf{U^n})\nabla\Psi_i\cdot\mathbf{n})\gamma\Psi_j\, ds, \quad [T_2^n(ij)] = -\sum_{e\in\Gamma} \int_e [\gamma\Psi_i]\cdot\langle D(\mathbf{U^n})\nabla\Psi_j\rangle ds,$$

$$[T_3^n(ij)] = \sum_{e\in\Gamma} \int_e [\gamma\Psi_j]\cdot\langle D(\mathbf{U^n})\nabla\Psi_i\rangle ds, \quad [T_4^n(ij)] = \sum_{e\in\Gamma} \frac{\alpha}{h_e} \int_e [\Psi_i]\cdot[\Psi_j] ds,$$

and

$$R^n = (r_{ij})_{i,j=1}^{N_h} = \int_{V_i^*} q^-\Psi_i\, \gamma\Psi_j\, dx, \qquad \mathbf{G}^n = (g_i^n)_{i=1}^{N_h} = \int_{V_i^*} \bar{c}q^+\, \gamma\Psi_i\, dx$$

.

3.6.1 Numerical experiments

For the test problem, we have taken the data from [80]. The spatial domain is $\Omega = (0, 1000)\times(0, 1000)$ ft^2 and the time period is [0, 3600] days, viscosity of oil is $\mu(0) = 1.0$ cp. The injection well is located at the upper right corner (1000, 1000) with injection rate

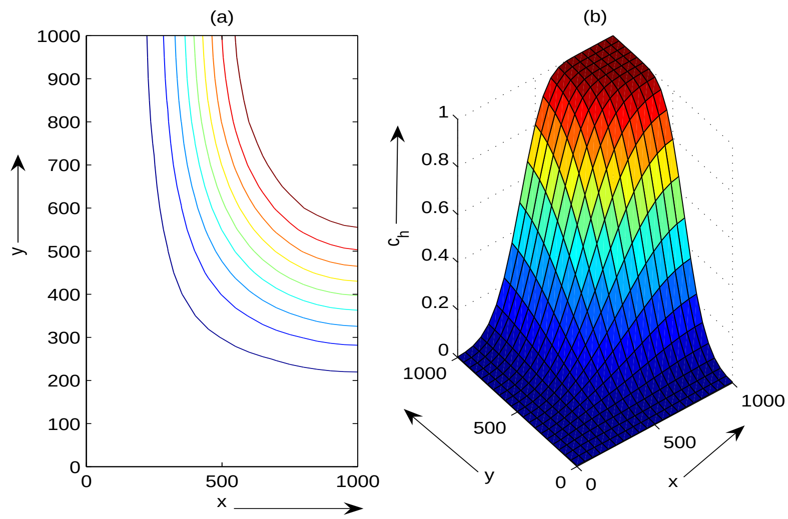

Figure 3.5: Surface (b) and contour plot (a) in Test 1 at $t = 3$ years.

$q^+ = 30\text{ft}^2/\text{day}$ and injection concentration $\bar{c} = 1.0$. The production well is located at the lower left corner with the production rate $q^- = 30\text{ft}^2/\text{day}$ and the initial concentration is $c(x, 0) = 0$. In the numerical simulation for spatial discretization we choose in 20 divisions on both x and y axes. For time discretization, we take $\Delta t_p = 360$ days and $\Delta t_c = 120$ days, i.e., we divide each pressure time interval into sub three intervals.

Test 1: We assume that the porous medium is homogeneous and isotropic. The permeability is $\kappa =$ is 80. The porosity of the medium is $\phi = .1$ and the mobility ratio between the resident and injected fluid is $M = 1$. Further more we assume that the molecular diffusion is $d_m = 1$ and dispersion coefficients are zero. The surface and contour plots for the concentration at $t = 3$ and $t = 10$ years are presented in Figure 3.5 and Figure 3.6, respectively.

Since only molecular diffusion is present and viscosity is also independent of the velocity, Figure 3.5, shows that the velocity is radial and the contour plots for the concentration is almost circular until the invading fluid reaches the production well. Figure 3.6 shows that when these plots are reached at production well, the invading fluid continues to fill the whole domain until $c = 1$.

Test 2: In this test we consider the numerical simulation of a miscible displacement

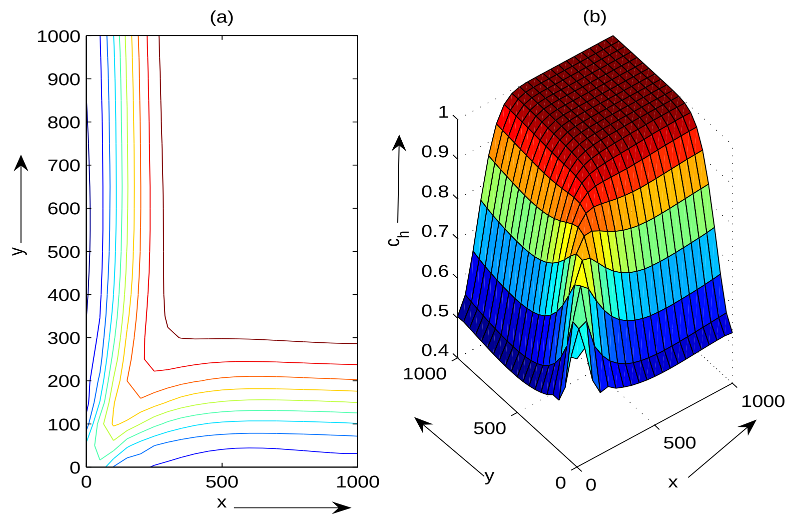

Figure 3.6: Surface (b) and contour plot (a) in Test 1 at $t = 10$ years.

problem with discontinuous permeability. Here, the data is same as given in Test 1 except the permeability of the medium $\kappa(x)$. We take $\kappa = 80$ on the sub domain $\Omega_L := (0, 1000) \times (0, 500)$ and $\kappa = 20$ on the sub domain $\Omega_U := (0, 1000) \times (500, 1000)$. The contour and surface plot at $t = 3$ and $t = 10$ years are given in Figure 3.7 and Figure 3.8 respectively.

In Test 2, the lower half domain has a larger permeability than the upper half. Figure 3.7 and Figure 3.8 shows that when the injecting fluid reaches the lower half domain, it starts moving much faster in the horizontal direction on this domain compared to the low permeability domain that is upper half domain. We observe that one should put the production well in a low permeability zone to increase the area swept by the injected fluid.

Order of convergence. We compute the order of convergence in the L^2- norm. Figure 3.9 shows that the computed order of convergence in L^2- norm is approximately 2, which matches our theoretical findings.

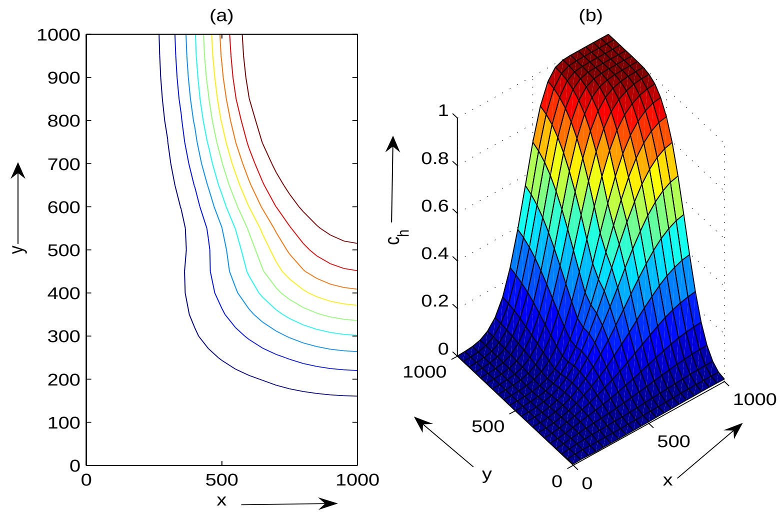

Figure 3.7: Contour (a) and surface plot (b) in Test 2 at $t = 3$ years.

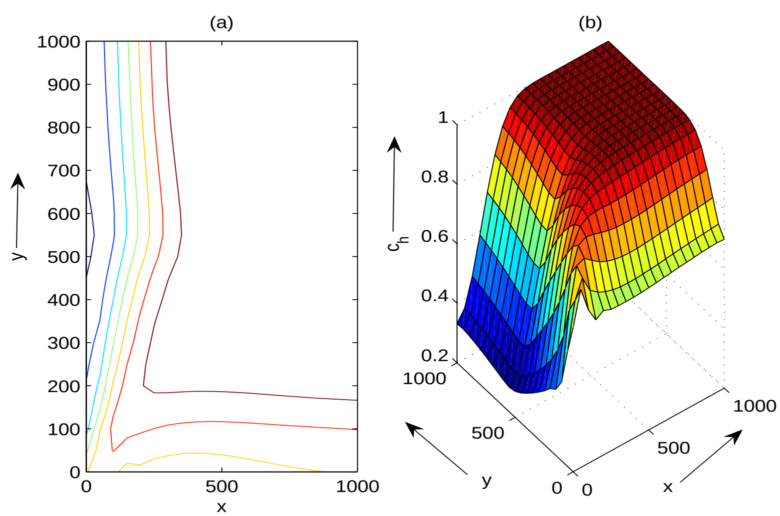

Figure 3.8: Contour (a) and surface plot (b) in Test 2 at $t = 10$ years.

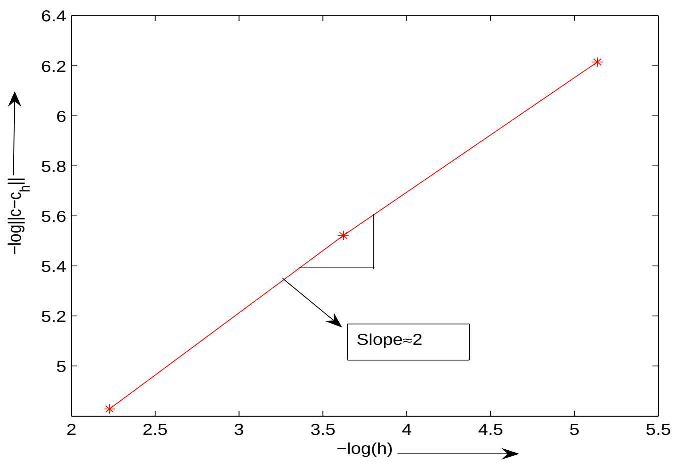

Figure 3.9: Order of convergence in L^2- norm

Chapter 4

The Modified Method of Characteristics Combined with FVEM

4.1 Introduction

The following system of coupled nonlinear partial differential equations describe the miscible displacement of one incompressible fluid by another. Find the pressure p, velocity $\mathbf{u}$ and the concentration c such that

$$\mathbf{u} = -\frac{\kappa(x)}{\mu(c)}\nabla p \qquad \forall (x,t) \in \Omega \times J, \tag{4.1.1}$$

$$\nabla \cdot \mathbf{u} = q \qquad \forall (x,t) \in \Omega \times J, \tag{4.1.2}$$

$$\phi(x)\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D(\mathbf{u})\nabla c) = (\tilde{c} - c)q \qquad \forall (x,t) \in \Omega \times J, \tag{4.1.3}$$

with boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{4.1.4}$$

$$D(\mathbf{u})\nabla c \cdot \mathbf{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times J, \tag{4.1.5}$$

and initial condition

$$c(x,0) = c_0(x) \qquad \forall x \in \Omega, \tag{4.1.6}$$

where $\tilde{c}$ is the concentration of the injective fluid. We assume that the domain Ω is a rectangle and all functions in (4.1.1)-(4.1.3) are spatially Ω-periodic. This assumption on Ω is physically reasonable, because the boundary condition (4.1.5) can be considered as a reflection boundary and in the reservoir simulation the boundary effect are of less interest compared to the inner flow.

As mentioned in Chapter 1, the concentration equation (4.1.3) is convection dominated diffusion type. The standard numerical schemes fail to provide a physically relevant solution because most of these methods suffer from grid orientation effects. The other way to minimize the grid orientation effect is to use modified methods of characteristics (MMOC). Douglas and Russell [41] introduced and analyzed MMOC for the approximation of convection dominated diffusion equations. The authors in [34, 44, 72] studied MMOC combined with Galerkin finite element methods for incompressible miscible displacement problems.

In this chapter, we apply MMOC combined with FVEM for the approximation of concentration equation (4.1.3) and mixed FVEM for pressure-velocity equations (4.1.1)-(4.1.2).

The basic idea behind the modified method of characteristics for approximating the concentration equation (4.1.3) is to set the hyperbolic part, i.e., $\phi\dfrac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c$, as a directional derivative.

Set

$$\psi(x,t) = (|\mathbf{u}(x,t)|^2 + \phi(x)^2)^{\frac{1}{2}} = (u_1(x,t)^2 + u_2(x,t)^2 + \phi(x)^2)^{\frac{1}{2}}.$$

The characteristic direction with respect to the operator $\phi\dfrac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ is the unit vector

$$\mathbf{s}(x,t) = \frac{(u_1(x,t), u_2(x,t), \phi(x))}{\psi(x,t)}.$$

The directional derivative of the concentration $c(x,t)$ in the direction of $\mathbf{s}$ is given by

$$\frac{\partial c}{\partial s} = \frac{\partial c}{\partial t}\frac{\phi(x)}{\psi(x,t)} + \frac{\mathbf{u}.\nabla c}{\psi(x,t)}$$

This implies that

$$\psi(x,t)\frac{\partial c}{\partial s} = \phi(x)\frac{\partial c}{\partial t} + \mathbf{u}.\nabla c,$$

where $\nabla c = \left(\dfrac{\partial c}{\partial x}, \dfrac{\partial c}{\partial y}\right)$.

Hence, (4.1.3) can be rewritten as

$$\psi(x,t)\frac{\partial c}{\partial s} - \nabla\cdot(D(\mathbf{u})\nabla c) = (\tilde{c} - c)q \quad \forall (x,t) \in \Omega\times J. \tag{4.1.7}$$

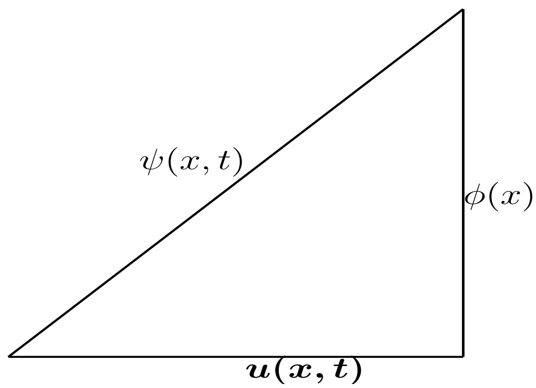

Figure 4.1: Direction of $\psi(x,t)$

Since (4.1.7) is in the form of heat equation, the behavior of the numerical solution of (4.1.7) should be better than (4.1.3) if the derivative term $\frac{\partial c}{\partial s}$ is approximated properly. We choose the same time steps for pressure and concentration for simplicity. However, the analysis can be extended to the case when different time steps are chosen for velocity and concentration through minor modifications.

Let $0 = t_0 < t_1 < \cdots t_N = T$ be a given partition of the time interval $[0, T]$ with the time step size Δt. For very small values of Δt, the characteristic direction starting from (x, t_{n+1}) crosses $t = t_n$ at (see Figure 4.2)

$$\check{x} = x - \frac{\mathbf{u^{n+1}}}{\phi(x)}\Delta t, \tag{4.1.8}$$

where $\mathbf{u^{n+1}} = \mathbf{u}(x, t_{n+1})$.

This suggests us to approximate the characteristic directional derivative at $t = t_{n+1}$ as

$$\begin{aligned} \frac{\partial c}{\partial s}\Big|_{t=t_{n+1}} &\approx \frac{c^{n+1} - c(\check{x}, t^n)}{\Delta s} \\ &= \frac{c^{n+1} - c(\check{x}, t_n)}{\left((x-\check{x})^2 + (t_{n+1} - t_n)^2\right)^{1/2}}, \end{aligned} \tag{4.1.9}$$

where $c^{n+1} = c(x, t_{n+1})$.

Using (4.1.8), we obtain

$$\psi(x,t)\frac{\partial c}{\partial s}\Big|_{t=t_{n+1}} \approx \phi(x)\frac{c^{n+1} - \check{c}^n}{\Delta t}, \tag{4.1.10}$$

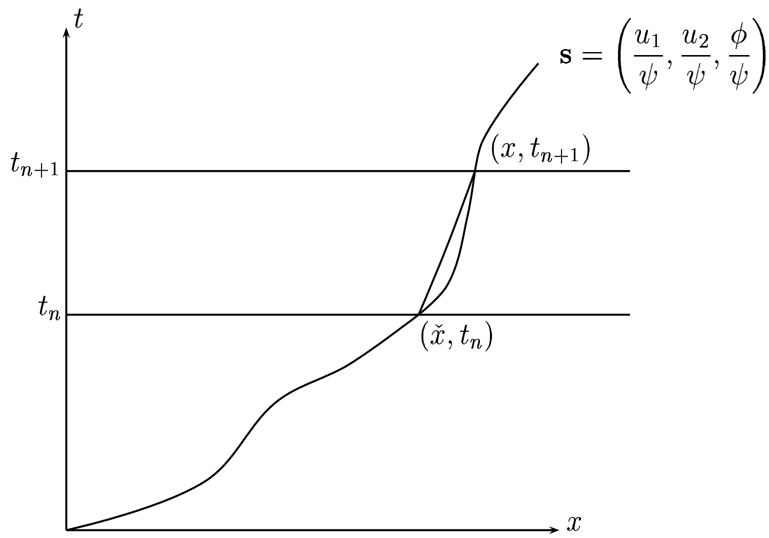

Figure 4.2: An illustration of the definition $\check{x}$

where $\check{c}^n = c(\check{x}, t_n)$.

This chapter is organized as follows. While Section 4.1 is introductory in nature, in Section 4.2, we discuss the FVEM formulation. In Section 4.3, we derive *a priori* error estimates for the velocity and for the concentration. Finally, we present the numerical procedure and the results of the numerical experiments in Section 4.4 to support our theoretical results.

4.2 Finite Volume element formulation

As mentioned in Chapter 2, the mixed FVE approximation corresponding to (4.1.1)-(4.1.2) can be written as: find $(\mathbf{u_h}, p_h) : \overline{J} \longrightarrow U_h \times W_h$ such that for $t \in (0, T]$,

$$(\kappa^{-1}\mu(c_h)\mathbf{u_h}, \gamma_h \mathbf{v_h}) + b(\gamma_h \mathbf{v_h}, p_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{4.2.1}$$

$$(\nabla \cdot \mathbf{u_h}, w_h) = (q, w_h) \qquad \forall w_h \in W_h, \tag{4.2.2}$$

where γ_h is the transfer operator defined in Chapter 2. Here, the spaces U_h, V_h and W_h are defined as follows:

$$U_h = \{\mathbf{v_h} \in U : \mathbf{v_h}|_T = (a + bx, c + by) \;\; \forall T \in \mathcal{T}_h\},$$

$$V_h = \left\{\mathbf{v_h} \in (L^2(\Omega))^2 : \mathbf{v_h}|_{T_M^*} \text{ is a constant vector } \forall T_M^* \in \mathcal{T}_h^* \text{ and } \; \mathbf{v_h} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}.$$

and

$$W_h = \{w_h \in W : w_h|_T \text{ is a constant } \forall T \in \mathcal{T}_h\}.$$

Let us define the discrete norm for $\mathbf{v_h} = (v_h^1, v_h^2) \in U_h$ as

$$\|\mathbf{v_h}\|_{1,h}^2 = \|\mathbf{v_h}\|_{(L^2(\Omega))^2}^2 + |\mathbf{v_h}|_{1,h}^2, \tag{4.2.3}$$

where $|\mathbf{v_h}|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v_h^1\|_{0,T}^2 + \|\nabla v_h^2\|_{0,T}^2$. For $v_h \in U_h$, it is straight forward to check that

$$\|\mathbf{v_h}\|_{1,h} \le C\|\mathbf{v_h}\|_{H(\text{div};\Omega)}, \tag{4.2.4}$$

where C is a constant independent of h. For $\mathbf{v_h} \in U_h$ the following inequality

$$\|\mathbf{v_h}\|_{(L^\infty(\Omega))^2} \le C\left(log\frac{1}{h}\right)^{1/2} \|\mathbf{v_h}\|_{1,h}, \tag{4.2.5}$$

holds true when Ω is in $\mathbb{R}^2$ and the triangulation $\mathcal{T}_h$ is quasi-uniform and can be proved using the same arguments as in the proof of Lemma 4 in [78, pp. 67].

For applying the standard finite volume element method to approximate the concentration, we define the trial space M_h on $\mathcal{T}_h$ and the test space L_h on $\mathcal{V}_h^*$ as follows:

$$M_h = \left\{z_h \in C^0(\bar{\Omega}) \;:\; z_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\right\}$$

$$L_h = \left\{w_h \in L^2(\Omega) \;:\; w_h|_{V_P^*} \text{ is a constant } \quad \forall V_P^* \in \mathcal{V}_h^*\right\}.$$

We now recall the transfer operator $\Pi_h^* : M_h \longrightarrow L_h$ which is defined as

$$\Pi_h^* z_h(x) = \sum_{j=1}^{N_h} z_h(P_j)\chi_j(x) \qquad \forall x \in \Omega, \tag{4.2.6}$$

where χ_j's are the characteristic functions corresponding to the control volume $V_{P_j}^*$, i.e.,

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in V_{P_j}^* \\ 0, & \text{elsewhere.} \end{cases}$$

For any given $\mathbf{v} \in U, \chi \in H^1(\Omega)$ and $\psi_h \in L_h$, we define the bilinear form $a_h(\mathbf{v};\cdot,\cdot)$ as

$$a_h(\mathbf{v};\chi,\psi_h) = -\sum_{j=1}^{N_h} \int_{\partial V_{P_j}^*} \Big(D(\mathbf{v})\nabla\chi \cdot \mathbf{n}_{P_j}\Big)\Pi_h^*\psi_h ds,$$

where $\mathbf{n}_{P_j}$ is the unit outward normal to the boundary of $V^*_{P_j}$.

Then, the FVE approximation corresponding to the concentration equation (4.1.7) is to find a solution $c_h : \overline{J} \longrightarrow M_h$ such that for $t \in (0, T]$,

$$\begin{aligned} (\psi \frac{\partial c_h}{\partial s}, \Pi_h^* z_h) + a_h(\mathbf{u_h}; c_h, z_h) + (c_h q, \Pi_h^* z_h) &= (\tilde{c} q, \Pi_h^* z_h) \quad \forall z_h \in M_h \qquad (4.2.7) \\ c_h(0) &= c_{0,h}, \end{aligned}$$

where $c_{0,h}$ is an approximation to c_0 to be defined later.

To approximate the concentration at any time say t_{n+1}, we use the approximation to the velocity at the previous time step. The fully discrete scheme corresponding to (4.2.1), (4.2.2) and (4.2.7) is defined as: For $n = 0, 1 \cdots N$, find $(c_h^n, p_h^n, \mathbf{u_h^n}) \in M_h \times W_h \times U_h$ such that

$$c_h^0 = R_h c(0), \qquad (4.2.8)$$

$$(\kappa^{-1} \mu(c_h^n) \mathbf{u_h^n}, \gamma_h \mathbf{v_h}) + b(\gamma_h \mathbf{v_h}, p_h^n) = 0 \qquad \forall \mathbf{v_h} \in U_h, \qquad (4.2.9)$$

$$(\nabla \cdot \mathbf{u_h^n}, w_h) = (q^n, w_h) \quad \forall w_h \in W_h, \qquad (4.2.10)$$

$$\begin{aligned} \left(\phi \frac{c_h^{n+1} - \hat{c}_h^n}{\Delta t}, \Pi_h^* \chi_h \right) &+ a_h(\mathbf{u_h^n}; c_h^{n+1}, \chi_h) + (q^{n+1} c_h^{n+1}, \Pi_h^* \chi_h) \\ &= (q^{n+1} \tilde{c}^{n+1}, \Pi_h^* \chi_h) \qquad \forall \chi_h \in M_h, \qquad (4.2.11) \end{aligned}$$

where $\hat{c}_h^n = c_h(\hat{x}, t_n) = c_h(x - \frac{\mathbf{u_h^n}}{\phi} \Delta t, t_n)$ and $R_h c$ is a projection of c onto M_h which will be defined in (4.3.3).

Note that in (4.1.9), we use the following notation for the exact velocity

$$\check{c}^n = c(\check{x}, t_n) = c(x - \frac{\mathbf{u^{n+1}}}{\phi} \Delta t, t_n).$$

4.3 A priori error estimates

In this section, we derive *a priori* error estimates for the concentration and the velocity. We have derived the following estimates for $\mathbf{u}$ and p in terms of the concentration in Chapter 2 (see Theorem 2.4.1).

THEOREM 4.3.1 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$, respectively, be the solutions of (4.1.1)-(4.1.2) and (4.2.1)-(4.2.2). Then, there exists a positive constant C, independent of h, but dependent on the bounds of κ^{-1} and μ such that*

$$\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + \|p - p_h\| \le C \left[\|c - c_h\| + h(\|\mathbf{u}\|_{(H^1(\Omega))^2} + \|p\|_1)\right], \tag{4.3.1}$$

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u_h})\| \le Ch\|\nabla \cdot \mathbf{u}\|_1, \tag{4.3.2}$$

provided $\mathbf{u}(t) \in (H^1(\Omega))^2,\ \nabla \cdot \mathbf{u}(t) \in H^1(\Omega)$ and $p(t) \in H^1(\Omega)$, for $t \in (0, T]$ a.e.

Let $R_h : H^1(\Omega) \longrightarrow M_h$ be the projection of c defined by

$$A(\mathbf{u}; c - R_h c, \chi) = 0 \qquad\qquad \forall \chi \in M_h, \tag{4.3.3}$$

where

$$A(\mathbf{u}; \psi, \chi) = a_h(\mathbf{u}; \psi, \chi) + (q\psi, \chi) + (\lambda\psi, \chi) \quad \forall \chi \in M_h. \tag{4.3.4}$$

The function λ will be chosen such that the coercivity of $A(\mathbf{u}; \cdot, \cdot)$ is assured.
The following lemma, which gives a bound for the error between the bilinear forms $a_h(\mathbf{u}; \cdot, \cdot)$ and $a(\mathbf{u}; \cdot, \cdot)$, can be proved using similar arguments as used in Lemma 2.4.2 of Chapter 2.

LEMMA 4.3.1 *There exists a positive constant C such that*

$$|\epsilon_a(\mathbf{u}; c - R_h c, \chi_h)| \le Ch^2 \left(|(c - \tilde{c})q|_1 + |\psi \frac{\partial c}{\partial s}|_1 + \|c\|_2 \right) |\chi_h|_1 \quad \forall \chi_h \in M_h. \tag{4.3.5}$$

We also state the following lemma, which gives us H^1- norm error estimates for the operator R_h and can be easily proved using the similar arguments as in proof of Lemma 2.4.1.

LEMMA 4.3.2 *There exists a positive constant C independent of h such that*

$$\|c - R_h c\|_1 \le Ch\|c\|_2, \tag{4.3.6}$$

provided $c(t) \in H^2(\Omega)$, for $t \in (0, T]$ a.e.

We also recall the following lemma from Chapter 2.

LEMMA 4.3.3 *There exists a positive constant C such that $\forall\theta \in M_h$,*

$$|a_h(\mathbf{u}; R_h c, \theta) - a_h(\mathbf{u_h}; R_h c, \theta)| \leq C\left(\|\mathbf{u} - \mathbf{u_h}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u} - \mathbf{u_h})\|\right)|\theta|_1. \quad (4.3.7)$$

In the next lemma, we state the error estimate for R_h in L^2- norm, the proof of which can be obtained by a modification of Lemma 2.4.3 in Chapter 2.

LEMMA 4.3.4 *There exists a positive constant C independent of h such that*

$$\|c - R_h c\| \leq Ch^2\left(|(c - \tilde{c})q|_1 + |\psi\frac{\partial c}{\partial s}|_1 + \|c\|_2\right), \quad (4.3.8)$$

provided $c \in H^2(\Omega)$ and $\psi\frac{\partial c}{\partial s} \in H^1(\Omega)$ for $t \in (0, T]$ a.e.

Before proving the main theorem, we prove the following two lemmas for our future use.

LEMMA 4.3.5 *For $f \in H^1(\Omega)$, there exists a positive constant C independent of h and Δt such that*

$$\left\|f - \check{f}\right\| \leq C\Delta t\|\nabla f^n\|, \quad (4.3.9)$$

where $\check{f} = f(\check{x}) = f\Big(x - \frac{\mathbf{u^{n+1}(x)}}{\phi(x)}\Delta t\Big)$.

Proof. Note that

$$\begin{aligned}\left\|f - \check{f}\right\|^2 &= \int_\Omega \left(f - \check{f}\right)^2 dx \\ &= \int_\Omega \left(\int_{\check{x}}^{x} \nabla f \cdot d\theta\right)^2 dx \quad (4.3.10)\end{aligned}$$

where $\theta = (1-\bar{z})\check{x} + \bar{z}x$, with $\bar{z} \in [0, 1]$ parametrizes the line segment joining $\check{x}$ to x. Then,

$$\left\|f - \check{f}\right\|^2 \leq \int_\Omega \left[\int_0^1 (x - \check{x})^2 d\bar{z}\right]^{1/2}\left[\int_0^1 \left(\frac{\partial f}{\partial z}((1-\bar{z})\check{x} + \bar{z}x)\right)^2 d\bar{z}|x\right]^{1/2} dx, \quad (4.3.11)$$

where $\mathbf{z}$ is the unit vector in the direction of $x - \check{x}$ and $\frac{\partial f^n}{\partial z}$ is the directional derivative of f in the direction of $\mathbf{z}$.

Since $|x - \check{x}| \leq \Delta t\left|\frac{\mathbf{u^{n+1}}(x)}{\phi(x)}\right|$, we find that

$$\left\|f - \check{f}\right\|^2 \leq (\Delta t)^2\left\|\frac{\mathbf{u^{n+1}}(x)}{\phi(x)}\right\|_\infty^2 \int_\Omega\int_0^1 \left(\frac{\partial f}{\partial z}((1-\bar{z})\check{x} + \bar{z}x)\right)^2 d\bar{z}dx \quad (4.3.12)$$

Define the transformation

$$y = G_{\bar{z}}(x) = (1-\bar{z})\check{x} + \bar{z}x = x - \frac{\mathbf{u^{n+1}}(x)}{\phi(x)}\Delta t(1-\bar{z}). \tag{4.3.13}$$

The Jacobian of the map $G_{\bar{z}}$, say $DG_{\bar{z}}$ with $\mathbf{u^{n+1}} = (u_1^{n+1}, u_2^{n+1})$ is given by

$$DG_{\bar{z}} = \begin{pmatrix} 1 - \frac{\partial}{\partial x_1}\left(\frac{u_1^{n+1}(x)}{\phi(x)}\Delta t(1-\bar{z})\right) & -\frac{\partial}{\partial x_2}\left(\frac{u_1^{n+1}(x)}{\phi(x)}\Delta t(1-\bar{z})\right) \\ -\frac{\partial}{\partial x_1}\left(\frac{u_2^{n+1}(x)}{\phi(x)}\Delta t(1-\bar{z})\right) & 1 - \frac{\partial}{\partial x_2}\left(\frac{u_2^{n+1}(x)}{\phi(x)}\Delta t(1-\bar{z})\right) \end{pmatrix}.$$

Since $\mathbf{u^{n+1}}$ and its first order partial derivatives are bounded, the determinant of $DG_{\bar{z}}$ is of the form $1 + O(\Delta t)$.

A change in the order of integration and variables in (4.3.12) yields

$$\left\|f^n - \check{f}\right\|^2 \le C(\Delta t)^2 \int_0^1 \sum_{T\in\mathcal{T}_h} \int_{G_{\bar{z}}(T)} \left|\frac{\partial f}{\partial z}(y)\right|^2 dx d\bar{z}. \tag{4.3.14}$$

For small Δt it can been seen that $G_{\bar{z}}$ is one-one mapping on each T. Moreover, for small h and Δt, $G_{\bar{z}}$ maps T into itself and its its immediate-neighbor element. Hence, $G_{\bar{z}}$ is globally at most finitely one-one and maps Ω into itself and its immediate-neighbor periodic copies. Thus the sum in (4.3.14) is bounded by finite many multiples of an Ω- integral. This implies that

$$\left\|f - \check{f}\right\| \le C\Delta t\|\nabla f\|. \tag{4.3.15}$$

LEMMA 4.3.6 *[41, pp. 875] If $\eta \in L^2(\Omega)$ and $\check{\eta}(x) = \eta(\check{x})$ with $\check{x} = x - \mathbf{r}(x)\Delta t$, for a nonzero function $\mathbf{r}(x)$ such that $\mathbf{r}$ and $\nabla\cdot\mathbf{r}$ are bounded, then*

$$\|\eta - \check{\eta}\|_{-1} \le C\|\eta\|\Delta t, \tag{4.3.16}$$

where C is a positive constant independent of h and Δt.

Proof. Let $z = G(x) = x - \mathbf{r}(x)\Delta t$. For sufficiently small Δt, the determinant of DG is nonzero. It can be easily seen that the determinant of DG and DG^{-1} are both of the form $1 + O(\Delta t)$.

Using the definition of $\|\cdot\|_{-1}$ norm, we have

$$\|\eta - \check{\eta}\|_{-1} = \sup_{f\in H^1(\Omega)/\{0\}} \left(\frac{1}{\|f\|_1}\int_\Omega [\eta(x) - \eta(x - \mathbf{r}(x)\Delta t)]\, f(x)dx\right). \tag{4.3.17}$$

Using a change of variable in the second term of the right hand side of (4.3.17), we obtain

$$\begin{aligned}
\|\eta - \check{\eta}\|_{-1} &= \sup_{f \in H^1(\Omega)/\{0\}} \left(\frac{1}{\|f\|_1} \left[\int_\Omega \eta(x) f(x) dx - \int_\Omega \eta(x) f(G^{-1}(x)) \det DG^{-1}(x) dx \right] \right) \\
&\leq \sup_{f \in H^1(\Omega)/\{0\}} \frac{1}{\|f\|_1} \left[\int_\Omega \eta(x) f(x) \left(1 - \det DG^{-1}(x)\right) dx \right] \\
&\quad + \sup_{f \in H^1(\Omega)/\{0\}} \frac{1}{\|f\|_1} \left[\int_\Omega \eta(x) \left(f(x) - f(G^{-1}(x)) \right) \det DG^{-1}(x) dx \right] \\
&= W_1 + W_2, \text{ say.} \qquad (4.3.18)
\end{aligned}$$

Since $|1 - \det DG^{-1}(x)| \leq C\Delta t$. Hence,

$$|W_1| \leq C\Delta t \sup_{f \in H^1(\Omega)/\{0\}} \frac{1}{\|f\|_1} \|f\| \|\eta\| \leq C\Delta t \|\eta\|. \qquad (4.3.19)$$

To bound W_2, we proceed as follows. Using the Cauchy-Schwarz inequality and the fact that $|\det DG^{-1}| \leq C$, we have

$$|W_2| \leq C \sup_{f \in H^1(\Omega)/\{0\}} \frac{1}{\|f\|_1} \|\eta(x)\| \; \|f(x) - f(G^{-1}(x))\|. \qquad (4.3.20)$$

Note that

$$\|f(x) - f(G^{-1}(x))\|^2 = \int_\Omega \left[f(x) - f(G^{-1}(x)) \right]^2 \, dx. \qquad (4.3.21)$$

Proceeding as in the proof of Lemma 4.3.5, we obtain

$$\|f(x) - f(G^{-1}(x)\| \leq C(\Delta t) \|f\|_1. \qquad (4.3.22)$$

Substituting (4.3.22) in (4.3.20), we obtain

$$|W_2| \leq C\Delta t \|\eta\|. \qquad (4.3.23)$$

Substitute (4.3.19) and (4.3.23) in (4.3.18) to complete the rest of the proof.

Now, we prove our main theorem.

THEOREM 4.3.2 *Let c^n and c_h^n be the solutions of (4.1.3) and (4.2.11) at $t = t_n$ respectively, and let $c_h(0) = c_{0,h} = R_h c(0)$. Further assume that $\Delta t = O(h)$. Then, for sufficiently small*

h, there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such that

$$\begin{aligned}\max_{0\le n\le N}\|c^n - c_h^n\|^2_{(L^2(\Omega))^2} \quad\le\quad & C\Big[h^4\Big(\|(c-\tilde{c})q\|^2_{L^\infty(0,T;H^1)} + \|\psi\frac{\partial c}{\partial s}\|^2_{L^\infty(0,T;H^1)} + \|c\|^2_{L^\infty(0,T;H^2)} \\ & +\|\nabla.\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|c_t\|^2_{L^\infty(0,T;H^2)} + \|\psi\frac{\partial^2 c}{\partial t\partial s}\|^2_{L^\infty(0,T;H^1)}\Big) \\ & +(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)} + \|(\nabla\cdot\mathbf{u})_t\|^2_{L^2(0,T;L^2)} + \|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(0,T,L^2)}\Big) \\ & +h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \qquad (4.3.24)\end{aligned}$$

Proof. Write $c^n - c_h^n = (c^n - R_hc^n) - (c_h^n - R_hc^n) = \rho^n - \theta^n$. Since the estimates of ρ^n are known, it is enough to estimate θ^n.

Multiplying (4.1.3) by $\Pi_h^*\chi_h$ and subtracting the resulting equation from (4.2.11) at $t = t_{n+1}$, we obtain

$$\begin{aligned}&\Big(\phi\frac{c_h^{n+1}-\hat{c}_h^n}{\Delta t},\Pi_h^*\chi_h\Big) + a_h(\mathbf{u_h^n}; c_h^{n+1},\chi_h) - a_h(\mathbf{u^{n+1}}; c^{n+1},\chi_h) + (q^{n+1}c_h^{n+1},\Pi_h^*\chi_h) \\ -\quad & (q^{n+1}c^{n+1},\Pi_h^*\chi) = \Big(\mathbf{u^{n+1}}\cdot\nabla c^{n+1} + \phi\frac{\partial c^{n+1}}{\partial t},\Pi_h^*\chi_h\Big) \quad \forall\chi_h\in M_h. \qquad (4.3.25)\end{aligned}$$

Choose $\chi_h = \theta^{n+1}$ in (4.3.25) and use the definition of R_h to obtain

$$\begin{aligned}\Big(\phi\frac{\theta^{n+1}-\theta^n}{\Delta t},\Pi_h^*\theta^{n+1}\Big) \quad + \quad & a_h(\mathbf{u_h^n};\theta^{n+1},\theta^{n+1}) = (\theta^{n+1}q^{n+1},\Pi_h^*\theta^{n+1}) \\ & +(\rho^{n+1},\theta^{n+1}) + (\rho^{n+1}q^{n+1},\theta^{n+1}-\Pi_h^*\theta^{n+1}) \\ & +\Big[a_h(\mathbf{u^{n+1}}; R_hc^{n+1},\theta^{n+1}) - a_h(\mathbf{u_h^{n+1}}; R_hc^{n+1},\theta^{n+1})\Big] \\ & +\Big(\mathbf{u^{n+1}}\cdot\nabla c^{n+1} + \phi\frac{\partial c^{n+1}}{\partial t} - \phi\frac{(c^{n+1}-\check{c}^n)}{\Delta t},\Pi_h^*\theta^{n+1}\Big) \\ & +\Big(\phi\frac{(\rho^{n+1}-\check{\rho}^n)}{\Delta t},\Pi_h^*\theta^{n+1}\Big) - \Big(\phi\frac{(\theta^n-\hat{\theta}^n)}{\Delta t},\Pi_h^*\theta^{n+1}\Big) \\ & +\Big(\phi\frac{(\hat{R_hc}^n - \check{R_hc}^n)}{\Delta t},\Pi_h^*\theta^{n+1}\Big) \\ & = T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8. \qquad (4.3.26)\end{aligned}$$

To estimate T_1, T_2 and T_3, we use the Cauchy-Schwarz inequality, boundedness of q and (2.3.47) to obtain

$$|T_1| = |(\theta^{n+1}q^{n+1}, \Pi_h^*\theta^{n+1})| \leq C\|\theta^{n+1}\|^2, \tag{4.3.27}$$

$$|T_2| = |(\rho^{n+1}, \theta^{n+1})| \leq C\|\rho^{n+1}\| \ \|\theta^{n+1}\|, \tag{4.3.28}$$

and

$$|T_3| = |(\rho^{n+1}q^{n+1}, \theta^{n+1} - \Pi_h^*\theta^{n+1})| \leq C\|\rho^{n+1}\| \ \|\theta^{n+1}\|. \tag{4.3.29}$$

To bound T_4, we use Lemma 4.3.3 to obtain

$$\begin{aligned} |T_4| &= |a_h(\mathbf{u^{n+1}}; R_hc^{n+1}, \theta^{n+1}) - a_h(\mathbf{u_h^{n+1}}; R_hc^{n+1}, \theta^{n+1})| \\ &\leq C\left(\|\mathbf{u^{n+1}} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u_h^{n+1}})\|\right)|\theta^{n+1}|_1 \\ &\leq C\Big(\|\mathbf{u^{n+1}} - \mathbf{u^n}\|_{(L^2(\Omega))^2} + \|\mathbf{u^n} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u^n})\| \\ &\quad +h\|\nabla\cdot(\mathbf{u^n} - \mathbf{u_h^n})\|\Big)|\theta^{n+1}|_1. \end{aligned} \tag{4.3.30}$$

From (2.5.22) and (2.5.23), we find that

$$\|\mathbf{u^{n+1}} - \mathbf{u^n}\|^2_{(L^2(\Omega))^2} \leq \Delta t\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)}, \tag{4.3.31}$$

and

$$\|\nabla\cdot(\mathbf{u^{n+1}} - \mathbf{u^n})\|^2_{L^2(\Omega)} \leq \Delta t\|(\nabla\cdot\mathbf{u})_t\|^2_{L^2(t_n,t_{n+1};L^2)}. \tag{4.3.32}$$

Hence,

$$\begin{aligned} |T_4| &\leq C\Big[(\Delta t)^{1/2}\Big(\|\mathbf{u}_t\|_{L^2(t_n,t_{n+1},(L^2(\Omega))^2)} + h\|(\nabla\cdot\mathbf{u})_t\|_{L^2(t_n,t_{n+1},L^2(\Omega))}\Big) \\ &\quad +\|\mathbf{u^n} - \mathbf{u_h^n}\|_{(L^2(\Omega))^2} + h\|\nabla\cdot(\mathbf{u^n} - \mathbf{u_h}^n)\|\Big]|\theta^{n+1}|. \end{aligned} \tag{4.3.33}$$

Using the Cauchy- Schwarz inequality and (2.3.47), we obtain

$$\begin{aligned} |T_5| &= \left|\left(\mathbf{u^{n+1}}\cdot\nabla c^{n+1} + \phi\frac{\partial c^{n+1}}{\partial t} - \phi\frac{(c^{n+1} - \hat{c}^n)}{\Delta t}, \Pi_h^*\theta^{n+1}\right)\right| \\ &\leq C\left\|\mathbf{u^{n+1}}\cdot\nabla c^{n+1} + \phi\frac{\partial c^{n+1}}{\partial t} - \phi\frac{(c^{n+1} - \hat{c}^n)}{\Delta t}\right\| \ \|\theta^{n+1}\|. \end{aligned} \tag{4.3.34}$$

To bound $\left\|\mathbf{u^{n+1}}\cdot\nabla c^{n+1}+\phi\frac{\partial c^{n+1}}{\partial t}-\phi\frac{(c^{n+1}-\hat{c}^n)}{\Delta t}\right\|$, we use the similar arguments given in [44]. Let $\sigma(x)=[\phi(x)^2+\mathbf{u^{n+1}}(x)^2]^{1/2}$, so that

$$\phi\frac{\partial c^{n+1}}{\partial t}+\mathbf{u^{n+1}}\cdot\nabla c^{n+1}=\sigma\frac{\partial c^{n+1}}{\partial \tau}. \tag{4.3.35}$$

where τ approximates the characteristic unit vector s. Let $\bar{\tau}\in[0,1]$ parametrize the approximate characteristic tangent from $(\check{x},t_n)[\bar{\tau}=0]$ to $(x,t_{n+1})[\bar{\tau}=1]$. A use of Taylor expansion along the characteristic gives us

$$\sigma\frac{\partial c^{n+1}}{\partial \tau}-\phi\frac{c^{n+1}-\check{c}^n}{\Delta t}=\frac{\phi}{\Delta t}\int_{(\check{x},t_n)}^{(x,t_{n+1})}[|x(\tau)-\check{x}|^2+(t(\tau)-t_n)^2]^{1/2}\frac{\partial^2 c}{\partial \tau^2}d\tau. \tag{4.3.36}$$

Taking the square of the $L^2(\Omega)$ norm, we obtain

$$\left\|\sigma\frac{\partial c^{n+1}}{\partial \tau}-\phi\frac{c^{n+1}-\check{c}^n}{\Delta t}\right\|^2\le\int_\Omega\left[\frac{\phi}{\Delta t}\right]^2\left[\frac{\sigma\Delta t}{\phi}\right]^2\left|\int_{(\check{x},t_n)}^{(x,t_{n+1})}\frac{\partial^2 c}{\partial\tau^2}d\tau\right|^2dx.$$

Using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}\left\|\sigma\frac{\partial c^{n+1}}{\partial \tau}-\phi\frac{c^{n+1}-\check{c}^n}{\Delta t}\right\|^2 &\le \Delta t\left\|\frac{\sigma^3}{\phi}\right\|_\infty\int_\Omega\int_{(\check{x},t_n)}^{(x,t_{n+1})}\left|\frac{\partial^2 c}{\partial\tau^2}\right|^2d\tau dx\\ &\le \Delta t\left\|\frac{\sigma^4}{\phi^2}\right\|_\infty\int_\Omega\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 c}{\partial\tau^2}(\bar{\tau}\check{x}+(1-\bar{\tau})x,t)\right|^2dtdx.\end{aligned} \tag{4.3.37}$$

For a fixed $\bar{\tau}$ consider the transformation

$$y=f_{\bar{\tau}}=\bar{\tau}\check{x}+(1-\bar{\tau})x=x-\frac{\mathbf{u^{n+1}}}{\phi}\Delta t\bar{\tau}.$$

Since $\mathbf{u},\phi$ and their first order partial derivatives are bounded, the determinant of $Df_{\bar{\tau}}$ is $1+O(\Delta t)$ and hence $Df_{\bar{\tau}}$ is invertible for sufficiently small Δt. By a change of variable argument, we have

$$\left\|\sigma\frac{\partial c^{n+1}}{\partial \tau}-\phi\frac{c^{n+1}-\check{c}^n}{\Delta t}\right\|^2\le C\Delta t\|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t_n,t_{n+1};L^2)}. \tag{4.3.38}$$

Hence, T_5 bounded as follows

$$|T_5|\le C(\Delta t)^{1/2}\|\frac{\partial^2 c}{\partial\tau^2}\|_{L^2(t_n,t_{n+1};L^2)}\|\theta^{n+1}\|. \tag{4.3.39}$$

To bound T_6, we proceed as follows

$$\begin{aligned}|T_6| &= \left|\left(\phi\frac{(\rho^{n+1}-\check{\rho}^n)}{\Delta t}, \Pi_h^*\theta^{n+1}\right)\right| \\ &\leq \left|\left(\phi\frac{(\rho^{n+1}-\check{\rho}^n)}{\Delta t}, \theta^{n+1}-\Pi_h^*\theta^{n+1}\right)\right| + \left|\left(\phi\frac{(\rho^{n+1}-\check{\rho}^n)}{\Delta t}, \theta^{n+1}\right)\right| \\ &= I_1 + I_2. \end{aligned} \tag{4.3.40}$$

Now I_1 can be written as

$$I_1 \leq \left|\left(\phi\frac{(\rho^{n+1}-\rho^n)}{\Delta t}, \theta^{n+1}-\Pi_h^*\theta^{n+1}\right)\right| + \left|\left(\phi\frac{(\rho^{n}-\check{\rho}^n)}{\Delta t}, \theta^{n+1}-\Pi_h^*\theta^{n+1}\right)\right|. \tag{4.3.41}$$

A use of Lemma 4.3.5 yields

$$\left\|\frac{(\rho^n-\check{\rho}^n)}{\Delta t}\right\| \leq C\|\nabla\rho^n\|. \tag{4.3.42}$$

It is easy to show that

$$\left\|\frac{(\rho^{n+1}-\rho^n)}{\Delta t}\right\| \leq C(\Delta t)^{-1/2}\|\frac{\partial\rho}{\partial t}\|_{L^2(t_n,t_{n+1},L^2)}. \tag{4.3.43}$$

Using (4.3.42), (4.3.43), (2.3.16) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_1 &\leq C\left[\|\theta^{n+1}-\Pi_h^*\theta^{n+1}\|\left((\Delta t)^{-1/2}\|\frac{\partial\rho}{\partial t}\|_{L^2(t_n,t_{n+1},L^2)} + \|\nabla\rho^n\|\right)\right] \\ &\leq Ch\left[(\Delta t)^{-1/2}\|\frac{\partial\rho}{\partial t}\|_{L^2(t_n,t_{n+1},L^2)} + \|\nabla\rho^n\|\right]\|\nabla\theta^{n+1}\|. \end{aligned} \tag{4.3.44}$$

I_2 can be bounded as follows:

$$I_2 \leq \left|\left(\phi\frac{(\rho^{n+1}-\rho^n)}{\Delta t}, \theta^{n+1}\right)\right| + \left|\left(\phi\frac{(\rho^{n}-\check{\rho}^n)}{\Delta t}, \theta^{n+1}\right)\right| = J_1 + J_2. \tag{4.3.45}$$

A use of Cauchy-Schwarz inequality yields

$$J_1 \leq C(\Delta t)^{-1/2}\|\frac{\partial\rho}{\partial t}\|_{L^2(t_n,t_{n+1},L^2)}\|\theta^{n+1}\|. \tag{4.3.46}$$

Using Lemma 4.3.6, we obtain

$$J_2 \leq C\left\|\frac{(\rho^n-\hat{\rho}^n)}{\Delta t}\right\|_{-1}\|\theta^{n+1}\|_1 \leq C\|\rho^n\|\|\theta^{n+1}\|_1. \tag{4.3.47}$$

This implies that

$$I_2 \leq C\left((\Delta t)^{-1/2}\|\frac{\partial\rho}{\partial t}\|_{L^2(t_n,t_{n+1},L^2)} + \|\rho^n\|\right)\|\theta^{n+1}\|. \tag{4.3.48}$$

Using (4.3.44), (4.3.48) and (4.3.40), we obtain the following bound for T_6:

$$|T_6| \leq C \left((\Delta t)^{-1/2} \| \frac{\partial \rho}{\partial t} \|_{L^2(t_n, t_{n+1}, L^2)} + h \|\nabla \rho^n\| + \|\rho^n\| \right) \|\theta^{n+1}\|. \tag{4.3.49}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |T_7| &\leq \left| \left(\frac{(\theta^n - \hat{\theta}^n)}{\Delta t}, \Pi_h^* \theta^{n+1} - \theta^{n+1} \right) \right| + \left| \left(\frac{(\theta^n - \hat{\theta}^n)}{\Delta t}, \theta^{n+1} \right) \right| \\ &\leq \left\| \frac{(\theta^n - \hat{\theta}^n)}{\Delta t} \right\| \|\Pi_h^* \theta^{n+1} - \theta^{n+1}\| + \left\| \frac{(\theta^n - \hat{\theta}^n)}{\Delta t} \right\|_{-1} \|\theta^{n+1}\|_1. \end{aligned} \tag{4.3.50}$$

We use Lemma 4.3.5 and Lemma 4.3.6 to bound $\left\| \frac{\theta^n - \hat{\theta}^n}{\Delta t} \right\|$ and $\left\| \frac{\theta^n - \hat{\theta}^n}{\Delta t} \right\|_{-1}$, respectively. For this, we need that $\mathbf{u_h^n}$ and its first derivative are bounded.

First let us make an induction hypothesis. Let there is a constant say $K^* \geq 2K$ with $\|\tilde{\mathbf{u}}_\mathbf{h}^\mathbf{n}\|_{(L^\infty(\Omega))^2} \leq K$ such that

$$\|\mathbf{u_h^n}\|_{(L^\infty(\Omega))^2} \leq K^*, \tag{4.3.51}$$

where $\tilde{\mathbf{u}}_\mathbf{h}$ is the projection of $\mathbf{u_h}$ at $t = t_n$ defined by (see Chapter 2, (2.4.8)-(2.4.9))

$$(\kappa^{-1}\mu(c)\tilde{\mathbf{u}}_h, \mathbf{v_h}) - (\nabla \cdot \mathbf{v_h}, \tilde{p}_h) = 0 \quad \forall \mathbf{v_h} \in U_h, \tag{4.3.52}$$

$$(\nabla \cdot \tilde{\mathbf{u}}_h, w_h) = (q, w_h) \qquad \forall w_h \in W_h. \tag{4.3.53}$$

To bound $\|\nabla . \mathbf{u_h^n}\|_\infty$, we use inverse inequality and (4.3.51)

$$\|\nabla . \mathbf{u_h^n}\|_\infty \Delta t \leq C h^{-1} \Delta t \|\mathbf{u_h^n}\|_\infty \leq C. \tag{4.3.54}$$

where we have used the assumption that $\Delta t = O(h)$.

Using Lemma 4.3.5, we have

$$\left\| \frac{\theta^n - \hat{\theta}^n}{\Delta t} \right\| \leq C(K^*) \|\nabla \theta^n\|. \tag{4.3.55}$$

Similar arguments as in the proof of Lemma 4.3.6, gives

$$\left\| \frac{\theta^n - \hat{\theta}^n}{\Delta t} \right\|_{-1} \leq C(K^*) \|\theta^n\|. \tag{4.3.56}$$

Now using (2.3.16), (4.3.55), inverse inequality (2.3.15) and (4.3.56), we obtain the following bound for T_7:

$$|T_7| \leq C(K^*)\|\theta^n\| \ \|\theta^{n+1}\|_1. \tag{4.3.57}$$

To bound T_8, we use maximum norm estimate of $\nabla R_h c$ (see (2.4.43)).

$$\left\| \frac{\hat{R_h c}^n - \check{R_h c}^n}{\Delta t} \right\| \leq \|R_h c\|_{1,\infty} \|\mathbf{u^n} - \mathbf{u^n_h}\| \leq C\|\mathbf{u^n} - \mathbf{u^n_h}\|_{(L^2(\Omega))^2}, \tag{4.3.58}$$

and hence, using (2.3.47), T_8 is bounded as follows

$$T_8 \leq C\|\theta^{n+1}\| \ \|\mathbf{u^n} - \mathbf{u^n_h}\|_{(L^2(\Omega))^2}. \tag{4.3.59}$$

Substitute the estimates of $T_1, T_2 \cdots T_8$ in (4.3.26) and use non-singular property of ϕ, kick back argument with the Young's inequality to obtain

$$\begin{aligned}
&\frac{1}{\Delta t}\left[(\theta^{n+1}, \Pi_h^* \theta^{n+1}) - (\theta^n, \Pi_h^* \theta^{n+1})\right] \leq C(K^*)\Big[\Delta t^{-1} \|\frac{\partial \rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)} + \|\theta^{n+1}\|^2 \\
&+\|\theta^{n+1}\|_1^2 + \|\theta^n\|^2 + \|\rho^{n+1}\|^2 + \Delta t\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)} + \|\nabla \cdot \mathbf{u})_t\|^2_{L^2(t_n,t_{n+1};L^2} \\
&+\|\frac{\partial^2 c}{\partial \tau^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big) + \|\mathbf{u^n} - \mathbf{u^n_h}\|^2_{(L^2(\Omega))^2} \\
&+h^2 \|\nabla.(\mathbf{u^n} - \mathbf{u^n_h})\|^2 + \|\rho^n\|^2 + h^2 \|\nabla \rho^n\|^2\Big].
\end{aligned} \tag{4.3.60}$$

Using (4.3.1) and (4.3.2), we obtain

$$\begin{aligned}
\|\mathbf{u^n} - \mathbf{u^n_h}\|_{(L^2(\Omega))^2} &\leq C\Big[\|c^n - c_h^n\| + h\left(\|\mathbf{u^n}\|_{(H^1(\Omega))^2} + \|p^n\|_1^2\right)\Big] \\
&\leq C\Big[\|\theta^n\| + \|\rho^n\| + h\left(\|\mathbf{u^n}\|_{(H^1(\Omega))^2} + \|p^n\|_1^2\right)\Big],
\end{aligned} \tag{4.3.61}$$

and

$$\|\nabla \cdot (\mathbf{u^n} - \mathbf{u^n_h})\| \leq Ch\|\nabla \cdot \mathbf{u^n}\|_1. \tag{4.3.62}$$

Substitute (4.3.61) and (4.3.62) in (4.3.60) to obtain

$$\begin{aligned}
|||\theta^{n+1}|||^2 - |||\theta^n|||^2 \leq\ & C(K^*)\Big[\Delta t\Big(\|\theta^{n+1}\|^2 + \|\theta^n\|^2 + \|\rho^n\|^2 + \|\rho^{n+1}\|^2 + h^2\|\nabla \rho^n\|^2\Big) \\
&+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)} + \|\nabla \cdot \mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)} \\
&+\|\frac{\partial^2 c}{\partial \tau^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big) + \|\frac{\partial \rho}{\partial t}\|^2_{L^2(t_n,t_{n+1},L^2)} \\
&+h^2 \Delta t\left(h^2 \|\nabla \cdot \mathbf{u^n}\|_1^2 + \|\mathbf{u^n}\|^2_{(H^1(\Omega))^2} + \|p^n\|_1^2\right)\Big].
\end{aligned} \tag{4.3.63}$$

Taking summation over $n = 0 \cdots m-1$, we obtain

$$\begin{aligned} |||\theta^m|||^2 &- |||\theta^0|||^2 \leq C(K^*)\Big[\sum_{n=0}^{m-1}\Big\{\Delta t\Big(\|\theta^{n+1}\|^2 + \|\theta^n\|^2 + \|\rho^n\|^2 + \|\rho^{n+1}\|^2 + h^2\|\nabla\rho^n\|^2\Big) \\ &+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(t_n,t_{n+1};L^2)} + \|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t_n,t_{n+1},L^2)}\Big) \\ &+\|\frac{\partial\rho}{\partial t}\|^2_{L^2(t_n,t_{n+1};L^2)} + h^2\Delta t\Big(h^2\|\nabla\cdot\mathbf{u^n}\|_1^2 + \|\mathbf{u^n}\|^2_{(H^1(\Omega))^2} + \|p^n\|_1^2\Big)\Big\}\Big]. \quad (4.3.64) \end{aligned}$$

Now using discrete Gronwall's (see Lemma 1.2.8), equivalence of the norms (2.3.46) and the estimates of ρ, we obtain

$$\begin{aligned} \|\theta^m\|^2 &\leq C(K^*)\Big[\|\theta^0\|^2 + h^4\Big(\|(c-\tilde{c})q\|^2_{L^\infty(0,T;H^1)} + \|\psi\frac{\partial c}{\partial s}\|^2_{L^\infty(0,T;H^1)} + \|c\|^2_{L^\infty(0,T;H^2)} \\ &+\|\nabla.\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|c_t\|^2_{L^\infty(0,T;H^2)} + \|\psi\frac{\partial^2 c}{\partial t\partial s}\|^2_{L^\infty(0,T;H^1)}\Big) \\ &+(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)} + \|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(0,T,L^2)}\Big) \\ &+h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \quad (4.3.65) \end{aligned}$$

Now it remains to show the induction hypothesis (4.3.51). Using (4.2.4) and (4.2.5), we have

$$\begin{aligned} \|\mathbf{u_h^n}\|_{(L^\infty(\Omega))^2} &\leq \|\mathbf{u_h^n} - \tilde{\mathbf{u}}\mathbf{_h^n}\|_{(L^\infty(\Omega))^2} + \|\tilde{\mathbf{u}}\mathbf{_h^n}\|_{(L^\infty(\Omega))^2} \\ &\leq C\Big(\log\frac{1}{h}\Big)^{1/2}\|\mathbf{u_h^n} - \tilde{\mathbf{u}}\mathbf{_h^n}\|_{H(div;\Omega)} + K. \end{aligned}$$

Using $\|\nabla.(\mathbf{u_h^n} - \tilde{\mathbf{u}}\mathbf{_h^n})\| = 0$, we have

$$\|\mathbf{u_h^n} - \tilde{\mathbf{u}}\mathbf{_h^n}\|_{(L^\infty(\Omega))^2} \leq C\Big(\log\frac{1}{h}\Big)^{1/2}\|\mathbf{u_h^n} - \tilde{\mathbf{u}}\mathbf{_h^n}\|_{(L^2(\Omega))^2}. \quad (4.3.66)$$

Now using (2.4.22) and (4.3.65), we obtain for small h

$$\|\mathbf{u_h^n}\|_{(L^\infty(\Omega))^2} \leq C(K^*)\log\frac{1}{h}\,(h+\Delta t) + K \leq 2K. \quad (4.3.67)$$

Here we have used $\Delta t = O(h)$ and $h\log\frac{1}{h} \longrightarrow 0$ as $h \to 0$ and this proves our induction hypothesis (4.3.51).

Now combine the estimates of ρ and θ and use triangle inequality to complete the rest of the proof. ■

Using (4.3.1) and Theorem 4.3.2, we obtain the following error estimates for velocity as well as pressure.

THEOREM 4.3.3 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Let $(\mathbf{u}, p)$ and $(\mathbf{u_h}, p_h)$ be respectively the solutions of (4.1.1)-(4.1.2) and (4.2.1)-(4.2.2) and let $c_h(0) = c_{0,h} = R_h c(0)$. Further assume that $\Delta t = O(h)$. Then for sufficiently small h there exists a positive constant $C(T)$ independent of h but dependent on the bounds of κ^{-1} and μ such that*

$$\begin{aligned}
\max_{0\le n\le N} \|\mathbf{u^n} - \mathbf{u^n_h}\|^2_{(L^2(\Omega))^2} \quad \le \quad & C\Big[h^4\Big(\|(c-\tilde{c})q\|^2_{L^\infty(0,T;H^1)} + \|\psi\frac{\partial c}{\partial s}\|^2_{L^\infty(0,T;H^1)} + \|c\|^2_{L^\infty(0,T;H^2)} \\
& +\|\nabla.\mathbf{u}\|^2_{L^\infty(0,T;H^1)} + \|c_t\|^2_{L^\infty(0,T;H^2)} + \|\psi\frac{\partial^2 c}{\partial t\partial s}\|^2_{L^\infty(0,T;H^1)}\Big) \\
& +(\Delta t)^2\Big(\|\mathbf{u}_t\|^2_{L^2(0,T;(L^2(\Omega))^2)} + \|\nabla\cdot\mathbf{u}_t\|^2_{L^2(0,T;L^2)} + \|\frac{\partial^2 c}{\partial \tau^2}\|^2_{L^2(0,T,L^2)}\Big) \\
& +h^2\Big(\|\mathbf{u}\|^2_{L^\infty(0,T;(H^1(\Omega))^2)} + \|p\|^2_{L^\infty(0,T;H^1)}\Big)\Big]. \qquad (4.3.68)
\end{aligned}$$

4.4 Numerical Experiments

As mentioned in Chapter 2, we use larger time steps for the pressure equation than the concentration equation. Let $0 = t_0 < t_1 < \cdots t_M = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t_m = t_{m+1} - t_m$ for the pressure equation and $0 = t^0 < t^1 < \cdots t^N = T$ be a given partition of the time interval $(0, T]$ with step length $\Delta t_n = t_{n+1} - t_n$ for the concentration equation. Let C^n, U_m and P_m be the approximation values of c_h, $\mathbf{u_h}$, p_h at $t = t^n$ and $t = t_m$, respectively.

If concentration step t^n relates to pressure steps by $t_{m-1} < t^n \le t_m$, we require a velocity approximation at $t = t^n$, which will be used in the concentration equation, based on $\mathbf{U}_{m-1}$ and earlier values. We define a velocity approximation [44, pp. 81] at $t = t^n$ by

$$E\mathbf{U^n} = \left(1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\right)\mathbf{U}_{m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{U}_{m-2} \qquad \text{for} \quad m \ge 2, \qquad (4.4.1)$$

$$E\mathbf{U^n} = \mathbf{U}_0 \qquad \text{for} \quad m = 1. \qquad (4.4.2)$$

Then the combined time stepping procedure is defined as: Find $C : \{t^0, t^1, \cdots t^N\} \longrightarrow M_h$ and $(\mathbf{U}, P) : \{t_0, t_1, \cdots t_M\} \longrightarrow U_h \times W_h$ such that

$$(\kappa^{-1}\mu(C_m)\mathbf{U_m}, \gamma_h \mathbf{v_h}) + b(\gamma_h \mathbf{v_h}, P_m) = 0 \qquad \forall \mathbf{v_h} \in U_h, \tag{4.4.3}$$

$$(\nabla \cdot \mathbf{U_m}, w_h) = (q^+ - q^-, w_h) \quad \forall w_h \in W_h, \quad m \geq 0, \tag{4.4.4}$$

$$\begin{aligned}\left(\phi \frac{C^{n+1} - \hat{C}^n}{\Delta t}, \Pi_h^* \chi_h\right) + a_h(E\mathbf{U^{n+1}}; C^{n+1}, \chi_h) + (q^- C^{n+1}, \Pi_h^* \chi_h) \\ = (q^+ \bar{c}, \Pi_h^* \chi_h) \qquad \forall \chi_h \in M_h,\end{aligned} \tag{4.4.5}$$

where $\hat{C}^n = C^n(x - (\frac{E\mathbf{U^n}}{\phi})\Delta t)$.

To solve the pressure equations, i.e., (4.4.3) and (4.4.4), we use the mixed finite volume element method and for concentration equation (4.4.5), we use the standard finite volume element method. We have already discussed the matrix formulation for the pressure equations in Chapter 2. Now we will discuss the matrix formulation and solution procedure for the concentration equation (4.4.5).

Let $\{\Psi_i\}_{i=1}^{N_h}$ be the basis functions for the finite dimensional space M_h. Then the approximate concentration at time level $t = t^n$ can be written as

$$C^n = \sum_{i=1}^{N_h} \gamma_i^n \Psi_i, \tag{4.4.6}$$

where $\gamma_i^n = C^n(P_i)$, i.e., the value of the nth level concentration at the vertices P_i.

Now using (4.4.6), the concentration equation (4.4.5) can be written in the following matrix form:

$$[D^n + \Delta t^n (H^n + R^n)] \boldsymbol{\gamma}^{n+1} = E^n \boldsymbol{\gamma}^n + \Delta t^n \mathbf{G}^n, \tag{4.4.7}$$

where

$$\boldsymbol{\gamma}^n = (C^n(P_i))_{i=1}^{N_h}, \quad D^n = (d_{ij})_{i,j=1}^{N_h} = \int_{V_i} \Psi_j dx,$$

$$E^n = (e_{ij})_{i,j=1}^{N_h} = \int_{V_i} \Psi\Big(x_j - \frac{E\mathbf{U^n}(x_j)}{\phi(x_j)}\Delta t^n\Big) dx, \quad H^n = (h_{ij})_{i,j=1}^{N_h} = -\int_{\partial V_i} D(E\mathbf{U^n})\nabla\Psi_j \cdot n_j ds,$$

and

$$R^n = (r_{ij})_{i,j=1}^{N_h} = \int_{V_i} q^- \Psi_j dx, \qquad \mathbf{G}^n = (g_i^n)_{i=1}^{N_h} = \int_{V_i} \bar{c} q^+ dx.$$

4.4.1 Test Problem

For the test problems, we take spatial domain as $\Omega = (0, 1000) \times (0, 1000)$ ft^2 and the time period is $[0, 3600]$ days, viscosity of oil as $\mu(0) = 1.0$ cp. The injection well is located at the upper right corner $(1000, 1000)$ with the injection rate $q^+ = 30\text{ft}^2/\text{day}$ and injection concentration $\bar{c} = 1.0$. The production well is located at the lower left corner with the production rate $q^- = 30\text{ft}^2/\text{day}$ and the initial concentration is $c(x, 0) = 0$. For time discretization, we take $\Delta t_p = 360$ days and $\Delta t_c = 120$ days, i.e., we divide each pressure time interval into three subintervals.

Test1: We assume that the porous medium is homogeneous and isotropic. The permeability $\kappa =$ is 80. The porosity of the medium is $\phi = .1$ and the mobility ratio between the resident and injected fluid is $M = 1$. Further more, we assume that the molecular diffusion is $d_m = 1$ and the dispersion coefficients are zero. In the numerical simulation for spatial discretization we divide in 20 number of divisions both along x and y axis. For time discretization, we take $\Delta t_p = 360$ days and $\Delta t_c = 120$ days, i.e., we divide each pressure time interval into three subintervals.

The surface and contour plots for the concentration at $t = 3$ and $t = 10$ years are presented in Figure 4.3 and Figure 4.4, respectively. Since only molecular diffusion is present and viscosity is also independent of the velocity, Figure 4.3, shows that the velocity is radial and the contour plots for the concentration is circular until the invading fluid reaches the production well. Figure 4.4 shows that when these plots are reached at production well, the invading fluid continues to fill the whole domain until $c = 1$.

Test 2: In this test, we take the permeability tensor is same as in Test 1 and $M = 41$ i.e., viscosity is dependent on the concentration. We assume that the physical diffusion and dispersion coefficients are given by $\phi d_m = 0.0\text{ft}^2/\text{day}$, $\phi d_t = 5.0\text{ft}$ and $\phi d_t = .5\text{ft}$. The difference between the longitudinal and the transverse dispersity coefficients implies that the fluid flow is much faster along the diagonal direction see Figures 4.5 and 4.6.

Test 3: In this test we consider the numerical simulation of a miscible displacement problem with discontinuous permeability. Here, the data is same as given in Test 1 except the permeability of the medium $\kappa(x)$. We take $\kappa = 80$ on the sub domain $\Omega_L :=$

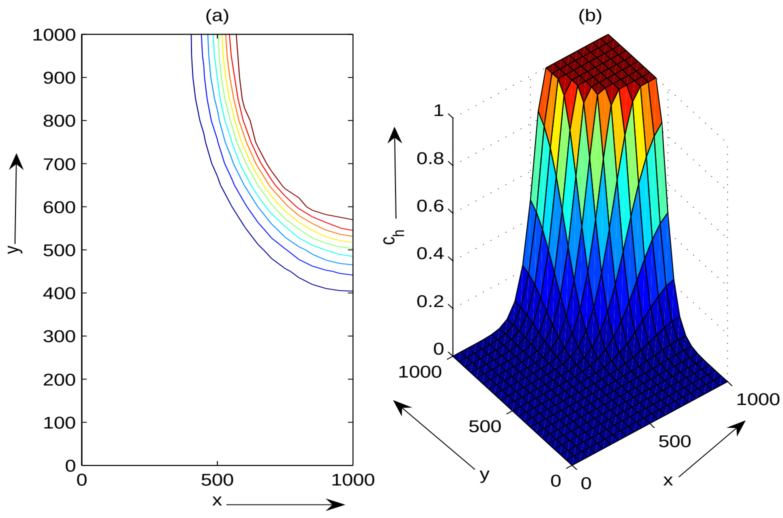

Figure 4.3: Contour (a) and surface plot (b) in Test 1 at $t = 3$ years.

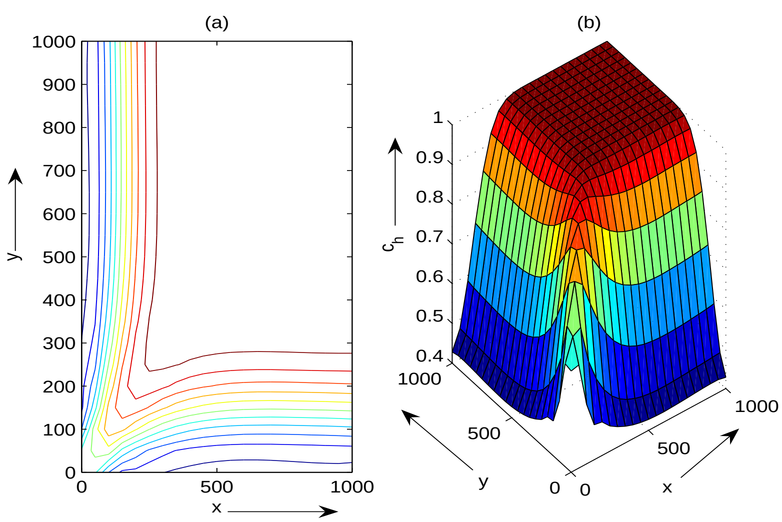

Figure 4.4: Contour (a) and surface plot (b) in Test 1 at $t = 10$ years.

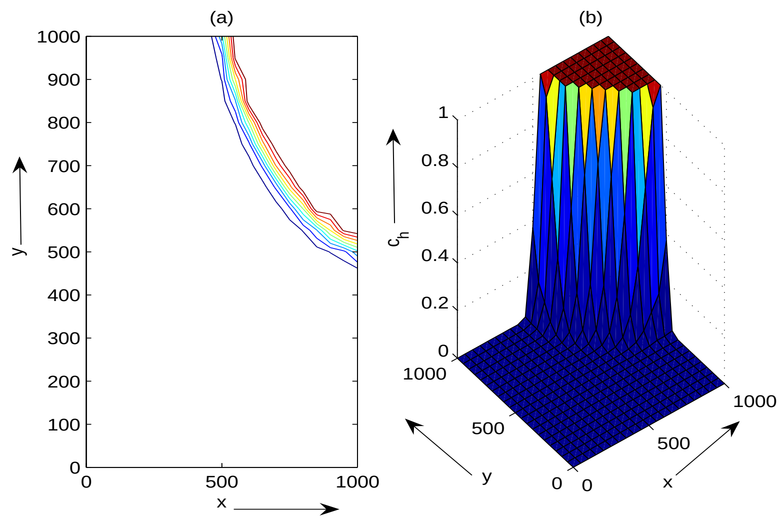

Figure 4.5: Contour (a) and surface plot (b) in Test 2 at $t = 3$ years.

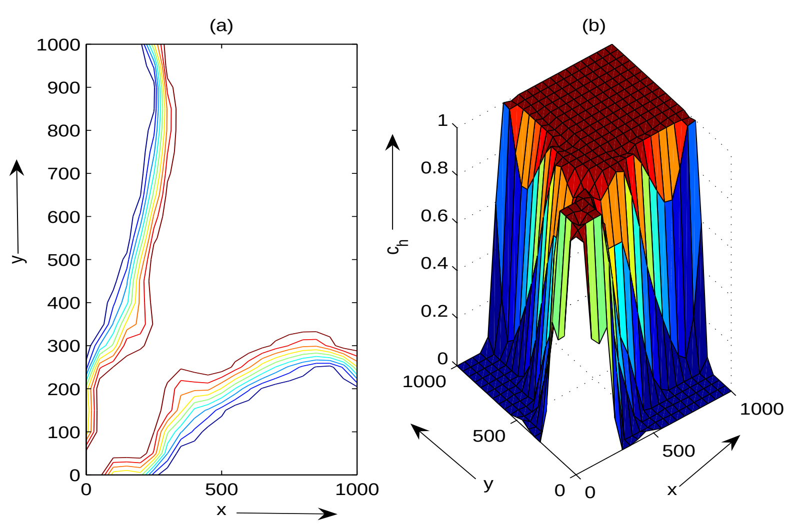

Figure 4.6: Contour (a) and surface plot (b) in Test 2 at $t = 10$ years.

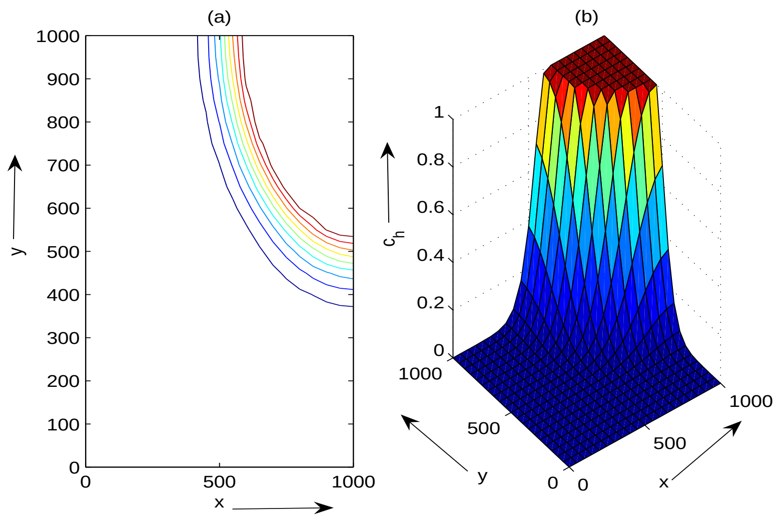

Figure 4.7: Contour (a) and surface plot (b) in Test 3 at $t = 3$ years

$(0, 1000) \times (0, 500)$ and $\kappa = 20$ on the sub domain $\Omega_U := (0, 1000) \times (500, 1000)$. The contour and surface plot at $t = 3$ and $t = 10$ years are given in Figure 4.7 and Figure 4.8 respectively.

Figure 4.7 and Figure 4.8 shows that when the injecting fluid reaches the lower half domain, it starts moving much faster in the horizontal direction on this domain compared to the low permeability domain that is upper half domain. We observe that one should put the production well in a low permeability zone to increase the area swept by the injected fluid.

Order of Convergence: In order to verify our theoretical results we also compute the order of convergence for the concentration for this particular test problem. We compute the order of convergence in L^2 norm. To discretize the time interval $[0, T]$, we take uniform time step $\Delta t = 360$ days for pressure and concentration equation. The computed order of convergence is given in Figure 4.9. Note that the computed order of convergence matches with the theoretical order of convergence derived in Theorem 4.3.2.

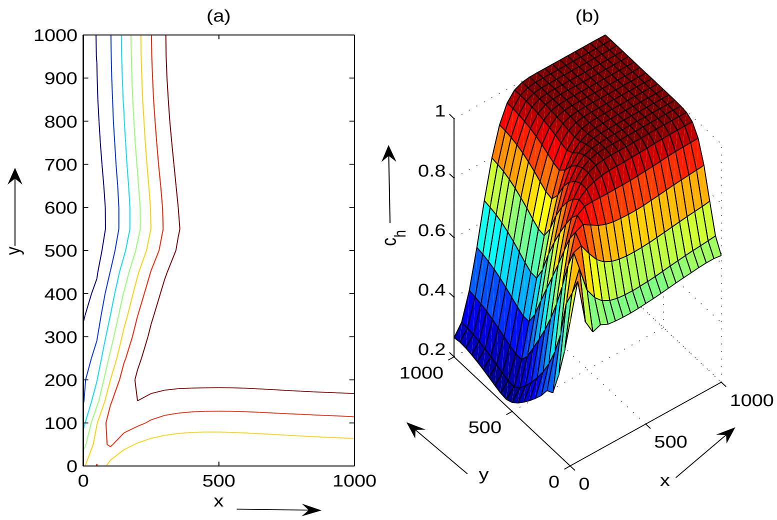

Figure 4.8: Contour (a) and surface plot (b) in Test 3 at $t = 10$ years

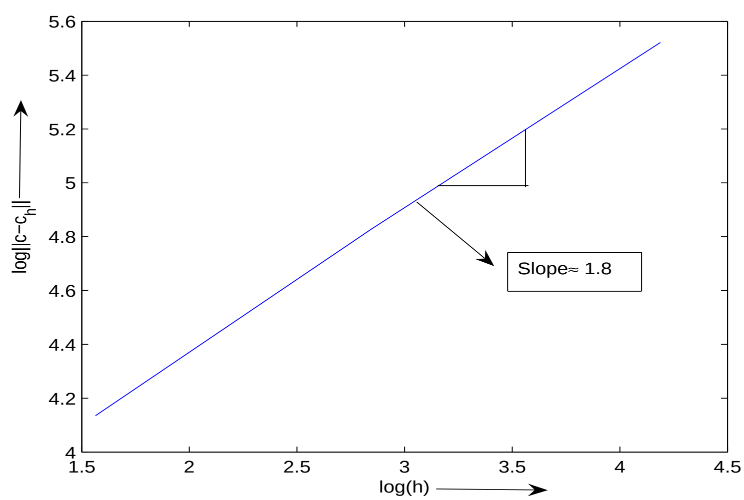

Figure 4.9: Order of convergence in L^2- norm

Chapter 5

Conclusions and Future Directions

In this thesis, an attempt has been made to study finite volume element methods for a coupled system of nonlinear elliptic and parabolic equations arising in the incompressible miscible displacement problems in porous media.

5.1 Summary

In Chapters 2, 3 and 4, we have applied a mixed finite volume element method for approximating the pressure equation and different kinds of methods for the approximation of the concentration equation. In Chapter 1, we have discussed briefly theoretical and computational issues related to the incompressible miscible displacement problems in porous media. Moreover, an extensive survey for the finite volume element methods is presented.
In Chapter 2, we have applied a mixed finite volume element method for the pressure equation and a standard finite volume element method for the approximation of the concentration equation. The trial space for velocity consists of the lowest order Raviart-Thomas element while the trial space for concentration is C^0- piecewise linear. We have obtained *a priori* error estimates in $L^\infty(L^2)$ norm for the concentration as well as for the velocity, see Theorem 2.4.2 and 2.4.3. We have also presented a couple of numerical experiments for the verification of our theoretical findings.
In Chapter 3, we have applied a discontinuous Galerkin finite volume element method for the approximation of the concentration equation and a mixed finite volume element method

for the approximation of the pressure equation. In Theorems 3.4.2 and 3.4.3, we have derived *a priori* $L^\infty(L^2)$ error estimates for the concentration and for velocity, respectively. Numerical experiments are also presented, which confirm our theoretical results.

As the modified method of characteristics reduces grid orientation effects, many researchers have studied modified methods of characteristics combined with finite element methods or finite difference methods for the approximation of the concentration equation, see [2, 34, 41, 42, 44, 72] etc. In Chapter 4 , we have analyzed a modified method of characteristics combined with the finite volume element method for approximating the concentration equation. Following the analysis of [41, 44, 72], we have derived *a priori* error bounds in $L^\infty(L^2)$- norm. Finally, we conclude this chapter with a couple of numerical examples.

5.2 Some Remarks

In [37, 44], the authors have chosen different mesh sizes; h_c and h_p for the concentration and the pressure equation, respectively. In the context of finite element method, they have derived order of convergence in $L^\infty(L^2)$- norm as $(h_c^2 + h_p)$ for both concentration and velocity when lowest order Raviart-Thomas elements and piecewise linear polynomials are used for approximating the velocity and concentration, respectively. Though, we have chosen the same spatial discretization parameter $'h'$ for both pressure as well concentration equation, it is possible to obtain error estimates in $L^\infty(L^2)$- norm depending on (h_c, h_p) as in [37, 44] in our analysis by making some minor modifications.
In Chapters 2 to 4, we have taken in the analysis the same time steps for the pressure as well as for the concentration equation. Since velocity is more smooth in time compared to concentration, it has been suggested in [47, 44] that one should take larger time steps for the pressure equation compared to the concentration equation. However in our numerical experiments, we have taken care to choose different time steps for pressure and concentration. The analysis can also be easily modified for this case by estimating the term $\|\mathbf{u^{n+1}} - E(\mathbf{u_h^{n+1}})\|$ in place of the term $\|\mathbf{u^{n+1}} - \mathbf{u_h^n}\|$, using the ideas given in [44].

5.3 Comparison

In Chapter 2, we have applied a mixed FVEM for approximating the pressure equation and a standard FVEM for approximation of the concentration equation. Since discontinuous Galerkin methods are easy to implement and also element wise conservative, in Chapter 3, we have applied DGFVEM for approximating the concentration equation. In numerical procedure presented in Chapter 2, for approximating the concentration equation, we have chosen the basis functions which are continuous through the inter element boundaries but, in Chapter 3, the basis functions are discontinuous. In DGFVEM, the support of the control volume is small compared to the support of the control volume in the standard FVEM. In addition, in DGFVEM the control volumes have support inside the triangle in which they belong to, but in the standard FVEM the control volumes have support in neighboring triangles also. The construction of the control volumes in DGFVEM is also similar to the construction of the control volumes in mixed FVEM which is not the case in the standard FVEM. We find that DGFVEM are also suitable for the approximation of the miscible displacement problems. But like DGFEMs, DGFVEMs are not well developed in the literature for miscible displacement problems. The main aim of Chapter 3 is to study DGFVEM for the miscible displacement problems in porous media. Using cut-off functions technique, we have derived *a priori* error estimates for the velocity and the concentration which match with the order of convergence obtained as in the case of the standard FVEM.

In Chapter 4, we have applied modified methods of characteristics combined with FVEM for the approximation of the concentration equation. After applying MMOC, the concentration equation behaves like the heat equation. It is expected that behavior of the numerical solution of the heat equation should be better compared to the convection dominated diffusion equation. If we compare the numerical results of this chapter with those obtained in the remaining chapters (Chapter 2 and 3) for the case when only the molecular diffusion is present and the effect of molecular diffusion is negligible, we observe that the contour plots are almost circular until the invading fluids reach to the production well. Figure 4.3 shows the contour plots are much circular compared to Figures 2.11 and 3.5. Essentially, we may infer that MMOC combined with FVEM yields better approximation for the concentration compared to standard FVEM and DGFVEM. Note that the derived order of convergence in $L^\infty(L^2)$- norm for the velocity as well concentration are same in the three chapters (Chapter 2, 3 and 4).

5.4 Future Directions

Since the concentration equation behaves more like a hyperbolic equation. It is expected that use of local discontinuous Galerkin (LDG) method with appropriate choice of numerical fluxes may provide a good approximation to the concentration equation. LDG finite element methods have been studied for elliptic problems by [11, 33] etc. In [11], the authors have considered the $P^k - P^k$ elements while in [33] $P^{k+1} - P^k$ elements are used, where k being degree of the polynomial. Now, we briefly discuss the local discontinuous Galerkin finite volume element method (LDGFVEM) for the approximation of the following elliptic problem based on [33].
Given f, find $(\mathbf{u}, p)$ such that

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega, \tag{5.4.1}$$

$$\nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega, \tag{5.4.2}$$

$$p = 0 \qquad \text{on } \Omega, \tag{5.4.3}$$

where Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $K = (k_{ij}(x))_{2\times 2} \in \left(W^{2,\infty}(\Omega)\right)^4$ is a real valued, symmetric and uniformly positive definite matrix, i.e., there exists a positive constant α_0 such that

$$\xi^T K \xi \geq \alpha_0 \xi^T \xi \qquad \forall \xi \in \mathbb{R}^2. \tag{5.4.4}$$

We also assume that K satisfy the following condition

$$K^* \geq K^{-1} \geq K_* > 0, \tag{5.4.5}$$

where K^* and K_* are constants.

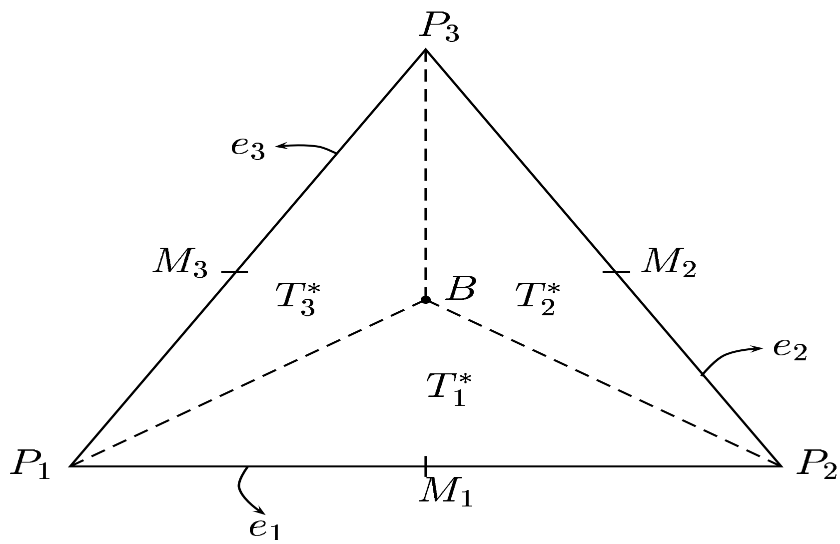

Figure 5.1: A triangular partition and its dual elements

5.4.1 P^1-P^0 LDGFVEM formulation

Let $\mathcal{T}_h$ be a regular, quasi-uniform triangulation of $\bar{\Omega}$ into closed triangles T with $h = \max_{T\in\mathcal{T}_h}(h_T)$, where h_T is the diameter of the triangle T. Let Γ denote the union of the boundaries of the triangles T of $\mathcal{T}_h$. The dual partition $\mathcal{T}_h^*$ corresponding to the primal partition $\mathcal{T}_h$ is constructed as follows: Divide each triangle $T \in \mathcal{T}_h$ into three triangles by joining the barycenter B and the vertices of T. In general, let T^* denote the dual element/control volume in $\mathcal{T}_h^*$, see Figure 5.1. Let the U_h and V_h be the trial and test spaces, respectively, associated with approximation of velocity defined by

$$U_h = \{\mathbf{v_h} \in L^2(\Omega)^2 : \mathbf{v_h}|_T \in (P_1(T))^2\} \quad \forall T \in \mathcal{T}_h$$

$$V_h = \{\mathbf{v_h} \in L^2(\Omega)^2 : \mathbf{v_h}|_T \in (P_0(T^*))^2\} \qquad \forall T^* \in \mathcal{T}_h^*$$

and the trial space W_h associated with the pressure is defined by

$$W_h = \{w_h \in L^2(\Omega) : w_h|_T \in P_0(T) \qquad \forall T \in \mathcal{T}_h\},$$

where $P_m(T)$(resp. $P_m(T^*)$) denotes the polynomials of degree less than or equal to m defined on T(resp. T^*).

Let $U(h) = U_h + H^2(\Omega)$. For connecting the trial and test spaces, define the transfer operator $\gamma : V(h) \longrightarrow U_h$ as follows:

$$\gamma \mathbf{v}|_{T^*} = \frac{1}{h_e} \int_e \mathbf{v}|_{T^*} ds, \qquad T^* \in \mathcal{T}_h^*, \tag{5.4.6}$$

where e is an edge in T, T^* is the dual element in $\mathcal{T}_h^*$ containing e and h_e is the length of the edge e.

Multiplying (5.4.1) by $\mathbf{v_h} \in V_h$, integrating over the control volumes $T^* \in \mathcal{T}_h^*$, applying Gauss's divergence theorem and summing up over all the control volumes, we obtain

$$(K^{-1}\mathbf{u}, \mathbf{v_h}) - \sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} p\mathbf{v_h} \cdot \mathbf{n} ds = 0 \qquad \forall \mathbf{v_h} \in V_h, \tag{5.4.7}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary ∂T^* of T^*. Let $T_j^* \in \mathcal{T}_h^* (j = 1, 2, 3)$ be the three triangles in $T \in \mathcal{T}_h$, (see Figure 5.1). Then, for $w_h \in W_h$

$$\begin{aligned} \sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} p\mathbf{v_h} \cdot \mathbf{n} ds &= \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{\partial T_j^*} p\mathbf{v_h} \cdot \mathbf{n} ds \\ = \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1}BA_j} p\mathbf{v_h} \cdot \mathbf{n} ds &+ \sum_{T \in \mathcal{T}_h} \int_{\partial T} p\mathbf{v_h} \cdot \mathbf{n} ds. \end{aligned} \tag{5.4.8}$$

In order to rewrite the last term on the right hand side of (5.4.8), we note that for any four real numbers a, b, c and d, we have

$$ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d). \tag{5.4.9}$$

Using (5.4.3), (5.4.9) and the fact that $[p] = 0$, (5.4.8) becomes

$$\sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} p\mathbf{v_h} \cdot \mathbf{n} ds = \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1}BA_j} p\mathbf{v_h} \cdot \mathbf{n} ds + \sum_{e \in \Gamma} \int_e [\mathbf{v_h}] \cdot \langle p \rangle ds. \tag{5.4.10}$$

Now we are in position to define our $P^1 - P^0$ LDGFVEM:

Find $(\mathbf{u_h}, p_h) \in U_h \times W_h$ such that

$$(\mathbf{u_h}, \gamma \mathbf{v_h}) + \sum_{T \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1}BA_j} p_h \gamma \mathbf{v_h} \cdot \mathbf{n} ds + \sum_{e \in \Gamma} \int_e \hat{p}_h [\gamma \mathbf{v_h}] ds = 0 \qquad \forall v_h \in U_h \tag{5.4.11}$$

$$(\nabla . \mathbf{u_h}) - \sum_{e \in \Gamma} \int_e \hat{w}_h [\gamma \mathbf{u_h}] = (f, w_h) \qquad \forall w_h \in W_h \tag{5.4.12}$$

where $\hat{w}_h$ are the approximation of w_h referred to as numerical flux, on the boundary of a triangle $T \in \mathcal{T}_h$.
In immediate future, we propose to develop the error estimates for the LDGFVEM applied to the concentration equation. Note that LDG methods also can be a good choice for the approximation of the convection-diffusion equations if the numerical fluxes are chosen suitably, see [12, 30].

5.4.2 Modified methods of characteristics with adjust advection (MMOCAA) procedure

The modified method of characteristics is not conservative in nature, but a modification of MMOC called MMOCAA, proposed by Douglas *et al.* [39] is conservative in nature. It also has been observed that this new scheme is naturally parallelizable and more accurate as it reduces the effects of grid orientation, the excessive smoothing of the sharp fronts and has low storage requirements. As in Chapter 4, the concentration equation can be written as

$$\psi(x,t)\frac{\partial c}{\partial s} - \nabla \cdot (D(\mathbf{u})\nabla c) = (\tilde{c} - c)q \quad \forall (x,t) \in \Omega \times J. \tag{5.4.13}$$

where $\psi(x,t) = (|\mathbf{u}(x,t)|^2 + \phi(x)^2)^{\frac{1}{2}} = (u_1(x)^2 + u_2(x)^2 + \phi(x)^2)^{\frac{1}{2}}$ and $\mathbf{s}$ be the unit vector in the direction of (u_1, u_2, ϕ) in $\Omega \times J$.
In MMOCAA procedure one splits the concentration equations into two sub equations namely, advection and transport equations and different time steps can be used for these equations. Essentially, in the MMOCCA procedure the advection term is treated as a primary variable and hence, more accurate approximation for the advection is excepted. Following this procedure, the concentration equation (5.4.13) can be split into a system of partial differential equations as

$$v = -D(\mathbf{u})\nabla c \qquad \forall (x,t) \in \Omega \times J, \tag{5.4.14}$$

$$\psi(x)\frac{\partial c}{\partial s} + \nabla \cdot v = (\tilde{c} - c)q \quad \forall (x,t) \in \Omega \times J, \tag{5.4.15}$$

It is also noted through experiments in [39] that there are advantages if one considers larger time step for advection term. We also plan to study MMOCAA for the approximation of the concentration equation. Since in LDG methods also, we split the second order equations into two first order equations, it would be good idea to combined LDG methods with MMOC.

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