# MORTAR FINITE ELEMENT METHODS FOR SECOND ORDER ELLIPTIC AND PARABOLIC PROBLEMS

**Thesis**

Submitted in partial fulfillment of the requirements
For the degree of

## Doctor of Philosophy

**by**

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**INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY**
**2008**

# Approval Sheet

The thesis entitled

**"MORTAR FINITE ELEMENT METHODS FOR SECOND ORDER ELLIPTIC AND PARABOLIC PROBLEMS"**

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is approved for the degree of

DOCTOR OF PHILOSOPHY

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*My Father*

*Sri Tarachand Patel*
&

*My Mother*
*Smt. Latika Patel*

INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY, INDIA

## CERTIFICATE OF COURSE WORK

This is to certify that Mr. Ajit Patel was admitted to the candidacy of the Ph.D. Degree on $18^{th}$ July 2002, after successfully completing all the courses required for the Ph.D. Degree programme. The details of the course work done are given below.

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# Abstract

The main objective of this thesis is to analyze mortar finite element methods for elliptic and parabolic initial-boundary value problems.

In Chapter 2 of this dissertation, we have discussed a standard mortar finite element method and a mortar element method with Lagrange multiplier for spatial discretization of a class of parabolic initial-boundary value problems. The introduction of a modified elliptic projection helps us to derive optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for semidiscrete methods. A completely discrete scheme using backward Euler scheme is also analyzed and optimal error estimates are derived in the framework of mortar element method. The results of numerical experiments support the theoretical results obtained in this thesis.

The basic requirement for the stability of the mortar element method is to construct finite element spaces which satisfy certain criteria known as inf-sup (well known as LBB, i.e., Ladyzhenskaya-Babuška-Brezzi) condition. Then many natural and convenient choices of finite element spaces ruled out as these spaces may not satisfy the inf-sup condition. In order to alleviate this problem stabilized multiplier techniques or Nitsche's method is used in Chapter 3 and 4. We have studied both stabilized symmetric and unsymmetric methods for second order elliptic boundary value problems under some assumptions on the penalty parameter, and established stability of the schemes with respect to the mesh dependent norm. The existence and uniqueness result of the discrete problem are discussed without using the discrete LBB condition. In Chapter 4, we have established optimal order of estimates with respect to broken $H^1$ and $L^2$-norm for both symmetric and unsymmetric cases with $\gamma = O(h)$. We have also analyzed the *Nitsche's mortaring element method* for parabolic initial-boundary value problems using semidiscrete and fully discrete schemes. Using the elliptic projection, with $\gamma = O(h)$, we have derived optimal order of estimates for both semidiscrete and fully discrete cases. Numerical experiments are conducted for support our theoretical results.

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# Chapter 1

# Introduction

## 1.1 Motivation

Domain decomposition methods provide a powerful technique for approximating solutions of partial differential equations (PDEs). The basic idea behind this method is to split the computational domain into smaller subdomains with or without overlap and then apply numerical techniques such as finite difference, finite volume or finite element schemes for approximately solving the underlying PDE on the subdomains independently. For extensive literature in this area, we refer to the proceedings of conferences on domain decomposition methods [27, 49, 59, 63]. In most of these numerical methods, a global mesh is initially constructed on the whole domain and then the meshes are subdivided into individual subdomains preserving the alignment of the nodes on the interior subdomain interfaces. Now a natural question that arises is whether we can discretize the subdomains independently, that is, whether we can allow incompatible grids on the interior subdomain interfaces. If this is possible, it allows us to change grids locally in one subdomain without changing the grids in other subdomains and this feature is quite useful in adaptive methods. As we have already pointed out, the approach of discretizing the subdomains independently may lead to nonmatching grids across the interfaces between adjacent subdomains. Furthermore, it requires transmission conditions to ensure the weak continuity of the traces on the subdomain interfaces. Now the major problem is "*how to achieve the continuity of the traces of the solution of the PDE across the interfaces of the subdomains*". In order to achieve continuity, mainly the following two different approaches have been proposed in the literature:

- direct procedure: the mortar element method by Bernardi, Maday, and Patera [23, 24], Belgacem [16] and Le Tallec and Sassi [84] with a suitable operator ensuring an optimal transmission condition across the adjacent subdomains.

- stabilized multiplier methods, or mesh-dependent penalty methods [12, 13, 15, 46, 82] to improve the stability of the method without compromising the consistency with the original problem.

So far, there is a great deal of literature devoted to mortar element methods for elliptic problems [16, 18, 23, 24, 89, 94], but there are only a few papers available for the mortar element method applied to parabolic problems. In this dissertation, we discuss mortar finite element methods with and without Lagrange multipliers and stabilized multiplier methods for parabolic initial and boundary value problems. Moreover, we have analyzed stabilized mortar element methods for both elliptic and parabolic problems and have derived optimal error estimates.

## 1.2 Preliminaries

In this section, we discuss standard Sobolev spaces with some properties which are used in the sequel. Moreover, we appeal to some results which will be useful in our subsequent chapters.

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$, the set of non-negative integers. Define a multi-valued index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, $\alpha_i \in \mathbb{N} \cup 0$, and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$. Let $\Omega$ denote an open bounded subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ [36]. Set

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} ... \partial {x_d}^{\alpha_d}}.$$

For $1 \leq p \leq \infty$, let $L^p(\Omega)$ denote the space of real-valued measurable functions $v$ on $\Omega$ for which $\int_\Omega |v(x)|^p dx \leq \infty$, associated with the norm

$$\|v\|_{L^p(\Omega)} := \left( \int_\Omega |v(x)|^p dx \right)^{1/p}.$$

In addition, let $L^\infty(\Omega)$ denote the space of real-valued measurable functions which are essential bounded in $\Omega$, associated with the norm

$$\|v\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

We denote the inner product and norm on $L^2(\Omega)$ as $(\cdot,\cdot)$ and $\|\cdot\|$ respectively, i.e.,

$$(v,w) = \int_\Omega v(x)w(x)dx \qquad \text{and} \qquad \|v\| := (v,v)^{1/2}.$$

Denote the set of all $r$ times continuously differentiable functions in $\Omega$ as $C^r$ and the set of all infinitely differentiable functions in $\Omega$ as $C^\infty$.

For a non-negative integer '$s$' and $1 < p < \infty$, the Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \{v \in L^p(\Omega) \colon D^\alpha v \in L^p(\Omega), \text{for all } |\alpha| \le s\},$$

where the derivatives are in the sense of distributions, and is equipped with the usual norm

$$\|v\|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|\le s} \int_\Omega |D^\alpha v|^p \; dx\right)^{1/p}.$$

Sometimes, we also use the following seminorm on $W^{s,p}(\Omega)$

$$|v|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|= s} \int_\Omega |D^\alpha v|^p \; dx\right)^{1/p}.$$

In particular, when $p = 2$, we denote $W^{s,2}(\Omega) = H^s(\Omega)$.

Let

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \colon v|_{\partial\Omega} = 0\}.$$

Note that $v_{|\partial\Omega}$ in the definition of $H_0^1(\Omega)$ should be understood in the sense of trace, see [4, 5]. For a given Banach space $Y$ with norm $\|\cdot\|_Y$, the space $H^s(0,T;Y)$ consists of all measurable functions $v\colon (0,T) \to Y$ such that

$$\sum_{j=0}^{s} \int_0^T \left\|\frac{\partial^j v}{\partial t^j}\right\|_Y^2 dt < \infty$$

and is equipped with the norm

$$\|v\|_{H^s(0,T;Y)} = \left(\sum_{j=0}^{s} \int_0^T \left\|\frac{\partial^j v}{\partial t^j}\right\|_Y^2 dt\right)^{1/2}.$$

In particular, $s = 0$ corresponds to the space $L^2(0,T;Y)$.
$L^\infty(0,T;Y)$ consists of all measurable functions $v\colon (0,T) \to Y$ with

$$\operatorname{ess}\sup_{0\le t\le T} \|v(t)\|_Y < \infty$$

and is equipped with the norm

$$\|v\|_{L^\infty(0,T;Y)} = \operatorname{ess}\sup_{0\le t\le T} \|v(t)\|_Y.$$

The space $H^{1/2}(\partial\Omega)$ be the range of $H^1(\Omega)$ by the trace operator [52]. Note that $H^{1/2}(\partial\Omega)$ is equipped with the norm

$$\|g\|_{H^{1/2}(\partial\Omega)} = \inf_{v\in H^1(\Omega), v_{|\partial\Omega}=g} \|v\|_{H^1(\Omega)}.$$

We denote by $H^{-1/2}(\partial\Omega)$, the dual space of $H^{1/2}(\partial\Omega)$ and equip it with the norm

$$\|\varphi\|_{H^{-1/2}(\partial\Omega)} = \sup_{\mu\in H^{1/2}(\partial\Omega),\mu\neq 0} \frac{|<\varphi,\mu>_{\partial\Omega}|}{\|\mu\|_{H^{1/2}(\partial\Omega)}},$$

where $<\cdot,\cdot>_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. With $\gamma \subset \partial\Omega$, let $\tilde{v}$ be an extension of $v \in H^{1/2}(\gamma)$ by zero to all of $\partial\Omega$. Then we set $H^{1/2}_{00}(\gamma)$, a subspace of $H^{1/2}(\gamma)$ as

$$H^{1/2}_{00}(\gamma) = \{v \in H^{1/2}(\gamma)\colon \tilde{v} \in H^{1/2}(\partial\Omega)\}.$$

The norm in $H^{1/2}_{00}(\gamma)$ is defined by:

$$\|g\|_{H^{1/2}_{00}(\gamma)} = \inf_{v\in H^1_{0,\partial\Omega\setminus\gamma}(\Omega), v_{|\gamma}=g} \|v\|_{H^1(\Omega)}.$$

Note that $H^{1/2}_{00}(\gamma)$ is strictly contained in $H^{1/2}(\gamma)$ and also continuously embedded in $H^{1/2}(\gamma)$, see [44, 60]. Let $H^{-1/2}_{00}(\gamma)$ be the dual space of $H^{1/2}_{00}(\gamma)$. Let $<\cdot,\cdot>_{00,1/2}$ denote the duality pairing between $H^{-1/2}_{00}(\gamma)$ and $H^{1/2}_{00}(\gamma)$ and let the norm in $H^{-1/2}_{00}(\gamma)$ be defined by

$$\|\varphi\|_{H^{-1/2}_{00}(\gamma)} = \sup_{\mu\in H^{1/2}_{00}(\gamma),\mu\neq 0} \frac{|<\varphi,\mu>_{00,1/2}|}{\|\mu\|_{H^{1/2}_{00}(\gamma)}}.$$

Note that $H_{00}^{1/2}(\gamma)$ can be characterized as the interpolation space [60] in between $H_0^1(\gamma)$ and $L^2(\gamma)$ with index $\frac{1}{2}$, i.e.,

$$H_{00}^{1/2}(\gamma) = [H_0^1(\gamma), L^2(\gamma)]_{1/2},$$

while

$$H^{1/2}(\gamma) = [H^1(\gamma), L^2(\gamma)]_{1/2}.$$

Moreover, let $s_1 < s_2$ and $s = \theta s_1 + (1-\theta)s_2,\ \theta \in (0,1)$, then for $v \in H^s(\gamma)$ the following interpolation inequality holds:

$$\|v\|_{H^s(\gamma)} \leq \|v\|_{H^{s_1}(\gamma)}^{\theta} \|v\|_{H^{s_2}(\gamma)}^{1-\theta}.$$

**Theorem 1.2.1 (Poincaré Inequality)[36].** *Let $\Omega$ be a bounded open set in $\mathbb{R}^d$. Then there exists a positive constant $C$ depending on $\Omega$ such that*

$$||v||_{L^2(\Omega)} \leq C|v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

**Theorem 1.2.2 (Trace Theorem)[52].** *Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ of class $C^{r+1}$ with boundary $\partial\Omega$. Then there exists a surjective map $\gamma = (\gamma_0, \gamma_1, ..., \gamma_{r-1})$*

$$\gamma\colon H^r(\Omega) \longrightarrow \prod_{j=0}^{r-1} H^{r-j-1/2}(\partial\Omega)$$

*such that, for $v \in C^\infty(\bar{\Omega})$, $\gamma_0(v) = v_{|\partial\Omega}, \gamma_1(v) = \dfrac{\partial v}{\partial \boldsymbol{n}}|_{\partial\Omega}, \cdots,$ and $\gamma_{r-1}(v) = \dfrac{\partial^{r-1}}{\partial \boldsymbol{n}^{r-1}}(v)|_{\partial\Omega}$, where $\boldsymbol{n}$ is the unit exterior normal to the boundary $\partial\Omega$.*

**Theorem 1.2.3 (Hölder Inequality)[4].** *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. If $v \in L^p(\Omega)$, $w \in L^{p'}(\Omega)$, then $vw \in L^1(\Omega)$ and*

$$\int_\Omega |v(x)w(x)|\ dx \leq \|v\|_{L^p(\Omega)} \|w\|_{L^{p'}(\Omega)}. \tag{1.2.1}$$

**Theorem 1.2.4 (Young's Inequality)[54].** *Let $a$ and $b$ be two positive real numbers, then the following inequality holds for all $\epsilon > 0$:*

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2. \tag{1.2.2}$$

**Theorem 1.2.5 (Hardy's Inequality)[47].** *If* $p > 1,\ f(x) \geq 0,$ *and* $F(x) = \int_0^x f(t)dt.$ *Then,*

$$\int_0^{\infty} \left(\frac{F(x)}{x}\right) dx < \left(\frac{p}{p-1}\right)^p \int_0^{\infty} (f(x))^p dx \tag{1.2.3}$$

*unless* $f \equiv 0$. *The constant* $\left(\frac{p}{p-1}\right)^p$ *is the best possible.*

**Theorem 1.2.6 (Gronwall Inequality)[70, Lemma 1.4.1],[68].** *Assume that the functions* $C(t) \geq 0$ *and* $F(t)$ *are absolutely integrable, and the integrable function* $y(t) \geq 0$ *satisfies the integral inequality for* $t \geq 0$

$$y(t) \leq \int_0^t C(s)y(s)\ ds + F(t),$$

*then*

$$y(t) \leq F(t) + \int_0^t C(\tau)F(\tau)exp\left(\int_0^t C(s)ds\right)\ d\tau.$$

**Theorem 1.2.7 (Discrete Gronwall Inequality)[70, Lemma 1.4.2],[68].** *If* $w_n \geq 0,\ f_n \geq 0,\ y_n \geq 0$ *and*

$$y_n \leq f_n + \sum_{j=0}^{n-1} w_j y_j, \text{for } n = 0, 1, 2, ...,\ \text{then for any } N \geq 1,$$

$$y_n \leq f_n + \sum_{j=0}^{n-1} exp\left(\sum_{j=n+1}^{N-1} w_j\right) w_n f_n,$$

*and*

$$y_n \leq exp\left(\sum_{i=0}^{N-1} w_i\right) \max_{0\leq n\leq N} f_n.$$

**Theorem 1.2.8 (Lax-Milgram lemma)[36].** *Let $V$ be a Hilbert space, $a(\cdot,\cdot)\colon V\times V\to \mathbb{R}$ be a continuous $V$-elliptic bilinear form, and let $f\colon V\to\mathbb{R}$ be a continuous linear form. Then the abstract variational problem: Find an element $u\in V$ such that*

$$a(u,v)=f(v)\qquad \forall\ v\in V, \tag{1.2.4}$$

*has one and only one solution.*

From time to time, we shall use $c$ and $C$ as generic positive constants which do not depend on the discretizing parameters.

## 1.3 Mortar Finite Element Method

In this section, we briefly describe mortar finite element methods [23, 24, 16, 17] in the context of second order elliptic boundary value problems.

Consider the following second order model problem with homogeneous Dirichlet condition: Find $u$ such that

$$-\nabla\cdot(a(x)\nabla u) = f(x) \ \text{ in } \ \Omega, \tag{1.3.1}$$

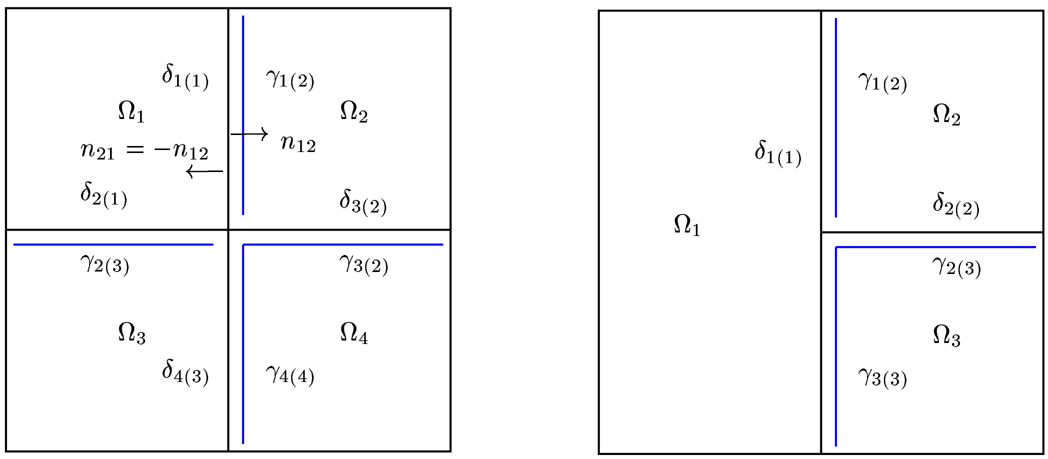
$$u = 0 \ \text{ on } \partial\Omega, \tag{1.3.2}$$

where $\Omega$ is a bounded convex polygon in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $\nabla\equiv(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$, and $f\in L^2(\Omega)$ is a given function. Assume that the coefficient $a(x)$ is smooth and satisfies $0<\alpha_0\le a(x)\le\alpha_1$, for some positive constants $\alpha_0$ and $\alpha_1$, and for all $x\in\overline{\Omega}$. The problem (1.3.1)-(1.3.2) has unique solution $u$ in $H^2(\Omega)$ by [55].

Mortar finite element methods deal with the decomposition of the domain $\Omega$ into $K$ nonoverlapping convex subdomains $\Omega_l,\ 1\le l\le K$ such that

$$\overline{\Omega}=\bigcup_{l=1}^{K}\overline{\Omega}_l.$$

Without loss of generality, we assume that each $\Omega_l$ is a convex polygon. Let $\partial\Omega_i\cap\partial\Omega_j=\gamma_{ij},\ 1\le i,j\le K$, and $n_{ij}$ be the unit normal oriented from $\Omega_i$ towards $\Omega_j$ such that $n_{ij}=-n_{ji}$ (See Figure 1.1 (a)). Geometrically, there are two versions of mortar methods; *geometrically conforming* and *geometrically non-conforming* methods. Here, we discuss the geometrically conforming version of the mortar method, i.e., the intersection of $\overline{\Omega}_i$ and $\overline{\Omega}_j$ for $i \neq j$ is either the empty set, a common edge, or a common vertex; see Figure 1.1 (a). If it fails to satisfy the above condition then, the mortar method is said to be geometrically non-conforming, see Figure 1.1 (b).




(a) Geometrically conforming (b) Geometrically non-conforming

Figure 1.1: Two different versions of mortar methods

Let

$$H^1_D(\Omega_l) = \{v \in H^1(\Omega_l) \colon v|_{\partial\Omega_l \cap \partial\Omega} = 0\}.$$

Now, define

$$X = \{v \in L^2(\Omega) \colon v_{|\Omega_l} \in H^1_D(\Omega_l), \forall 1 \leq l \leq K\},$$

which is equipped with the norm and seminorm

$$\|v\|_X = \left(\sum_{l=1}^{K} \|v\|^2{}_{H^1(\Omega_l)}\right)^{1/2} \quad \text{and} \quad |v|_X = \left(\sum_{l=1}^{K} \|\nabla v\|^2{}_{L^2(\Omega_l)}\right)^{1/2},$$

respectively.

The weak formulation corresponding to (1.3.1)-(1.3.2) is to find $u \in H^1_0(\Omega)$ such that

$$a(u, v) = l(v) \qquad \forall v \in H^1_0(\Omega), \tag{1.3.3}$$

where

$$a(u,v) = \sum_{l=1}^{K} \int_{\Omega_l} a(x)\nabla u_l \cdot \nabla v_l \; dx,$$

and

$$l(v) = \sum_{l=1}^{K} \int_{\Omega_l} fv \; dx.$$

Here, $u_l$ and $v_l$ are the restrictions of $u$ and $v$, respectively, to $\Omega_l$. From the properties of the coefficient $a(\cdot)$, it is easy to show that the bilinear form $a(\cdot,\cdot)$ satisfies the following boundedness property: for $v$ and $w$ in $X$,

$$|a(v,w)| \;\; \leq \;\; \beta|v|_X|w|_X \leq \beta\|v\|_X\|w\|_X. \qquad (1.3.4)$$

Now, we are in a position to discuss the mortar finite element method to approximate the solution of (1.3.3). In each subdomain $\Omega_l$, associate a triangulation $\mathcal{T}_h(\Omega_l)$ consisting of triangles of different mesh sizes $h_l$, i.e.,

$$\bar{\Omega}_l = \bigcup_{T\in\mathcal{T}_h(\Omega_l)} \bar{T}.$$

The discretization parameter is defined by $h = \max_{1\leq l\leq K}\{h_l\}$, where $h_l = \max_{T\in\mathcal{T}_h(\Omega_l)} h_T$ and $h_T = \sup_{x,y\in T} d(x,y)$ where $d(x,y)$ is the distance between any two points $x$ and $y$ in $T$. Assume that the family of triangulations associated with each $\Omega_l$ is regular in the sense that for all $T \in \mathcal{T}_h(\Omega_l)$

$$h_T/\rho_T \leq C \qquad (1.3.5)$$

for some positive constant $C$, where $\rho_T$ is the diameter of largest ball contained in $T$.

The mortar element method first deals with a skeleton of the decomposition, i.e., the union of all edges (interfaces)

$$\Gamma = \bigcup_{l=1}^{K} \overline{\partial\Omega_l \backslash \partial\Omega} \qquad (1.3.6)$$

and consists of choosing one of the decompositions of $\Gamma$ that is made up of disjoint open segments (that are edges of subdomains) denoted by $\gamma_m,\ 1 \leq m \leq m_0$ as mortars, i.e.,

$$\Gamma = \bigcup_{m=1}^{m_0} \bar{\gamma}_m \qquad \gamma_m \cap \gamma_l = \emptyset \text{ if } m \neq l.$$

Let $\gamma_{m(i)}$ denote an edge of $\Omega_i$ that is a mortar (master) and let $\delta_{m(j)}$ denote an edge of $\Omega_j$ that occupies physically the same place, called as nonmortar (slave). Note that on each mortar edge $\gamma_{m(i)}$, there is a natural triangulation which is generated by the triangulation $\mathcal{T}_h(\Omega_i)$. Similarly, there is also a partition on each non-mortar edge $\delta_{m(j)}$ due to the triangulation $\mathcal{T}_h(\Omega_j)$.

Once a triangulation $\mathcal{T}_h(\Omega_l)$ is chosen over each $\Omega_l$, the finite element subspaces in the subdomains and the interfaces can be defined. While it is possible to choose locally the finite element method that is best suited to the local properties of the solution, for the present problem, we assume conforming linear finite elements defined on each triangulation $\mathcal{T}_h(\Omega_l), 1 \leq l \leq K$ that is, to introduce

$$X_h(\Omega_l) \quad = \quad \{v_{l,h} \in C^0(\bar{\Omega}_l) \colon v_{l,h_{|T}} \in P_1(T) \ \forall\ T \in \mathcal{T}_h(\Omega_l), v_{l,h_{|\partial\Omega \cap \partial\Omega_l}} = 0\},$$

where $P_1(T)$ is the set of all linear polynomials over the triangle $T$ in $\mathcal{T}_h(\Omega_l)$. The global finite element approximation space $X_h(\Omega)$ consists of square integrable functions whose restriction over each $\Omega_l$ belongs to $X_h(\Omega_l)$, that is,

$$X_h(\Omega) \quad = \quad \{v_h \in L^2(\Omega) \colon v_{h_{|\Omega_l}} \in X_h(\Omega_l) \ \forall 1 \leq l \leq K\}.$$

For notational purpose, we write $X_h$ in place of $X_h(\Omega)$. In general, a function $v_h \in X_h$ is typically discontinuous across the common interfaces of the subdomains.
Let $\gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ and $W^{h_j}(\gamma_{ij})$ be the restriction of $X_h(\Omega_j)$ to $\gamma_{ij}$. Since the triangulations on two adjacent subdomains are independent, the interface $\gamma_{ij} = \gamma_{m(i)} = \delta_{m(j)}$ is provided with two different and independent $(1D)$ triangulations and two different spaces $W^{h_i}(\gamma_{m(i)})$ and $W^{h_j}(\delta_{m(j)})$. Additionally, define an auxiliary test space $M^{h_j}(\delta_{m(j)})$ which is a subspace of the nonmortar space $W^{h_j}(\delta_{m(j)})$ such that its functions are constants on the elements which intersect the ends of $\delta_{m(j)}$, see Figure 1.2.

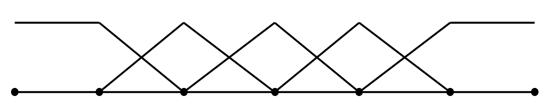


Figure 1.2: Basis functions over a non-mortar

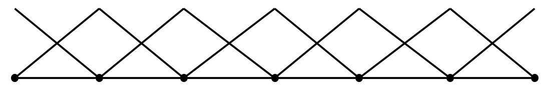


Figure 1.3: Basis functions over a mortar

The dimension of $M^{h_j}(\delta_{m(j)})$ is equal to dimension of $W^{h_j}(\delta_{m(j)})$ minus two. In order to achieve the continuity across the common interfaces of subdomains, in the non-mortar side, we impose the following matching condition on each element $v_h \in X_h$ as

$$\forall \delta_{m(j)} \subset \Gamma, \int_{\delta_{m(j)}} (v_{i,h} - v_{j,h})_{|\delta_{m(j)}} \psi d\tau = 0 \qquad \forall \psi \in M^{h_j}(\delta_{m(j)}), \tag{1.3.7}$$

where $v_{i,h}$ and $v_{j,h}$ in the integral are, respectively, the traces of $v_h$ onto the common interface $\gamma_{m(i)}$ and $\delta_{m(j)}$ of $\Omega_i$ and $\Omega_j$. The integral condition (1.3.7) is well known as the **mortar condition** in literature, see [23]. Thus, we define a discrete space $V_h$ as

$$V_h = \{v_h \in X_h(\Omega) \colon \forall \delta_{m(j)} \subset \Gamma, \quad \int_{\delta_{m(j)}} (v_{i,h} - v_{j,h})_{|\delta_{m(j)}} \psi d\tau = 0, \\ \forall \psi \in M^{h_j}(\delta_{m(j)})\}. \tag{1.3.8}$$

From (1.3.7), it follows that the interior nodes of the non-mortar sides are not associated with genuine degrees of freedom in the finite element space $V_h$. For notational convenience, we denote the non-mortar side $\delta_{m(j)}$ by $\delta_m$. Since $V_h$ is not a subspace of $H_0^1(\Omega)$, a consistency error appears in the error estimate, see (1.3.42).

The mortar finite element formulation for the problem (1.3.3) is to seek $u_h \in V_h$ such that

$$a(u_h, v_h) = l(v_h) \qquad \forall\ v_h \in V_h, \tag{1.3.9}$$

where

$$a(v_h, w_h) = \sum_{l=1}^{K} \int_{\Omega_l} a(x) \nabla v_{l,h} \cdot \nabla w_{l,h}\ dx,$$

and

$$l(v_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h\ dx.$$

Here, $v_{l,h}$ and $w_{l,h}$ are, respectively, the restrictions of $v_h$ and $w_h$ to $\Omega_l$. We now recall the following version of Poincaré inequality on broken Sobolev spaces which will be used in our subsequent analysis.

**Lemma 1.3.1 (Generalized Poincaré Inequality)[23]** *There exists a positive constant $c$ such that for $v_h \in V_h$ the following relation holds true:*

$$\sum_{l=1}^{K} \|v_{|\Omega_l}\|^2_{L^2(\Omega_l)} \leq c \sum_{l=1}^{K} |v_{|\Omega_l}|^2_{H^1(\Omega_l)}. \tag{1.3.10}$$

Using Poincaré inequality for elements in $V_h$, see [23, Proposition A.3, Page 45], we note that the seminorm $|\cdot|_X$ is indeed a norm on $V_h$ which is equivalent to $\|\cdot\|_X$ norm. From the properties of the coefficient $a$ and Lemma 1.3.1, the $V_h$-ellipticity of the bilinear form $a(\cdot,\cdot)$ follows, i.e. for $v_h$ in $V_h$,

$$a(v_h, v_h) \;\; \geq \;\; \alpha_0 |v_h|^2_X \geq \alpha \|v_h\|^2_X, \tag{1.3.11}$$

where $\alpha$ depends on $\alpha_0$ and the constant in the generalized Poincaré inequality. Therefore, the wellposedness of the problem (1.3.9) follows.

We now recall the following results for our future use. Let $I_{h_i}$ be the Lagrange interpolation operator defined on $X_h(\Omega_i)$ for $1 \leq i \leq K$. Then the following estimate [36] holds true. For any $\sigma > 1$,

$$\forall u \in H^\sigma(\Omega_i), \qquad \|u - I_{h_i} u\|_{H^1(\Omega_i)} \leq C h^{\sigma-1} \|u\|_{H^\sigma(\Omega_i)}. \tag{1.3.12}$$

Also we have the following approximation properties for $I_{h_i}$ for $1 \leq i \leq K$ on the edge of triangles in $\Omega_l$ (see Page 57 of [29]).

**Lemma 1.3.2** *Let $u_{|\Omega_l} \in H^2(\Omega_l),\ 1 \leq l \leq K$, then there exists positive constant $C$ independent of discretization parameter, such that*

$$\|(u - I_{h_i} u)|_{\Omega_l}\|_{L^2(\gamma_{ij})} \leq C h_l^{3/2} \|u\|_{H^2(\Omega_l)} \text{for } l = i \text{ or } j. \tag{1.3.13}$$

Now we define the mortar projection which is one of the most important features in the mortar element method. Assume that the triangulation on $\gamma_{ij}$ induced either from the mortar side (say from $\Omega_i$) or from the nonmortar side (say from $\Omega_j$) is quasiuniform in the sense that each segment $e$ from the triangulation $\mathcal{T}_h(\gamma_{ij})$ over $\gamma_{ij}$ satisfies the following relation: For all $e, e' \in \mathcal{T}_h(\gamma_{ij})$, there exists a positive constant $c$ independent of discretization parameters $h_e$ and $h_{e'}$, such that

$$h_e \leq c h_{e'}. \tag{1.3.14}$$

Now, we define the mortar projection operator

$$\Pi_{h_i} \colon L^2(\gamma_{ij}) \longrightarrow W^{h_i}(\gamma_{ij}) \cap H_{00}^{1/2}(\gamma_{ij})$$

by

$$\int_{\gamma_{ij}} (\varphi - \Pi_{h_i}\varphi)\psi \; d\tau = 0 \qquad \forall \psi \in M^{h_j}(\delta_{m(j)}). \tag{1.3.15}$$

Stability of $\Pi_{h_i}$ has been shown in [16, 23, 24] for the two dimensional case. Stability for the three dimensional case is discussed in Belgacem, Maday [17], and Braess, Dahmen [30]. In the following lemma, we recall from [16, p.181] the stability of $\Pi_{h_i}$. For a quick exposure, we provide a proof. For a proof of the stability result, we need the following inverse inequality [36, Page 140]:

**Lemma 1.3.3** *Let the family of triangulation over the finite element space be quasiuniform (1.3.14), then with $l, m \geq 0$ and $1 \leq r, q \leq \infty$, there exists a constant such that, for all $\varphi_h \in W^h(\Gamma)$,*

$$|\varphi_h|_{W^{m,q}(e)} \leq C \frac{(h_e^n)^{1/q - 1/r}}{h_e^{m-l}} |\varphi_h|_{W^{l,r}(e)}. \tag{1.3.16}$$

**Lemma 1.3.4** *The projection operator $\Pi_{h_i}$ is $L^2$-stable, i.e.,*

$$\|\Pi_{h_i}\varphi\|_{L^2(\gamma_{ij})} \leq C\|\varphi\|_{L^2(\gamma_{ij})} \qquad \forall \varphi \in L^2(\gamma_{ij}). \tag{1.3.17}$$

*Moreover, when $\varphi \in H_{00}^{1/2}(\gamma_{ij})$,*

$$\|\Pi_{h_i}\varphi\|_{H_{00}^{1/2}(\gamma_{ij})} \leq C\|\varphi\|_{H_{00}^{1/2}(\gamma_{ij})}. \tag{1.3.18}$$

*Further, the following estimate holds for any* $s(1/2 < s \leq 2)$ *and for* $\varphi \in H^s(\gamma_{ij}) \cap H_{00}^{1/2}(\gamma_{ij})$*:*

$$\|\varphi - \Pi_{h_i}\varphi\|_{H_{00}^{1/2}(\gamma_{ij})} \leq C h_i^{s-1/2} \|\varphi\|_{H^s(\gamma_{ij})}. \tag{1.3.19}$$

**Proof.** We prove the result (1.3.18) for the reference interval $\gamma_{ij} = (-1, 1)$. The triangulation $\mathcal{T}_h(\gamma_{ij})$ associated with $\gamma_{ij}$ is assumed to be quasiuniform. Let $(y_i)_{i=0}^{n_k}$ form a quasiuniform partition of $\gamma_{ij}$, i.e.,

$$\gamma_{ij} = \bigcup_{l=0}^{n_k-1} (y_l, y_{l+1}) = \bigcup_{l=0}^{n_k-1} e_l \tag{1.3.20}$$

with $y_0 = -1$ and $y_{n_k} = 1$.

Let $\mathcal{D}(\gamma_{ij})$ be the class of infinitely differentiable functions with compact support on $\gamma_{ij}$. For $\varphi \in \mathcal{D}(\gamma_{ij})$, we define the function $\psi$ as below

$$\psi|_{e_0} = \frac{1+y_1}{1+y} \Pi_{h_i}\varphi|_{e_0} \tag{1.3.21}$$

$$\psi|_{e_l} = \Pi_{h_i}\varphi|_{e_l} \qquad \forall 1 \leq l \leq n_k - 2 \tag{1.3.22}$$

$$\psi|_{e_{n_k-1}} = \frac{1-y_{n_k-1}}{1-y} \Pi_{h_i}\varphi|_{e_{n_k-1}}. \tag{1.3.23}$$

The function $\psi$ is continuous on $\gamma_{ij}$ and piecewise linear in each interval except in the first and last intervals $e_0$ and $e_{n_k-1}$, where it is a constant, so that $\psi$ is in $M^{h_i}(\gamma_{ij})$. Now from (1.3.15), we find that

$$\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi \; dy = \int_{\gamma_{ij}} \varphi\psi \; dy. \tag{1.3.24}$$

Using the definition of $\psi$, (1.3.21)-(1.3.23), we write

$$\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi \; dy = \int_{e_0} (\Pi_{h_i}\varphi)^2 \frac{1+y_1}{1+y} \; dy + \sum_{l=1}^{n_k-2} \int_{e_l} (\Pi_{h_i}\varphi)^2 dy + \int_{e_{n_k-1}} (\Pi_{h_i}\varphi)^2 \frac{1-y_{n_k-1}}{1-y} dy. \tag{1.3.25}$$

Note that, $\dfrac{1+y_1}{1+y} \geq 1$ over $e_0$ and $\dfrac{1-y_{n_k-1}}{1-y} \geq 1$ over $e_{n_k-1}$. Therefore, we find that

$$\left| \int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi \; dy \right| \geq \sum_{l=0}^{n_k-1} \int_{e_l} (\Pi_{h_i}\varphi)^2 dy = \|\Pi_{h_i}\varphi\|^2_{L^2(\gamma_{ij})}. \tag{1.3.26}$$

On the other hand from (1.3.24) and the definition of $\psi$, (1.3.21)-(1.3.23), we derive

$$\begin{aligned} \int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi\ dy = \int_{\gamma_{ij}} \varphi\psi\ dy &= \int_{e_0} \varphi\Pi_{h_i}\varphi\frac{1+y_1}{1+y}\ dy + \sum_{l=1}^{n_k-2}\int_{e_l}\varphi\Pi_{h_i}\varphi dy \\ &+ \int_{e_{n_k-1}} \varphi\Pi_{h_i}\varphi\frac{1-y_{n_k-1}}{1-y}dy. \end{aligned} \tag{1.3.27}$$

Apply Cauchy Schwarz inequality to obtain

$$\begin{aligned} \left|\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi\ dy\right| &\le (1+y_1)\|\varphi\|_{L^2(e_0)}\left\|\frac{\Pi_{h_i}\varphi}{1+y}\right\|_{L^2(e_0)} \\ &+ \sum_{l=1}^{n_k-2}\|\varphi\|_{L^2(e_l)}\|\Pi_{h_i}\varphi\|_{L^2(e_l)} \\ &+(1-y_{n_k-1})\|\varphi\|_{L^2(e_{n_k-1})}\left\|\frac{\Pi_{h_i}\varphi}{1-y}\right\|_{L^2(e_{n_k-1})}. \end{aligned} \tag{1.3.28}$$

An application of Hardy's inequality (Lemma 1.2.5) yields

$$\begin{aligned} \left|\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi\ dy\right| &\le (1+y_1)\|\varphi\|_{L^2(e_0)}\left\|\frac{d}{dy}\Pi_{h_i}\varphi\right\|_{L^2(e_0)} \\ &+ \sum_{l=1}^{n_k-2}\|\varphi\|_{L^2(e_l)}\|\Pi_{h_i}\varphi\|_{L^2(e_l)} \\ &+(1-y_{n_k-1})\|\varphi\|_{L^2(e_{n_k-1})}\left\|\frac{d}{dy}\Pi_{h_i}\varphi\right\|_{L^2(e_{n_k-1})}. \end{aligned} \tag{1.3.29}$$

Use the inverse inequality (1.3.16) for the first interval $e_0$ and the last interval $e_{n_k-1}$ to obtain

$$\begin{aligned} \left|\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi\ dy\right| &\le C(1+y_1)(1+y_1)^{-1}\|\varphi\|_{L^2(e_0)}\|\Pi_{h_i}\varphi\|_{L^2(e_0)} \\ &+ \sum_{l=1}^{n_k-2}\|\varphi\|_{L^2(e_l)}\|\Pi_{h_i}\varphi\|_{L^2(e_l)} \\ &+(1-y_{n_k-1})(1-y_{n_k-1})^{-1}\|\varphi\|_{L^2(e_{n_k-1})}\|\Pi_{h_i}\varphi\|_{L^2(e_{n_k-1})}, \end{aligned} \tag{1.3.30}$$

and hence, we find that

$$\begin{aligned} \left|\int_{\gamma_{ij}} \Pi_{h_i}\varphi\psi\ dy\right| &\le C\sum_{l=0}^{n_k-1}\|\varphi\|_{L^2(e_l)}\|\Pi_{h_i}\varphi\|_{L^2(e_l)} \\ &\le C\|\varphi\|_{L^2(\gamma_{ij})}\|\Pi_{h_i}\varphi\|_{L^2(\gamma_{ij})}. \end{aligned} \tag{1.3.31}$$

From (1.3.26), (1.3.31) and the fact that $\mathcal{D}(\gamma_{ij})$ is dense in $L^2(\gamma_{ij})$, we derive the following $L^2$-stability of $\Pi_{h_i}$:

$$\|\Pi_{h_i}\varphi\|_{L^2(\gamma_{ij})} \leq C\|\varphi\|_{L^2(\gamma_{ij})}. \tag{1.3.32}$$

For $H_0^1$-stability of $\Pi_{h_i}$, apply the inverse inequality (1.3.16) and (1.3.31) to obtain for $\varphi \in H_0^1(\gamma_{ij})$,

$$\begin{aligned}
\|\Pi_{h_i}\varphi\|_{H^1(\gamma_{ij})} &\leq C\|\varphi - \Pi_{h_i}\varphi\|_{H^1(\gamma_{ij})} + \|\varphi\|_{H^1(\gamma_{ij})} \\
&\leq C\left(\max_{0\leq l\leq n_k-1} |e_l|\right)^{-1} \|\varphi - \Pi_{h_i}\varphi\|_{L^2(\gamma_{ij})} + \|\varphi\|_{H^1(\gamma_{ij})} \\
&\leq C\|\varphi\|_{H^1(\gamma_{ij})}.
\end{aligned} \tag{1.3.33}$$

The $H_{00}^{1/2}$-stability of $\Pi_{h_i}$, that is, (1.3.18) can be establish by interpolating (1.3.32) and (1.3.33). For the estimate (1.3.19), we now proceed in the following way: For all $\varphi \in H^s(\gamma_{ij}) \cap H_{00}^{1/2}(\gamma_{ij})$,

$$\begin{aligned}
\|\varphi - \Pi_{h_i}\varphi\|_{H_{00}^{1/2}(\gamma_{ij})} &\leq C \inf_{\chi\in W^{h_i}(\gamma_{ij})\cap H_{00}^{1/2}(\gamma_{ij})} \|\varphi - \chi\|_{H_{00}^{1/2}(\gamma_{ij})} \\
&\leq Ch_i^{s-1/2}\|\varphi\|_{H^s(\gamma_{ij})}.
\end{aligned} \tag{1.3.34}$$

Hence the lemma follows, and this completes the rest of the proof. ■

Below, we define a lifting operator $R_{h_i}$ for the traces of the finite element functions on $\gamma_{ij}$ and discuss its properties. The existence and stability of such an operator is discussed in [26, Lemma 5.1].

**Lemma 1.3.5** *There exists a lifting operator $R_{h_i}$ from $W^{h_i}(\gamma_{ij})\cap H_0^1(\gamma_{ij})$ into $X_h(\Omega_i)$ such that, for any $\varphi \in W^{h_i}(\gamma_{ij}) \cap H_0^1(\gamma_{ij})$, $R_{h_i}\varphi$ is equal to $\varphi$ on $\gamma_{ij}$, vanishes over each side of $\Omega_i$ except $\gamma_{ij}$ and satisfies*

$$\|R_{h_i}\varphi\|_{H^1(\Omega_i)} \leq C\|\varphi\|_{H_{00}^{1/2}(\gamma_{ij})}. \tag{1.3.35}$$

Following [23], define the operator $Q_h$ as

$$Q_h v = I_h v + \sum_{l=1}^{K} \sum_{\gamma_{lj}\subset\Gamma} r_h^{l,j}, \tag{1.3.36}$$

where the function $I_h v$ equals $I_{h_l} v$ on each $\Omega_l$, $1 \le l \le K$ and $r_h^{l,j}$ is equal to zero, if $\gamma_{lj}$ is a mortar edge otherwise it equals $R_{h_l}(\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}})$ on $\Omega_l$ in case of $\gamma_{lj}$ is a nonmortar edge. Here, the mortar function $\varphi$ is equal to the trace of $I_{h_l} v$ on the mortar side.
The central issue is how to find a good approximation $Q_h v \in V_h$ such that it satisfies the desired approximation properties. Here, we state and prove the result from [23, p 43].

**Lemma 1.3.6** *For any real number $\sigma$, $\frac{3}{2} < \sigma \le 2$, there exists a positive constant $C$, independent of $h_l$, such that for any function $v \in H_0^1(\Omega)$ with $v_{|\Omega_l} \in H^\sigma(\Omega_l)$*

$$\|v - Q_h v\|_X \le C \sum_{l=1}^{K} h_l^{\sigma-1} \|v_{|\Omega_l}\|_{H^\sigma(\Omega_l)}. \tag{1.3.37}$$

**Proof.** From the definition (1.3.36) of $Q_h$, we note that

$$\begin{aligned} \|v - Q_h v\|_{H^1(\Omega_l)} &= \|v - I_h v\|_{H^1(\Omega_l)} + \|r_h^{l,j}\|_{H^1(\Omega_l)} \\ &= \|v - I_h v\|_{H^1(\Omega_l)} + \|R_{h_l}(\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}})\|_{H^1(\Omega_l)}. \end{aligned} \tag{1.3.38}$$

Since the estimate for $\|v - I_h v\|_{H^1(\Omega_l)}$ is known from (1.3.12), it is enough to find an estimate for the second term.

For the second term on the right hand side of (1.3.38), use the property (1.3.35) of lifting operator $R_{h_l}$ to arrive at

$$\|R_{h_l}(\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}})\|_{H^1(\Omega_l)} \le C \|\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}}\|_{H_{00}^{1/2}(\gamma_{lj})} \tag{1.3.39}$$

Note that, here $\gamma_{lj}$ is a nonmortar side and $\varphi$ matches with trace of $I_h v$ on the opposite side i.e., on the mortar side of $\gamma_{lj}$. Since the triangulation on $\gamma_{lj}$ is assumed to be quasiuniform, we apply here the inverse inequality (1.3.16) to obtain

$$\|R_{h_l}(\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}})\|_{H^1(\Omega_l)} \le C h_l^{-1/2} \|\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}}\|_{L^2(\gamma_{lj})}. \tag{1.3.40}$$

Now use the stability property (Lemma 1.3.4) to find

$$\begin{aligned} \|R_{h_l}(\Pi_{h_l}(\varphi - I_{h_l} v)_{|\gamma_{lj}})\|_{H^1(\Omega_l)} &\le C h_l^{-1/2} \|\varphi - I_{h_l} v_{|\gamma_{lj}}\|_{L^2(\gamma_{lj})} \\ &\le C h_l^{-1/2} \left( \|\varphi - v_{|\gamma_{lj}}\|_{L^2(\gamma_{lj})} + \|v_{|\gamma_{lj}} - I_{h_l} v_{|\gamma_{lj}}\|_{L^2(\gamma_{lj})} \right) \end{aligned} \tag{1.3.41}$$

Since the function $\varphi$ matches with trace of $I_h v$ on the mortar side of $\gamma_{lj}$ , summing up (1.3.38) over all $1 \leq l \leq K$, the rest of the proof follows. ■

**Error estimate**

For the error estimate, thanks to Strang's lemma so that we obtain

$$\|u - u_h\|_X \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_X + \sup_{w_h \in V_h} \sum_{m=1}^{m_0} \frac{\int_{\delta_m} a \frac{\partial u}{\partial n} [[w_h]]|_{\delta_m} d\tau}{\|w_h\|_X} \right). \qquad (1.3.42)$$

where $[[v]]|_{\gamma_{ij}} = (v_{|\Omega_i} - v_{|\Omega_j})$ denotes the jump of $v$ across $\gamma_{ij}$. For a proof of (1.3.42), add and subtract $v_h \in V_h$ on the error term $\|u - u_h\|_X$ so that we obtain

$$\|u - u_h\|_X \leq \|u - v_h\|_X + \|v_h - u_h\|_X. \qquad (1.3.43)$$

Now use coercivity of $a(\cdot, \cdot)$ to find that

$$\begin{aligned} \alpha \|v_h - u_h\|_X^2 &\leq a(v_h - u_h, v_h - u_h) \\ &\leq |a(u - u_h, v_h - u_h)| + |a(u - v_h, v_h - u_h)| \\ &\leq C \left( \sum_{m=1}^{m_0} \int_{\delta_m} a \frac{\partial u}{\partial n} [[v_h - u_h]]|_{\delta_m} d\tau + \|u - v_h\|_X \|v_h - u_h\|_X \right). \end{aligned} \qquad (1.3.44)$$

Divide by $\|v_h - u_h\|_X$ on both sides of (1.3.44) to obtain (1.3.42).

The first term of right hand side of (1.3.42) is the *approximation error* and the second term is *consistency error* due to the non-conformity of the method. Since the estimate of the first term on the right hand side of (1.3.42) is known from Lemma 1.3.6, we discuss below the consistency error term, that is, the second term on the right hand side of (1.3.42).

**Consistency error**

We now define an $L^2$-orthogonal projection $\pi_{h_j}$ from $L^2(\delta_{m(j)})$ into $M^{h_j}(\delta_{m(j)})$ as below:

$$\pi_{h_j} \colon L^2(\delta_{m(j)}) \longrightarrow M^{h_j}(\delta_{m(j)})$$

by

$$\int_{\delta_{m(j)}} (\varphi - \pi_{h_j}\varphi)\psi \; d\tau = 0 \qquad \forall \psi \in M^{h_j}(\delta_{m(j)}). \qquad (1.3.45)$$

Note that, we can establish a result similar to Lemma 1.3.4 for the stability of $\pi_{h_j}$. More over, $\pi_{h_j}$ satisfies following properties ([23, Page 37, Lemma 4.1]).

**Lemma 1.3.7** *For any real number $s$, $0 \leq s \leq 1$, the following estimate holds true for any function $\varphi \in H^s(\delta_{m(j)})$:*

$$\|\varphi - \pi_{h_j}\varphi\|_{L^2(\delta_{m(j)})} + h_j^{-1/2}\|\varphi - \pi_{h_j}\varphi\|_{H^{-1/2}(\delta_{m(j)})} \leq Ch_j^s\|\varphi\|_{H^s(\delta_{m(j)})}. \tag{1.3.46}$$

Now, a use of the mortar condition (1.3.7) yields

$$\left|\sum_{m=1}^{m_0}\int_{\delta_m\subset\Gamma} a\frac{\partial u}{\partial n}[[w_h]]d\tau\right| \leq \sum_{m=1}^{m_0}\left|\int_{\delta_m\subset\Gamma}\left(a\frac{\partial u}{\partial n} - \pi_{h_j}\left(a\frac{\partial u}{\partial n}\right)\right)[[w_h]]d\tau\right|, \tag{1.3.47}$$

and hence,

$$\left|\sum_{m=1}^{m_0}\int_{\delta_m\subset\Gamma} a\frac{\partial u}{\partial n}[[w_h]]d\tau\right| \leq \sum_{m=1}^{m_0}\left\|a\frac{\partial u}{\partial n} - \pi_{h_j}\left(a\frac{\partial u}{\partial n}\right)\right\|_{H^{-1/2}(\delta_m)}\|[[w_h]]\|_{H^{1/2}(\delta_m)}.$$

Finally, use Lemma 1.3.7 to arrive at

$$\begin{aligned}\left|\sum_{m=1}^{m_0}\int_{\delta_m\subset\Gamma} a\frac{\partial u}{\partial n}[[w_h]]d\tau\right| &\leq C\sum_{m=1}^{m_0} h_j\left\|a\frac{\partial u}{\partial n}\right\|_{H^{1/2}(\delta_m)}\|[[w_h]]\|_{H^{1/2}(\delta_m)} \\ &\leq C\sum_{j=1}^{K} h_j\|u\|_{H^2(\Omega_j)}\|w_h\|_X. \end{aligned} \tag{1.3.48}$$

Hence, from Lemma 1.3.6, (1.3.42) and (1.3.48), we derive the following error estimates, see also [16, 23].

**Theorem 1.3.1** *Let $u$ be the solution of (1.3.3). Moreover, assume $u|_{\Omega_l} \in H^\sigma(\Omega_l)$, $\frac{3}{2} < \sigma \leq 2$, then there exists a positive constant $C$, independent of $h_l$, such that*

$$\|u - u_h\|_X \leq C\sum_{l=1}^{K} h_l^{\sigma-1}\|u_{|\Omega_l}\|_{H^\sigma(\Omega_l)}. \tag{1.3.49}$$

By an application of the Aubin-Nitsche duality argument to the problem (1.3.9), we can derive an optimal $L^2$- error estimate [22]:

$$\|u - u_h\|_{L^2(\Omega)} \leq C\sum_{l=1}^{K} h_l^{\sigma}\|u_{|\Omega_l}\|_{H^\sigma(\Omega_l)}. \tag{1.3.50}$$

**Remark 1.3.1** *The estimates here only depend upon the local regularity of the original solution $u$ in each subdomain and don't require the global regularity of $u$ on the whole domain $\Omega$. Therefore, with low regularity one achieves optimal order of estimates which is an advantage over the standard finite element method. On this basis, we can apply the method in certain problems where singularities of the solution arise due to to the geometry of domain. For instance, this method can be applied to the driven cavity problem, a particular case of the Navier-Stokes equation, where there are singularities in the corners and one needs to do refinements near the corners. Further, it is possible to apply this method to elliptic problems with discontinuous coefficients specially if the discontinuity occurs across the subdomain interfaces.*

### 1.3.1 Mortar finite element method with a Lagrange multiplier

Instead of imposing the constraint (1.3.7) in the finite dimensional space $X_h$, it is possible to impose the weak continuity condition across the subdomain interfaces in the variational formulation. We observe, in this section, that the Lagrange multiplier is a good approximation to the normal derivative along the interfaces of subdomains. We define the following spaces for our use in future.

$$H(\text{div};\Omega) \;=\; \{\mathbf{q} \in (L^2(\Omega))^n \colon \text{ div } \mathbf{q} \in L^2(\Omega)\}$$

and

$$H_0(\text{div};\Omega) \;=\; \{\mathbf{q} \in (L^2(\Omega))^n \colon \text{ div } \mathbf{q} \in L^2(\Omega),\; \mathbf{q}\cdot n_{|\Gamma} = 0\},$$

equipped with the norm

$$\|\mathbf{q}\|_{H(\text{div};\Omega)} = (\|\mathbf{q}\|^2_{0,\Omega} + \|\text{ div } \mathbf{q}\|^2_{0,\Omega})^{1/2},$$

where $n$ is the unit outward normal along $\partial\Omega$.

We define the Lagrange multiplier space $M$ as follows:

$$M = \{\psi \in \prod_{l=1}^{K} H_D^{-1/2}(\partial\Omega_l) \colon \text{there exists a function } \mathbf{q} \in H_0(\text{div};\Omega) \text{ such that } \psi_k = \mathbf{q}\cdot n_l\}$$

where $\mathbf{q} \cdot n_l \in H_D^{-1/2}(\partial\Omega_l)$ is normal component of $\mathbf{q}$ and $n_l$ is the unit outward normal along $\partial\Omega_l$. Here, $H_D^{-1/2}(\partial\Omega_l)$ represents the dual space of $H_D^{1/2}(\partial\Omega_l)$ while $H_D^{1/2}(\partial\Omega_l)$ is the range of $H_D^1(\Omega)$ by the trace operator. The space $M$ is equipped with the norm

$$\|\psi\|_M = \inf_{\mathbf{q}\in H_0(\mathrm{div};\Omega):\ \mathbf{q}\cdot n_l=\psi_l \forall l} \|\mathbf{q}\|_{H(\mathrm{div};\Omega)}.$$

Also, we need the auxiliary space

$$\tilde{M} = \prod_{m=1}^{m_0} H_{00}^{-1/2}(\delta_m), \tag{1.3.51}$$

equipped with the norm

$$\|\varphi\|_{\tilde{M}} = \left( \sum_{m=1}^{m_0} \|\varphi_m\|^2_{H_{00}^{-1/2}(\delta_m)} \right)^{1/2}. \tag{1.3.52}$$

Setting the flux $a(x)\dfrac{\partial u}{\partial n}$ as $\lambda$, we obtain the following weak formulation of the problem (1.3.1)-(1.3.2) : Find $(u, \lambda) \in X \times M$ such that

$$a(u,v) + b(v,\lambda) = l(v) \qquad \forall\, v \in X, \tag{1.3.53}$$

$$b(u,\mu) = 0 \qquad \forall \mu \in M, \tag{1.3.54}$$

where

$$\begin{aligned} a(u,v) &= \sum_{l=1}^{K} \int_{\Omega_l} a(x)\nabla u_l \cdot \nabla v_l \, dx, \\ b(v,\mu) &= -\sum_{m=1}^{m_0} < \mu, [[v]]_{|\delta_m} >_{\delta_m}, \end{aligned}$$

and

$$l(v) = \sum_{l=1}^{K} \int_{\Omega_l} f v \, dx.$$

Now we define the discrete space for the Lagrange multiplier over all nonmortars as follows:

$$M_h = \prod_{m=1}^{m_0} M^{h_j}(\delta_m), \tag{1.3.55}$$

where $M^{h_j}(\delta_m)$ is the subspace of $W^{h_j}(\delta_m)$ with codimension two. Now, the mortar element formulation with Lagrange multiplier corresponding to (1.3.53)-(1.3.54) is to seek $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = l(v_h) \qquad \forall v_h \in X_h \tag{1.3.56}$$

$$b(u_h, \mu_h) = 0 \qquad \forall \mu_h \in M_h. \tag{1.3.57}$$

Note that

$$\begin{aligned} a(u_h, v_h) &= \sum_{l=1}^{K} \int_{\Omega_l} a(x) \nabla u_{h_l} \cdot \nabla v_{h_l} \, dx, \\ b(v_h, \mu_h) &= -\sum_{m=1}^{m_0} \int_{\gamma_m \subset \Gamma} \mu_h [[v_h]]_{|\gamma_m} \, d\tau, \end{aligned}$$

and

$$l(v_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h \, dx.$$

Contrary to the standard classical mortar formulation where the constraint (1.3.57) is imposed in the mortar finite element space $V_h$, the condition (1.3.57) is the weak continuity constraint which appears in the formulation. Since the problem (1.3.56)-(1.3.57) is equivalent to a system of linear equations with finite dimension, the existence of a pair of solution $(u_h, \lambda_h)$ follows from the uniqueness. For uniqueness, take $f = 0$ in (1.3.56) and with the help of (1.3.57) and (1.3.11), we find $u_h = 0$. Therefore,

$$b(v_h, \lambda_h) = 0 \qquad\qquad \forall v_h \in X_h,$$

and hence, $\lambda_h = 0$ if and only if

$$\{\mu_h \in M_h \colon b(v_h, \mu_h) = 0 \ \forall v_h \in X_h\} = \{0\}. \tag{1.3.58}$$

Moreover, note that the solution $u_h$ of (1.3.56)-(1.3.57) is also the unique solution of (1.3.9) [16, Page 185] and [73, Page 396]. Now, we state below without proof a theorem on error estimate. For a proof, we refer to [16, Theorem 2.8].

**Theorem 1.3.2** *Let $(u, \lambda)$ be the solution of the problem (1.3.53)-(1.3.54). Moreover, assume $u|_{\Omega_l} \in H^\sigma(\Omega_l)$ for any real number $\sigma$, $\frac{3}{2} < \sigma \leq 2$, then there exists a positive constant $C$, independent of $h_l$, such that*

$$\|u - u_h\|_X + \|\lambda - \lambda_h\|_{\tilde{M}} \leq C \sum_{l=1}^{K} h_l^{\sigma-1} \|u_{|\Omega_l}\|_{H^\sigma(\Omega_l)}. \tag{1.3.59}$$

By an application of Aubin-Nitsche duality argument, we can derive an optimal $L^2$- error estimate [22]:

$$\|u - u_h\|_{L^2(\Omega)} \leq C \sum_{l=1}^{K} h_l^{\sigma} \|u_{|\Omega_l}\|_{H^\sigma(\Omega_l)}. \tag{1.3.60}$$

## 1.4 Literature Survey

The mortar element method firstly introduced by Bernardi, Maday, and Patera [23, 24] in 1987, which designed for solving $2^{nd}$ order partial differential equations with very few restrictions on the domain and grid related to the discretization procedure. Bernardi, Maday and Patera [23, 24] have discussed the *standard* mortar finite element method without Lagrange multiplier and also discussed the mortar element method in the framework of *spectral* elements. Later on Bernardi et al. [22] discussed the coupling of finite elements with spectral elements. The original computational domain is subdivided into two subdomains; a finite element approximation is used on the first domain and a spectral discretization is used in the second domain. In the mortar element method, a good transmission of information between the interfaces of adjacent subdomains is achieved by imposing the condition that the interface jumps are orthogonal to a suitably chosen multiplier space, which is known in literature as the mortaring condition. This method seems to be an efficient tool in this framework due to its flexibility of choosing independent discretization parameters in nonoverlapping subdomains. We refer to Bernardi, Maday, and Patera [23, 24], Ben Belgacem [16], Ben Belgacem and Maday [17] for a general presentation of the mortar element method applied to elliptic problems. Restoring almost all the advantages of domain decomposition methods, the mortar finite element method has the following extra advantages because of its nonconforming property:

- The triangulations need not align across subdomain interfaces and thereby, independent discretization over subdomains employ locally-conforming but globally nonconforming method.
- Due to non-matching of grids on the inter subdomain interfaces, it is possible to refine the mesh in one subdomain without affecting the other and this flexibility is an attractive feature in adaptive procedure.
- It allows to couple different variational schemes such as finite element method and spectral element method over different subdomains to take advantages of both the methods.

In order to achieve the interface continuity weakly, two kinds of matching conditions are discussed and compared by Bernardi et al. [22]. The natural one is the *pointwise matching condition* at the subdomain interfaces and this leads to a pointwise matching of the approximating functions. In this case, there should be one globally defined mesh on the entire domain. Moreover, the pointwise matching of the approximating functions gives rise to a non-optimality of the approximation error of the method and hence, is not used in general. On the other hand, the *integral matching condition* leads to optimal schemes and is very much used in practice. In mortar element method, the later one is adopted. The integral matching condition ensures that the traces of the solutions across the interfaces are made orthogonal to a suitably chosen multiplier space. To achieve stability of the corresponding scheme, we need to choose the multiplier space appropriately so that the discrete spaces for the primal variable and the multiplier satisfy the inf-sup condition, also known as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition. In literature, mostly two types of multiplier spaces are chosen. The first one is subspaces of functions of traces of primal variables with codimension two, for example, see, Bernardi, Maday and Patera [23, 24], and Belgacem [16, 17]. The second choice is using dual spaces for the multipliers, see, Wohlmuth [89], Lamichhane and Wohlmuth [58], and Braess, Dahmen and Wieners [29].

In the standard mortar finite element method [23], the continuity at the vertices is considered and the variational formulation leads to a positive definite system on the constrained mortar space. On the other hand, in the second generation of mortar finite element methods by Ben Belgacem [16], the continuity is not imposed at the common vertices of the subdomains. Belgacem analyzed the mortar element method with Lagrange multiplier by setting under a primal hybrid formulation and generalized the previous work of Raviart and Thomas [72]. He also emphasized the importance of $H_{00}^{1/2}$ space in the derivation of error analysis for the mortar finite element method with Lagrange multipliers. The discrete LBB condition for the discretization space for the primal variable and multipliers is derived by choosing some appropriate spaces. While the Lagrange multiplier is used to relax the mortaring condition on the finite element spaces, the corresponding discrete formulation gives rise to an indefinite system. He also discussed the parallel implementation complexity for the mortar element method. Subsequently, Belgacem and Maday [17], and Braess and Dahmen [30] analyzed the mortar element method in three dimensional cases. They discussed the stability of mortar projection operator individually in three dimensional case. Braess and Dahmen [30] used the mesh-dependent norms in their analysis and they discussed some properties (see, below) of the bilinear form $b(\cdot,\cdot)$ used in general:

**(i)** If $[[v]]_{|\gamma_{ij}} \in H^{1/2}(\gamma_{ij})$ and $\mu \in (H^{1/2}(\gamma_{ij}))'$, then

$$|b(v,\mu)| \leq \|[[v]]\|_{H^{1/2}(\gamma_{ij})}\|\mu\|_{(H^{1/2}(\gamma_{ij}))'}, \tag{1.4.1}$$

Taking the advantage of the trace theorem, it is easy to prove the following estimate:

$$\|[[v]]\|_{H^{1/2}(\gamma_{ij})} \leq c\left(\|v\|_{H^1(\Omega_i)} + \|v\|_{H^1(\Omega_j)}\right). \tag{1.4.2}$$

**(ii)** Assume the continuity at the corners of the subdomains, then $[[v]]_{|\gamma_{ij}} \in H_{00}^{1/2}(\gamma_{ij})$ and $\mu \in (H_{00}^{1/2}(\gamma_{ij}))'$. In this case we can not take the advantage of the trace theorem as the estimate (1.4.2) does not hold in general. Therefore, it was suggested in [30] to make use of the following mesh-dependent norms:

$$\|v\|_{1/2,h,\gamma_{ij}} := h^{-1/2}\|v\|_{L^2(\gamma_{ij})} \qquad \text{and} \qquad \|\mu\|_{-1/2,h,\gamma_{ij}} := h^{1/2}\|\mu\|_{L^2(\gamma_{ij})}.$$

Then

$$\left|\int_{\gamma_{ij}} \mu[[v]]\, d\tau\right| \leq \|v\|_{1/2,h,\gamma_{ij}}\|\mu\|_{-1/2,h,\gamma_{ij}}.$$

Using the mesh-dependent norm, they proved the LBB condition in 3D and also discussed the $L^2$-error estimates.

The mortar element method using dual spaces for the Lagrange multipliers has been studied in [29, 58, 89]. The Lagrange multiplier space is replaced by a dual space without losing the optimality of the method. The advantage of this approach is, all the basis functions are locally supported in a few elements. Compare to the standard mortar method where a linear system of equations for the mortar projection must be solved; in this case the matrix associated with mortar projection is represented by a diagonal matrix. For the construction of such a dual basis function, we refer to [88], [89, Page 993]. Lamichhane, Stevenson and Wohlmuth in [57] generalized the concept of dual Lagrange multiplier bases by relaxing the condition that the trace space of the approximation space at the slave side with zero boundary condition on the interface and the Lagrange multiplier space have the same dimension. They provided a theoretical framework within this relaxed setting, which was a simpler way to construct dual Lagrange multiplier bases for higher order finite element spaces.

A residual based error estimator for the approximation of linear boundary value problems by nonconforming finite element methods which are based on Crouzeix-Raviart elements of lowest order was analyzed by Wohlmuth [90] and compared with the error estimator obtained in the more general mortar situation. In [92], Wohlmuth also discussed hierarchical a posteriori error estimators for mortar finite element methods.

Since mortar element method is nonconforming in most cases, the matrix system arising from the finite element discretization has a large condition number and hence, the system becomes ill-conditioned. Efforts have been made in literature for developing algorithms to solve efficiently the corresponding algebraic systems. Achdou, Maday and Widlund [1, 3] discussed iterative substructuring algorithms for the algebraic systems arising from the discretization of symmetric, second-order, elliptic equations in two dimensions. Both spectral and finite element methods, for geometrically conforming as well as nonconforming domain decompositions, were studied. In both the cases, they obtained a polylogarithmic bound on the condition number of the preconditioned matrix.

Generally, the mortar approach has the disadvantage that even when the boundary value problem is elliptic, the arising linear system is of saddle point type, usually for which iterative methods are known to be less efficient than for symmetric positive definite systems. However, when working with dual Lagrange multiplier bases, the degrees of freedom associated with the multiplier can be locally eliminated leading to a sparse, positive definite system, on which, for example, efficient multigrid methods can be applied, see [91, 93]. For the lowest order finite elements in 3D, dual Lagrange multiplier bases are constructed in [53, 91]. A multigrid algorithm for the system of equations arising from the mortar finite element discretization of second order elliptic boundary value problems has been developed and analyzed by Braess et al. [29]. They have also highlighted the important role of $H_{00}^{1/2}$ space from the numerical analysis point of view. They have revisited the concept of mortar element method by employing suitable mesh-dependent norms and verified the validity of LBB condition. Further, they have verified the optimal multigrid efficiency based on a smoother which is applied to the whole coupled system of equations. Later Gopalakrishnan and Pasciak [42] have discussed and analyzed a multigrid technique for uniformly preconditioning linear systems arising from a mortar finite element discretization of second order elliptic boundary value problems. These problems are posed on domains partitioned into subdomains, each of which is independently triangulated in a multilevel fashion. Suitable grid transfer operators and smoothers are developed which lead to a variable V-cycle preconditioner resulting in a uniformly preconditioned algebraic system.

Bjørstad, Dryja and Rahman [25] have designed and analyzed two variants of the additive Schwarz method for solving linear systems arising from the mortar finite element discretization on nonmatching meshes of second order elliptic problems with discontinuous coefficients. The methods are defined on non-overlapping subdomains, and they have used special coarse spaces, resulting in algorithms that are well suited for parallel computation. They have discussed the condition number estimate for the preconditioned systems which is independent of the discontinuous jumps of the coefficients. Dryja et al. [39] have analyzed a multilevel preconditioner for mortar finite element method. The analysis is carried out within the abstract framework of the additive Schwarz methods.

The finite element tearing and interconnecting (FETI) method is an iterative substructuring method using Lagrange multipliers to enforce the continuity of the finite element solution across the subdomain interface. Stefanica [79] has presented a numerical study of FETI algorithms for an elliptic self-adjoint equation discretized by mortar finite elements. Several preconditioners which have been successful for the case of conforming finite elements are considered and experiments are carried out for both two and three dimensional problems. He has also included a study of the relative costs of applying different preconditioners for the mortar elements.

The $hp$ version of mortar finite element methods for elliptic problems has been studied by Padmanabhan and Suri [75, 76]. They have discussed and analyzed uniform convergence results for the mortar finite element method in case of $h$, $p$ and $hp$ discretizations over general meshes and derived optimal rates of convergence even for highly nonquasiuniform meshes over the subdomains. They established optimality for the non-quasiuniform $h$ discretizations that include, among others, radical and geometric meshes needed for the treatment of the singularities. Also for $p$ version, where the degree $p$ is allowed to increase, while the mesh is kept fixed is shown to be optimal up to $O(p^{3/4})$ and the $hp$ version over geometric meshes, which leads to exponential convergence.

Mixed finite element methods for second order elliptic equations on nonmatching multiblock grids are discussed and analyzed by Arbogast, Cowsar, Wheeler and Yotov [6]. A mortar finite element space is introduced on the nonmatching interfaces for approximating the trace of the solution. A standard mixed finite element method is used within each block. Optimal order of convergence is obtained for the solution and its flux. Moreover, at certain discrete points, superconvergence is obtained for the solution and also for the flux in certain cases. Recently, mortar finite element discretization for the flow in a nonhomogeneous porous medium is discussed and analyzed by Bernardi, Hecht and Mghazli in [21].

Because of the nonconformity and flexible nature of mortar element method, it turns out to be well adapted to handle mesh adaptivity in finite elements. Mesh adaptivity in the mortar finite element method has been studied by Bernardi and Hecht [20]. In [19], they extended the numerical analysis of residual error indicators to mesh adaptive methods for a model problem.

Marcinkowski [65] discussed domain decomposition methods for mortar finite element discretizations of the plate problem. A mortar finite element method for the fourth order problems in two dimensions with Lagrange multipliers can be found in [64].

In [35] and [94], the authors have applied a finite difference scheme to discretize the parabolic equation in temporal direction to obtain an elliptic equation at each time step, and then mortar finite element method is used to discretize the resulting elliptic equation in spatial direction. In [78], domain decomposition methods for the mortar mixed finite element method have been applied to parabolic problems in the framework of splitting method. The approach adopted in [78] makes use of the parabolic structure and a non-iterative scheme is proposed to solve the problem once in each subdomain via an operator splitting. An *a posteriori* error estimate for the mortar finite element method for parabolic problems has been developed in [18]. In particular, the authors have studied the residual spatial error indicators at each time step using mortar elements to discretize in spatial direction. A multigrid method for mortar finite element method has been applied to parabolic problems in [94]. The error analysis is developed essentially in the framework of elliptic problems. Note that the cumulative effect of the time discretization is missing in their analysis. In this thesis, we discussed and analyzed a standard mortar finite element method and a mortar element method with Lagrange multiplier which is used for the spatial discretization. We derive optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for the semidiscrete methods for both the cases. A mixed finite element discretization on nonmatching multiblock grids for a degenerate parabolic equation arising in porous media flow is discussed by Yotov [95].

We have seen that a basic requirement for the mortar element method is to construct multiplier spaces which satisfy certain criteria known as the inf-sup properties for the scheme to be stable. Many natural and convenient choices of these spaces are ruled out as these spaces do not satisfy the inf-sup condition. In order to alleviate this problem, stabilized multiplier techniques or Nitsche's method [82] is used. This was originally introduced for solving non-homogeneous Dirichlet problems without enforcing the boundary condition on the finite element spaces [10, 11, 67]. In this method, the original bilinear forms of the problem are modified by adding suitable stabilized terms in order to improve stability without compromising on the consistency of the method. We refer to [9, 12, 13, 14] for the various penalty methods applied to elliptic problems and discuss how to circumvent the inf-sup condition in order to achieve the consistency and stability of the methods.

In [15, 46, 48, 82], the various possible Nitsche's mortaring methods have been introduced and analyzed for elliptic problems. In [82], Stenberg pointed out the close connection between Nitsche's method and stabilized methods and called it as mortaring Nitsche method. Becker, Hansbo and Stenberg [15] extended the previous work and discussed in more details the analysis of this technique where independent discretizations were used in different subdomains. The continuity of the solution along the common interface is imposed without disturbing the consistency of the resulting scheme with the original problem.

The drawback of most of the stabilized methods is that they use the jump in the primal variables as one of stabilized term across the subdomain interfaces. This means that piecewise polynomials on unrelated, unstructured meshes have to be integrated which is quite expensive to implement, especially in three or higher dimensions. To mitigate this problem, Hansbo et al. [46] proposed a stabilization method which avoids the cumbersome integration of products of unrelated mesh functions. While Hansbo et al. discussed $H^1$-estimate for the elliptic problem, optimal $L^2$-estimate was missing in their analysis. In Chapter 4, we have proposed and established the error estimates for a stabilized Nitsche mortar formulation which is consistent with the original problem. We obtain optimal order estimates in both $L^2$- and $H^1$-norms for the stabilized Nitsche's mortaring method for second order elliptic and parabolic problems.

## 1.5 Outline of the Thesis

The organization of the thesis is as follows. Chapter 1 is introductory in nature. After recalling some inequalities and results, we have briefly described the mortar finite element methods with and without Lagrange multipliers for the second order elliptic problems. Further, we have discussed a literature survey for the mortar finite element method.

In Chapter 2, an effort has been made to apply mortar element methods with and without Lagrange multipliers for the parabolic initial and boundary value problems. We have discussed mortar finite element methods for the second order parabolic problems. A standard mortar finite element method and a mortar element method with Lagrange multiplier are used for the spatial discretization. Optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for semidiscrete methods for both the cases are established. The key feature that we have adopted here is to introduce a modified elliptic projection. In the standard mortar element method, a completely discrete scheme using backward Euler scheme is discussed and optimal error estimates are derived. Finally, numerical experiments that support the theoretical results are obtained.

In order to alleviate the LBB condition, in Chapter 3, we have proposed a Nitsche's mortaring element method for the elliptic and parabolic problems. We added a stabilized term which contains $\epsilon$ as a parameter which helped us to show the existence and uniqueness of the resulting discrete scheme without using the discrete LBB condition. We have obtained a priori error estimates. Since, this proposed scheme is not consistent, we have obtained only sub-optimal $L^2$ and $H^1$-error estimates.

In Chapter 4, we have proposed and analyzed stabilized Nitsche mortar formulation which is consistent. Under a mild assumption on the penalty parameter, the method is shown to be stable. Further, we derive *a priori* error estimates for the Nitsche's mortaring method applied to a second-order parabolic problems with discontinuous coefficients in a polygonal region $\Omega$ with Lipschitz boundary. Moreover, we have analyzed the error estimates in both $L^2$- and $H^1$-norms for the Nitsche's mortaring method for both semidiscrete and completely discrete schemes.

Finally, in Chapter 5, we have summarized the results obtained and have done a critical assessment of the work. Further, we have discussed the possible extensions of the results derived in this thesis. Also we have discussed the future work in the direction of mortar finite element method applied to nonlinear problems.

# Chapter 2

# Mortar Finite Element Methods with and without Lagrange multipliers

## 2.1 Introduction

In this chapter, a standard mortar finite element method and a mortar element method with Lagrange multipliers are applied for the spatial discretization of the following class of parabolic initial-boundary value problems:

$$u_t - \nabla \cdot (a(x)\nabla u) = f(x,t) \quad \text{in} \quad \Omega \times (0,T], \tag{2.1.1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \tag{2.1.2}$$

$$u(0) = u_0(x) \quad \text{in } \Omega, \tag{2.1.3}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $T$ is the fixed final time, $u_t = \frac{\partial u}{\partial t}$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, and $f \in L^2(\Omega)$, $u_0$ are given functions. Assume that the coefficient $a(x)$ is smooth and satisfies $0 < \alpha_0 \leq a(x) \leq \alpha_1$, for some positive constants $\alpha_0$ and $\alpha_1$ and for all $x \in \bar{\Omega}$. For $u_0(x)$ to be in $L^2(\Omega)$, the problem (2.1.1)-(2.1.3) has unique solution in $H^2(\Omega)$ by [56].

The related work for the parabolic initial-boundary value problem (2.1.1)-(2.1.3) can be found in [18, 35, 78, 94]. In [35] and [94], finite difference scheme is applied to discretize the parabolic equation in temporal direction to obtain an elliptic equation at each time step, and then mortar finite element method is used to discretize the resulting elliptic equation in spatial direction. In [94], a multigrid method for mortar finite element method has been applied to parabolic problems. In [78], mortar mixed finite element method have been analyzed for parabolic problems and a non-iterative scheme is proposed to solve the problem once in each subdomain via an operator splitting. In [18], the authors have studied the residual spatial error indicators at each time step using mortar elements to discretize in spatial direction.

In this chapter, we establish optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for the semidiscrete scheme for mortar element methods with and without Lagrange multipliers. The key feature that we have adopted here is the introduction of a modified elliptic projection. Using backward Euler method, a completely discrete scheme is analyzed for the standard mortar finite element method and error estimates are derived.

A brief outline of this chapter is as follows. In Section 2, we define the problem and discuss the mortar finite element discretization. In Section 3, some approximation properties are stated and an auxiliary projection which is used in the subsequent error analysis is defined. In Section 4, error estimates for both semidiscrete and fully discrete schemes are developed. In Section 5, the mortar finite element method with a Lagrange multiplier is discussed. Finally in Section 6, the results of some numerical experiments that support our theoretical results are presented.

## 2.2 A Mortar finite element method

The main objective in this section is to provide an approximation to the solution of (2.1.1)-(2.1.3). The starting point of this method is to use a natural decomposition of the domain $\Omega$ into $K$ nonoverlapping convex subdomains $\Omega_l,\ 1 \leq l \leq K$. Denote $\partial\Omega_i \cap \partial\Omega_j = \gamma_{ij},\ 1 \leq i, j \leq K$. Let $n_{ij}$ be the unit normal oriented from $\Omega_i$ towards $\Omega_j$ so that $n_{ij} = -n_{ji}$. Here, we discuss the geometrically conforming version of the mortar method, i.e., the intersection of $\bar{\Omega}_i$ and $\bar{\Omega}_j$ for $i \neq j$ is either empty set, a common edge or a common vertex. Let

$$H_D^1(\Omega_l) = \{v \in H^1(\Omega_l) \colon v|_{\partial\Omega_l \cap \partial\Omega} = 0\}.$$

Now, define

$$X = \{v \in L^2(\Omega) \colon v_{|\Omega_l} \in H_D^1(\Omega_l)\ \forall 1 \leq l \leq K\},$$

which is equipped with a norm and seminorm

$$\|v\|_X = \left( \sum_{l=1}^{K} \|v\|^2_{H^1(\Omega_l)} \right)^{1/2} \quad \text{and} \quad |v|_X = \left( \sum_{l=1}^{K} \|\nabla v\|^2_{L^2(\Omega_l)} \right)^{1/2},$$

respectively.

The weak formulation corresponding to (2.1.1)-(2.1.3) is to find $u : [0,T] \longrightarrow H^1_0(\Omega)$ such that

$$(u_t, v) + a(u,v) = l(v) \qquad \forall v \in H^1_0(\Omega), \tag{2.2.1}$$

$$u(0) = u_0. \tag{2.2.2}$$

Here, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$,

$$a(u,v) = \sum_{l=1}^{K} \int_{\Omega_l} a(x) \nabla u_l \cdot \nabla v_l \, dx,$$

and

$$l(v) = \sum_{l=1}^{K} \int_{\Omega_l} f v \, dx,$$

where $u_l$ and $v_l$ are restrictions of $u$ and $v$, respectively, to $\Omega_l$. From the properties of the coefficient $a$, it is easy to show that the bilinear form $a(\cdot,\cdot)$ satisfies the following boundedness property: for $v$ and $w$ in $X$,

$$|a(v,w)| \leq \alpha_1 |v|_X \, |w|_X \leq \alpha_1 \|v\|_X \, \|w\|_X, \tag{2.2.3}$$

Now, we recall from Chapter 1, that the discrete space $V_h$ is defined as

$$V_h = \{ v_h \in X_h(\Omega) \colon \forall \delta_{m(j)} \subset \Gamma, \quad \int_{\delta_{m(j)}} (v_{i,h} - v_{j,h})_{|\delta_{m(j)}} \psi d\tau = 0, \quad \forall \psi \in M^{h_j}(\delta_{m(j)}) \}.$$

For notational convenience, we denote the non-mortar side $\delta_{m(j)}$ by $\delta_m$.
The mortar finite element formulation for the problem (2.2.1)-(2.2.2) is to seek $u_h : (0,T] \longrightarrow V_h$ such that

$$(u_{ht}, v_h) + a(u_h, v_h) = l(v_h) \qquad \forall \; v_h \in V_h, \tag{2.2.4}$$

$$u_h(0) = u_{0,h}, \tag{2.2.5}$$

where $u_{0,h}$ is an approximation of $u_0$ in $V_h$ to be defined later,

$$a(v_h, w_h) = \sum_{l=1}^{K} \int_{\Omega_l} a(x)\nabla v_{l,h} \cdot \nabla w_{l,h}\, dx,$$

and

$$l(v_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h\, dx.$$

Here, $v_{l,h}$ and $w_{l,h}$ are respectively, the restrictions of $v_h$ and $w_h$ to $\Omega_l$.

Note that, Poincaré inequality holds true on the broken Sobolev spaces, which is stated as below:

**Lemma 2.2.1 (Generalized Poincaré Inequality)[23]** *There exists a positive constant $c$ such that for $v_h \in V_h$ the following relation holds true:*

$$\sum_{l=1}^{K} \|v_{|\Omega_l}\|^2_{L^2(\Omega_l)} \leq c \sum_{l=1}^{K} |v_{|\Omega_l}|^2_{H^1(\Omega_l)}. \tag{2.2.6}$$

Using Poincaré inequality for elements in $V_h$, ( Lemma 2.2.1), we note that the seminorm $|\cdot|_X$ is indeed a norm on $V_h$ which is equivalent to the $\|\cdot\|_X$ norm. From the properties of the coefficient $a$, we note that the bilinear form $a(\cdot,\cdot)$ satisfies the following coercive property: for $v_h$ in $V_h$

$$a(v_h, v_h) \;\; \geq \;\; \alpha_0 |v_h|^2_X \geq \alpha \|v_h\|^2_X, \tag{2.2.7}$$

where $\alpha$ depends on $\alpha_0$ and the constant in the Poincaré inequality. Note that, (2.2.4) is equivalent to a linear system of ordinary differential equations and the matrix involved with the bilinear form is positive definite. Therefore Picard's theorem ensures the existence of a unique solution $u_h(t) \in V_h$ for $t \in [0, T]$ of the problem (2.2.4)-(2.2.5).

## 2.3 A Modified Elliptic Projection

For our error analysis, we now introduce a modified elliptic projection $P_h$ from $X$ onto $V_h$, which is defined as follows: For a given $u \in X$, find $P_h u \in V_h$ such that

$$a(u - P_h u, \chi) - \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[\chi]] d\tau = 0 \qquad \forall \chi \in V_h. \tag{2.3.1}$$

where $[[v]] = (v_i - v_j)_{|\gamma_{ij}}$ denotes the jump of $v$ over the common interface $\gamma_{ij}$ of $\Omega_i$ and $\Omega_j$, with $v_i, v_j$ being the restrictions of $v$ to $\Omega_i$ and $\Omega_j$, respectively. Now for a given $u$, the problem (2.3.1) has a unique solution $P_h u \in V_h$ as the bilinear form $a(\cdot, \cdot)$ satisfies the coercive property (2.2.7). Observe that (2.3.1) is a modification of the standard elliptic projection used in the context of parabolic problems, see Thomeé [85]. Below, we discuss *a priori* estimates for $u - P_h u$ in broken $H^1$-norm and $L^2$-norm.

**Lemma 2.3.1** *Assume that for $t \in (0, T]$, $u(t) \in H_0^1(\Omega)$ and $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^\sigma(\Omega_l)$. For any real number $\sigma$, with $\frac{3}{2} < \sigma \leq 2$, there exists a positive constant $C$, independent of $h_l$, such that*

$$\|u - P_h u\| + h\|u - P_h u\|_X \leq Ch^\sigma \sum_{l=1}^{K} \|u\|_{H^\sigma(\Omega_l)}. \tag{2.3.2}$$

*Further,*

$$\|u_t - P_h u_t\| + h\|u_t - P_h u_t\|_X \leq Ch^\sigma \sum_{l=1}^{K} \|u_t\|_{H^\sigma(\Omega_l)}. \tag{2.3.3}$$

**Proof.** Using the definition of $Q_h$ in (1.3.36), we rewrite the equation (2.3.1) as

$$a(Q_h u - P_h u, \chi) = -a(u - Q_h u, \chi) + \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[\chi]] d\tau \quad \forall \chi \in V_h. \tag{2.3.4}$$

Set $\chi = Q_h u - P_h u \in V_h$ in (2.3.4). Using the coercivity (2.2.7) and boundedness (2.2.3) of the bilinear form $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} \alpha\|Q_h u - P_h u\|_X^2 &\leq a(Q_h u - P_h u, Q_h u - P_h u) \\ &\leq \alpha_1 \|u - Q_h u\|_X \, \|Q_h u - P_h u\|_X + \sum_{m=1}^{m_0} |\int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[Q_h u - P_h u]] \, d\tau|. \end{aligned} \tag{2.3.5}$$

Now using the mortar condition (1.3.7) and Lemma 1.3.7 with the trace inequality, we proceed as in [23], Lemma 3.5 of [29] to obtain

$$\sum_{m=1}^{m_0} \Big| \int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[Q_h u - P_h u]]d\tau \Big| = \sum_{m=1}^{m_0} \Big| \int_{\delta_m \subset \Gamma} (a\nabla u \cdot n - \lambda_h)[[Q_h u - P_h u]]d\tau \Big| \quad \forall \lambda_h \in M^{h_j}(\delta_m)$$

$$\leq C \|a\nabla u \cdot n - \lambda_h\|_{H^{-1/2}(\Gamma)} \|[[Q_h u - P_h u]]\|_{H^{1/2}(\Gamma)}$$

$$\leq C \sum_{j=1}^{K} h_j^{\sigma-1} \|u\|_{H^\sigma(\Omega_j)} \|Q_h u - P_h u\|_{H^1(\Omega_j)}. \tag{2.3.6}$$

Substituting (2.3.6) in (2.3.5) and using Lemma 1.3.6 , we arrive at

$$\|Q_h u - P_h u\|_X \leq C(\alpha, \alpha_1)\|u - Q_h u\|_X + \sum_{l=1}^{K} h_l^{\sigma-1} \|u\|_{H^\sigma(\Omega_l)}. \tag{2.3.7}$$

Since

$$\|u - P_h u\|_X \leq \|u - Q_h u\|_X + \|Q_h u - P_h u\|_X,$$

again using Lemma 1.3.6 with (2.3.7), we obtain

$$\|u - P_h u\|_X \leq C \sum_{l=1}^{K} h_l^{\sigma-1} \|u\|_{H^\sigma(\Omega_l)}. \tag{2.3.8}$$

For the $L^2$ error estimate, we appeal to Aubin-Nitsche duality argument. Let $\psi_l = \psi_{|\Omega_l} \in H^2(\Omega_l) \cap H_0^1(\Omega), 1 \leq l \leq K$, be the solution of the transmission problem :

$$-\nabla \cdot (a(x)\nabla \psi_l) = u_l - P_h u_l \text{ in } \Omega_l, \tag{2.3.9}$$

$$\psi_l = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_l, \tag{2.3.10}$$

$$[[\psi]]_\Gamma = 0, \quad \left[\left[a\frac{\partial \psi}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along } \Gamma, \tag{2.3.11}$$

which satisfies following regularity condition

$$\sum_{l=1}^{K} \|\psi_l\|_{H^2(\Omega_l)} \leq C\|u - P_h u\|. \tag{2.3.12}$$

We now refer to Theorem 1.1 of [11], Theorem 2.1 of [34] and the references therein for the proof of the elliptic regularity (2.3.12). Multiplying both sides of (2.3.9) by $u_l - P_h u_l$ and summing over all $l$, $1 \le l \le K$, we find that

$$\begin{aligned}\|u - P_h u\|^2 &= \sum_{l=1}^{K} \|u_l - P_h u_l\|^2 \\ &= a(u - P_h u, \psi) - \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} a\nabla\psi \cdot n[[u - P_h u]]d\,\tau.\end{aligned}$$

Using (2.3.1), and the mortar condition (1.3.7), we now arrive at

$$\begin{aligned}\|u - P_h u\|^2 &= a(u - P_h u, \psi - Q_h\psi) + a(u - P_h u, Q_h\psi) \\ &\quad - \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} (a\nabla\psi \cdot n - \mu_h)[[u - P_h u]]d\,\tau \quad \forall \mu_h \in M^{h_j}(\delta_m) \\ &= a(u - P_h u, \psi - Q_h\psi) + \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[Q_h\psi]]d\,\tau \\ &\quad - \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} (a\nabla\psi \cdot n - \mu_h)[[u - P_h u]]d\,\tau. \qquad (2.3.13)\end{aligned}$$

Now boundedness of $a(\cdot,\cdot)$ with Lemma 1.3.6 yields

$$\begin{aligned}|a(u - P_h u, \psi - Q_h\psi)| &\le \alpha_1 \|u - P_h u\|_X \|\psi - Q_h\psi\|_X \\ &\le C(\alpha_1) \sum_{l=1}^{K} h_l \|\psi\|_{H^2(\Omega_l)} \|u - P_h u\|_X. \qquad (2.3.14)\end{aligned}$$

To estimate the second term on the right-hand side of (2.3.13), we use (1.3.7), and $[[\psi]] = 0$ on $\Gamma$. Then, we apply Cauchy-Schwarz inequality and Lemma 1.3.7 to obtain for $\lambda_h \in M^{h_j}(\delta_m)$

$$\begin{aligned}\sum_{m=1}^{m_0} |\int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[Q_h\psi]]d\tau| &= \sum_{m=1}^{m_0} |\int_{\delta_m \subset \Gamma} (a\nabla u \cdot n - \lambda_h)[[\psi - Q_h\psi]]d\tau| \\ &\le \sum_{m=1}^{m_0} \inf_{\lambda_h \in M^{h_j}(\delta_m)} \|a\nabla u \cdot n - \lambda_h\|_{L^2(\delta_m)} \|[[\psi - Q_h\psi]]\|_{L^2(\delta_m)} \\ &\le C \sum_{m=1}^{m_0} h_j^{1/2} \|a\nabla u \cdot n\|_{H^{1/2}(\delta_m)} \|[[\psi - Q_h\psi]]\|_{L^2(\delta_m)}. \qquad (2.3.15)\end{aligned}$$

From the definition (1.3.36) of $Q_h$, we note that $\|[[\psi - Q_h\psi]]\|_{L^2(\delta_m)}$ can be written in terms of $\|[[\psi - I_h\psi]]\|_{L^2(\delta_m)}$. Now using (1.3.13), (2.3.15) leads to

$$\sum_{m=1}^{m_0} |\int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[Q_h\psi]]d\tau| \le C \sum_{j=1}^{K} h_j^2 \|u\|_{H^2(\Omega_j)} \|\psi\|_{H^2(\Omega_j)}. \qquad (2.3.16)$$

For the last term on the right-hand side of (2.3.13), we proceed similarly as in (2.3.6) to obtain

$$\begin{aligned}\sum_{m=1}^{m_0} \Big|\int_{\delta_m\subset\Gamma}(a\nabla\psi\cdot n-\mu_h)[[u-P_hu]]d\tau\Big| &= \sum_{m=1}^{m_0} \Big|\int_{\delta_m\subset\Gamma}(a\nabla\psi\cdot n-\mu_h)[[u-P_hu]]d\tau\Big| \\ &\le C\sum_{j=1}^{K} h_j\|\psi\|_{H^2(\Omega_j)}\|u-P_hu\|_{H^1(\Omega_j)}. \quad (2.3.17)\end{aligned}$$

Substituting (2.3.14)-(2.3.17) in (2.3.13) and using the elliptic regularity result (2.3.12), we find that

$$\|u-P_hu\| \le C\sum_{l=1}^{K} h_l^{\sigma}\|u\|_{H^{\sigma}(\Omega_l)}. \qquad (2.3.18)$$

With $h=\max_{1\le l\le K} h_l$, the proof of (2.3.2) follows from (2.3.8) and (2.3.18). To prove (2.3.3), differentiate (2.3.1) with respect to time to obtain

$$a(u_t-P_hu_t,\chi)-\sum_{m=1}^{m_0}\int_{\delta_m\subset\Gamma} a(x)\nabla u_t\cdot n[[\chi]]d\tau=0 \qquad \forall\chi\in V_h. \qquad (2.3.19)$$

Since the equation remains invariant under time differentiation, $(P_hu)_t=P_hu_t$, replacing $u_t$ by $v$, we obtain exactly same equation (2.3.1) for $v$. Then proceed similarly to derive estimates for $u_t-P_hu_t$. ■

## 2.4 Error estimates for the mortar finite element method

In this section, optimal error estimates in $L^\infty(H^1)$- and $L^\infty(L^2)$-norms are discussed.

### 2.4.1 Error estimates for the semidiscrete method

Note that for $v\in X$ and for $t\in(0,T]$, from (2.1.1)-(2.1.3), we obtain

$$\begin{aligned}(u_t,v)+a(u,v) &= \sum_{l=1}^{K}\int_{\Omega_l} fv\,dx+\sum_{m=1}^{m_0}\int_{\gamma_m\subset\Gamma} a\nabla u\cdot n[[v]]d\tau \;\; \forall v\in X \qquad (2.4.1)\\ u(0) &= u_0. \qquad (2.4.2)\end{aligned}$$

**Theorem 2.4.1** *Assume that for $t \in (0,T]$, $u(t) \in H_0^1(\Omega)$ and $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^2(\Omega_l)$. Let $u$ and $u_h$ be the solutions of (2.4.1)-(2.4.2) and (2.2.4)-(2.2.5), respectively. Further, let $u_{0,h} = I_h u_0$ or $P_h u_0$. Then, there exists a positive constant $C$, independent of $h_l$, such that for $t \in (0,T]$, the following estimates hold:*

$$\|(u-u_h)(t)\| \le C \sum_{l=1}^{K} h_l^2 \left( \|u_0\|_{H^2(\Omega_l)} + \|u_t\|_{L^2(0,T;H^2(\Omega_l))} \right), \tag{2.4.3}$$

*and*

$$\|(u-u_h)(t)\|_X \le C \sum_{l=1}^{K} h_l \left( \|u_0\|_{H^2(\Omega_l)} + \|u_t\|_{L^2(0,T;H^2(\Omega_l))} \right). \tag{2.4.4}$$

**Proof.** Using the definition of $P_h$, we now split $u - u_h$ as

$$u - u_h = (u - P_h u) + (P_h u - u_h) = \rho + \theta. \tag{2.4.5}$$

Since the estimates of $\rho$ are known from Lemma 2.3.1, it is enough to estimate $\theta$. From (2.2.4), (2.3.1) and (2.4.1), we obtain

$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi) \qquad \forall \chi \in V_h. \tag{2.4.6}$$

Substituting $\chi = \theta$ in (2.4.6), applying coercivity of $a(\cdot,\cdot)$ with coercivity constant $\alpha$ and using the Young's inequality (1.2.2), we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \alpha\|\theta\|_X^2 &\le \|\rho_t\|\|\theta\| \\ &\le \frac{1}{2\alpha}\|\rho_t\|^2 + \frac{\alpha}{2}\|\theta\|_X^2. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|\theta\|^2 + \alpha\|\theta\|_X^2 \le \frac{1}{\alpha}\|\rho_t\|^2.$$

Integrating from 0 to $t$, we find that

$$\|\theta(t)\|^2 \le \|\theta(0)\|^2 + \frac{1}{\alpha}\int_0^t \|\rho_t\|^2 ds. \tag{2.4.7}$$

If $u_{0,h} = P_h u_0$, then $\theta(0) = 0$, otherwise with $u_{0,h} = I_h u_0$,

$$\begin{aligned} \|\theta(0)\| = \|P_h u_0 - u_{0,h}\| &\le \|u_0 - I_h u_0\| + \|P_h u_0 - u_0\| \\ &\le C \sum_{l=1}^{K} h_l^2 \|u_0\|_{H^2(\Omega_l)}. \end{aligned} \tag{2.4.8}$$

For the second term on the right-hand side of (2.4.7), we apply (2.3.3) to obtain

$$\|\rho_t\| = \|u_t - P_h u_t\| \leq C(\alpha) \sum_{l=1}^{K} h_l^2 \|u_t\|_{H^2(\Omega_l)}. \tag{2.4.9}$$

Substituting (2.4.8) and (2.4.9) in (2.4.7), we find that

$$\|\theta(t)\|^2 \leq C(\alpha) \sum_{l=1}^{K} h_l^4 \left( \|u_0\|^2_{H^2(\Omega_l)} + \int_0^t \|u_t\|^2_{H^2(\Omega_l)} \, ds \right). \tag{2.4.10}$$

A use of triangle inequality with Lemma 2.3.1 and

$$\psi(t) = \psi(0) + \int_0^t \psi_t(s) \, ds \tag{2.4.11}$$

yields (2.4.3).

For a bound in $X$-norm, substitute $\chi = \theta_t$ in (2.4.6) and apply Cauchy-Schwarz inequality to obtain

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt} a(\theta, \theta) \leq \|\rho_t\| \|\theta_t\| \leq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\theta_t\|^2,$$

and hence,

$$\|\theta_t\|^2 + \frac{d}{dt} a(\theta, \theta) \leq \|\rho_t\|^2. \tag{2.4.12}$$

Integrating both sides of (2.4.12) from 0 to $t$, using coercivity and boundedness of $a(\cdot, \cdot)$, we arrive at

$$\|\theta(t)\|_X^2 \leq C(\alpha) \left( \|\theta(0)\|_X^2 + \int_0^t \|\rho_t\|^2 ds \right). \tag{2.4.13}$$

When $u_{0,h} = P_h u_0$, $\theta(0) = 0$, otherwise with $u_{0,h} = I_h u_0$,

$$\begin{aligned} \|\theta(0)\|_X = \|P_h u_0 - u_{0,h}\|_X &\leq \|u_0 - I_h u_0\|_X + \|P_h u_0 - u_0\|_X \\ &\leq C \sum_{l=1}^{K} h_l \|u_0\|_{H^2(\Omega_l)}. \end{aligned} \tag{2.4.14}$$

Substituting (2.4.9) and (2.4.14) in (2.4.13), we obtain

$$\|\theta(t)\|_X^2 \leq C \sum_{l=1}^{K} \left( h_l^2 \|u_0\|^2_{H^2(\Omega_l)} + h_l^4 \int_0^t \|u_t\|^2_{H^2(\Omega_l)} \, ds) \right). \tag{2.4.15}$$

An application of triangle inequality with Lemma 2.3.1 and (2.4.11) yields (2.4.4). This completes the rest of the proof. ■

**Remark 2.4.1** *If we choose $u_h(0) = P_h u(0)$, then $\theta(0) = 0$ in (2.4.13). Therefore, using Lemma 2.3.1, we derive the following super-convergent result in $\theta$:*

$$\|\theta(t)\|_X^2 \leq Ch^4 \sum_{l=1}^{K} \int_0^t \|u_t\|^2_{H^2(\Omega_l)} ds. \tag{2.4.16}$$

## 2.4.2 Completely discrete scheme

Let $k$ be the time step parameter such that $N = T/k$ and $t_n = nk$. For a continuous function $\varphi \in C[0,T]$, we set the backward difference quotient as $\bar{\partial}_t \varphi^n = \dfrac{\varphi^n - \varphi^{n-1}}{k}$. The backward Euler approximation is to seek a function $U^n \in V_h$ so that $U^n,\ n \geq 1$, satisfies

$$\begin{aligned} (\bar{\partial}_t U^n, \chi) + a(U^n, \chi) &= (f(t_n), \chi) \qquad \forall \chi \in V_h, \\ U^0 &= u_{0,h}, \end{aligned} \tag{2.4.17}$$

where $u_{0,h}$ is chosen either as $I_h u_0$ or $P_h u_0$.

Note that at each step $t = t_n$, (2.4.17) leads to a system of linear algebraic equations. It is easy to check that this system has a unique solution. We discuss below the a priori error estimates for the solution $U^n$ of (2.4.17).

**Theorem 2.4.2** *Assume that for $t \in (0,T]$, $u(t) \in H_0^1(\Omega)$, $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^2(\Omega_l)$ and $u_{tt}(t) \in L^2(\Omega)$. Let $u(t_n)$ be the solution of (2.4.1)-(2.4.2) and let $U^n \in V_h$ be an approximation of $u(t)$ at $t = t_n$ is given by (2.4.17). Then with $u_{0,h} = I_h u_0$ or $P_h u_0$, there exists positive constants $C$, independent of $h_l$ and $k$, such that*

$$\|u(t_n) - U^n\|^2 \leq C \left( \sum_{l=1}^{K} h_l^4 \left[ \|u_0\|^2_{H^2(\Omega_l)} + \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_l)} ds \right] + k^2 \int_0^{t_n} \|u_{tt}\|^2 ds \right). \tag{2.4.18}$$

*and,*

$$\|u(t_n) - U^n\|_X^2 \leq C \left( \sum_{l=1}^{K} h_l^2 \left[ \|u_0\|^2_{H^2(\Omega_l)} + \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_l)} ds \right] + k^2 \int_0^{t_n} \|u_{tt}\|^2 ds \right). \tag{2.4.19}$$

**Proof.** Set

$$u(t_n) - U^n = \rho^n + \theta^n = (u(t_n) - P_h u(t_n)) + (P_h u(t_n) - U^n). \tag{2.4.20}$$

Since the estimates of $\rho^n$ are known from Lemma 2.3.1 at $t = t_n$, it is sufficient to estimate $\theta^n$. Using elliptic projection (2.3.1), (2.4.1)- (2.4.2) at $t = t_n$ and (2.4.17), we obtain

$$(\bar{\partial}_t \theta^n, \chi) + a(\theta^n, \chi) \;=\; (w^n, \chi) \qquad \forall \chi \in V_h, \tag{2.4.21}$$

where

$$\begin{aligned} w^n = \bar{\partial}_t P_h u(t_n) - u_t(t_n) &= -\bar{\partial}_t \rho^n + (\bar{\partial}_t u(t_n) - u_t(t_n)) \\ &= w_1^n + w_2^n. \end{aligned} \tag{2.4.22}$$

Choose $\chi = \theta^n$ in (2.4.21). Note that

$$(\bar{\partial}_t \theta^n, \theta^n) = \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2 + \frac{k}{2}\|\bar{\partial}_t \theta^n\|^2 \geq \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2.$$

Using the coercivity of $a(\cdot,\cdot)$, Cauchy-Schwarz inequality and Young's inequality (1.2.2), we find that

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2 + \alpha \|\theta^n\|_X^2 &\leq \|w^n\| \|\theta^n\| \leq C\|w^n\| \|\theta^n\|_X \\ &\leq \frac{1}{2\alpha}\|w^n\|^2 + \frac{\alpha}{2}\|\theta^n\|_X^2, \end{aligned}$$

and hence,

$$\bar{\partial}_t \|\theta^n\|^2 + \alpha \|\theta^n\|_X^2 \leq C(\alpha)\|w^n\|^2.$$

Using the definition of $\bar{\partial}_t$, we arrive at

$$\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + C(\alpha) k \|w^n\|^2,$$

and hence, by repeated application, we obtain

$$\|\theta^n\|^2 \leq \|\theta^0\|^2 + Ck\Big(\sum_{j=1}^{n} \|w_1^j\|^2 + \sum_{j=1}^{n} \|w_2^j\|^2\Big). \tag{2.4.23}$$

With $u_{0,h} = P_h u_0, \theta(0) = 0$, otherwise we have with $u_{0,h} = I_h u_0$

$$\begin{aligned} \|\theta^0\| = \|P_h u_0 - U^0\| &\leq \|u_0 - I_h u_0\| + \|P_h u_0 - u_0\| \\ &\leq C \sum_{l=1}^{K} h_l^2 \|u_0\|_{H^2(\Omega_l)}. \end{aligned} \tag{2.4.24}$$

Since

$$w_1^j = -\bar{\partial}_t \rho^n = -k^{-1} \int_{t_{j-1}}^{t_j} \rho_t(s)\ ds,$$

we use (2.3.3) to arrive at

$$k \sum_{j=1}^{n} \|w_1^j\|^2 \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\rho_t(s)\|^2 ds \ \leq \ C \sum_{l=1}^{K} h_l^4 \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_l)} ds. \qquad (2.4.25)$$

To estimate $w_2^j$, we apply Taylor series expansion, to find that

$$\begin{aligned} w_2^j = \bar{\partial}_t u(t_j) - u_t(t_j) &= k^{-1}(u(t_j) - u(t_{j-1})) - u_t(t_j) \\ &= -k^{-1} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s) ds \end{aligned}$$

and, hence,

$$k \sum_{j=1}^{n} \|w_2^j\|^2 \leq \sum_{j=1}^{n} \left( \int_{t_{j-1}}^{t_j} |s - t_{j-1}| \|u_{tt}\| ds \right)^2 \leq Ck^2 \int_0^{t_n} \|u_{tt}\|^2 ds. \qquad (2.4.26)$$

Substitute (2.4.24)-(2.4.26) in (2.4.23) to obtain

$$\|\theta^n\|^2 \leq C \left( \sum_{l=1}^{K} h_l^4 \left[ \|u_0\|^2_{H^2(\Omega_l)} + \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_l)}\ ds \right] + k^2 \int_0^{t_n} \|u_{tt}\|^2 ds \right). \qquad (2.4.27)$$

With an application of triangle inequality, (2.4.27), Lemma 2.3.1 and (2.4.11), the estimate (2.4.18) follows. In order to obtain (2.4.19), substitute $\chi = \bar{\partial}_t \theta^n$ in (2.4.21) and then proceed in a similar way to derive an estimate of $\|u(t_n) - U^n\|_X$. This completes the proof of the theorem. ∎

**Remark 2.4.2** *Higher order schemes like Crank-Nicolson scheme and second order backward difference methods can be applied to discretize in time direction. The corresponding error analysis follows closely the analysis in Thomeé [85] and hence, we prefer to skip these results.*

## 2.5 A Mortar finite element method with a Lagrange multiplier

In the previous section, we note that the mortar condition (1.3.7) is imposed on the mortar finite element space which is computationally cumbersome. Instead, in this section, we use Lagrange multiplier method. By doing so, although we avoid imposing mortar condition on the mortar element space, but we obtain a larger system of equations. However, by adopting a Lagrange multiplier method, we also obtain an estimate for the Lagrange multiplier, which is an approximation to $a\frac{\partial u}{\partial n}$ on the interfaces of the subdomains along with the solution which is useful for domain decomposition methods.

In this section, we discuss the mortar method with a Lagrange multiplier for the problem (2.1.1)-(2.1.3).

Now we define Lagrange multiplier space $M$ as follows:

$$M = \{\psi \in \prod_{l=1}^{K} H_D^{-1/2}(\partial\Omega_l) \colon \text{there exists a function } \mathbf{q} \in H_0(\text{div};\Omega) \text{ such that } \psi_k = \mathbf{q}\cdot n_l\}, \tag{2.5.1}$$

and an auxiliary space is defined as:

$$\tilde{M} = \prod_{m=1}^{m_0} H_{00}^{-1/2}(\delta_m). \tag{2.5.2}$$

Multiply the equation (2.1.1) by $v \in X$ and then integrate by parts over $\Omega_l$. An introduction of the flux $\lambda = a\frac{\partial u}{\partial n}$, now yields the following weak formulation of the problem (2.1.1)-(2.1.3) : Find $(u,\lambda)\colon (0,T] \longrightarrow X \times M$ such that for $t \in (0,T]$

$$(u_t, v) + a(u,v) + b(v,\lambda) = \sum_{l=1}^{K}\int_{\Omega_l} fv\ dx \qquad \forall\ v \in X, \tag{2.5.3}$$

$$b(u,\mu) = 0 \qquad \forall \mu \in M, \tag{2.5.4}$$

$$u(0) = u_0, \tag{2.5.5}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, and

$$a(u,v) = \sum_{l=1}^{K}\int_{\Omega_l} a(x)\nabla u_l \cdot \nabla v_l\ dx,$$

and

$$b(v,\mu) = -\sum_{m=1}^{m_0} \int_{\gamma_m \subset \Gamma} \mu[[v]]_{|\gamma_m} \, d\tau.$$

Here, we follow the same notations and definitions which are defined in the earlier sections. Define the space

$$M_h = \prod_{m=1}^{m_0} M^{h_j}(\delta_m) \tag{2.5.6}$$

over all nonmortars. Now, the mortar element formulation with Lagrange multiplier corresponding to (2.2.1)-(2.2.2) is to seek $(u_h, \lambda_h)\colon (0,T] \longrightarrow X_h \times M_h$ such that

$$(u_{h,t}, v_h) + a(u_h, v_h) + b(v_h, \lambda_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h \, dx \qquad \forall v_h \in X_h \tag{2.5.7}$$

$$b(u_h, \mu_h) = 0 \qquad \forall \mu_h \in M_h \tag{2.5.8}$$

$$u_h(0) = u_{0,h}, \tag{2.5.9}$$

where $u_{0,h} \in X_h$ is a suitable approximation of $u_0$ to be defined later. Note that

$$a(u_h, v_h) = \sum_{l=1}^{K} \int_{\Omega_l} a(x) \nabla u_{h_l} \cdot \nabla v_{h_l} \, dx,$$

and

$$b(v_h, \mu_h) = -\sum_{m=1}^{m_0} \int_{\gamma_m \subset \Gamma} \mu_h [v_h]_{|\gamma_m} \, d\tau.$$

For given $u$ and $\lambda$, we now define $P_h^* u \in X_h$ and $\Pi_h^* \lambda \in M_h$ by

$$a(u - P_h^* u, \chi) - b(\chi, \lambda - \Pi_h^* \lambda) = 0 \qquad \forall \chi \in X_h, \tag{2.5.10}$$

and

$$b(u - P_h^* u, \mu_h) = 0 \qquad \forall \mu_h \in M_h. \tag{2.5.11}$$

Note that for a given $u$ and $\lambda$, (2.5.10)-(2.5.11) has a unique solution $(P_h^* u, \Pi_h^* \lambda) \in X_h \times M_h$. Based on the analysis in Theorem 2.8 of [16], we obtain easily the following optimal error estimates. We recall the following result from Chapter 1.

**Lemma 2.5.1** *For any real number $\sigma$, $\frac{3}{2} < \sigma \leq 2$, there exists positive constants $C$, independent of $h_l$, such that for any function $u \in H_0^1(\Omega)$ with $u_{|\Omega_l}$ and $u_{t|\Omega_l} \in H^\sigma(\Omega_l)$,*

$$\sum_{i=0}^{2}\left(\left\|\frac{\partial^i}{\partial t^i}(u - P_h^* u)\right\| + h\left[\left\|\frac{\partial^i}{\partial t^i}(u - P_h^* u)\right\|_X + \left\|\frac{\partial^i}{\partial t^i}(\lambda - \Pi_h^* \lambda)\right\|_M\right]\right) \leq Ch^\sigma \sum_{i=0}^{2}\sum_{l=1}^{K}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{H^\sigma(\Omega_l)}, \quad (2.5.12)$$

*where $h = \max_l h_l$.*

Note that for time derivative we first differentiate (2.5.10)-(2.5.11) with respect to time and then proceed as in Theorem 2.8 of [16] to complete the proof.

## 2.6 Error estimates for the semidiscrete method

In this section, we discuss optimal error estimates for the error $u - u_h$ and $\lambda - \lambda_h$.

**Theorem 2.6.1** *Let $u$ and $u_h$ be the solutions of (2.5.3)-(2.5.5) and (2.5.7)-(2.5.9), respectively. Further, let $u|_{\Omega_i}, u_t|_{\Omega_i} \in H^2(\Omega_i)$ for $1 \leq i \leq K$. Then with $u_{0,h} = I_h u_0$ or $P_h^* u_0$ there exists positive constants $C$, independent of $h_l$, such that for $t \in (0, T]$,*

$$\|(u - u_h)(t)\| \leq C \sum_{l=1}^{K} h_l^2 \left[\|u_0\|_{H^2(\Omega_l)} + \|u_t\|_{L^2(0,T;H^2(\Omega_l))}\right] \quad (2.6.1)$$

*and,*

$$\|(u - u_h)(t)\|_X \leq C \sum_{l=1}^{K} h_l \left[\|u_0\|_{H^2(\Omega_l)} + \|u(t)\|_{H^2(\Omega_l)} + \|u_t\|_{L^2(0,T;H^1(\Omega_l))}\right]. \quad (2.6.2)$$

**Proof.** Using $P_h^* u$ and $\Pi_h^* \lambda$, we write

$$u - u_h = (u - P_h^* u) + (P_h^* u - u_h) = \rho + \theta \quad (2.6.3)$$

and

$$\lambda - \lambda_h = (\lambda - \Pi_h^* \lambda) + (\Pi_h^* \lambda - \lambda_h) = \eta + \xi. \quad (2.6.4)$$

From (2.5.3)-(2.5.4), (2.5.7)-(2.5.8), (2.5.10) and (2.5.11), we obtain

$$(\theta_t, \chi) + a(\theta, \chi) + b(\chi, \xi) = -(\rho_t, \chi) \qquad \forall \chi \in X_h \tag{2.6.5}$$

$$b(\theta, \mu_h) = 0 \qquad \forall \mu_h \in M_h. \tag{2.6.6}$$

Substituting $\chi = \theta$ in (2.6.5), $\mu_h = \xi$ in (2.6.6), using Lemma 2.5.1 and proceeding as in the proof of Theorem 2.4.1 we derive the $L^2$-error estimate (2.6.1). In order to derive (2.6.2), we substitute $\chi = \theta_t$ in (2.6.5), differentiate (2.6.6) with respect to $t$ and set $\mu_h = \xi$ to obtain

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}a(\theta, \theta) = -(\rho_t, \theta_t).$$

To complete the rest of the proof, proceed as in the proof of Theorem 2.4.1. ■

**Remark 2.6.1** *As a consequence of Theorem 2.6.1, we obtain*

$$\|\theta_t\|^2_{L^2(0,T;L^2(\Omega))} \leq C\left[\|u_{0,h} - P_h^* u_0\|_X^2 + \int_0^t \sum_{l=1}^K h_l^2 \|u_t\|^2_{H^1(\Omega_l)} ds\right]. \tag{2.6.7}$$

*Further, differentiating (2.6.5), (2.6.6), we find that*

$$(\theta_{tt}, \chi) + a(\theta_t, \chi) + b(\chi, \xi_t) = -(\rho_{tt}, \chi) \qquad \forall \chi \in X_h \tag{2.6.8}$$

$$b(\theta_t, \mu_h) = 0 \qquad \forall \mu_h \in M_h. \tag{2.6.9}$$

*Substituting $\chi = \theta_t$ in (2.6.8), $\mu_h = \xi_t$ in (2.6.9) and multiplying both sides of (2.6.8) by $t$, we arrive at*

$$t(\theta_{tt}, \theta_t) + ta(\theta_t, \theta_t) = -t(\rho_{tt}, \theta_t),$$

*and hence*

$$\frac{1}{2}\frac{d}{dt}(t\|\theta_t\|^2) + ta(\theta_t, \theta_t) = -t(\rho_{tt}, \theta_t) + \frac{1}{2}\|\theta_t\|^2. \tag{2.6.10}$$

*Integrating (2.6.10) from 0 to $t$ with respect to $t$, using coercivity of $a(\cdot,\cdot)$ and Cauchy-Schwarz inequality, we obtain*

$$t\|\theta_t\|^2 + \alpha\int_0^t s\|\theta_t\|_X^2\, ds \leq C(\alpha)\left(\int_0^t s\|\rho_{tt}\|^2\, ds + \int_0^t \|\theta_t\|^2\, ds\right). \tag{2.6.11}$$

*Using (2.6.7), we now find that*

$$\|\theta_t\|^2 \leq \frac{C(\alpha)}{t}\sum_{l=1}^{K} h_l^2 \left[\|u_0\|^2_{H^2(\Omega_l)} + \int_0^t \left(\|u_t\|^2_{H^1(\Omega_l)}ds + s\|u_{tt}\|^2_{H^1(\Omega_l)}\right) ds\right]. \tag{2.6.12}$$

**Error estimate for the Lagrange multiplier**

For the error bound in Lagrange multiplier, we need the inf-sup (well known as Ladyzhenskaya-Babuška-Brezzi) condition for the bilinear form $b(\cdot,\cdot)$. Since it is difficult to prove the inf-sup condition in $X$, following [16], we now define an auxiliary space $X_{00}$ as

$$X_{00} = \{v \in X\colon [[v]]_{|\delta_m} \in H^{1/2}_{00}(\delta_m) \forall \delta_m \subset \Gamma\} \tag{2.6.13}$$

with the norm

$$\|v\|^2_{X_{00}} = \|v\|^2_X + \sum_{\delta_m \subset \Gamma} \|[[v]]\|^2_{H^{1/2}_{00}(\delta_m)}.$$

Now define the new approximation spaces as

$$\tilde{X}_h = X_h \cap X_{00},$$

and

$$\tilde{M}_h = M_h \cap \tilde{M},$$

where,

$$\tilde{M} = \prod_{m=1}^{m_0} H^{-1/2}_{00}(\delta_m).$$

Now, consider the bilinear form $\tilde{b}(v_h, \mu_h)$ defined over $\tilde{X}_h \times \tilde{M}_h$ as below:

$$\tilde{b}(v_h, \mu_h) = -\sum_{m=1}^{m_0} \int_{\gamma_m \subset \Gamma} \mu_h [[v_h]]_{|\gamma_m}\, d\tau.$$

With this modification the discrete spaces satisfy the inf-sup condition. Below, we present a Proposition on inf-sup condition satisfied by $\tilde{b}(\cdot,\cdot)$. For a proof, see (Proposition 2.6, [16]).

**Proposition 2.6.1** *The bilinear form $\tilde{b}(\cdot,\cdot)$ verifies the following uniform inf-sup condition over $\tilde{X}_h \times \tilde{M}_h$: there exists a constant $c_0 > 0$ independent of $h_l$, $1 \le l \le K$ such that*

$$\sup_{v_h \in \tilde{X}_h} \frac{\tilde{b}(v_h, \mu_h)}{\|v\|_{X_{00}}} \ge c_0 \|\mu_h\|_{\tilde{M}} \qquad \forall \mu_h \in \tilde{M}_h. \tag{2.6.14}$$

In the following theorem, we discuss the error estimate for the Lagrange multiplier.

**Theorem 2.6.2** *Assume that for $t \in (0,T]$, $u(t) \in H_0^1(\Omega)$, $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^2(\Omega_l)$ and $u_{tt}(t)_{|\Omega_l} \in H^1(\Omega_l)$. Let $u_{0,h} = I_h u_0$ or $P_h^* u_0$. Then there exists a positive constant $C$ which is independent of $h_l$, $1 \le l \le K$, such that for $t \in (0,T]$,*

$$\begin{aligned} \|(\lambda - \lambda_h)(t)\|_{\tilde{M}}^2 \;\le\; & \frac{C(c_0,\alpha)}{t} h^2 \sum_{l=1}^{K} \Big( \|u_0\|_{H^2(\Omega_l)}^2 + \int_0^t \|u_t\|_{H^1(\Omega_l)}^2 ds \\ & + \int_0^t s \|u_{tt}\|_{H^1(\Omega_l)}^2 ds \Big) + Ch^2 \sum_{l=1}^{K} \|u\|_{H^2(\Omega_l)}^2. \end{aligned} \tag{2.6.15}$$

**Proof.** Since $\tilde{X}_h \subset X_h$, from the error equation (2.6.5), we can write for all $\chi \in \tilde{X}_h$,

$$\tilde{b}(\chi, \xi) = -(\rho_t, \chi) - (\theta_t, \chi) - a(\theta, \chi). \tag{2.6.16}$$

A use of the inf-sup condition (2.6.14) and Cauchy-Schwarz inequality leads to

$$c_0 \|\xi\|_{\tilde{M}} \le \|\rho_t\| + \|\theta_t\| + \|\theta\|_X. \tag{2.6.17}$$

Now

$$\|\lambda - \lambda_h\|_{\tilde{M}} \le \|\eta\|_{\tilde{M}} + \|\xi\|_{\tilde{M}}. \tag{2.6.18}$$

Using (2.6.17), we obtain

$$\|\lambda - \lambda_h\|_{\tilde{M}} \le \|\eta\|_{\tilde{M}} + \frac{1}{c_0}\{\|\rho_t\| + \|\theta_t\| + \|\theta\|_X\}. \tag{2.6.19}$$

From Lemma 2.5.1, (2.6.12) and Theorem 2.6.1, the rest of the proof follows. ■

**Remark 2.6.2**

1. *We can also easily obtain an estimate for* $\|\lambda - \lambda_h\|_{L^2(0,T;\tilde{M})}$ *from the equations (2.6.17), (2.6.18) and (2.6.7). Moreover, choosing* $u_h(0) = P_h^* u(0)$, *we obtain a super-convergent result in* $\|\xi\|_{L^2(0,T;\tilde{M})}$. *Similarly, we can also obtain a super convergent result for* $\|\xi(t)\|_{\tilde{M}}$.

2. *Note that from Theorem 4.1, Theorem 4.2 and Theorem 6.1 we obtain* $O(h)$ *estimate with respect to* $H^1$*-norm and* $O(h^2)$ *estimate in* $L^2$*-norm for both the semidiscrete and fully-discrete cases. These estimates are optimal as in the case of elliptic problems. Also, Theorem 6.2 yields* $O(h)$ *estimate for the Lagrange multiplier as in the elliptic case.*

3. *The analysis can be carried out in an exactly similar manner for the case when the coefficient* $a(x)$ *is replaced by* $a(x,t)$ *with* $0 < \alpha_0 \leq a(x,t) \leq M \ \forall \ x \in \bar{\Omega}, t \in [0,T]$. *Numerical experiments illustrating this has been included in the next section.*

4. *Parabolic equations with discontinuous coefficients can occur in many physical problems, such as in material sciences, fluid dynamics, where the original domain of interest consists of materials with the different conductivities, permeabilities or densities, which lead to discontinuity of the coefficients across the interfaces. Note that, our analysis is also valid when the coefficient is discontinuous along the subdomain interface but is piecewise smooth in each subdomain with the condition that the coefficient is bounded below and above by positive constants.*

## 2.7 Numerical Experiments

In this section, we discuss the implementation procedure for the discrete problem (2.4.17) for the geometrically conforming case. For construction of the nodal basis functions for the mortar element space we refer to [25], [33], [66]. Here, the implementation is done using MATLAB. The basis functions are defined with the help of the following sets of nodes:

- the nodes which lie in the interior of the subdomains,
- the nodes which lie in the interior to the mortars, and

- the nodes of vertices of subdomains except those on $\partial\Omega$.

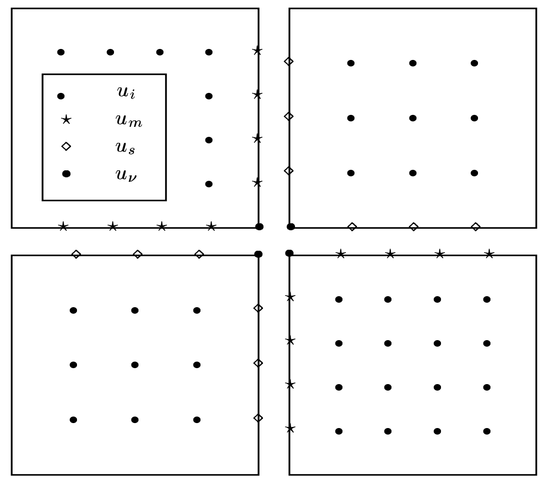



Figure 2.1: Degrees of Freedom over the subdomains

Now we construct the canonical basis functions involved with each of the nodal points defined above. The functions $\varphi_i^l$ corresponding to the interior nodes $x_i^l, 1 \leq l \leq K, 1 \leq i \leq K_l$ are canonical nodal basis functions as in the conforming finite element discretization, where $K$ is the number of non-overlapping subdomains and $K_l$ is the total number of interior nodes in the subdomain $\Omega_l$. Here, we have taken piecewise linear polynomial functions on each triangle $T \in \mathcal{T}_h(\Omega_l),\ 1 \leq l \leq K$.

The basis functions corresponding to the nodes interior to the mortars are defined as follows. The function $\varphi_i^m$ associated with the node $x_i^m, 1 \leq m \leq m_0, 1 \leq i \leq m_1$, where $m_0$ is the total number of mortar edges and $m_1$ is the total number of interior nodal points in the mortar edge is a continuous piecewise linear function which takes value one at $x_i^m$, zero at $x_i^l, 1 \leq l \leq K, 1 \leq i \leq K_l$ and at the nodes of vertices of subdomains. The values of this function $\varphi_j^s$ at the interiors of the nonmortars are determined by the mortar condition with zero values at the end points of the non-mortars. From this it follows that the interior nodes of the nonmortar sides are not associated with the genuine degrees of freedom in the finite element space. From this point of view, we present, here, the matrix formulation of the mortar conditions as follows. Let $\delta_m$ be an arbitrary nonmortar side, and let $u_s$ be the vector of the interior nodal values of $u_h$ on $\delta_m$. For the sake of convenience, assume that the mesh is uniform on $\delta_m$ with mesh size $h_j$. Further, let $u_m$ be the vector of the values of $u_h$ at the interior nodal points of the other side $\gamma_m$, which is a mortar side. Then $u_s$ is uniquely determined by $u_m$ using the mortar conditions and this can be written in a matrix form as :

$$M_s u_s - M_m u_m = 0, \tag{2.7.1}$$

where $M_s$ and $M_m$ are given by $(m_s)_{ij} = \int_{\delta_m} \varphi_j^s \psi_{h_i} d\tau$ and $(m_m)_{ij} = \int_{\delta_m} \varphi_j^m \psi_{h_i} d\tau$, respectively. Here, $\psi_{h_i}$ are the nodal basis functions for $M^{h_j}(\delta_m)$, $\varphi_j^s$ and $\varphi_j^m$ denote the basis functions for $W^h(\Gamma)$ associated with the nodes corresponding to nonmortar and mortar sides, respectively. Since $\mathcal{M}_s$ is a tridiagonal, symmetric and positive definite matrix, we can write $u_s$ as :

$$u_s = M_s^{-1} M_m u_m. \tag{2.7.2}$$

Finally, we define the basis functions associated with vertices of subdomains in the following way. Let $S$ denote the set of vertices of the subdomains which are associated with degrees of freedom of $V_h$. Each crosspoint of $\Gamma$ corresponds to several nodes of $S$ and, hence, the mortar finite element functions are typically multivalued at the cross points of the subdomain and contribute one degree of freedom for each of the subregion that coincides at that point; which are in same physical position, but are assigned to different subregions. Let $\Phi_{y_\nu}$ be the basis function associated with a vertex $y_\nu \in S$ of $\Omega_i$ which takes the value one at $y_\nu$ and zero at all other vertices of $S$ and all vertices interior to all mortars and nonmortars.

We first assume the vertex $y_\nu$ is a common end of two mortars $\gamma_\nu$ and $\gamma_m$, $\Phi_{y_\nu}$ restricted to $\gamma_\nu$ and $\gamma_m$ is the standard nodal basis function corresponding to $y_\nu$, i.e., one at $y_\nu$ and zero at the remaining nodes of both the mortars. The basis function $\Phi_{y_\nu}$ is determined by the mortar conditions (1.3.7) on the nonmortars $\delta_\nu = \gamma_\nu$ and $\delta_m = \gamma_m$ with zero values at the ends of $\delta_\nu$ and $\delta_m$, respectively. If $y_\nu$ is a common end of two nonmortars $\delta_p$ and $\delta_q$, then $\Phi_{y_\nu}$ restricted to the mortars $\gamma_p = \delta_p$ and $\gamma_q = \delta_q$ is zero and on the nonmortars $\delta_p$ and $\delta_q$ is determined by the mortar condition with one at $y_\nu$ and zero at the other ends of the $\delta_p$ and $\delta_q$. Finally, suppose $y_\nu$ is a common end of a mortar $\gamma_r$ and a nonmortar $\delta_s$, then $\Phi_{y_\nu}$ is defined on the mortar $\gamma_r$ as in the first case and on the nonmortar $\delta_s$ as in the second case. In all cases, $\Phi_{y_\nu}$ is defined as zero on the remaining mortars and nonmortars.

With these above sets of basis functions, the fully discrete problem (2.4.17) can be expressed in the matrix form as :

$$(B + kA)\alpha^n = B\alpha^{n-1} + kF(t_n), \qquad n \geq 1, \tag{2.7.3}$$

with $\alpha = (u_i^l, u_i^m, u_\nu)^T$.
Here $A$, $B$ and $F$ will have the following forms:

$$A = \begin{pmatrix} A_{ii} & \mathcal{A}_{im} & A_{i\nu} \\ A_{mi} & \mathcal{A}_{mm} & A_{m\nu} \\ A_{\nu i} & \mathcal{A}_{\nu m} & A_{\nu\nu} \end{pmatrix}, \tag{2.7.4}$$

where

$$\begin{aligned}
\mathcal{A}_{im} &= A_{is}M_s^{-1}M_m + A_{im} \\
\mathcal{A}_{mm} &= A_{ms}M_s^{-1}M_m + A_{mm} \\
\mathcal{A}_{nm} &= A_{sm}M_s^{-1}M_m + A_{nm} \\
A_{ii} &= \{a(\varphi_i^{(l)}, \varphi_j^{(l)})\} \qquad x_i^l, x_j^l \in \mathcal{T}_h(\Omega_l),\ 1 \leq l \leq K, \\
A_{im} &= \{a(\varphi_i^{(l)}, \varphi_m)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_m \in \gamma_m, \\
A_{is} &= \{a(\varphi_i^{(l)}, \varphi_s)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_s \in \delta_m, \\
A_{in} &= \{a(\varphi_i^{(l)}, \Phi_n)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_n \in \nu.
\end{aligned}$$

Further,

$$B = \begin{pmatrix} B_{ii} & \mathcal{B}_{im} & B_{in} \\ B_{mi} & \mathcal{B}_{mm} & B_{mn} \\ B_{ni} & \mathcal{B}_{nm} & B_{nn} \end{pmatrix}, \tag{2.7.5}$$

where

$$
\begin{aligned}
\mathcal{B}_{im} &= B_{is} M_s^{-1} M_m + B_{im} \\
\mathcal{B}_{mm} &= B_{ms} M_s^{-1} M_m + B_{mm} \\
\mathcal{B}_{nm} &= B_{sm} M_s^{-1} M_m + B_{nm} \\
B_{ii} &= \{(\varphi_i^{(l)}, \varphi_j^{(l)})\} \qquad x_i^l, x_j^l \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le K, \\
B_{im} &= \{(\varphi_i^{(l)}, \varphi_m)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_m \in \gamma_m, \\
B_{is} &= \{B(\varphi_i^{(l)}, \varphi_s)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_s \in \delta_m, \\
B_{in} &= \{(\varphi_i^{(l)}, \Phi_n)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_n \in \nu
\end{aligned}
$$

and

$$
F = \begin{pmatrix} F_i & \mathcal{F}_m & F_\nu \end{pmatrix}^T .
$$

where

$$
\begin{aligned}
\mathcal{F}_i &= F_s M_s^{-1} M_m + F_m \\
F_i &= \{(f_i^{(l)}, \varphi_j^{(l)})\} \qquad x_i^l, x_j^l \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le K, \\
F_m &= \{(f_i^{(l)}, \varphi_m)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_m \in \gamma_m, \\
F_s &= \{(f_i^{(l)}, \varphi_s)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l), x_s \in \delta_m, \\
F_n &= \{(f_i^{(l)}, \Phi_n)\} \qquad x_i^l \in \mathcal{T}_h(\Omega_l) x_n \in \nu.
\end{aligned}
$$

### 2.7.1 Numerical Result

Choose the following parabolic initial boundary value problem on the square domain $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary condition:

$$
\begin{aligned}
u_t - \nabla \cdot (a(x) \nabla u) &= f \ \text{ in } \ \Omega \times (0,1], \\
u(x,t) &= 0 \ \text{ on } \ \partial\Omega \times (0,1], \\
u(0) &= u_0 \ \text{ in } \ \Omega.
\end{aligned}
$$

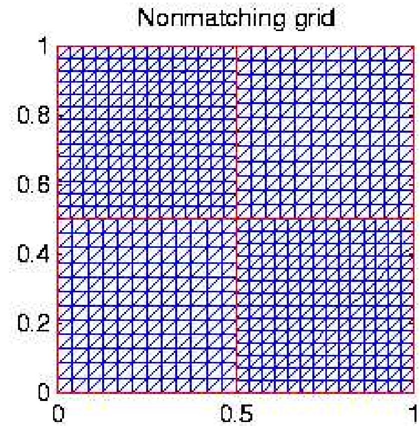



Figure 2.2: Independent discretizations over subdomains

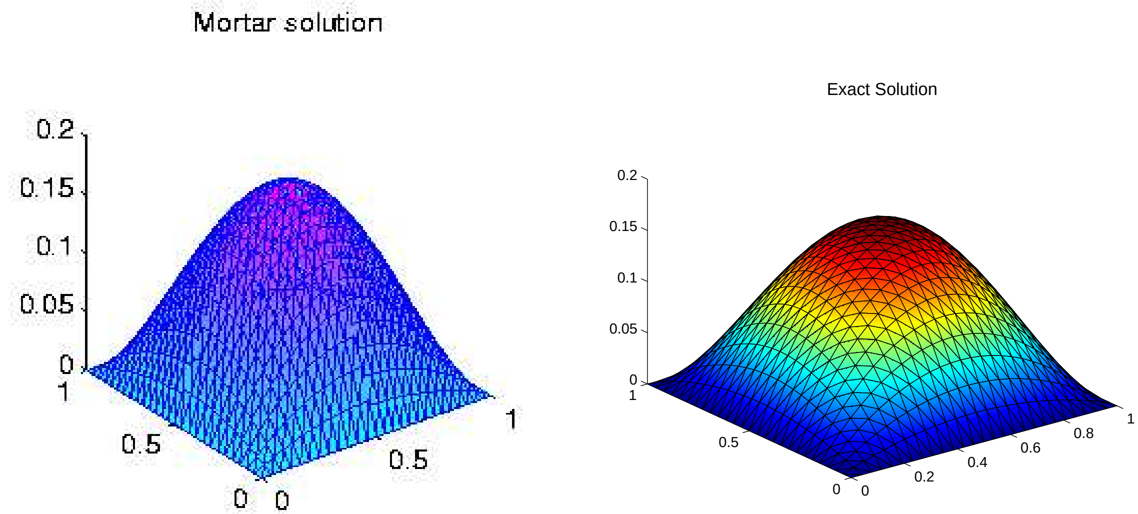
We divide $\Omega$ into four equal subdomains $\Omega_l, 1 \leq l \leq 4$ (See Figure 2.2). Each subdomain is triangulated into triangular elements with different mesh sizes $h_i$. In Figure 2.2, $h_1 = 1/12, h_2 = 1/14, h_3 = 1/14, h_4 = 1/12$ in the four subdomains $\Omega_l, 1 \leq l \leq 4$, respectively.

In our first example we take $u_0 = 0$, $g = 0$ with constant coefficient $a(x) = 1$. We choose $f$ such that the exact solution is $u(x,t) = x(x-1)y(y-1)e^t$. Choosing the time step parameter $k = O(h^2)$, we obtain the mortar solution. The mortar and exact solutions at $t = 1$ are shown in Figure 2.3(a) and Figure 2.3(b) respectively.

Table 2.1: Order of convergence $p$ w.r.t. space variable $h$ and $q$ w.r.t. time variable $k$

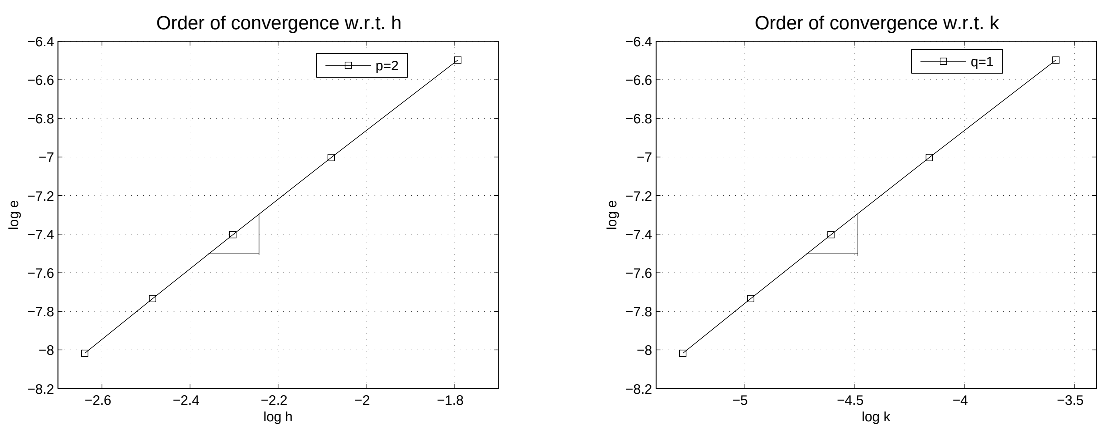
| $(h_1, h_2, h_3, h_4)$ | $h = \max_l h_l$ | $k$ | Error $e$ | $p$ | $q$ |
|---|---|---|---|---|---|
| $(\frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \frac{1}{6})$ | 1/6 | 1/36 | 0.00150650 | | |
| $(\frac{1}{8}, \frac{1}{10}, \frac{1}{10}, \frac{1}{8})$ | 1/8 | 1/64 | 0.00090908 | 1.7558 | 0.8779 |
| $(\frac{1}{10}, \frac{1}{12}, \frac{1}{12}, \frac{1}{10})$ | 1/10 | 1/100 | 0.00060966 | 1.7905 | 0.8952 |
| $(\frac{1}{12}, \frac{1}{14}, \frac{1}{14}, \frac{1}{12})$ | 1/12 | 1/144 | 0.00043774 | 1.8170 | 0.9085 |
| $(\frac{1}{14}, \frac{1}{16}, \frac{1}{16}, \frac{1}{14})$ | 1/14 | 1/196 | 0.00032975 | 1.8377 | 0.9189 |

The order of convergence at $t = 1$ for the error $e = (u - u_h)$ in $L^2$ norm '$p$' with respect to the space variable $h$ and '$q$' with respect to the time step $k$ has been computed in Table 2.1. Figure 2.4(a) and 2.4(b) shows the computed order of convergence with respect to $h$ and $k$, respectively, for $\|u - u_h\|$ in the log-log scale. The computed order of convergence




(a) Mortar Solution (b) Exact Solution

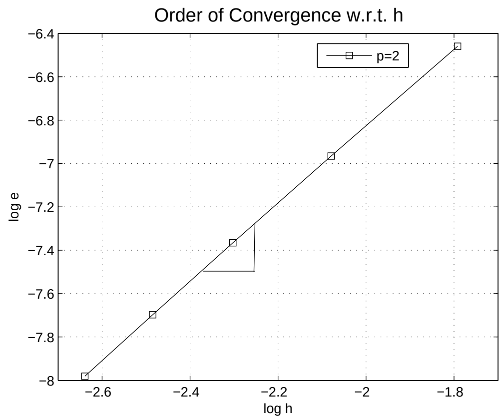
Figure 2.3: Solution Figures



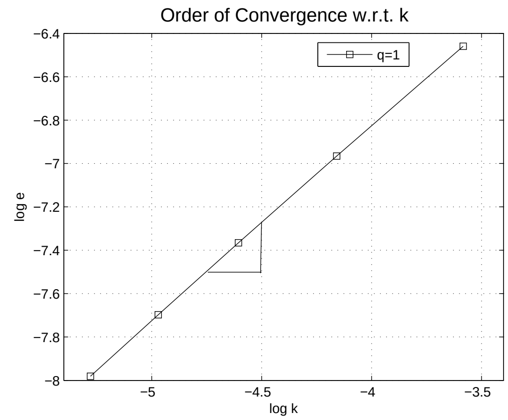

(a) Order of Convergence w.r.t. discretization parameter $h$

(b) Order of Convergence w.r.t. $k$

Figure 2.4: Order of Convergence with constant coefficient




(a) Order of Convergence w.r.t. discretization parameter $h$ with variable coefficient




(b) Order of Convergence w.r.t. time parameter $k$ with variable coefficient

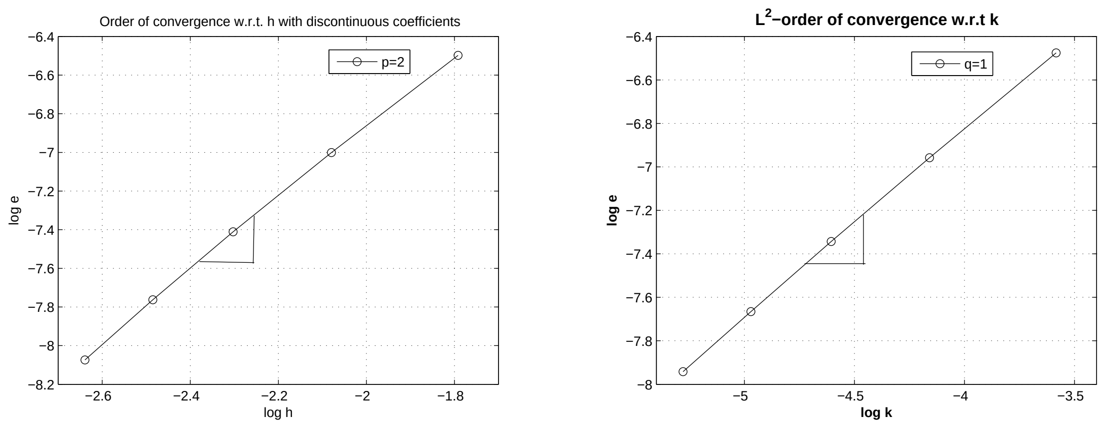
Figure 2.5: Order of Convergence for variable coefficient

matches with the theoretical order of convergence derived in Theorem 2.4.2. In the second example, we take $u_0 = 0$, $g = 0$ with variable coefficient $a(x,t) = e^t$. In this case also, we choose $f$ such that the exact solution is $u(x,t) = x(x-1)y(y-1)e^t$. Figure 2.5(a) and 2.5(b) shows the computed order of convergence with respect to $h$ and $k$ respectively for $\|u - u_h\|$ in the log-log scale. We conducted the experiment by taking time step parameter $k = O(h^2)$ corresponding to space discretization parameters $h = 1/6, 1/8, 1/10, 1/12, 1/14$.

In the third example, with $u_0 = 0$, $g = 0$, we consider discontinuous coefficients along the common interfaces of subdomain. We take the coefficients $(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 10, 10, 1)$. We choose $f$ in such a way that the exact solution is $u(x,t) = x(x-1)y(y-1)e^t$. The order of convergence at $t = 1$ for the error $e = (u - u_h)$ in $L^2$ norm '$p$' with respect to the space variable parameter $h$ and '$q$' with respect to the time parameter $k$ has been computed in Table 2.2. Figure 2.6(a) and 2.6(b) shows the computed order of convergence with respect to $h$ and $k$, respectively, for $\|u - u_h\|$ in the log-log scale. The computed result illustrates the validity of our result as we stated in Remark 2.6.2 also in the case with discontinuous coefficients across the interfaces.

Table 2.2: Order of convergence $p$ w.r.t. space variable $h$ and $q$ w.r.t. time variable $k$ in case of discontinuous coefficients

| $(h_1, h_2, h_3, h_4)$ | $h = \max_l h_l$ | $k$ | Error $e$ | $p$ | $q$ |
|---|---|---|---|---|---|
| $(\frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \frac{1}{6})$ | 1/6 | 1/36 | 0.0015066 | | |
| $(\frac{1}{8}, \frac{1}{10}, \frac{1}{10}, \frac{1}{8})$ | 1/8 | 1/64 | 0.00091132 | 1.7475 | 0.8381 |
| $(\frac{1}{10}, \frac{1}{12}, \frac{1}{12}, \frac{1}{10})$ | 1/10 | 1/100 | 0.00060446 | 1.8399 | 0.8640 |
| $(\frac{1}{12}, \frac{1}{14}, \frac{1}{14}, \frac{1}{12})$ | 1/12 | 1/144 | 0.00042524 | 1.9289 | 0.8831 |
| $(\frac{1}{14}, \frac{1}{16}, \frac{1}{16}, \frac{1}{14})$ | 1/14 | 1/196 | 0.00031147 | 2.0199 | 0.8975 |




(a) Order of Convergence w.r.t. discretization parameter $h$

(b) Order of Convergence w.r.t. $k$

Figure 2.6: Order of Convergence with discontinuous coefficients

# Chapter 3

# Nitsche Mortaring Element Method

## 3.1 Introduction

In our earlier chapters, we have discussed mortar element methods with and without Lagrange multipliers which fall under the category of direct methods. While in the standard mortar element method, the mortar condition is imposed on the finite element space, in the mortar element method with Lagrange multipliers, the constraint on the space is relaxed by imposing it in the formulation. One of the drawbacks of the method is that the stability of the method is guaranteed if the discrete spaces corresponding to the primal variable and the Lagrange multiplier satisfy the "*discrete LBB condition*."

In order to alleviate this problem, *stabilized multiplier techniques* or simply *Nitsche's method* [82] is used in the literature. This was originally introduced for solving Dirichlet problems without enforcing the boundary condition in the finite element spaces. Nitsche has introduced penalty term on the boundary to derive optimal error estimates. In [15, 46, 82], Nitsche's technique has been extended to mortar element method and a penalty term involving jump on the subdomain interfaces is added in the original bilinear forms of the problem to improve stability.

In this chapter, we have proposed *stabilized mortar finite element methods* containing a small penalty parameter for both elliptic and parabolic problems and have discussed *a priori* error bounds.

A brief outline of this chapter is as follows. In Section 3.2, we formulate the elliptic interface problem and in Section 3.3, we introduce Nitsche's mortaring element method with a perturbation. In Section 3.4, we extend the method to parabolic initial and boundary value problems and analyze the error estimates for both semidiscrete and fully discrete schemes. Finally, Section 3.5 deals with some numerical experiments to support our theoretical results.

## 3.2 Elliptic Interface Problem

In this section, we introduce an elliptic interface problem. After introducing Lagrange multiplier space, a saddle point formulation is discussed. Then a continuous perturbed saddle point problem which forms a basis for the Nitsche mortaring method with perturbation is proposed and analyzed.

We now consider a second order model problem with discontinuous coefficients. Let $\Omega \subset \mathcal{R}^2$ be a bounded convex domain with boundary $\partial\Omega$. We consider only the case where the domain $\bar{\Omega}$ is subdivided into two non-overlapping, convex and polygonal subdomains $\Omega_1$ and $\Omega_2$, i.e., $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. Denote the common interface as $\partial\bar{\Omega}_1 \cap \partial\bar{\Omega}_2 = \Gamma$ and $\Gamma_i = \partial\Omega_i \cap \Gamma$. Further, let $n_i$ be the unit normal oriented from $\Omega_i$ towards $\Omega_j,\ 1 \le i < j \le 2$ such that $n := n_1 = -n_2$ (See, Figure 3.1). Now consider the following elliptic interface problem: Given $f \in L^2(\Omega)$, find $u_i,\ i = 1, 2$, such that

$$
\begin{aligned}
-\nabla \cdot (\beta_i(x)\nabla u_i) &= f \ \text{ in } \ \Omega_i, && (3.2.1)\\
u_i &= 0 \ \text{ on } \ \partial\Omega \cap \partial\Omega_i, && (3.2.2)\\
[[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma &= 0 \quad \text{along } \ \Gamma, && (3.2.3)
\end{aligned}
$$

where $\beta_{|\Omega_i} = \beta_i$, is discontinuous along the interface $\Gamma$, but is piecewise smooth in each subdomain $\Omega_i$, that is $\beta_i$ is smooth in each subdomain. Further, we assume that $\beta$ or each $\beta_i$ is bounded below by a positive constant say $\alpha_0$ and bounded above by a positive constant $\alpha_1$. The problem (3.2.1)-(3.2.3) has unique solution in $H^2(\Omega)$ by [55]. Along the interface, $\Gamma$ we denote

$$[[v]]_\Gamma = (v_1 - v_2)_{|\Gamma}$$

for the jump, where $v_i = v_{|\Omega_i}$ and

$$\{v\} = \frac{1}{2}v_1 + \frac{1}{2}v_2$$

for the average.

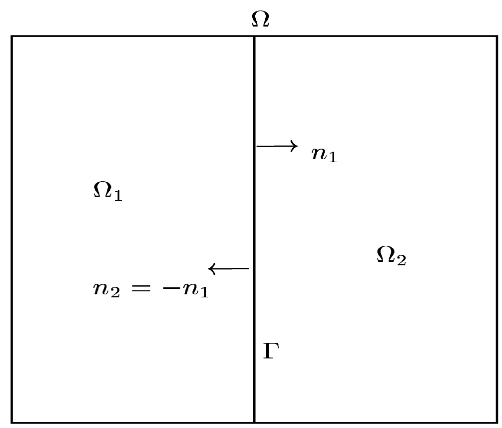



Figure 3.1: $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $n_1$ and $n_2$ represents the outward normal components.

With the notations described above, we note that

$$\left\{\frac{\partial v}{\partial n}\right\} = \frac{1}{2}\frac{\partial v_1}{\partial n} + \frac{1}{2}\frac{\partial v_2}{\partial n} = \frac{1}{2}\frac{\partial v_1}{\partial n_1} - \frac{1}{2}\frac{\partial v_2}{\partial n_2}$$

and hence from (3.2.3),

$$\left\{\beta\frac{\partial u}{\partial n}\right\} = \beta_1\frac{\partial u_1}{\partial n_1} = -\beta_2\frac{\partial u_2}{\partial n_2}.$$

Now, let

$$M = H_{00}^{-1/2}(\Gamma),$$

and

$$X = \{v \in L^2(\Omega)\colon v_{|\Omega_l} \in H_D^1(\Omega_l),\ l = 1, 2\}.$$

Introducing the Lagrange multiplier $\lambda = \beta_1\dfrac{\partial u_1}{\partial n_1} = -\beta_2\dfrac{\partial u_2}{\partial n_2}$, the saddle point formulation for the problem (3.2.1)-(3.2.3) is to find $(u, \lambda) \in X \times M$ such that

$$a(u, v) + b(v, \lambda) = l(v) \qquad \forall v \in X, \tag{3.2.4}$$

$$b(u, \mu) = 0 \qquad \forall \mu \in M, \tag{3.2.5}$$

where

$$a(u,v) = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_i \cdot \nabla v_i dx, \tag{3.2.6}$$

and

$$b(v,\lambda) = < \lambda, [[v]] >_\Gamma, \qquad l(v) = \sum_{i=1}^{2} \int_{\Omega_i} f v dx. \tag{3.2.7}$$

Here $< \cdot, \cdot >_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. For the existence of a unique solution $(u,\lambda) \in X \times M$ of (3.2.4)-(3.2.5), we refer to [16, 31, 73].

## 3.3 Nitsche's Mortaring method

In each subdomain $\Omega_i$, we associate a regular triangulation $\mathcal{T}_h(\Omega_i)$ consisting of elements of different mesh sizes $h_i$, i.e.,

$$\bar{\Omega}_i = \bigcup_{T \in \mathcal{T}_h(\Omega_i)} \bar{T}.$$

Here, the discretization parameter is $(h_1, h_2)$ over $\Omega_1$ and $\Omega_2$, where $h_i = \max_{\mathcal{T}_h(\Omega_i)} h_T$ and $h_T = \text{diam } T$. Once the triangulation $\mathcal{T}_h(\Omega_i)$ is chosen over each $\Omega_i$, the finite element subspaces in the subdomains and on the interface can be defined. We choose locally the finite element method that is best suited to the local properties of the solution. Let us assume that we work with the simple generic case of linear finite elements. We now introduce the space for $i = 1, 2$

$$X_h(\Omega_i) \;=\; \{v_{i,h} \in C(\bar{\Omega}_i) \colon v_{i,h|_{\partial\Omega\cap\partial\Omega_i}} = 0,\ v_{i,h|_T} \in P_1(T)\ \forall\ T \in \mathcal{T}_h(\Omega_i)\},$$

where $P_1(T)$ is the set of all linear polynomials over the triangle $T$ in $\mathcal{T}_h(\Omega_i)$. The global finite element approximation $X_h(\Omega)$ consists of functions whose restriction over each $\Omega_i$ belongs to $X_h(\Omega_i)$ and is defined as

$$X_h(\Omega) \;=\; \{v_h \in L^2(\Omega) \colon v_{h|_{\partial\Omega}} = 0,\ v_{h|_{\Omega_i}} \in X_h(\Omega_i) \ \ i = 1, 2\}.$$

Let $W^h(\Gamma_i)$ be the restriction of $X_h(\Omega_i)$ to $\Gamma_i = \partial\Omega_i \cap \Gamma$. Since the triangulations on two adjacent subdomains are independent, the interfaces $\Gamma_1$ and $\Gamma_2$ are provided with two different and independent ($1D$) triangulations $\mathcal{T}_h(\Gamma_1)$, $\mathcal{T}_h(\Gamma_2)$ and correspondingly two different spaces $W^h(\Gamma_1)$ and $W^h(\Gamma_2)$. The natural choice for the multiplier space over the common interface $\Gamma$ for our purpose is either $W^h(\Gamma_1)$ or $W^h(\Gamma_2)$ with the mesh parameter $h_{e_1}$ and $h_{e_2}$. For our convenience, let us choose $W^h(\Gamma)$ to be $W^h(\Gamma_2)$. For the analysis purpose, we also define

$$h = max\{h_T, h_e \colon T \in \mathcal{T}_h(\Omega_i), e \in \mathcal{T}_h(\Gamma_i), i = 1, 2\}.$$

The *Nitsche's mortar finite element method* is to find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_h \cdot \nabla v_h \; dx + \int_\Gamma \lambda_h [[v_h]] \; d\tau = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \; dx \quad \forall v_h \in X_h, \tag{3.3.1}$$

$$\int_\Gamma [[u_h]] \mu_h \; d\tau - \epsilon \int_\Gamma \lambda_h \mu_h \; d\tau = 0 \qquad \forall \mu_h \in W^h(\Gamma), \tag{3.3.2}$$

where, $\epsilon$ is a suitably chosen penalty parameter. Now using the Gelfand triplet $H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$ i.e. for $\mu \in L^2(\Gamma)$ and $[[v]] \in H^{1/2}(\Gamma)$, we have

$$b(v, \mu) = \int_\Gamma [[v]] \mu d\tau. \tag{3.3.3}$$

Equivalently, (3.3.1)-(3.3.2) can be written as: Find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that

$$\mathcal{A}(u_h, \lambda_h; v_h, \mu_h) = \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \mu_h \in W^h(\Gamma) \tag{3.3.4}$$

where

$$\begin{aligned} \mathcal{A}(v, \mu; w, \nu) = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla v \cdot \nabla w \; dx \quad &+ \quad \int_\Gamma \mu [[w]] \; d\tau \\ &- \quad \int_\Gamma [[v]] \nu \; d\tau + \epsilon \int_\Gamma \mu \nu \; d\tau, \end{aligned} \tag{3.3.5}$$

for all $(v, \mu), (w, \nu) \in X \times L^2(\Gamma)$, and

$$\mathcal{F}(v_h) = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \; dx. \tag{3.3.6}$$

Subtracting (3.3.1)-(3.3.2) from (3.2.4)-(3.2.5), we obtain

$$a(u - u_h, v_h) + b(v_h, \lambda - \lambda_h) - b(u - u_h, \mu_h) = -\epsilon \int_\Gamma \lambda_h \mu_h d\tau \quad \forall v_h \in X_h, \mu_h \in W^h(\Gamma), \tag{3.3.7}$$

which can be written as

$$\mathcal{A}(u-u_h,\lambda-\lambda_h;v_h,\mu_h)=\epsilon\int_\Gamma \lambda\mu_h d\tau \quad \forall v_h\in X_h, \mu_h\in W^h(\Gamma). \tag{3.3.8}$$

Note that, the interface $\Gamma$ here can be chosen either $\Gamma_1$ or $\Gamma_2$. For our convenience, we choose $\Gamma$ to be $\Gamma_2$. For the existence and uniqueness of the solution $(u_h,\lambda_h)\in X_h\times W^h(\Gamma)$ of (3.3.1)-(3.3.2), the discrete spaces may not satisfy the inf-sup condition.

**Lemma 3.3.1** *There exists a unique solution* $(u_h,\lambda_h)\in X_h\times W^h(\Gamma)$ *to the problem (3.3.1)-(3.3.2).*

**Proof.** Since (3.3.1)-(3.3.2) gives rise to a system of linear algebraic equations, uniqueness implies existence of the solution. With $f=0$ in (3.3.1)-(3.3.2), we claim that $u_h\equiv 0$ in each $\Omega_i,\ i=1,2$ and $\lambda_h\equiv 0$ on $\Gamma$. Substitute $v_h=u_h$ and $\mu_h=\lambda_h$ in (3.3.1) and (3.3.2) respectively, and then subtract (3.3.2) from (3.3.1) to obtain

$$\sum_{i=1}^{2}\|\beta_i^{1/2}\nabla u_h\|^2_{L^2(\Omega_i)}+\|\epsilon^{1/2}\lambda_h\|^2_{L^2(\Gamma)}=0. \tag{3.3.9}$$

Since $u_h$ vanishes on $\partial\Omega$, $u_h=0$ in each $\Omega_i,\ i=1,2$ and $\lambda_h=0$ on $\Gamma$, and hence, uniqueness follows. This completes the rest of the proof. ∎

For the stability of the scheme (3.3.4), we define here the mesh-dependent norm as below.

$$|||(v_h,\mu_h)|||^2=\sum_{i=1}^{2}\|\nabla v_h\|^2_{L^2(\Omega_i)}+\|\epsilon^{1/2}\mu_h\|^2_{L^2(\Gamma)}. \tag{3.3.10}$$

Below, we prove the coercivity of $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$.

**Lemma 3.3.2** *For all* $v_h\in X_h$ *and* $\mu_h\in W^h(\Gamma)$*,* $\mathcal{A}$ *satisfies the following coercivity property:*

$$\mathcal{A}(v_h,\mu_h;v_h,\mu_h)\geq\alpha|||(v_h,\mu_h)|||^2, \tag{3.3.11}$$

*for positive* $\alpha$*, where* $\alpha=\min(\alpha_0,1)$*, with* $\alpha_0$ *being the lower bound for the coefficients* $\beta_i,\ i=1,2.$

**Proof.** From (3.3.5),

$$\begin{aligned}\mathcal{A}(v_h,\mu_h;v_h,\mu_h) &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla v_h\cdot\nabla v_h\ dx+\int_{\Gamma}\mu_h[[v_h]]\ d\tau \\ &\quad -\int_{\Gamma}[[v_h]]\mu_h\ d\tau+\epsilon\int_{\Gamma}\mu_h\mu_h d\tau \\ &= \sum_{i=1}^{2}\|\beta_i^{1/2}\nabla v_h\|^2_{L^2(\Omega_i)}+\|\epsilon^{1/2}\mu_h\|^2_{L^2(\Gamma)}. \end{aligned} \tag{3.3.12}$$

Hence, using the lower bound of the coefficients $\beta_i,\ i=1,2$, we derive the desired coercivity property (3.3.11). ■

Now we prove the boundedness property for $\mathcal{A}$.

**Lemma 3.3.3** *Let $(v,\mu)\in X\times L^2(\Gamma)$ and $(w_h,\mu_h)\in X_h\times M_h$. Then the following inequality holds true:*

$$\mathcal{A}(v,\mu;w_h,\mu_h) \ \le\ C\left(|||(v,\mu)|||+\|\mu\|_{H^{-1/2}(\Gamma)}+\|\epsilon^{-1/2}[[v]]\|_{L^2(\Gamma)}\right)|||(w_h,\mu_h)|||. \tag{3.3.13}$$

**Proof.** Applying Cauchy Schwarz inequality, the duality between $H^{-1/2}$ and $H^{1/2}$, we obtain

$$\begin{aligned}\mathcal{A}(v,\mu;w_h,\mu_h) &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla v\cdot\nabla w_h\ dx+\int_{\Gamma}[[w_h]]\mu d\tau-\int_{\Gamma}[[v]]\mu_h d\tau+\epsilon\int_{\Gamma}\mu\mu_h d\tau \\ &\le C\left(\sum_{i=1}^{2}\|\nabla v\|_{L^2(\Omega_i)}\|\nabla w_h\|_{L^2(\Omega_i)}+\|\mu\|_{H^{-1/2}(\Gamma)}\|[[w_h]]\|_{H^{1/2}(\Gamma)}\right. \\ &\quad \left.+\|\epsilon^{-1/2}[[v]]\|_{L^2(\Gamma)}\|\epsilon^{1/2}\mu_h\|_{L^2(\Gamma)}+\|\epsilon^{1/2}\mu\|_{L^2(\Gamma)}\|\epsilon^{1/2}\mu_h\|_{L^2(\Gamma)}\right). \end{aligned} \tag{3.3.14}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{A}(v,\mu;w_h,\mu_h) &\le C\left(|||(v,\mu)|||^2+\|\mu\|^2_{H^{-1/2}(\Gamma)}+\|\epsilon^{-1/2}[[v]]\|^2_{L^2(\Gamma)}\right)^{1/2} \\ &\quad \left(\sum_{i=1}^{2}\|\nabla w_h\|^2_{L^2(\Omega_i)}+\|[[w_h]]\|^2_{H^{1/2}(\Gamma)}+\|\epsilon^{1/2}\mu_h\|^2_{L^2(\Gamma)}\right)^{1/2}. \end{aligned} \tag{3.3.15}$$

Using the trace inequality (Theorem 1.2.2), we obtain an estimate as below

$$\begin{aligned}\mathcal{A}(v,\mu;w_h,\mu_h) &\le C\left(|||(v,\mu)|||+\|\mu\|_{H^{-1/2}(\Gamma)}+\|\epsilon^{-1/2}[[v]]\|_{L^2(\Gamma)}\right) \\ &\quad \left(\sum_{i=1}^{2}\|\nabla w_h\|^2_{L^2(\Omega_i)}+\|\epsilon^{1/2}\mu_h\|^2_{L^2(\Gamma)}\right)^{1/2}, \end{aligned} \tag{3.3.16}$$

and, hence, the boundedness (3.3.13) of $\mathcal{A}$ follows from (3.3.10). ∎

Let us recall here from Chapter 1 that $I_h u_i \in X_h(\Omega_i)$ is the nodal interpolant of $u_i$ in $\Omega_i$ for $i = 1, 2$ which satisfies the following approximation property :

$$\|u_i - I_h u_i\|_{H^1(\Omega_i)} \leq Ch\|u_i\|_{H^2(\Omega_i)} \quad \text{for } u_i \in H^2(\Omega_i). \tag{3.3.17}$$

We also define the $L^2$ projection $\tilde{\Pi}_h$ from $M$ to $W^h(\Gamma)$ as follows:

$$\int_\Gamma (\varphi - \tilde{\Pi}_h\varphi)\psi_h = 0 \qquad \forall \psi_h \in W^h(\Gamma). \tag{3.3.18}$$

Moreover, the operator $\tilde{\Pi}_h$ satisfies the property [23, Lemma 2.4]: For $\lambda \in H^{1/2}(\Gamma)$,

$$\|\lambda - \tilde{\Pi}_h\lambda\|_{L^2(\Gamma)} + h^{-1/2}\|\lambda - \tilde{\Pi}_h\lambda\|_{H^{-1/2}(\Gamma)} \leq Ch^{1/2}\|\lambda\|_{H^{1/2}(\Gamma)}. \tag{3.3.19}$$

**Theorem 3.3.1** *Let $(u, \lambda) \in X \times M$ be the solution of (3.2.4)-(3.2.5) and $(u_h, \lambda_h) \in X_h \times M_h$ be the solution of (3.3.1)-(3.3.2). Moreover, assume $u|_{\Omega_i} \in H^2(\Omega_i)$ for $i = 1, 2$. Then for $\epsilon = O(h)$ the following estimates hold true:*

$$|||(u - u_h, \lambda - \lambda_h)||| \leq Ch^{1/2}\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}, \tag{3.3.20}$$

*and*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}. \tag{3.3.21}$$

**Proof.** Using $I_h u$ and $\tilde{\Pi}_h\lambda$, split the error term and then apply the triangle inequality to obtain

$$|||(u - u_h, \lambda - \lambda_h)||| \leq |||(u - I_h u, \lambda - \tilde{\Pi}_h\lambda)||| + |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)|||. \tag{3.3.22}$$

Since the estimates for the first term on the right hand side of (3.3.22) are known from the standard approximation properties (3.3.17) and(3.3.19), it is enough to derive an estimate for the second term on the right hand side of (3.3.22). Now using Lemma 3.3.2, we obtain

$$\begin{aligned} |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)|||^2 &= \frac{1}{\alpha}\mathcal{A}(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h; I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h) \\ &\leq \frac{1}{\alpha}\Big(\mathcal{A}(u - u_h, \lambda - \lambda_h; I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h) \\ &\quad -\mathcal{A}(u - I_h u, \lambda - \tilde{\Pi}_h\lambda; I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)\Big) \end{aligned} \tag{3.3.23}$$

From (3.3.8), we arrive at

$$\begin{aligned} |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)|||^2 &\leq \frac{1}{\alpha}\epsilon \left| \int_\Gamma \lambda(\tilde{\Pi}_h\lambda - \lambda_h) d\tau \right| \\ &+ \frac{1}{\alpha}\left| \mathcal{A}(u - I_h u, \lambda - \tilde{\Pi}_h\lambda; I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h) \right| \end{aligned} \quad (3.3.24)$$

Here, we note that for $u|_{\Omega_i} \in H^2(\Omega_i)$, $\lambda$ is in $H^{1/2}(\Gamma)$. Now, use Cauchy Schwarz inequality for the first term on right hand side of (3.3.24). To estimate the second term on the right hand side of (3.3.24), we apply Lemma 3.3.3 and the trace inequality. Now altogether we obtain

$$\begin{aligned} |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)|||^2 &\leq C(\alpha)\Big( \|\epsilon^{1/2}\lambda\|_{L^2(\Gamma)} + |||(u - I_h u, \tilde{\Pi}_h\lambda - \lambda)||| \\ &+ \|\epsilon^{-1/2}[[u - I_h u]]\|_{L^2(\Gamma)} + \|\tilde{\Pi}_h\lambda - \lambda\|_{H^{-1/2}(\Gamma)} \Big) \\ &\Big( |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)||| \Big), \end{aligned} \quad (3.3.25)$$

and hence, we arrive at

$$\begin{aligned} |||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)||| &\leq C\Big( |||(u - I_h u, \tilde{\Pi}_h\lambda - \lambda)||| + \|\epsilon^{-1/2}[[u - I_h u]]\|_{L^2(\Gamma)} \\ &+ \|\tilde{\Pi}_h\lambda - \lambda\|_{H^{-1/2}(\Gamma)} + \|\epsilon^{1/2}\lambda\|_{L^2(\Gamma)} \Big). \end{aligned} \quad (3.3.26)$$

By taking $\epsilon = O(h)$, and then using the standard approximation properties (3.3.17) and (3.3.19), we find an estimate for the first term on the right hand side of (3.3.26) as

$$|||(u - I_h u, \tilde{\Pi}_h\lambda - \lambda)||| \leq Ch\left( \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} + \|\lambda\|_{L^2(\Gamma)} \right). \quad (3.3.27)$$

For the second term on the right hand side of (3.3.26), use Lemma 1.3.2 to obtain an estimate

$$\begin{aligned} \|\epsilon^{-1/2}[[u - I_h u]]\|_{L^2(\Gamma)} &\leq \|\epsilon^{-1/2}(u_1 - I_h u_1)\|_{L^2(\Gamma)} + \|\epsilon^{-1/2}(u_2 - I_h u_2)\|_{L^2(\Gamma)} \\ &\leq C\epsilon^{-1/2}h^{3/2}\sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} \quad \text{for } u_i \in H^2(\Omega_i) \\ &= Ch\sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} \quad \text{if } \epsilon = O(h). \end{aligned} \quad (3.3.28)$$

For the third term on the right hand side of (3.3.26), use (3.3.19) to find

$$\|\tilde{\Pi}_h\lambda - \lambda\|_{H^{-1/2}(\Gamma)} \leq Ch\|\lambda\|_{H^{1/2}(\Gamma)}. \tag{3.3.29}$$

For the last term on the right hand side of (3.3.26), we only obtain $h^{1/2}$ by taking $\epsilon = O(h)$

$$\|\epsilon^{1/2}\lambda\|_{L^2(\Gamma)} \leq Ch^{1/2}\|\lambda\|_{L^2(\Gamma)}. \tag{3.3.30}$$

Note that,

$$\|\lambda\|_{L^2(\Gamma)} \leq \|\lambda\|_{H^{1/2}(\Gamma)} \leq C\sum_{i=1}^{2}\|u_i\|_{H^2(\Omega_i)}. \tag{3.3.31}$$

Substituting (3.3.27)-(3.3.31) in (3.3.26), we find that

$$|||(I_h u - u_h, \tilde{\Pi}_h\lambda - \lambda_h)||| \leq Ch^{1/2}\sum_{i=1}^{2}\|u_i\|_{H^2(\Omega_i)}. \tag{3.3.32}$$

Hence, from (3.3.22), we arrive at

$$|||(u - u_h, \lambda - \lambda_h)||| \;\leq\; Ch^{1/2}\sum_{i=1}^{2}\|u_i\|_{H^2(\Omega_i)}. \tag{3.3.33}$$

Observe that here, we only obtain $\|\lambda - \lambda_h\|_{L^2(\Gamma)} \leq C$.

Now we appeal to Aubin-Nitsche duality argument for the $L^2$ error estimate. Let $\psi_i = \psi_{|\Omega_i} \in H^2(\Omega_i) \cap H_0^1(\Omega), i = 1, 2$ be the solution of the interface problem

$$-\nabla\cdot(\beta_i(x)\nabla\psi_i) \;=\; u_i - u_{hi} \;\text{ in } \Omega_i, \tag{3.3.34}$$

$$\psi_i \;=\; 0 \quad \text{on } \partial\Omega\cap\partial\Omega_i, \tag{3.3.35}$$

$$[[\psi]]_\Gamma = 0, \quad \left[\left[\beta\frac{\partial\psi}{\partial n}\right]\right]_\Gamma \;=\; 0 \quad \text{along } \Gamma, \tag{3.3.36}$$

which satisfies the following regularity condition [11, Theorem 1.1], [34, Theorem 2.1]

$$\sum_{i=1}^{2}\|\psi_i\|_{H^2(\Omega_i)} \leq c\|u - u_h\|_{L^2(\Omega)}, \tag{3.3.37}$$

where

$$\|u - u_h\|_{L^2(\Omega)}^2 = \sum_{i=1}^{2}\|u_i - u_{hi}\|_{L^2(\Omega_i)}^2.$$

With $e_i = u_i - u_{hi}$, we multiply both the sides of (3.3.34) by $e_i$ and sum over $i = 1, 2$ to obtain

$$\|e\|^2 = \sum_{i=1}^{2} \|e_i\|^2_{L^2(\Omega_i)} = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla \psi_i \; dx - \int_\Gamma \beta \frac{\partial \psi}{\partial n} [[e]] \; d\tau. \qquad (3.3.38)$$

Note that by (3.3.8),

$$\begin{aligned} \epsilon \int_\Gamma \lambda \tilde{\Pi}_h \left( \beta \frac{\partial \psi}{\partial n} \right) d\tau &= \mathcal{A}\left( u - u_h, \lambda - \lambda_h; I_h \psi, \tilde{\Pi}_h \left( \beta \frac{\partial \psi}{\partial n} \right) \right) \\ &= \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla I_h \psi_i \; dx + \int_\Gamma (\lambda - \lambda_h) [[I_h \psi]] d\tau \\ &\quad - \int_\Gamma [[e]] \Pi_h \left( \beta \frac{\partial \psi}{\partial n} \right) d\tau + \epsilon \int_\Gamma (\lambda - \lambda_h) \tilde{\Pi}_h \left( \beta \frac{\partial \psi}{\partial n} \right) d\tau \end{aligned} \qquad (3.3.39)$$

Since $[[\psi]] = 0$ along $\Gamma$, from (3.3.38) and (3.3.39), we arrive at

$$\begin{aligned} \|e\|^2 &= \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla (\psi_i - I_h \psi_i) \; dx + \int_\Gamma (\lambda - \lambda_h) [[\psi - I_h \psi]] \; d\tau \\ &\quad - \int_\Gamma [[e]] \left( \beta \frac{\partial \psi}{\partial n} - \Pi_h \left( \beta \frac{\partial \psi}{\partial n} \right) \right) d\tau + \epsilon \int_\Gamma (\lambda - \lambda_h) \tilde{\Pi}_h \left( \beta \frac{\partial \psi}{\partial n} \right) d\tau \\ &\quad + \epsilon \int_\Gamma \lambda \tilde{\Pi}_h \left( \beta \frac{\partial \psi}{\partial n} \right) d\tau \end{aligned} \qquad (3.3.40)$$

For the first term on the right hand side of (3.3.40), apply Cauchy Schwarz inequality, trace inequality, (3.3.33) and (3.3.17), to find the estimate given below:

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla (\psi_i - I_h \psi_i) \; dx \leq C h^{3/2} \left( \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \right) \left( \sum_{i=1}^{2} \|\psi_i\|_{H^2(\Omega_i)} \right). \qquad (3.3.41)$$

From (3.3.33), Lemma 1.3.2 and $\epsilon = O(h)$, we obtain an estimate for second term of (3.3.40) as

$$\begin{aligned} \int_\Gamma (\lambda - \lambda_h) [[\psi - I_h \psi]] \; d\tau &\leq C \|\epsilon^{1/2} (\lambda - \lambda_h)\|_{L^2(\Gamma)} \|\epsilon^{-1/2} [[\psi - I_h \psi]]\|_{L^2(\Gamma)} \\ &\leq C h^{3/2} \left( \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \right) \left( \sum_{i=1}^{2} \|\psi_i\|_{H^2(\Omega_i)} \right). \end{aligned} \qquad (3.3.42)$$

For the third term on the right hand side of (3.3.40), use Cauchy-Schwarz inequality and apply the trace inequality. Then from (3.3.33) and (3.3.19), we obtain

$$\int_{\Gamma}[[e]]\left(\beta\frac{\partial\psi}{\partial n}-\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right)d\tau \le Ch^{3/2}\left(\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}\right)\left(\sum_{i=1}^{2}\|\psi_i\|_{H^2(\Omega_i)}\right) \quad (3.3.43)$$

For the fourth term on the right hand side of (3.3.40), apply Cauchy Schwarz inequality, use (3.3.19), (3.3.33) to obtain

$$\begin{aligned}
\epsilon\int_{\Gamma}(\lambda-\lambda_h)\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)d\tau &= \epsilon\int_{\Gamma}(\lambda-\lambda_h)\left(\beta\frac{\partial\psi}{\partial n}-\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right)d\tau-\epsilon\int_{\Gamma}(\lambda-\lambda_h)\beta\frac{\partial\psi}{\partial n}d\tau\\
&\le \|\epsilon^{1/2}(\lambda-\lambda_h)\|_{L^2(\Gamma)}\left\|\epsilon^{1/2}\left(\beta\frac{\partial\psi}{\partial n}-\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right)\right\|_{L^2(\Gamma)}\\
&\quad+\|\epsilon^{1/2}(\lambda-\lambda_h)\|_{L^2(\Gamma)}\left\|\epsilon^{1/2}\left(\beta\frac{\partial\psi}{\partial n}\right)\right\|_{L^2(\Gamma)}\\
&\le Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}\sum_{i=1}^{2}\|\psi_i\|_{H^2(\Omega_i)} \qquad (3.3.44)
\end{aligned}$$

Finally, for the last term on the right hand side of (3.3.40), we proceed in the similar way as in (3.3.44) to arrive at

$$\epsilon\int_{\Gamma}\lambda\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)d\tau \le Ch\left(\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}\right)\left(\sum_{i=1}^{2}\|\psi_i\|_{H^2(\Omega_i)}\right). \quad (3.3.45)$$

Now using the regularity condition (3.3.37), from (3.3.40)-(3.3.45), we obtain the estimate:

$$\|u-u_h\|_{L^2(\Omega)} \le Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}, \quad (3.3.46)$$

and this completes the proof. ■

**Remark 3.3.1** *Note that, here we have not been able to improve the order of convergence in (3.3.32) even if we choose* $\epsilon = O(h^m)$ *for* $m > 1$ *with* $u_i \in H^2(\Omega_i),\ i = 1,2$.

## 3.4 Nitsche's method for parabolic problem

Here, we extend the method to parabolic initial and boundary value problems with discontinuous coefficients and observe the effect of inconsistency of the method which we have discussed earlier. We follow the same assumptions and notations as in the last section.

Let $\Omega \subset \mathcal{R}^2$ be a bounded domain with boundary $\partial\Omega$. Now consider the following parabolic initial-boundary value problem: Given $f \in L^2(\Omega)$, find $u_i = u_i(x,t)$ satisfying

$$\begin{aligned} u_{i_t} - \nabla \cdot (\beta(x)\nabla u_i) &= f \ \text{ in } \ \Omega_i \times (0,T], && (3.4.1)\\ u_i(x,t) &= 0 \ \text{ on } \ (\partial\Omega_i \cap \partial\Omega) \times [0,T], && (3.4.2)\\ u_i(x,0) &= u_0(x) \ \text{ in } \ \Omega && (3.4.3)\\ [[u]]_\Gamma = 0, \ \left[\left[\beta\frac{\partial u}{\partial n}\right]\right]_\Gamma &= 0 \quad \text{along } \ \Gamma. && (3.4.4) \end{aligned}$$

The problem (3.4.1)-(3.4.3) has unique solution in $H^2(\Omega)$ by [56].

### 3.4.1 Semidiscrete method

Integrating by parts, and introducing the flux $\lambda = \beta_1\frac{\partial u_1}{\partial n_1} = -\beta_2\frac{\partial u_2}{\partial n_2}$, we derive the following Lagrange multiplier method for the interface problem (3.4.1)-(3.4.4): Find $(u(t),\lambda(t)) \in X \times M$ such that for $t \in (0,T]$,

$$\begin{aligned} (u_t,v) + a(u,v) + b(v,\lambda) &= \sum_{i=1}^{2}\int_{\Omega_i} fv\ dx \qquad \forall\ v \in X, && (3.4.5)\\ b(u,\mu) &= 0 \qquad \forall \mu \in M, && (3.4.6)\\ (u(0),v) &= (u_0,v), && (3.4.7) \end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are defined in (3.2.6) and (3.2.7) respectively.

**Nitsche's Mortaring method**

We propose the Nitsche's Mortaring method for the parabolic problem (3.4.5)-(3.4.7) as follows: Find $(u_h(t),\lambda_h(t)) \in X_h \times W^h(\Gamma)$ such that for $t \in (0,T]$ and for all $(v_h,\mu_h) \in X_h \times W^h(\Gamma)$,

$$\begin{aligned} \sum_{i=1}^{2}\int_{\Omega_i} u_{h,t}v dx + \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla u_h\cdot\nabla v_h dx + \int_\Gamma \lambda_h[[v_h]]\ d\tau &= \sum_{i=1}^{2}\int_{\Omega_i} fv_h dx, && (3.4.8)\\ \int_\Gamma [[u_h]]\mu_h\ d\tau - \epsilon\int_\Gamma \lambda_h\mu_h\ d\tau &= 0, && (3.4.9)\\ u_h(0) &= u_{h0}. && (3.4.10) \end{aligned}$$

Equivalently (3.4.8)-(3.4.10) can be written as: Find $(u_h(t), \lambda_h(t)) \in X_h \times W^h(\Gamma)$ such that for $t \in (0, T]$,

$$\begin{aligned}(u_{h,t}, v) + \mathcal{A}(u_h, \lambda_h; v_h, \mu_h) &= \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \mu_h \in W^h(\Gamma) &(3.4.11)\\ u_h(0) &= u_{h0}, &(3.4.12)\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}(v, \mu; w, \nu) = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla v \cdot \nabla w \, dx \quad &+ \quad \int_{\Gamma} \mu[[w]] \, d\tau \\ &- \quad \int_{\Gamma} [[v]]\nu \, d\tau + \epsilon \int_{\Gamma} \mu\nu \, d\tau,\end{aligned}$$

for all $(v, \mu), (w, \nu) \in X \times L^2(\Gamma)$, and

$$\mathcal{F}(v_h) \quad = \quad \sum_{i=1}^{2} \int_{\Omega_i} f v_h \, dx.$$

Since (3.4.12) forms a system of linear ordinary differential equations, Picard's existence theorem ensures the existence of a unique solution $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$.

We discuss below the error estimates for (3.4.11)-(3.4.12).

**Theorem 3.4.1** *Let $(u, \lambda)$ and $(u_h, \lambda_h)$ be the solutions of (3.4.5)-(3.4.7) and (3.4.8)-(3.4.10), respectively. Further, let $u_{|\Omega_i}, u_{t|\Omega_i} \in H^2(\Omega_i)$. Then there exists a positive constant $C$, independent of $h$, such that for $t \in (0, T]$ and $\epsilon = O(h)$*

$$\|u(t) - u_h(t)\| \leq Ch \left( \sum_{i=1}^{2} \left[ \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)} \, ds \right] \right)^{1/2}. \qquad (3.4.13)$$

*and*

$$\begin{aligned}|||((u - u_h, \lambda - \lambda_h)(t))|||^2 \leq \frac{C}{t} h \sum_{i=1}^{2} \Big[ \|u_0\|^2_{H^2(\Omega_i)} \quad &+ \quad \int_0^t \Big( s\|u_t\|^2_{H^2(\Omega_i)} \\ &+ \|u_t\|^2_{H^2(\Omega_i)} \Big) \, ds \Big]. \qquad (3.4.14)\end{aligned}$$

**Proof.** Subtracting (3.4.8)-(3.4.9) from (3.4.5)-(3.4.6), we obtain the error equation

$$\begin{aligned}(u_t - u_{h,t}, v_h) + a(u - u_h, v_h) + b(v_h, \lambda - \lambda_h) &= 0, \qquad \forall v_h \in X_h, &(3.4.15)\\ b(u - u_h, \mu_h) + \epsilon \int_{\Gamma} \lambda_h \mu_h d\tau &= 0 \qquad \forall \mu_h \in W^h(\Gamma). &(3.4.16)\end{aligned}$$

We now define mixed elliptic projection as follows. For given $u$ and $\lambda$ set $(\hat{u}_h, \hat{\lambda}_h)$ as the solution of the corresponding elliptic part of (3.4.15)-(3.4.16).

$$\mathcal{A}(u - \hat{u}_h, \lambda - \hat{\lambda}_h; v_h, \mu_h) = \epsilon \int_\Gamma \lambda \mu_h d\tau \qquad \forall v_h \in X_h, \mu_h \in W^h(\Gamma). \tag{3.4.17}$$

Set

$$u - u_h = (u - \hat{u}_h) + (\hat{u}_h - u_h) = \rho + \theta \tag{3.4.18}$$

and

$$\lambda - \lambda_h = (\lambda - \hat{\lambda}_h) + (\hat{\lambda}_h - \lambda_h) = \eta + \xi. \tag{3.4.19}$$

Since estimates of $\rho$ and $\eta$ are known from Theorem 3.3.1, we need to bound $\theta$ and $\xi$. From the equations (3.4.15)-(3.4.19) and using the elliptic projection (3.4.17), we find

$$(\theta_t, v_h) + a(\theta, v_h) + b(v_h, \xi) \;=\; -(\rho_t, v_h) \qquad \forall v_h \in X_h, \tag{3.4.20}$$

$$b(\theta, \mu_h) - \epsilon \int_\Gamma \xi \mu_h d\tau \;=\; 0 \qquad \forall \mu_h \in W^h(\Gamma). \tag{3.4.21}$$

Set $v_h = \theta$ in (3.4.20), $\mu_h = \xi$ in (3.4.21) and then apply coercivity of $\mathcal{A}(\cdot, \cdot; \cdot, \cdot)$. Then, using Young's inequality (1.2.2), we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \alpha |||(\theta, \xi)|||^2 &\le \|\rho_t\| \|\theta\| \\ &\le \frac{1}{2\alpha}\|\rho_t\|^2 + \frac{\alpha}{2}\|\theta\|_X^2, \end{aligned}$$

and hence,

$$\frac{d}{dt}\|\theta\|^2 + \alpha |||(\theta, \xi)|||^2 \le C(\alpha)\|\rho_t\|^2 \tag{3.4.22}$$

Integrating (3.4.22) from 0 to $t$, we find

$$\|\theta(t)\|^2 + \alpha \int_0^t |||(\theta, \xi)|||^2 d\tau \le C(\alpha) \left( \|\theta(0)\|^2 + \int_0^t \|\rho_t\|^2 d\tau \right). \tag{3.4.23}$$

Now choose $u_{0,h} = \hat{u}_h(0)$, then $\theta(0) = 0$, otherwise with $u_{0,h} = I_h u_0$

$$\begin{aligned} \|\theta(0)\| = \|\hat{u}_h(0) - u_{0,h}\| &\le \|u_0 - I_h u_0\| + \|\hat{u}_h(0) - u_0\| \\ &\le Ch \sum_{i=1}^{2} \|u_0\|_{H^2(\Omega_i)}, \end{aligned} \tag{3.4.24}$$

provided $\epsilon = O(h)$. From (3.3.21), we obtain

$$\|\rho_t\| = \|u_t - \hat{u}_t\| \leq Ch \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}. \tag{3.4.25}$$

Substituting (3.4.24) and (3.4.25) in (3.4.23), we find that

$$\|\theta(t)\|^2 + \alpha \int_0^t |||(\theta, \xi)|||^2 d\tau \leq Ch^2 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)} \, ds \right). \tag{3.4.26}$$

Using triangle inequality and (3.3.21) with $\epsilon = O(h)$, we derive

$$\|u(t) - u_h(t)\|^2 \leq Ch^2 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)} \, ds \right). \tag{3.4.27}$$

with $\epsilon = O(h)$.

For a bound in $|||(\cdot, \cdot)|||$-norm, substitute $v_h = \theta_t$ in (3.4.20), differentiate the equation (3.4.21), put $\mu_h = \xi$ in (3.4.21) and apply Cauchy-Schwarz inequality, Young's inequality (1.2.2) to obtain

$$\|\theta_t\|^2 + \frac{\alpha}{2} \frac{d}{dt} |||(\theta, \xi)|||^2 \leq \|\rho_t\| \|\theta_t\| \leq \frac{1}{2\alpha} \|\rho_t\|^2 + \frac{\alpha}{2} \|\theta_t\|^2,$$

and hence,

$$\|\theta_t\|^2 + \frac{d}{dt} |||(\theta, \xi)|||^2 \leq C(\alpha) \|\rho_t\|^2. \tag{3.4.28}$$

Multiply both sides of (3.4.28) by $t$. Since

$$\frac{d}{dt}(t|||(\theta, \xi)|||^2) = |||(\theta, \xi)|||^2 + t \frac{d}{dt} |||(\theta, \xi)|||^2, \tag{3.4.29}$$

the equation (3.4.28) can be written as

$$t\|\theta_t\|^2 + \frac{d}{dt}(t|||(\theta, \xi)|||^2) \leq Ct\|\rho_t\|^2 + |||(\theta, \xi)|||^2. \tag{3.4.30}$$

Integrating both sides of (3.4.30) from 0 to $t$, using (3.4.23), we arrive at

$$t|||(\theta(t), \xi(t))|||^2 \leq C \left( \|\theta(0)\|^2 + \int_0^t \tau \|\rho_t\|^2 d\tau + \int_0^t \|\rho_t\|^2 d\tau \right). \tag{3.4.31}$$

From (3.4.24) and (3.4.25), we obtain the estimate

$$|||(\theta(t),\xi(t))|||^2 \le \frac{C}{t}h^2\sum_{i=1}^{2}\left(\|u_0\|^2_{H^2(\Omega_i)} + \int_0^t\left(\tau\|u_t\|^2_{H^2(\Omega_i)} + \|u_t\|^2_{H^2(\Omega_i)}\right)d\tau\right). \quad (3.4.32)$$

An application of triangle inequality yields

$$|||((u-u_h,\lambda-\lambda_h)(t))|||^2 \le \frac{C}{t}h\sum_{i=1}^{2}\Big[\|u_0\|^2_{H^2(\Omega_i)} \quad + \quad \int_0^t\Big(\tau\|u_t\|^2_{H^2(\Omega_i)} + \|u_t\|^2_{H^2(\Omega_i)}\Big)\,d\tau\Big]. \quad (3.4.33)$$

and hence rest of the proof follows. ∎

**Remark 3.4.1** *Due to the inconsistency of the method (3.3.1)-(3.3.2), we loose* $h^{1/2}$ *order for* $|||(\cdot,\cdot)|||$*-norm in case of elliptic problem. And hence we also observe a loss of order one in case of* $L^2$*-order estimate.*

In the next section, we discuss a completely discrete scheme which is based on backward Euler method.

### 3.4.2 Fully discrete method

Let $k$ be the time step parameter $T = Nk$ and $t_n = nk$. For a continuous function $\varphi \in C[0,T]$, we set the backward difference quotient as $\bar{\partial}_t\varphi^n = \dfrac{\varphi^n - \varphi^{n-1}}{k}$. The backward Euler approximation of $(u,\lambda)$ is to seek a pair of functions $(U^n,\Lambda^n) \in X_h \times W^h(\Gamma)$ so that the pair $(U^n,\Lambda^n)$, $n \ge 1$, satisfies

$$(\bar{\partial}_t U^n, v_h) + \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla U^n\cdot\nabla v_h dx + \int_\Gamma \Lambda^n[[v_h]]\;d\tau \;=\; \sum_{i=1}^{2}\int_{\Omega_i} f v_h dx\;\forall v_h \in X_h, \quad (3.4.34)$$

$$\int_\Gamma [[U^n]]\mu_h\;d\tau - \epsilon\int_\Gamma \Lambda^n\mu_h\;d\tau \;=\; 0\;\forall \mu_h \in W^h(\Gamma). \quad (3.4.35)$$

$$U^0(0) \;=\; u_{h_0}, \quad (3.4.36)$$

where $u_{h_0}$ is an approximation to $u(0)$ in $X_h$ to be chosen later.

Equivalently (3.4.34)-(3.4.35) can be written as: For $n \ge 1$, find $(U^n,\Lambda^n) \in X_h \times W^h(\Gamma)$

such that

$$(\bar{\partial}_t U^n, v_h) + \mathcal{A}(U^n, \Lambda^n; v_h, \mu_h) \;=\; \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \mu_h \in W^h(\Gamma) \tag{3.4.37}$$

For the error estimate we decompose the error terms as follows:

$$u(t_n) - U^n = (u(t_n) - \hat{u}_h(t_n)) + (\hat{u}_h(t_n) - U^n) = \rho^n + \theta^n. \tag{3.4.38}$$

and

$$\lambda(t_n) - \Lambda^n = (\lambda(t_n) - \hat{\lambda}_h(t_n)) + (\hat{\lambda}_h(t_n) - \Lambda^n) = \eta^n + \xi^n. \tag{3.4.39}$$

Since the estimates of $\rho^n$ and $\eta^n$ are known, it is sufficient to estimate $\theta^n$ and $\xi^n$. Using the elliptic projection (3.4.17), (3.4.38)-(3.4.39) at $t = t_n$ and (3.4.34)-(3.4.35), we obtain

$$(\bar{\partial}_t \theta^n, v_h) + a(\theta^n, v_h) + b(v_h, \xi^n) \;=\; (w^n, v_h) \qquad \forall v_h \in X_h \tag{3.4.40}$$

$$b(\theta^n, \mu_h) - \epsilon \int_{\Gamma} \xi^n \mu_h \, d\tau \;=\; 0 \qquad \forall \mu_h \in W^h(\Gamma) \tag{3.4.41}$$

where

$$\begin{aligned} w^n = \bar{\partial}_t \hat{u}_h(t_n) - u_t(t_n) &= (\bar{\partial}_t \hat{u}_h(t_n) - \bar{\partial}_t u(t_n)) + (\bar{\partial}_t u(t_n) - u_t(t_n)) \\ &= w_1^n + w_2^n. \end{aligned} \tag{3.4.42}$$

Choose $v_h = \theta^n$ in (3.4.40) and $\mu_h = \xi^n$ in (3.4.41). Note that

$$(\bar{\partial}_t \theta^n, \theta^n) = \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2 + \frac{k}{2}\|\bar{\partial}_t \theta^n\|^2 \geq \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2.$$

Using the Cauchy-Schwarz inequality and Young's inequality (1.2.2), we find that

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t \|\theta^n\|^2 + \alpha |||(\theta^n, \xi^n)|||^2 &\leq \|w^n\| \|\theta^n\| \leq \|w^n\| \|\theta^n\| \\ &\leq \frac{1}{2\alpha}\|w^n\|^2 + \frac{\alpha}{2}\|\theta^n\|^2 \\ &\leq \frac{1}{2\alpha}\|w^n\|^2 + \frac{\alpha}{2}|||(\theta^n, \xi^n)|||^2 \end{aligned}$$

and hence,

$$\bar{\partial}_t \|\theta^n\|^2 + \alpha |||(\theta^n, \xi^n)|||^2 \leq C(\alpha)\|w^n\|^2.$$

Using the definition of $\partial_t$, we arrive at

$$\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + C(\alpha)k\|w^n\|^2, \tag{3.4.43}$$

and hence, by repeated application, we obtain

$$\|\theta^n\|^2 \leq \|\theta^0\|^2 + Ck(\sum_{j=1}^{n} \|w_1^j\|^2 + \sum_{j=1}^{n} \|w_2^j\|^2). \tag{3.4.44}$$

Choose $u_{0_h} = \hat{u}_h(0)$, then $\theta^0 = 0$, otherwise with $u_{0_h} = I_h u_0$,

$$\begin{aligned} \|\theta^0\| = \|\hat{u}_h(0) - u_h\| &\leq \|u_0 - I_h u_0\| + \|\hat{u}_h(0) - u_0\| \\ &\leq Ch \sum_{i=1}^{2} \|u_0\|_{H^2(\Omega_i)}. \end{aligned} \tag{3.4.45}$$

Since

$$w_1^j = \bar{\partial}_t \hat{u}_h(t_j) - \bar{\partial}_t u(t_j) = k^{-1} \int_{t_{j-1}}^{t_j} (\hat{u}_{ht} - u_t) ds,$$

we now find that

$$k \sum_{j=1}^{n} \|w_1^j\|^2 \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|(\hat{u}_{ht} - u_t)\|^2 ds \quad \leq \quad C \sum_{i=1}^{K} h^2 \int_{t=0}^{t_n} \|u_t\|^2_{H^2(\Omega_i)} ds \tag{3.4.46}$$

To estimate $w_2^j$, we note using Taylor series expansion that

$$\begin{aligned} w_2^j = \bar{\partial}_t u(t_j) - u_t(t_j) &= k^{-1}(u(t_j) - u(t_{j-1})) - u_t(t_j) \\ &= -k^{-1} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s) ds \end{aligned}$$

$$k \sum_{j=1}^{n} \|w_2^j\|^2 \leq \sum_{j=1}^{n} \left( \int_{t_{j-1}}^{t_j} |s - t_{j-1}| \|u_{tt}\| ds \right)^2 \leq Ck^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds. \tag{3.4.47}$$

Substituting (3.4.45)-(3.4.47) in (3.4.44), we obtain

$$\|\theta^n\|^2 \leq C(\alpha) \left( h^2 \left( \sum_{i=1}^{2} \|u_0\|^2_{H^2(\Omega_i)} + \int_{t=0}^{t_n} \|u_t\|^2_{H^2(\Omega_i)} \right) + k^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds \right). \tag{3.4.48}$$

With an application of triangle inequality with (3.4.48), we derive the estimate:

$$\|u(t_n) - U^n\|^2 \leq C \left( h^2 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_{t=0}^{t_n} \|u_t\|^2_{H^2(\Omega_i)} \right) + k^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds \right). \tag{3.4.49}$$

In order to estimate in $\|\cdot\|_X$-norm, substitute $v_h = \bar{\partial}_t\theta^n$ in (3.4.40), and then use (3.4.41) to obtain

$$k\|\bar{\partial}_t\theta^n\|^2 + \||(\theta^n,\xi^n)\||^2 - a(\theta^n,\theta^{n-1}) - b(\theta^{n-1},\xi^n) = k(w^n,\bar{\partial}_t\theta^n)$$

Using the equation (3.4.40) for $v_h = \theta^{n-1}$, Cauchy Schwarz inequality, we arrive at

$$k\|\bar{\partial}_t\theta^n\|^2 + \||(\theta^n,\xi^n)\||^2 = (w^n,\theta^{n-1}) - (\bar{\partial}_t\theta^n,\theta^{n-1}) + k(w^n,\bar{\partial}_t\theta^n) \tag{3.4.50}$$

Now an application of Cauchy Schwarz inequality, Young's inequality (1.2.2) gives,

$$\||(\theta^n,\xi^n)\||^2 \leq C\|\theta^{n-1}\|^2 + k\|w^n\|^2. \tag{3.4.51}$$

Since from (3.4.43)

$$\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + Ck\|w^n\|^2,$$

by repeated application

$$\||(\theta^n,\xi^n)\||^2 \leq \|\theta^0\|^2 + Ck(\sum_{j=1}^{n}\|w_1^j\|^2 + \sum_{j=1}^{n}\|w_2^j\|^2) \tag{3.4.52}$$

Now proceed similar way as in (3.4.46)-(3.4.49), to obtain

$$\begin{aligned}\|u(t_n) - U^n\|_X^2 + \|\epsilon^{1/2}(\lambda(t_n) - \Lambda^n)\|_{L^2(\Gamma)}^2 \;\leq\; & C\left(h\sum_{i=1}^{2}\left(\|u_0\|_{H^2(\Omega_i)}^2 + \int_{t=0}^{t_n}\|u_t\|_{H^2(\Omega_i)}^2\right)\right.\\ & \left. + k^2\int_{t=0}^{t_n}\|u_{tt}\|^2 ds\right). \end{aligned} \tag{3.4.53}$$

The result we proved can be stated as:

**Theorem 3.4.2** *Let $(u(t_n),\lambda(t_n))$ be the solution of (3.4.34)-(3.4.36) and let $(U^n,\Lambda^n) \in X_h \times W^h(\Gamma)$ be an approximation of $(u(t),\lambda(t))$ at $t = t_n$ is given by (3.4.37). Further, assume that for $t \in (0,T]$, $u(t) \in H_0^1(\Omega)$, $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^2(\Omega_l)$ and $u_{tt}(t) \in L^2(\Omega)$. Then with $u_{0,h} = \hat{u}_0$, there exist positive constants $C$, independent of $h$ and $k$, such that for $\epsilon = O(h)$,*

$$\|u(t_n) - U^n\|^2 \leq C\left(h^2\sum_{i=1}^{2}\left(\|u_0\|_{H^2(\Omega_i)}^2 + \int_{t=0}^{t_n}\|u_t\|_{H^2(\Omega_i)}^2 ds\right) + k^2\int_{t=0}^{t_n}\|u_{tt}\|^2 ds\right). \tag{3.4.54}$$

*and*

$$\|u(t_n) - U^n\|_X^2 + \|\epsilon^{1/2}(\lambda(t_n) - \Lambda^n)\|_{L^2(\Gamma)}^2 \leq C\left(h\sum_{i=1}^{2}\left(\|u_0\|_{H^2(\Omega_i)}^2 + \int_{t=0}^{t_n}\|u_t\|_{H^2(\Omega_i)}^2 ds\right) + k^2\int_{t=0}^{t_n}\|u_{tt}\|^2 ds\right). \quad (3.4.55)$$

**Remark 3.4.2** *Note that, here we obtain suboptimal order of convergence in the case of $H^1$ and $L^2$ for $\epsilon = O(h)$ and this is the effect of the inconsistency of the method (3.4.34)-(3.4.36). In Chapter 4, we have modified the method in such way that the resulting method is consistent and we achieve optimal order of estimates in both the cases.*

## 3.5 Matrix Formulation

In this section, we discuss the algebraic formulation arising from the discrete formulation (3.3.1)-(3.3.2). Construction of basis functions are similar to that in Section 2.7 of Chapter 2, with a slight modification for the Lagrange multiplier space. As in Chapter 2, in order to find a matrix formulation for (3.3.1)-(3.3.2), we need to provide a matrix form of (3.3.2). We can write the matrix form associated with (3.3.2) as:

$$M_s u_s - M_m u_m = \epsilon M_{mm}\lambda_m, \quad (3.5.1)$$

where $M_s$, $M_m$ and $M_{mm}$ are given by $(m_s)_{ij} = \int_{\Gamma_1}\varphi_j^s\psi_{h_i}d\tau$, $(m_m)_{ij} = \int_{\Gamma_2}\varphi_j^m\psi_{h_i}d\tau$ and $(\tilde{m}_m)_{ij} = \int_{\Gamma_2}\psi_{h_i}\psi_{h_j}d\tau$ respectively. Here, $\psi_{h_i}$ are the nodal basis functions for $W^h(\Gamma_2)$, $\varphi_j^s$ and $\varphi_j^m$ denote the basis functions for $W^h(\Gamma_1)$ and $W^h(\Gamma_2)$, respectively.

Now, the matrix representation of (3.3.1)-(3.3.2) can be given as:

$$A\alpha = F, \quad (3.5.2)$$

where $\alpha = (u_i^1, u_s^1, u_i^2, u_m^2, \lambda_m)^T$. Note that here $u_i^1$ and $u_i^2$ represent the unknowns associated with all the internal nodal points in $\Omega_1$ and $\Omega_2$, respectively. Further, $u_s^1$ and $u_m^2$ are unknowns associated with $\Gamma_1$ and $\Gamma_2$ and $\lambda_m$ is the unknown for the Lagrange multipliers associated with $W^h(\Gamma)$. Note that

$$A = \begin{pmatrix} A^1_{ii} & A^1_{is} & 0 & 0 & 0 \\ A^1_{si} & A^1_{ss} & 0 & 0 & M_s \\ 0 & 0 & A^2_{ii} & A^2_{im} & 0 \\ 0 & 0 & A^2_{mi} & A^2_{mm} & -(M_m)^T \\ 0 & M_s & 0 & -M_m & -\epsilon M_{mm} \end{pmatrix}, \tag{3.5.3}$$

with

$$\begin{aligned} A^l_{ii} &= \{a(\varphi^{(l)}_i, \varphi^{(l)}_j)\} \qquad x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ A^l_{is} &= \{a(\varphi^{(l)}_i, \varphi_s)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1. \\ A^1_{ss} &= \{a(\varphi_s, \varphi_s)\} \qquad x_s \in \Gamma_1. \\ A^l_{im} &= \{a(\varphi^{(l)}_i, \varphi_m)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2, \\ A^2_{mm} &= \{a(\varphi_m, \varphi_m)\} \qquad x_m \in \Gamma_2, \end{aligned}$$

and

$$F = \begin{pmatrix} F^1_i & F^1_s & F^2_i & F^2_m & 0 \end{pmatrix}^T, \tag{3.5.4}$$

where

$$\begin{aligned} F^l_i &= \{(f^{(l)}_i, \varphi^{(l)}_j)\} \qquad x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ F^l_s &= \{(f^{(l)}_i, \varphi_s)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1, \\ F^l_m &= \{(f^{(l)}_i, \varphi_m)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2, \end{aligned}$$

Now, for fully discrete formulation (3.4.34)-(3.4.35), the associated matrix formulation can be given as:

$$(B + kA)\alpha^n = B\alpha^{n-1} + kF(t_n), \qquad n \ge 1, \tag{3.5.5}$$

with $\alpha = (u^1_i, u^1_s, u^2_i, u^2_m, \lambda_m)^T$, where $A$ and $F$ are same as in (3.5.3) and (3.5.4), respectively and $B$ has the following form:

$$B = \begin{pmatrix} B^1_{ii} & B^1_{is} & 0 & 0 & 0 \\ B^1_{si} & B^1_{ss} & 0 & 0 & M_s \\ 0 & 0 & B^2_{ii} & B^2_{im} & 0 \\ 0 & 0 & B^2_{mi} & B^2_{mm} & -(M_m)^T \\ 0 & M_s & 0 & -M_m & -\epsilon M_{mm} \end{pmatrix}, \qquad (3.5.6)$$

where

$$\begin{aligned} B^l_{ii} &= \{(\varphi^{(l)}_i, \varphi^{(l)}_j)\} && x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ B^l_{is} &= \{B(\varphi^{(l)}_i, \varphi_s)\} && x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1, \\ B^1_{ss} &= \{B(\varphi_s, \varphi_s)\} && x_s \in \Gamma_1, \\ B^l_{im} &= \{(\varphi^{(l)}_i, \varphi_m)\} && x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2. \\ B^2_{mm} &= \{(\varphi_m, \varphi_m)\} && x_m \in \Gamma_2. \end{aligned}$$

### 3.5.1 Numerical Experiments

We choose the following second order elliptic problem on the unit square domain $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary condition and homogeneous jump conditions as follows:

$$\begin{aligned} -\nabla \cdot (\beta_i(x)\nabla u_i) &= f \ \text{ in } \ \Omega_i, \\ u_i &= 0 \ \text{ on } \ \partial\Omega \cap \partial\Omega_i, \\ [[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma &= 0 \quad \text{along } \ \Gamma. \end{aligned}$$

The computational domain $\Omega$ is subdivided into two equal subdomain $\Omega_i,\ i = 1, 2$. In each subdomains $\Omega_i$, we decompose it into a family of linear triangular elements of different mesh sizes $h_i$. Here, we take $h_1 = 1/14$ and $h_2 = 1/16$. With the penalty parameter $\epsilon = O(h)$, we choose discontinuous coefficients with $\beta_1 = 1$ and $\beta_2 = 10$ in two subdomains. We choose $f$ such that the exact solution is $u(x, y) = x(x-1)y(y-1)$.

The order of convergence for the error $e = (u - u_h)$ in $L^2$-norm with respect to the space variable parameter $h$ has been computed in Table 3.1 in the log-log scale. The computed

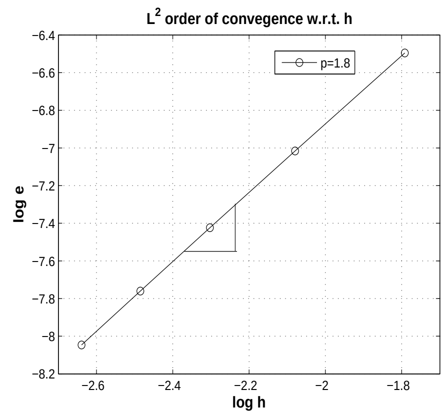



Figure 3.2: Order of Convergence w.r.t. $h$

Table 3.1: $L^2$ Order of Convergence

| $(h_1, h_2)$ | $h = \max_i h_i$ | $\|u - u_h\|_{L^2(\Omega)}$ | Order |
|---|---|---|---|
| $(\frac{1}{6}, \frac{1}{8})$ | 1/6 | 0.0015099 | |
| $(\frac{1}{8}, \frac{1}{10})$ | 1/8 | 0.0008971 | 1.80974559675578 |
| $(\frac{1}{10}, \frac{1}{12})$ | 1/10 | 0.0005967 | 1.82731188795612 |
| $(\frac{1}{12}, \frac{1}{14})$ | 1/12 | 0.00042635 | 1.84374176729892 |
| $(\frac{1}{14}, \frac{1}{16})$ | 1/14 | 0.00032016 | 1.85818015641332 |

order of convergence is better than the theoretical order of convergence derived in Theorem 3.3.1.

For the second order parabolic initial and boundary value problem, we consider again the unit square domain $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary condition and homogeneous jump conditions: Find $u_i = u_i(x,t)$ satisfying

$$u_{i_t} - \nabla \cdot (\beta(x)\nabla u_i) = f \quad \text{in} \quad \Omega_i \times (0,1], \tag{3.5.7}$$

$$u_i(x,t) = 0 \quad \text{on} \quad (\partial\Omega_i \cap \partial\Omega) \times [0,1], \tag{3.5.8}$$

$$u_i(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{3.5.9}$$

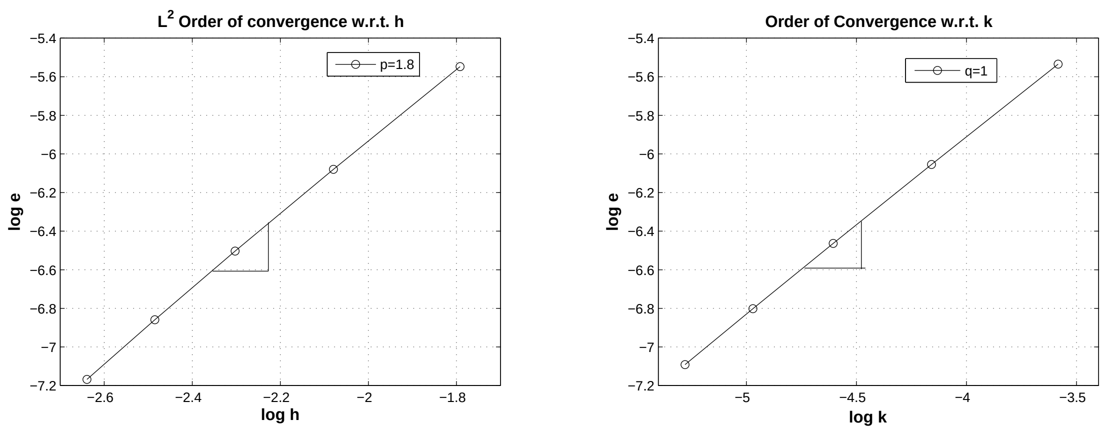
$$[[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along} \quad \Gamma. \tag{3.5.10}$$

The computational domain $\Omega$ is subdivided into two equal subdomains $\Omega_i$, $i = 1, 2$. In each subdomains $\Omega_i$, we decomposed into a family of linear triangular elements of different mesh sizes $h_i$. With $u_0 = 0$, we consider the discontinuous coefficients along the common interface $\Gamma$ of subdomain. We take the coefficients $(\beta_1, \beta_2) = (1, 10)$ and choose $f$ in such a way that the exact solution is $u(x,t) = x(x-1)y(y-1)e^t$. By choosing the penalty parameter $\epsilon = O(h)$, the order of convergence at $t = 1$ for the error $e = (u - u_h)$ in $L^2$-norm '$p$' with respect to the space variable parameter $h$ and '$q$' with respect to the time parameter $k$ has been computed in Table 3.2. Figure 3.3(a) and 3.3(b) show that the computed order of convergence with respect to $h$ and $k$, respectively, in the log-log scale. The computed result illustrates an improved order of convergence in comparison to our theoretical result (See, Theorem 3.4.2).

Table 3.2: Order of convergence $p$ w.r.t. space variable $h$ and $q$ w.r.t. time variable $k$

| $(h_1, h_2)$ | $h = \max_l h_l$ | $k$ | Error $e$ | $p$ | $q$ |
|---|---|---|---|---|---|
| $(\frac{1}{6}, \frac{1}{8})$ | 1/6 | 1/36 | 0.0038965 | | |
| $(\frac{1}{8}, \frac{1}{10})$ | 1/8 | 1/64 | 0.0022893 | 1.8487 | 0.9029 |
| $(\frac{1}{10}, \frac{1}{12})$ | 1/10 | 1/100 | 0.0014986 | 1.8988 | 0.9164 |
| $(\frac{1}{12}, \frac{1}{14})$ | 1/12 | 1/144 | 0.0010499 | 1.9517 | 0.9287 |
| $(\frac{1}{14}, \frac{1}{16})$ | 1/14 | 1/196 | 0.0007706 | 2.0064 | 0.9394 |

In the next chapter, we would like to modify the scheme so that it is possible to derive optimal order of convergence.




(a) Order of Convergence w.r.t. discretization parameter $h$

(b) Order of Convergence w.r.t. $k$

Figure 3.3: Order of Convergence with discontinuous coefficients

# Chapter 4

# Stabilized Mortaring Element Method

## 4.1 Introduction

In Chapter 3, we have proposed and analyzed Nitsche's mortaring method with a parameter for both elliptic and parabolic problems with discontinuous coefficients. Since this method is not consistent, we could only derive suboptimal error estimates that are of order $O(h^{1/2})$ in the broken $H^1$-norm and $O(h)$ in the case of $L^2$-norm under the assumption that the perturbation parameter $\epsilon$ is of $O(h)$. In literature, mortar finite element has been studied by [15, 46, 82] in case of elliptic problems. The close connection between Nitsche's method and mortaring method has been shown in [82]. In [15], a priori and a posteriori error estimates have been derived for second order elliptic problems. In their analysis, the formulation contains a penalty term which involves an integral term of products of piecewise polynomials on unrelated meshes. This is very expensive to implement in higher dimension cases. The interface Lagrange multiplier [46] is chosen with the purpose of avoiding the cumbersome integration of products of functions on unrelated meshes (for example, global polynomials as multipliers).

In this chapter, we introduce a *stabilized Nitsche's mortaring method* which is consistent with the original problem and derive optimal error bounds in both the norms when $\epsilon$ is of $O(h)$. In the first part of this chapter, we deal with the elliptic boundary value problems with discontinuous coefficients and in second part, we extend the method to parabolic initial and boundary value problems with discontinuous coefficients.

A brief outline of this chapter is as follows. In Section 2, we formulate the interface problem with discontinuous coefficients. In Section 3, we describe the *stabilized Nitsche's mortaring method* and discuss the error analysis for second order elliptic problems. In Section 4, we apply the method to parabolic initial and boundary value problems and analyze the error estimates for both semidiscrete and fully discrete schemes. In Section 5, we present the result of some numerical experiments.

## 4.2 Elliptic Boundary Value Problem with Discontinuous Coefficients

Let $\Omega \subset \mathcal{R}^2$ be a bounded domain with boundary $\partial\Omega$. We consider the case when the domain $\bar{\Omega}$ is subdivided into two non-overlapping, convex and polygonal subdomains $\Omega_1$ and $\Omega_2$, i.e., $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. Let the common interface be denoted by $\partial\bar{\Omega}_1 \cap \partial\bar{\Omega}_2 = \Gamma$ and let $n_i$ be the unit normal oriented from $\Omega_i$ towards $\Omega_j$ for $1 \leq i < j \leq 2$ such that $n := n_1 = -n_2$. Now recall the following interface problem defined in Chapter 3: for $i = 1, 2$

$$-\nabla \cdot (\beta_i(x)\nabla u_i) = f \quad \text{in} \quad \Omega_i, \tag{4.2.1}$$

$$u_i = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_i, \tag{4.2.2}$$

$$[[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along} \quad \Gamma, \tag{4.2.3}$$

where $\beta_{|\Omega_i} = \beta_i$ is discontinuous along the interface $\Gamma$, but is piecewise smooth in each subdomain $\Omega_i$. Further, we assume that $\beta$ or each $\beta_i$ is bounded below by a positive constant say $\alpha_0$ and bounded above by a positive constant $\alpha_1$.

With

$$M = H_{00}^{-1/2}(\Gamma)$$

and

$$X = \{v \in L^2(\Omega) \colon v_{|\Omega_l} \in H_D^1(\Omega_l),\ l = 1, 2\},$$

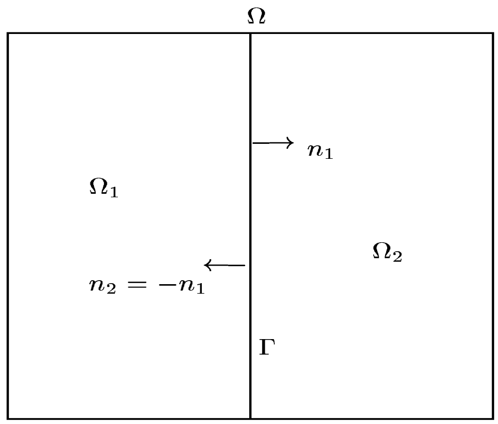



Figure 4.1: $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $n_1$ and $n_2$ represent the outward normal components

we now introduce the Lagrange multiplier $\lambda = \beta_1 \dfrac{\partial u_1}{\partial n_1} = -\beta_2 \dfrac{\partial u_2}{\partial n_2}$ and write the weak formulation for the problem (4.2.1)-(4.2.3) as: Find $(u, \lambda) \in X \times M$ such that

$$a(u,v) + b(v,\lambda) = l(v) \quad \forall v \in X, \tag{4.2.4}$$

$$b(u,\mu) = 0 \qquad \forall \mu \in M, \tag{4.2.5}$$

where

$$a(u,v) = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_i \cdot \nabla v_i dx, \tag{4.2.6}$$

$$b(v,\lambda) = <\lambda, [[v]]>_\Gamma, \qquad l(v) = \sum_{i=1}^{2} \int_{\Omega_i} fv \; dx, \tag{4.2.7}$$

and $<\cdot,\cdot>_\Gamma$ denotes the duality pairing between $H^{-1/2}$ and $H^{1/2}$.

## 4.3 Stabilized Nitsche's Mortaring method

Let us recall the mortar finite element spaces which we defined in Chapter 3. Let

$$X_h(\Omega_i) = \{v_{i,h} \in C(\bar{\Omega}_i) \colon v_{i,h_{|\partial\Omega\cap\partial\Omega_i}} = 0, \; v_{i,h_{|T}} \in P_1(T) \; \forall \; T \in \mathcal{T}_h(\Omega_i)\}$$

be defined on each $\Omega_i$ and let the global finite element space $X_h$ be given by

$$X_h(\Omega) = \{v_h \in L^2(\Omega) \colon v_{h_{|\partial\Omega}} = 0, \; v_{h_{|\Omega_i}} \in X_h(\Omega_i) \;\; i = 1, 2\}.$$

Let $W^h(\Gamma_i)$ be the restriction of $X_h(\Omega_i)$ to $\Gamma_i$. Note that, the choice of the multiplier space over the common interface $\Gamma$ for our purpose is either $W^h(\Gamma_1)$ or $W^h(\Gamma_2)$ with the mesh parameter $h_{e_1}$ and $h_{e_2}$, respectively. For convenience, let us choose $W^h(\Gamma)$ as $W^h(\Gamma_2)$. Further, we assume that there exist positive constants $c_1, c_2$ such that

$$c_1 h_{e_1} \leq h_{e_2} \leq c_2 h_{e_1} \tag{4.3.1}$$

holds for all pairs $(e_1, e_2) \in (\mathcal{T}_h(\Gamma_1), \mathcal{T}_h(\Gamma_2))$, with $e_1 \cap e_2 \neq \phi$. For the analysis purpose, we also define

$$h = max\{h_T, h_e \colon T \in \mathcal{T}_h(\Omega_i), e \in \mathcal{T}_h(\Gamma_i), i = 1, 2\}.$$

**Nitsche's Method:** The *stabilized Nitsche's mortaring approximation* is to find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that for all $v_h \in X_h$ and $\mu_h \in W^h(\Gamma)$,

$$\begin{aligned}\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_h \cdot \nabla v_h \; dx \;&+\; \int_\Gamma \lambda_h [[v_h]] \; d\tau + \sigma \int_\Gamma \gamma \lambda_h \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau \\ &-\; \sigma \int_\Gamma \gamma \left\{ \beta \frac{\partial u_h}{\partial n} \right\} \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \; dx, \qquad (4.3.2)\\ \int_\Gamma [[u_h]] \mu_h \; d\tau \;&+\; \int_\Gamma \gamma \left\{ \beta \frac{\partial u_h}{\partial n} \right\} \mu_h \; d\tau - \int_\Gamma \gamma \lambda_h \mu_h d\tau = 0. \qquad (4.3.3)\end{aligned}$$

When $\sigma = 0$, this method is unsymmetric and for $\sigma = 1$, this method is symmetric. Here, $\gamma$ is at our disposal and will be chosen later.

Equivalently, (4.3.2)-(4.3.3) can be written as: Find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that

$$\mathcal{A}(u_h, \lambda_h; v_h, \mu_h) = \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \; \mu_h \in W^h(\Gamma) \tag{4.3.4}$$

where

$$\begin{aligned}\mathcal{A}(v, \mu; w, \nu) \;=\; & \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla v \cdot \nabla w \; dx + \int_\Gamma \mu [[w]] \; d\tau \\ & - \int_\Gamma [[v]] \nu \; d\tau + \sigma \int_\Gamma \gamma \mu \left\{ \beta \frac{\partial v}{\partial n} \right\} d\tau \\ & -\sigma \int_\Gamma \gamma \left\{ \beta \frac{\partial v}{\partial n} \right\} \left\{ \beta \frac{\partial w}{\partial n} \right\} d\tau - \int_\Gamma [[v]] \nu \; d\tau \\ & - \int_\Gamma \gamma \left\{ \beta \frac{\partial v}{\partial n} \right\} \nu \; d\tau + \int_\Gamma \gamma \mu \nu d\tau, \qquad (4.3.5)\end{aligned}$$

for all $(v,\mu),(w,\nu) \in X \times L^2(\Gamma)$, and

$$\mathcal{F}(v_h) = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \, dx. \tag{4.3.6}$$

For both the cases, i.e., $\sigma = 0$ and $\sigma = 1$, this method is consistent with the original problem (4.2.4)-(4.2.5). In order to verify the consistency, substitute $(u,\lambda)$ in place of $(u_h,\lambda_h)$ in (4.3.2)-(4.3.3). Since $\lambda = \beta_1 \frac{\partial u_1}{\partial n_1} = -\beta_2 \frac{\partial u_2}{\partial n_2}$ and

$$\left\{\frac{\partial u}{\partial n}\right\} = \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2},$$

we have

$$\int_\Gamma \gamma\lambda \left\{\beta \frac{\partial v_h}{\partial n}\right\} d\tau - \int_\Gamma \gamma \left\{\beta \frac{\partial u}{\partial n}\right\}\left\{\beta \frac{\partial v_h}{\partial n}\right\} d\tau = 0$$

and

$$-\int_\Gamma \gamma \left\{\beta \frac{\partial u}{\partial n}\right\} \mu_h \, d\tau + \int_\Gamma \gamma\lambda\mu_h \, d\tau = 0$$

We state this in the form of a lemma as follows.

**Lemma 4.3.1** *The problem (4.3.4) is consistent with the original problem (4.2.4)-(4.2.5). Moreover, if $(u,\lambda)$ is a solution of (4.2.4)-(4.2.5) and $(u_h,\lambda_h)$ is a solution of (4.3.4), then*

$$\mathcal{A}(u-u_h, \lambda-\lambda_h; v_h, \mu_h) = 0 \qquad \forall v_h \in X_h,\ \mu_h \in W^h(\Gamma). \tag{4.3.7}$$

In order to verify stability of (4.3.4), we define here the following weighted norm: for $(v,\mu) \in X \times L^2(\Gamma)$

$$|||(v,\mu)|||^2 = \sum_{i=1}^{2} \|\nabla v\|^2_{L^2(\Omega_i)} + \|\gamma^{1/2}\mu\|^2_{L^2(\Gamma)}. \tag{4.3.8}$$

We also need the following result (See, [15, Lemma 2.4], [82, Lemma 2]) for our subsequent analysis.

**Lemma 4.3.2** *Consider $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$ as given in 4.3.4. There exists a positive constant $C_I$ such that*

$$\left\| h_i^{1/2} \frac{\partial v_i}{\partial n_i} \right\|_{L^2(\Gamma_i)} \leq C_I \|\nabla v_i\|_{L^2(\Omega_i)} \qquad \forall v_i \in X_h(\Omega_i). \tag{4.3.9}$$

Below, we prove the stability of the method (4.3.4).

**Lemma 4.3.3** *There exists a positive constant $\alpha$ independent of $h$ such that*

$$\mathcal{A}(v_h,\mu_h;v_h,\mu_h) \geq \alpha |||(v_h,\mu_h)|||^2 \qquad \forall v_h \in X_h,\ \ \mu_h \in W^h(\Gamma); \tag{4.3.10}$$

*for $\gamma = \gamma_0 h$ with $\gamma_0 < \dfrac{4\alpha_0}{C_I\alpha_1^2}$; $\alpha_0, \alpha_1, C_I$ being positive constants.*

**Proof.** From (4.3.5) and the Cauchy Schwarz's inequality, we obtain

$$\begin{aligned}\mathcal{A}(v_h,\mu_h;v_h,\mu_h) &= \sum_{i=1}^{2} \|\beta_i^{1/2}\nabla v_h\|^2_{L^2(\Omega_i)} + (\sigma-1)\int_\Gamma \gamma\left\{\beta\frac{\partial v_h}{\partial n}\right\}\mu_h\, d\tau \\ &- \sigma\left\|\gamma^{1/2}\left\{\beta\frac{\partial v_h}{\partial n}\right\}\right\|^2_{L^2(\Gamma)} + \|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)}. \end{aligned}\tag{4.3.11}$$

Now another application of Cauchy Schwarz's inequality, Young's inequality together with Lemma 4.3.2 yield a bound for the second term on the right hand side of (4.3.11) as

$$\begin{aligned}\left|\int_\Gamma \gamma\left\{\beta\frac{\partial v_h}{\partial n}\right\}\mu_h\, d\tau\right| &\leq \left\|\gamma^{1/2}\left\{\beta\frac{\partial v_h}{\partial n}\right\}\right\|_{L^2(\Gamma)} \|\gamma^{1/2}\mu_h\|_{L^2(\Gamma)} \\ &\leq \frac{1}{2\epsilon}\left\|\gamma^{1/2}\left\{\beta\frac{\partial v_h}{\partial n}\right\}\right\|^2_{L^2(\Gamma)} + \frac{\epsilon}{2}\|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)} \\ &\leq \frac{C_I\gamma_0\alpha_1^2}{2\epsilon}\sum_{i=1}^{2}\|\nabla v_i\|^2_{L^2(\Omega_i)} + \frac{\epsilon}{2}\|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)}. \end{aligned}\tag{4.3.12}$$

Since (4.3.1) holds, we choose $\gamma = \gamma_0 h$, where $\gamma_0 > 0$ is independent of $h$ . Now substitute (4.3.12) in (4.3.11), use the bounds of the coefficients $\beta$ to derive

$$\begin{aligned}\mathcal{A}(v_h,\mu_h;v_h,\mu_h) &\geq \alpha_0\sum_{i=1}^{2}\|\nabla v_h\|^2_{L^2(\Omega_i)} + \|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)} \\ &+(\sigma-1)\left(\frac{C_I\gamma_0\alpha_1^2}{2\epsilon}\sum_{i=1}^{2}\|\nabla v_i\|^2_{L^2(\Omega_i)} + \frac{\epsilon}{2}\|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)}\right) \\ &-\sigma C_I\gamma_0\alpha_1^2\sum_{i=1}^{2}\|\nabla v_i\|^2_{L^2(\Omega_i)}. \end{aligned}\tag{4.3.13}$$

Note that here $\alpha_0$ and $\alpha_1$ are the lower and upper bound for the coefficient $\beta$. Now for $\sigma = 0$, choose $\alpha_0 - \dfrac{C_I\alpha_1^2\gamma_0}{2\epsilon} > 0$ and $\epsilon < 2$. And hence for $\gamma_0 < \dfrac{4\alpha_0}{C_I\alpha_1^2}$, there exists some positive constant $\alpha$ such that (4.3.10) holds.

For $\sigma = 1$, choose $\alpha_0 - C_I\alpha_1^2\gamma_0 > 0$. Therefore, for $\gamma_0 < \dfrac{\alpha_0}{C_I\alpha_1^2}$, there exists some positive constant $\alpha$ such that the stability (4.3.10) holds. This completes the proof. ■

We need also the boundedness of $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$ with respect to $|||(\cdot,\cdot)|||$-norm.

**Lemma 4.3.4** *For all* $(v, \mu) \in X \times L^2(\Gamma)$ *and* $(w_h, \mu_h) \in X_h \times W^h(\Gamma)$*, the following relation holds true for* $\gamma = O(h)$*:*

$$\mathcal{A}(v,\mu,w_h,\mu_h) \leq C\left( |||(v,\mu)||| + \|\gamma^{-1/2}[[v]]\|_{L^2(\Gamma)} + \|\mu\|_{H^{-1/2}(\Gamma)} \right) |||(w_h,\mu_h)||| \quad (4.3.14)$$

**Proof.** Apply Cauchy Schwarz's inequality and the duality between $H^{-1/2}$ and $H^{1/2}$ to derive

$$\begin{aligned}
\mathcal{A}(v,\mu;w_h,\mu_h) &= \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla v \cdot \nabla w_h \, dx + \int_\Gamma \mu[[w_h]] \, d\tau + \sigma \int_\Gamma \gamma\mu \left\{ \beta \frac{\partial w_h}{\partial n} \right\} d\tau \\
&\quad -\sigma \int_\Gamma \gamma \left\{ \beta \frac{\partial v}{\partial n} \right\} \left\{ \beta \frac{\partial w_h}{\partial n} \right\} d\tau - \int_\Gamma [[v]]\mu_h \, d\tau \\
&\quad - \int_\Gamma \gamma \left\{ \beta \frac{\partial v}{\partial n} \right\} \mu_h \, d\tau + \int_\Gamma \gamma \mu \mu_h d\tau \\
&\leq C \Bigg( \sum_{i=1}^{2} \|\nabla v\|_{L^2(\Omega_i)} \|\nabla w_h\|_{L^2(\Omega_i)} + \|\mu\|_{H^{-1/2}(\Gamma)} \|[[w_h]]\|_{H^{1/2}(\Gamma)} \\
&\quad + \sigma \|\gamma^{1/2}\mu\|_{L^2(\Gamma)} \left\| \gamma^{1/2} \left\{ \beta \frac{\partial w_h}{\partial n} \right\} \right\|_{L^2(\Gamma)} \\
&\quad + \sigma \left\| \gamma^{1/2} \left\{ \beta \frac{\partial v}{\partial n} \right\} \right\|_{L^2(\Gamma)} \left\| \gamma^{1/2} \left\{ \beta \frac{\partial w_h}{\partial n} \right\} \right\|_{L^2(\Gamma)} \\
&\quad + \|\gamma^{-1/2}[[v]]\|_{L^2(\Gamma)} \|\gamma^{1/2}\mu_h\|_{L^2(\Gamma)} \\
&\quad + \left\| \gamma^{1/2} \left\{ \beta \frac{\partial v}{\partial n} \right\} \right\|_{L^2(\Gamma)} \|\gamma^{1/2}\mu_h\|_{L^2(\Gamma)} + \|\gamma^{1/2}\mu\|_{L^2(\Gamma)} \|\gamma^{1/2}\mu_h\|_{L^2(\Gamma)} \Bigg)
\end{aligned} \quad (4.3.15)$$

Using Lemma 4.3.2, we obtain for $\gamma = O(h)$

$$\begin{aligned}
\mathcal{A}(v,\mu;w_h,\mu_h) &\leq C \left( |||(v,\mu)|||^2 + \|\mu\|^2_{H^{-1/2}(\Gamma)} + \|\gamma^{-1/2}[[v]]\|^2_{L^2(\Gamma)} \right)^{1/2} \\
&\quad \left( \sum_{i=1}^{2} \|\nabla w_h\|^2_{L^2(\Omega_i)} + \|[[w_h]]\|^2_{H^{1/2}(\Gamma)} + \|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)} \right)^{1/2}.
\end{aligned} \quad (4.3.16)$$

Using the trace inequality (Theorem 1.2.2) and Poincaré inequality (Theorem 1.2.1), we arrive at

$$\begin{aligned}
\mathcal{A}(v,\mu;w_h,\mu_h) &\leq C \left( |||(v,\mu)||| + \|\mu\|_{H^{-1/2}(\Gamma)} + \|\gamma^{-1/2}[[v]]\|_{L^2(\Gamma)} \right) \\
&\quad \left( \sum_{i=1}^{2} \|\nabla w_h\|^2_{L^2(\Omega_i)} + \|\gamma^{1/2}\mu_h\|^2_{L^2(\Gamma)} \right)^{1/2}.
\end{aligned} \quad (4.3.17)$$

The boundedness (4.3.14) of $\mathcal{A}$ follows from (4.3.8) and this completes the rest of the proof. ■

The coercive property (4.3.10) and the boundedness (Lemma 4.3.14) of $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$ ensure the validity of Lax-Milgram lemma (Theorem 1.2.8) and hence, there exists a unique pair of solution $(u_h,\lambda_h) \in X_h \times W^h(\Gamma)$ to the problem (4.3.4).

### 4.3.1 Error Analysis

In this section, we derive *a priori* error estimates for the elliptic problem (4.2.4)-(4.2.5).

**Theorem 4.3.1** *Let $(u,\lambda)$ be the solution of (4.2.4)-(4.2.5) and $(u_h,\lambda_h)$ be the solution of (4.3.4). Then, there exists a positive constant $C$ independent of $h$ and $\gamma$ such that*

$$\begin{aligned} |||(u-u_h,\lambda-\lambda_h)||| \;&\le\; C \inf_{w_h\in X_h,\mu_h\in W^h(\Gamma)} \Big(|||(u-w_h,\lambda-\mu_h)||| \\ &\quad + \|\gamma^{-1/2}[[u-w_h]]\|_{L^2(\Gamma)} + \|\lambda-\mu_h\|_{H^{-1/2}(\Gamma)}\Big). \end{aligned} \tag{4.3.18}$$

**Proof.** By adding and subtracting $w_h$ and $\mu_h$ in the error term, we obtain as a consequence of triangle inequality:

$$|||(u-u_h,\lambda-\lambda_h)||| \le |||(u-w_h,\lambda-\mu_h)||| + |||(w_h-u_h,\mu_h-\lambda_h)|||. \tag{4.3.19}$$

Now using (4.3.10), orthogonality (4.3.7) and boundedness (4.3.14) of $\mathcal{A}$ we obtain

$$\begin{aligned} |||(w_h-u_h,\mu_h-\lambda_h)|||^2 \;&\le\; \frac{1}{\alpha}\mathcal{A}(w_h-u_h,\mu_h-\lambda_h;w_h-u_h,\mu_h-\lambda_h) \\ &\le\; \frac{1}{\alpha}\mathcal{A}(u-u_h,\lambda-\lambda_h;w_h-u_h,\mu_h-\lambda_h) \\ &\quad -\frac{1}{\alpha}\mathcal{A}(u-w_h,\lambda-\mu_h;w_h-u_h,\mu_h-\lambda_h) \\ &\le\; C\,\Big(|||(u-w_h,\lambda-\mu_h)||| \\ &\quad + \|\gamma^{-1/2}[[u-w_h]]\|_{L^2(\Gamma)} + \|\lambda-\mu_h\|_{H^{-1/2}(\Gamma)}\Big) \\ &\qquad (|||(w_h-u_h,\mu_h-\lambda_h)|||)\,. \end{aligned} \tag{4.3.20}$$

From (4.3.19) and (4.3.20), (4.3.18) follows and this completes the proof. ■

**Theorem 4.3.2** *Let $(u, \lambda)$ be the solution of (4.2.4)- (4.2.5) and $(u_h, \lambda_h)$ be the solution of (4.3.4). Further, assume $u \in H^2(\Omega_i)$, then with $\gamma = O(h)$ there exists a positive constant $C$ such that*

$$|||(u - u_h, \lambda - \lambda_h)||| \leq Ch \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} \tag{4.3.21}$$

*and*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)}. \tag{4.3.22}$$

**Proof.** From Theorem 4.3.1, using the nodal interpolant $I_i$ of $u_i$ in $\Omega_i$ and the $L^2$-orthogonal projection $\tilde{\Pi}_h$ on $W^h(\Gamma)$ defined by (3.3.17) and (3.3.19) in Chapter 3, we obtain

$$\begin{aligned} |||(u - u_h, \lambda - \lambda_h)||| \;&\leq\; C \Big( |||(u - I_h u, \lambda - \tilde{\Pi}_h \lambda)||| \\ &+\; \|\gamma^{-1/2}[[u - I_h u]]\|_{L^2(\Gamma)} + \|\lambda - \tilde{\Pi}_h \lambda\|_{H^{-1/2}(\Gamma)} \Big). \end{aligned} \tag{4.3.23}$$

For the first term on right hand side of (4.3.23), we use the standard interpolation estimate and $L^2(\Gamma)$ estimates

$$\|u_i - I_h u_i\|_{H^1(\Omega_i)} \leq Ch \|u_i\|_{H^2(\Omega_i)} \quad \text{for } u_i \in H^2(\Omega_i), \tag{4.3.24}$$

and

$$\|\lambda - \tilde{\Pi}_h \lambda\|_{L^2(\Gamma)} \leq Ch^{1/2} \|\lambda\|_{H^{1/2}(\Gamma)} \text{for } \lambda \in H^{1/2}(\Gamma). \tag{4.3.25}$$

Hence, we obtain

$$|||(u - I_h u, \lambda - \tilde{\Pi}_h \lambda)||| \leq C(h + \gamma^{1/2} h^{1/2}) \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)}. \tag{4.3.26}$$

For the second term on the right hand side of (4.3.23), we use Lemma 1.3.2 to derive

$$\begin{aligned} \|\gamma^{-1/2}[[u - I_h u]]\|_{L^2(\Gamma)} \;&\leq\; \|\gamma^{-1/2}(u_1 - I_h u_1)\|_{L^2(\Gamma)} + \|\gamma^{-1/2}(u_2 - I_h u_2)\|_{L^2(\Gamma)} \\ &\leq\; C\gamma^{-1/2} h^{3/2} \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} \quad \text{for } u_i \in H^2(\Omega_i). \end{aligned} \tag{4.3.27}$$

For the last term on the right hand side of (4.3.23), we use (3.3.19) to arrive at

$$\|\lambda - \tilde{\Pi}_h \lambda\|_{H^{-1/2}(\Gamma)} \leq Ch\|\lambda\|_{H^{1/2}(\Gamma)} \leq Ch \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)}. \tag{4.3.28}$$

Substituting (4.3.26)-(4.3.28) in (4.3.23), and choosing $\gamma \cong O(h)$, the estimate (4.3.21) follows.

For the $L^2$-error estimate, we appeal to Aubin-Nitsche duality argument. Let $\psi_i = \psi_{|\Omega_i} \in H^2(\Omega_i) \cap H_0^1(\Omega), i = 1, 2$ be the solution of the interface problem

$$-\nabla \cdot (\beta_i(x) \nabla \psi_i) = u_i - u_{h_i} \quad \text{in } \Omega_i, \tag{4.3.29}$$

$$\psi_i = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_i, \tag{4.3.30}$$

$$[[\psi]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial \psi}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along } \Gamma, \tag{4.3.31}$$

which satisfies the regularity condition[11, Theorem 1.1], [34, Theorem 2.1]

$$\sum_{i=1}^{2} \|\psi_i\|_{H^2(\Omega_i)} \leq c\|u - u_h\|_{L^2(\Omega)}. \tag{4.3.32}$$

Setting $e_i = u_i - u_{h_i}$ and multiplying both the sides of (4.3.29) by $e_i$, and summing up over $i = 1, 2$, we obtain

$$\|e\|^2 = \sum_{i=1}^{2} \|e_i\|^2_{L^2(\Omega_i)} = \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla \psi_i \, dx - \int_\Gamma \beta \frac{\partial \psi}{\partial n} [[e]] \, d\tau. \tag{4.3.33}$$

With the help of (4.3.7), we find that

$$\begin{aligned} 0 &= \mathcal{A}\left(u - u_h, \lambda - \lambda_h; I_h\psi, \tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right)\right) \\ &= \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla e_i \cdot \nabla I_h \psi_i \, dx + \int_\Gamma (\lambda - \lambda_h)[[I_h\psi]] \, d\tau \\ &\quad + \sigma \int_\Gamma \gamma(\lambda - \lambda_h) \left\{\beta \frac{\partial I_h \psi}{\partial n}\right\} d\tau - \sigma \int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \left\{\beta \frac{\partial I_h \psi}{\partial n}\right\} d\tau \\ &\quad - \int_\Gamma [[e]] \tilde{\Pi}_h \left(\beta \frac{\partial \psi}{\partial n}\right) d\tau - \int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \tilde{\Pi}_h \left(\beta \frac{\partial \psi}{\partial n}\right) d\tau \\ &\quad + \int_\Gamma \gamma(\lambda - \lambda_h) \tilde{\Pi}_h \left(\beta \frac{\partial \psi}{\partial n}\right) d\tau. \end{aligned} \tag{4.3.34}$$

Subtracting (4.3.34) from (4.3.33), and making use of $[[\psi]] = 0$ along $\Gamma$, we obtain

$$\begin{aligned}\|e\|^2 &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla e_i\cdot\nabla(\psi_i - I_h\psi_i)\,dx - \int_\Gamma \gamma^{1/2}(\lambda-\lambda_h)\gamma^{-1/2}[[\psi - I_h\psi]]\,d\tau \\ &\quad -\sigma\int_\Gamma \gamma(\lambda-\lambda_h)\left\{\beta\frac{\partial I_h\psi}{\partial n}\right\}d\tau + \sigma\int_\Gamma\gamma\left\{\beta\frac{\partial e}{\partial n}\right\}\left\{\beta\frac{\partial I_h\psi}{\partial n}\right\}d\tau \\ &\quad -\int_\Gamma [[e]]\left(\beta\frac{\partial\psi}{\partial n} - \tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right)d\tau + \int_\Gamma\gamma\left\{\beta\frac{\partial e}{\partial n}\right\}\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\,d\tau \\ &\quad -\int_\Gamma\gamma(\lambda-\lambda_h)\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)d\tau. \end{aligned} \tag{4.3.35}$$

Note that, using trace inequality (Theorem 1.2.2), we find that

$$\begin{aligned}\left\|\left\{\beta\frac{\partial e}{\partial n}\right\}\right\|_{L^2(\Gamma)} &\le \frac{1}{2}\left\|\beta_1\frac{\partial e_1}{\partial n}\right\|_{L^2(\Gamma)} + \frac{1}{2}\left\|\beta_2\frac{\partial e_2}{\partial n}\right\|_{L^2(\Gamma)} \\ &\le C\sum_{i=1}^{2}\|\nabla e_i\|_{L^2(\Omega_i)} \le Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}. \end{aligned} \tag{4.3.36}$$

For the first term on the right-hand side of (4.3.35), use the Cauchy Schwarz inequality, (4.3.21) and then the approximation property (3.3.17) to obtain

$$\left|\sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla e_i\cdot\nabla(\psi_i - I_h\psi_i)\,dx\right| \le Ch^2\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}\|\psi\|_{H^2(\Omega_i)}. \tag{4.3.37}$$

For the second term on right hand-side of (4.3.35), use Cauchy Schwarz's inequality, (4.3.21) and then Lemma 1.3.2 to derive

$$\begin{aligned}\left|\int_\Gamma\gamma^{1/2}(\lambda-\lambda_h)\gamma^{-1/2}[[\psi - I_h\psi]]\,d\tau\right| &\le \|\gamma^{1/2}(\lambda-\lambda_h)\|_{L^2(\Gamma)}\|\gamma^{-1/2}[[\psi - I_h\psi]]\|_{L^2(\Gamma)} \\ &\le Ch^2\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}\|\psi\|_{H^2(\Omega_i)}, \end{aligned} \tag{4.3.38}$$

when $\gamma = O(h)$.

For the fourth term on right-hand side of (4.3.35), an application of Cauchy Schwarz's inequality, the trace inequality (4.3.35) yields

$$
\begin{aligned}
\left|\int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \left\{\beta \frac{\partial I_h \psi}{\partial n}\right\} d\tau\right| &\leq C \left|\int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \left\{\beta \frac{\partial}{\partial n}(I_h\psi - \psi)\right\} d\tau\right| \\
&\quad + \left|\int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \left\{\beta \frac{\partial \psi}{\partial n}\right\} d\tau\right| \\
&\leq C\gamma \left\|\left\{\beta \frac{\partial e}{\partial n}\right\}\right\|_{L^2(\Gamma)} \\
&\quad \left(\left\|\left\{\beta \frac{\partial}{\partial n}(I_h\psi - \psi)\right\}\right\|_{L^2(\Gamma)} + \left\|\left\{\beta \frac{\partial \psi}{\partial n}\right\}\right\|_{L^2(\Gamma)}\right) \\
&\leq C\gamma \sum_{i=1}^{2} \|e\|_{H^1(\Omega_i)} \|\psi\|_{H^2(\Omega_i)} \\
&\leq Ch^2 \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|\psi\|_{H^2(\Omega_i)}. \qquad (4.3.39)
\end{aligned}
$$

For the fifth term on right-hand side of (4.3.35), use the duality pairing between $H^{1/2}$ and $H^{-1/2}$, the trace inequality, (4.3.21) and the approximation property (3.3.19) to obtain

$$
\begin{aligned}
\left|\int_\Gamma [[e]] \left(\beta \frac{\partial \psi}{\partial n} - \tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right)\right) d\tau\right| &\leq C \|[[e]]\|_{H^{1/2}(\Gamma)} \left\|\left(\beta \frac{\partial \psi}{\partial n} - \tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right)\right)\right\|_{H^{-1/2}(\Gamma)} \\
&\leq C \sum_{i=1}^{2} \|e\|_{H^1(\Omega_i)} \left\|\left(\beta \frac{\partial \psi}{\partial n} - \tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right)\right)\right\|_{H^{-1/2}(\Gamma)} \\
&\leq Ch^2 \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|\psi\|_{H^2(\Omega_i)}. \qquad (4.3.40)
\end{aligned}
$$

An application of Cauchy Schwarz's inequality together with the trace inequality and $\gamma = O(h)$ yields an estimate for the sixth term on the right hand side of (4.3.35) as

$$
\begin{aligned}
\left|\int_\Gamma \gamma \left\{\beta \frac{\partial e}{\partial n}\right\} \tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right) d\tau\right\| &\leq C\gamma \left\|\left\{\beta \frac{\partial e}{\partial n}\right\}\right\|_{L^2(\Gamma)} \left\|\tilde{\Pi}_h\left(\beta \frac{\partial \psi}{\partial n}\right)\right\|_{L^2(\Gamma)} \\
&\leq C\gamma \sum_{i=1}^{2} \|e\|_{H^1(\Omega_i)} \left\|\left(\beta \frac{\partial \psi}{\partial n}\right)\right\|_{L^2(\Gamma)} \\
&\leq Ch^2 \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|\psi\|_{H^2(\Omega_i)}. \qquad (4.3.41)
\end{aligned}
$$

Finally, for the third and last term on the right-hand side of (4.3.35), we proceed in the following way. From (4.2.4)-(4.2.5) and (4.3.2)-(4.3.3), we arrive at

$$\sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla(u-u_h)\cdot\nabla v_h\ dx+\int_\Gamma(\lambda-\lambda_h)[[v_h]]\ d\tau+\sigma\int_\Gamma\gamma(\lambda-\lambda_h)\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau$$
$$-\sigma\int_\Gamma\gamma\left\{\beta\frac{\partial}{\partial n}(u-u_h)\right\}\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau=0, \quad (4.3.42)$$
$$\int_\Gamma[[u-u_h]]\mu_h\ d\tau+\int_\Gamma\gamma\left\{\beta\frac{\partial}{\partial n}(u-u_h)\right\}\mu_h\ d\tau-\int_\Gamma\gamma(\lambda-\lambda_h)\mu_h=0. \quad (4.3.43)$$

With $v_h=(v_{h_1},0)$, we derive from (4.3.42)

$$\begin{aligned}\int_{\Omega_1}\beta_1\nabla(P_hu-u_h)\cdot\nabla v_{h_1}\ dx &+ \int_\Gamma(\tilde{\Pi}_h\lambda-\lambda_h)v_{h_1}\ d\tau+\sigma\int_\Gamma\gamma(\tilde{\Pi}_h\lambda-\lambda_h)\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau\\ &- \sigma\int_\Gamma\gamma\left\{\beta\frac{\partial}{\partial n}(u-u_h)\right\}\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau\\ =\int_{\Omega_1}\beta_1\nabla(P_hu-u)\cdot\nabla v_{h_1}\ dx &+ \int_\Gamma(\tilde{\Pi}_h\lambda-\lambda)v_{h_1}\ d\tau+\sigma\int_\Gamma\gamma(\tilde{\Pi}_h\lambda-\lambda)\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau,\end{aligned} \quad (4.3.44)$$

and hence,

$$\begin{aligned}\int_\Gamma(\tilde{\Pi}_h\lambda-\lambda_h)v_{h_1}\ d\tau &= \int_{\Omega_1}\beta_1\nabla(u_h-P_hu)\cdot\nabla v_{h_1}\ dx+\sigma\int_\Gamma\gamma(\lambda_h-\tilde{\Pi}_h\lambda)\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau\\ &+ \int_{\Omega_1}\beta_1\nabla(P_hu-u)\cdot\nabla v_{h_1}\ dx+\int_\Gamma(\tilde{\Pi}_h\lambda-\lambda)v_{h_1}\ d\tau\\ &+ \sigma\int_\Gamma\gamma(\tilde{\Pi}_h\lambda-\lambda)\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau\\ &+ \sigma\int_\Gamma\gamma\left\{\beta\frac{\partial}{\partial n}(u-u_h)\right\}\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau.\end{aligned} \quad (4.3.45)$$

Note that by using Lemma 4.3.2, we obtain an estimate for the second term on the right hand side of (4.3.45) as

$$\begin{aligned}\left|\int_\Gamma\gamma(\lambda_h-\tilde{\Pi}_h\lambda)\left\{\beta\frac{\partial v_{h_1}}{\partial n}\right\}d\tau\right| &\leq C\|\gamma^{1/2}(\lambda_h-\tilde{\Pi}_h\lambda)\|_{L^2(\Gamma)}\left\|\gamma^{1/2}\left(\beta\frac{\partial v_{h_1}}{\partial n}\right)\right\|_{L^2(\Gamma)}\\ &\leq Ch\sum_{i=1}^{2}\|u_i\|_{H^2(\Omega_i)}\|\nabla v_{h_1}\|_{L^2(\Omega_1)}.\end{aligned} \quad (4.3.46)$$

Now for the rest of the terms on the right-hand side of (4.3.45) we use Cauchy Schwarz inequality, duality pairing between $H^{-1/2}$ and $H^{1/2}$, the trace inequality (Theorem 1.2.2) along with the standard approximation properties (3.3.17), (3.3.19), (4.3.21), and similarly as in (4.3.46) we arrive at

$$\int_\Gamma (\tilde{\Pi}_h\lambda - \lambda_h) v_{h_1} d\tau \le Ch \sum_{i=1}^{2} \|u_i\|_{H^2(\Omega_i)} \|v_{h_1}\|_{H^1(\Omega_1)}. \tag{4.3.47}$$

Applying (4.3.47), for the third term on the right-hand side of (4.3.35), with $v_h = (v_{h_1}, 0) \in X_h$ and $v_{h_1} = R_h\left(\beta\frac{\partial I_h\psi}{\partial n}\right)$, where $R_h$ is the continuous lifting operator defined in Lemma 1.3.5, we obtain for $\gamma = O(h)$

$$\begin{aligned} \gamma\int_\Gamma (\lambda - \lambda_h)\left(\beta\frac{\partial I_h\psi}{\partial n}\right) d\tau &\le C\gamma h \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \left\|\left(\beta\frac{\partial I_h\psi}{\partial n}\right)\right\|_{H^{1/2}(\Gamma)} \\ &\le C\gamma h \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|I_h\psi_i\|_{H^1(\Omega_i)} \\ &\le Ch^2 \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|\psi_i\|_{H^2(\Omega_i)}. \end{aligned} \tag{4.3.48}$$

Similarly for the last term on the right-hand side of (4.3.35), $v_{h_1} = R_h\left(\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right)$, we arrive at the estimate for $\gamma = O(h)$

$$\begin{aligned} \gamma\int_\Gamma (\lambda - \lambda_h)\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right) d\tau &\le C\gamma h \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \left\|\tilde{\Pi}_h\left(\beta\frac{\partial\psi}{\partial n}\right)\right\|_{H^{1/2}(\Gamma)} \\ &\le Ch^2 \sum_{i=1}^{2} \|u\|_{H^2(\Omega_i)} \|\psi_i\|_{H^2(\Omega_i)}. \end{aligned} \tag{4.3.49}$$

From (4.3.35)-(4.3.41), (4.3.48) and using the regularity condition (4.3.32), we obtain $L^2$-error estimate (4.3.22). This completes the proof. ■

## 4.4 Parabolic Initial and Boundary Value Problems

In this section, we study both the semidiscrete and fully discrete methods for the parabolic initial and boundary value problem with discontinuous coefficients. We follow the same assumptions and notations as in the last sections.

Consider the following parabolic initial-boundary value problem with discontinuous coefficients: Given $f \in L^2(\Omega)$, find $u_i = u_i(x,t)$ satisfying

$$u_{i_t} - \nabla \cdot (\beta(x)\nabla u_i) = f \ \text{ in } \ \Omega_i \times (0,T], \tag{4.4.1}$$

$$u_i(x,t) = 0 \ \text{ on } \ (\partial\Omega_i \cap \partial\Omega) \times [0,T], \tag{4.4.2}$$

$$u_i(x,0) = u_{i0}(x) \ \text{ in } \ \Omega \tag{4.4.3}$$

$$[[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along } \ \Gamma. \tag{4.4.4}$$

### 4.4.1 Semidiscrete method

By integrating by parts, introducing the flux $\lambda = -\beta_1 \dfrac{\partial u_1}{\partial n_1} = \beta_2 \dfrac{\partial u_2}{\partial n_2}$, we can derive the Lagrange multiplier method for the interface problem (4.4.1)-(4.4.4) as follows. Find $(u(\cdot,t), \lambda(\cdot,t)) \in X \times M$ such that for $t \in (0,T]$

$$(u_t, v) + a(u,v) + b(v,\lambda) = \sum_{i=1}^{2} \int_{\Omega_i} fv \ dx \qquad \forall \ v \in X, \tag{4.4.5}$$

$$b(u,\mu) = 0 \qquad \forall \mu \in M, \tag{4.4.6}$$

$$(u(0),v) = (u_0, v). \tag{4.4.7}$$

Now we propose the *stabilized Nitsche's Mortaring method* for the parabolic problem (4.4.5)-(4.4.7): Find $(u_h(t), \lambda_h(t)) \in X_h \times W^h(\Gamma)$ such that for $t \in (0,T]$

$$\begin{aligned} (u_{h_t}, v_h) &+ \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_h \cdot \nabla v_h \ dx + \int_\Gamma \lambda_h [[v_h]] \ d\tau + \int_\Gamma \gamma \lambda_h \left\{\beta \frac{\partial v_h}{\partial n}\right\} d\tau \\ &- \int_\Gamma \gamma \left\{\beta \frac{\partial u_h}{\partial n}\right\} \left\{\beta \frac{\partial v_h}{\partial n}\right\} d\tau = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \ dx \ \forall v_h \in X_h, \end{aligned} \tag{4.4.8}$$

$$\int_\Gamma [[u_h]]\mu_h \ d\tau + \int_\Gamma \gamma \left\{\beta \frac{\partial u_h}{\partial n}\right\} \mu_h \ d\tau - \int_\Gamma \gamma \lambda_h \mu_h d\tau = 0. \qquad \forall \mu_h \in W^h(\Gamma) \tag{4.4.9}$$

$$u_h(0) = u_{h0}, \tag{4.4.10}$$

where $u_{h0}$ is an approximation of $u$ in $X_h$ to be defined later. Equivalently, (4.4.8)-(4.4.9) can be written as: Find $(u_h(t), \lambda_h(t)) \in X_h \times W^h(\Gamma)$ such that such that for $t \in (0,T]$

$$(u_{h_t}, v_h) + \mathcal{A}(u_h, \lambda_h; v_h, \mu_h) = \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \ \ \mu_h \in W^h(\Gamma). \tag{4.4.11}$$

where

$$\begin{aligned}\mathcal{A}(w_h,\nu_h;v_h,\mu_h) &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla w_h\cdot\nabla v_h\ dx+\int_\Gamma \nu_h[[v_h]]\ d\tau+\int_\Gamma\gamma\nu_h\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau\\ &- \int_\Gamma\gamma\left\{\beta\frac{\partial w_h}{\partial n}\right\}\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau-\int_\Gamma[[w_h]]\mu_h\ d\tau\\ &- \int_\Gamma\gamma\left\{\beta\frac{\partial w_h}{\partial n}\right\}\mu_h\ d\tau+\int_\Gamma\gamma\nu_h\mu_h d\tau, \end{aligned}\tag{4.4.12}$$

and

$$\mathcal{F}(v_h)=\sum_{i=1}^{2}\int_{\Omega_i}fv_h\ dx. \tag{4.4.13}$$

(4.4.11) leads to a system of linear ordinary equations and an application of Picard's existence theorem yields the existence and uniqueness of solution on $[0,T]$.

### 4.4.2 Error Analysis

In this section, we discuss the error estimates for the scheme (4.4.11). Subtracting (4.4.8)-(4.4.9) from (4.4.5)-(4.4.6), we obtain the error equation such that for all $(v_h,\mu_h)\in X_h\times W^h(\Gamma)$,

$$\begin{aligned}(u_t-u_{h_t},v_h)+\sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla(u-u_h)\cdot\nabla v_h\ dx+\int_\Gamma(\lambda-\lambda_h)[[v_h]]\ d\tau\\ +\int_\Gamma\gamma(\lambda-\lambda_h)\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau-\int_\Gamma\gamma\left\{\beta\frac{\partial(u-u_h)}{\partial n}\right\}\left\{\beta\frac{\partial v_h}{\partial n}\right\}d\tau=0,\end{aligned}\tag{4.4.14}$$

$$\int_\Gamma[[(u-u_h)]]\mu_h\ d\tau+\int_\Gamma\gamma\left\{\beta\frac{\partial(u-u_h)}{\partial n}\right\}\mu_h\ d\tau-\int_\Gamma\gamma(\lambda-\lambda_h)\mu_h\ d\tau=0. \tag{4.4.15}$$

Equivalently (4.4.14)-(4.4.15) can be written as

$$(u_t-u_{h,t},v_h)+\mathcal{A}(u-u_h,\lambda-\lambda_h;v_h,\mu_h)=0\qquad\forall v_h\in X_h,\mu_h\in W^h(\Gamma). \tag{4.4.16}$$

For the error estimates, we now define mixed elliptic projection as follows. For given $u$ and $\lambda$, set $(\hat{u}_h,\hat{\lambda}_h)$ as solution of elliptic part of (4.4.11), i.e.,

$$\mathcal{A}(u-\hat{u}_h,\lambda-\hat{\lambda}_h;v_h,\mu_h)=0\qquad\forall v_h\in X_h,\mu_h\in W^h(\Gamma). \tag{4.4.17}$$

Set

$$u - u_h = (u - \hat{u}_h) + (\hat{u}_h - u_h) = \rho + \theta \tag{4.4.18}$$

and

$$\lambda - \lambda_h = (\lambda - \hat{\lambda}_h) + (\hat{\lambda}_h - \lambda_h) = \eta + \xi. \tag{4.4.19}$$

Since estimates of $\rho$ and $\eta$ are known from Theorem 4.3.2, it is enough to estimate $\theta$ and $\xi$. From the equations (4.4.18), (4.4.19), (4.4.16) and using the elliptic projection (4.4.17), we obtain

$$(\theta_t, v_h) + \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla\theta \cdot \nabla v_h \; dx + \int_\Gamma \xi[[v_h]] \; d\tau + \int_\Gamma \gamma\xi \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau$$
$$- \int_\Gamma \gamma \left\{ \beta \frac{\partial \theta}{\partial n} \right\} \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau = -(\rho_t, v_h), \tag{4.4.20}$$
$$\int_\Gamma [[\theta]]\mu_h \; d\tau + \int_\Gamma \gamma \left\{ \beta \frac{\partial \theta}{\partial n} \right\} \mu_h \; d\tau - \int_\Gamma \gamma\xi\mu_h \; d\tau = 0. \tag{4.4.21}$$

Substitute $v_h = \theta$ in (4.4.20), $\mu_h = \xi$ in (4.4.21). Then subtract (4.4.21) from (4.4.20) and apply coercivity of $\mathcal{A}(\cdot, \cdot; \cdot, \cdot)$. Then using Young's inequality (1.2.2) and Poincaré inequality (Theorem 1.2.1), we now arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \alpha |||(\theta, \xi)|||^2 &\leq \|\rho_t\|\|\theta\| \\ &\leq C(\alpha)\|\rho_t\|^2 + \frac{\alpha}{2}|||(\theta, \xi)|||^2, \end{aligned}$$

and hence,

$$\frac{d}{dt}\|\theta\|^2 + \alpha |||(\theta, \xi)|||^2 \leq C(\alpha)\|\rho_t\|^2. \tag{4.4.22}$$

Integrating (4.4.22) from 0 to $t$, we find

$$\|\theta(t)\|^2 + \alpha \int_0^t |||(\theta, \xi)|||^2 d\tau \leq C(\alpha) \left( \|\theta(0)\|^2 + \int_0^t \|\rho_t\|^2 d\tau \right). \tag{4.4.23}$$

Now choose $u_{0,h} = \hat{u}_h(0)$, then $\theta(0) = 0$, otherwise with $u_{0,h} = I_h u_0$

$$\begin{aligned} \|\theta(0)\| = \|\hat{u}_h(0) - u_{0,h}\| &\leq \|u_0 - I_h u_0\| + \|\hat{u}_h(0) - u_0\| \\ &\leq Ch^2 \sum_{i=1}^{2} \|u_0\|_{H^2(\Omega_i)}. \end{aligned} \tag{4.4.24}$$

From Theorem 4.3.2, we obtain

$$\|\rho_t\| = \|u_t - \hat{u}_t\| \le Ch^2 \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}. \tag{4.4.25}$$

Substituting (4.4.24) and (4.4.25) in (4.4.23), we find that

$$\|\theta(t)\|^2 + \alpha \int_0^t |||(\theta,\xi)|||^2 d\tau \le Ch^4 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)}\, ds \right). \tag{4.4.26}$$

Using triangle inequality and Theorem 4.3.2, we arrive at

$$\|u(t) - u_h(t)\|^2 \le Ch^4 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)}\, ds \right), \tag{4.4.27}$$

For a bound in $|||(\cdot,\cdot)|||$-norm, substitute $v_h = \theta_t$ in (4.4.20), differentiate the equation (4.4.21), put $\mu_h = \xi$ in (4.4.21) and proceed in similar way as in the proof of Theorem 3.4.2 to arrive at

$$|||(\theta(t),\xi(t))|||^2 \le \frac{C}{t} h^4 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \left( \tau \|u_t\|^2_{H^2(\Omega_i)} + \|u_t\|^2_{H^2(\Omega_i)} \right) d\tau \right). \tag{4.4.28}$$

Finally, we apply triangle inequality. The result obtained is stated as a theorem below.

**Theorem 4.4.1** *Let $(u,\lambda)$ and $(u_h,\lambda_h)$ be the solutions of (4.4.5)-(4.4.7) and (4.4.8)-(4.4.10), respectively. Further, let $u_{|\Omega_i}, u_{t|\Omega_i} \in H^2(\Omega_i)$. Then with $u_{0,h} = \hat{u}_0$ or $I_h u_0$, there exists a positive constant $C$, independent of $h$, such that for $t \in (0,T]$,*

$$\|u(t) - u_h(t)\|^2 \le Ch^4 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \|u_t\|^2_{H^2(\Omega_i)}\, ds \right), \tag{4.4.29}$$

*and,*

$$|||((u-u_h,\lambda-\lambda_h)(t))|||^2 \le \frac{C}{t} h^2 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^t \left( \tau \|u_t\|^2_{H^2(\Omega_i)} + \|u_t\|^2_{H^2(\Omega_i)} \right) d\tau \right). \tag{4.4.30}$$

### 4.4.3 Fully discrete method

Let $k$ be the time step parameter $k = \dfrac{T}{N}$ and $t_n = nk$. For a continuous functions $\varphi \in C[0,T]$, we set the backward difference quotient as $\bar{\partial}_t \varphi^n = \dfrac{\varphi^n - \varphi^{n-1}}{k}$. The backward Euler approximation of $(u, \lambda)$ is to seek a pair of functions $(U^n, \Lambda^n) \in X_h \times W^h(\Gamma)$ so that $(U^n, \Lambda^n),\ n \geq 1$, satisfies

$$
\begin{aligned}
(\bar{\partial}_t U^n, v_h) \quad &+ \quad \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla U^n \cdot \nabla v_h \, dx + \int_{\Gamma} \Lambda^n [[v_h]] \, d\tau + \int_{\Gamma} \gamma \Lambda^n \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau \\
&- \quad \int_{\Gamma} \gamma \left\{ \beta \frac{\partial U^n}{\partial n} \right\} \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \, dx \ \forall v_h \in X_h \qquad (4.4.31) \\
\int_{\Gamma} [[U^n]] \mu_h \, d\tau \quad &+ \quad \int_{\Gamma} \gamma \left\{ \beta \frac{\partial U^n}{\partial n} \right\} \mu_h \, d\tau - \int_{\Gamma} \gamma \Lambda^n \mu_h = 0. \qquad \forall \mu_h \in W^h(\Gamma) \qquad (4.4.32) \\
&U^0 = u_{h0} \qquad (4.4.33)
\end{aligned}
$$

Equivalently, (4.4.31)-(4.4.32) can be written as: for $n \geq 1$, find $(U^n, \Lambda^n) \in X_h \times W^h(\Gamma)$ such that

$$
(\bar{\partial}_t U^n, v_h) + \mathcal{A}(U^n, \Lambda^n; v_h, \mu_h) \ = \ \mathcal{F}(v_h) \qquad \forall v_h \in X_h, \mu_h \in W^h(\Gamma). \qquad (4.4.34)
$$

**Theorem 4.4.2** *Let $(u(t_n), \lambda(t_n))$ be the solution of (4.4.5)-(4.4.7) and let $(U^n, \Lambda^n) \in X_h \times W^h(\Gamma)$ be an approximation of $(u(t), \lambda(t))$ at $t = t_n$ as given by (4.4.34). Further, assume that for $t \in (0, T]$, $u(t) \in H_0^1(\Omega)$, $u(t)_{|\Omega_l}, u_t(t)_{|\Omega_l} \in H^2(\Omega_l)$ and $u_{tt}(t) \in L^2(\Omega)$. Then with $u_{0,h} = \hat{u}_{h0}$ or $I_h u_0$, there exists a positive constants $C$, independent of $h$ and $k$, such that*

$$
\|u(t_n) - U^n\|^2 \leq C \left( h^4 \sum_{i=1}^{2} \left( \|u_0\|^2_{H^2(\Omega_i)} + \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_i)} \right) + k^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds \right) \qquad (4.4.35)
$$

*and,*

$$
\begin{aligned}
|||(u(t_n) - U^n, \lambda(t_n) - \Lambda^n)|||^2 \leq C \Bigg( h^2 \sum_{i=1}^{2} \Big( \|u_0\|^2_{H^2(\Omega_i)} \quad &+ \quad \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_i)} \Big) \\
&+ \quad k^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds \Bigg) \qquad (4.4.36)
\end{aligned}
$$

**Proof.** For the error estimate we split the error terms as follows:

$$u(t_n) - U^n = (u(t_n) - \hat{u}_h(t_n)) + (\hat{u}_h(t_n) - U^n) = \rho^n + \theta^n. \tag{4.4.37}$$

and

$$\lambda(t_n) - \Lambda^n = (\lambda(t_n) - \hat{\lambda}_h(t_n)) + (\hat{\lambda}_h(t_n) - \Lambda^n) = \eta^n + \xi^n. \tag{4.4.38}$$

Since the estimates of $\rho^n$ and $\eta^n$ are known from Theorem 4.3.2, it is sufficient to estimate $\theta^n$ and $\xi^n$. Using (4.4.37)-(4.4.38) in (4.4.31)-(4.4.32), elliptic projection (4.4.16), we arrive at

$$\begin{aligned}(\bar{\partial}_t\theta^n, v_h) \; &+ \; \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\nabla\theta^n\cdot\nabla v_h\, dx + \int_\Gamma \xi^n[[v_h]]\, d\tau + \int_\Gamma \gamma\xi^n\left\{\beta\frac{\partial v_h}{\partial n}\right\} d\tau \\ &- \; \int_\Gamma \gamma\left\{\beta\frac{\partial\theta^n}{\partial n}\right\}\left\{\beta\frac{\partial v_h}{\partial n}\right\} d\tau = (w^n, v_h) \qquad \forall v_h \in X_h \qquad (4.4.39)\\ \int_\Gamma [[\theta^n]]\mu_h\, d\tau \; &+ \; \int_\Gamma \gamma\left\{\beta\frac{\partial\theta^n}{\partial n}\right\}\mu_h\, d\tau - \int_\Gamma \gamma\xi^n\mu_h = 0. \qquad \forall\mu_h W^h(\Gamma) \qquad (4.4.40)\end{aligned}$$

where,

$$\begin{aligned} w^n = \bar{\partial}_t\hat{u}_h(t_n) - u_t(t_n) &= (\bar{\partial}_t\hat{u}_h(t_n) - \bar{\partial}_t u(t_n)) + (\bar{\partial}_t u(t_n) - u_t(t_n)) \\ &= w_1^n + w_2^n. \end{aligned} \tag{4.4.41}$$

Choose $v_h = \theta^n$ in (4.4.39), $\mu_h = \xi^n$ in (4.4.40) and subtract the resulting equation. Note that

$$(\bar{\partial}_t\theta^n, \theta^n) = \frac{1}{2}\bar{\partial}_t\|\theta^n\|^2 + \frac{k}{2}\|\bar{\partial}_t\theta^n\|^2 \geq \frac{1}{2}\bar{\partial}_t\|\theta^n\|^2.$$

Using the Cauchy-Schwarz inequality, ellipticity (4.3.10) and Young's inequality (1.2.2), we find that

$$\begin{aligned}\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2 + |||(\theta^n, \xi^n)|||^2 &\leq C\|w^n\|\|\theta^n\| \leq \|w^n\|\|\theta^n\|_X \\ &\leq C(\alpha)\|w^n\|^2 + \frac{\alpha}{2}|||(\theta^n, \xi^n)|||^2,\end{aligned}$$

and hence,

$$\bar{\partial}_t\|\theta^n\|^2 + \alpha|||(\theta^n, \xi^n)|||^2 \leq C(\alpha)\|w^n\|^2.$$

Using the definition of $\bar{\partial}_t$, we arrive at

$$\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + Ck\|w^n\|^2, \tag{4.4.42}$$

and hence, by repeated application, we obtain

$$\|\theta^n\|^2 \leq \|\theta^0\|^2 + Ck(\sum_{j=1}^{n}\|w_1^j\|^2 + \sum_{j=1}^{n}\|w_2^j\|^2). \tag{4.4.43}$$

Choose $u_{0,h} = \hat{u}_h(0)$, then $\theta^0 = 0$, otherwise with $u_{0,h} = I_h u_0$

$$\begin{aligned}\|\theta^0\| = \|\hat{u}_h(0) - u_{0,h}\| &\leq \|u_0 - I_h u_0\| + \|\hat{u}_h(0) - u_0\| \\ &\leq Ch^2 \sum_{i=1}^{2} \|u_0\|_{H^2(\Omega_i)}. \end{aligned} \tag{4.4.44}$$

Since

$$w_1^j = \bar{\partial}_t \hat{u}_h(t_j) - \bar{\partial}_t u(t_j) = k^{-1}\int_{t_{j-1}}^{t_j} (\hat{u}_{ht} - u_t)ds,$$

we now find that

$$k\sum_{j=1}^{n}\|w_1^j\|^2 \leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|(\hat{u}_{ht} - u_t)\|^2 ds \quad \leq \quad C\sum_{l=1}^{K} h^4 \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_i)} ds. \tag{4.4.45}$$

To estimate $w_2^j$, we note using Taylor series expansion that

$$\begin{aligned} w_2^j = \bar{\partial}_t u(t_j) - u_t(t_j) &= k^{-1}(u(t_j) - u(t_{j-1})) - u_t(t_j) \\ &= -k^{-1}\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)ds, \end{aligned}$$

and hence,

$$k\sum_{j=1}^{n}\|w_2^j\|^2 \leq \sum_{j=1}^{n}\left(\int_{t_{j-1}}^{t_j} |s - t_{j-1}|\|u_{tt}\|ds\right)^2 \leq Ck^2\int_0^{t_n}\|u_{tt}\|^2 ds. \tag{4.4.46}$$

Substituting (4.4.44)-(4.4.46) in (4.4.43), we obtain

$$\|\theta^n\|^2 \leq C\left(h^4\sum_{i=1}^{2}\|u_0\|^2_{H^2(\Omega_i)} + \int_0^{t_n}\|u_t\|^2_{H^2(\Omega_i)} + k^2\int_{t=0}^{t_n}\|u_{tt}\|^2 ds\right). \tag{4.4.47}$$

An application of triangle inequality with (4.4.47) yields

$$\|u(t_n) - U^n\|^2 \leq C\left(h^4 \sum_{i=1}^{2}\left(\|u_0\|^2_{H^2(\Omega_i)} + \int_0^{t_n} \|u_t\|^2_{H^2(\Omega_i)}\right) + k^2 \int_{t=0}^{t_n} \|u_{tt}\|^2 ds\right). \quad (4.4.48)$$

In order to estimate in $|||(\cdot,\cdot)|||$-norm, substitute $v_h = \bar{\partial}_t \theta^n$ in (4.4.39), then proceed similar way as in Theorem 3.4.2 to obtain (4.4.36). ■

**Remark 4.4.1** *Since our scheme is exactly consistent with the original problem, we derive the optimal estimates in $L^2$-norm as well as in $|||(\cdot,\cdot)|||$-norm when $\gamma = O(h)$.*

## 4.5 Matrix Formulation

In this section, we discuss the algebraic formulation arising from the discrete formulation (4.3.2)-(4.3.3). Here in this case also, construction of basis functions are similar to that in Section 2.7 of Chapter 2, with a slight modification for the Lagrange multiplier space. The matrix representation for (4.3.3) can be given by:

$$M_s u_s + \frac{\gamma}{2} N^s u_s + \frac{\gamma}{2} N^m u_m - M_m u_m = \epsilon M_{mm} \lambda_m. \quad (4.5.1)$$

Here, $M_s$, $M_m$ and $M_{mm}$ are given by $(m_s)_{ij} = \int_{\Gamma_1} \varphi^s_j \psi_{h_i} d\tau$, $(m_m)_{ij} = \int_{\Gamma_2} \varphi^m_j \psi_{h_i} d\tau$ and $(\tilde{m}_m)_{ij} = \int_{\Gamma_2} \psi_{h_i} \psi_{h_j} d\tau$ respectively. Here, $\psi_{h_i}$ are the nodal basis functions for $W^h(\Gamma_2)$, $\varphi^s_j$ and $\varphi^m_j$ denote the basis functions for $W^h(\Gamma_1)$ and $W^h(\Gamma_2)$, respectively. Moreover, let $N^s$, $N^m$, $N^s_n$ and $N^m_n$ are defined by $N^s_{ij} = \int_{\Gamma_1} \varphi^s_j \cdot n_1 \psi_{h_i} d\tau$, $N^m_{ij} = \int_{\Gamma_2} \varphi^m_j \cdot n_2 \psi_{h_i} d\tau$, $N^s_{n_{ij}} = \int_{\Gamma_1} \varphi^s_i \cdot n_1 \varphi^s_j \cdot n_1$, and $N^m_{n_{ij}} = \int_{\Gamma_2} \varphi^m_i \cdot n_2 \varphi^m_j \cdot n_2$, respectively.

Now, the matrix representation of (4.3.2)-(4.3.3) can be given as:

$$A\alpha = F, \quad (4.5.2)$$

where $\alpha = (u^1_i, u^1_s, u^2_i, u^2_m, \lambda_m)^T$. Note that here $u^1_i$ and $u^2_i$ represents the unknowns associated with all the internal nodal points in $\Omega_1$ and $\Omega_2$, respectively. Further, $u^1_s$ and $u^2_m$ are the unknowns associated with $\Gamma_1$ and $\Gamma_2$ and $\lambda'_m$s are the unknowns for Lagrange multipliers associated with $W^h(\Gamma)$.

$$A = \begin{pmatrix} A^1_{ii} & A^1_{is} & 0 & 0 & 0 \\ A^1_{si} & A^1_{ss} & 0 & 0 & M_s + \frac{\gamma\sigma}{2}(N^s) + \frac{\gamma\sigma}{2}N^s_n \\ 0 & 0 & A^2_{ii} & A^2_{im} & 0 \\ 0 & 0 & A^2_{mi} & A^2_{mm} & -(M_m)^T + \frac{\gamma\sigma}{2}(N^m)^T + \frac{\gamma\sigma}{2}(N^m_n)^T \\ 0 & M_s + \frac{\gamma}{2}N^s & 0 & -M_m + \frac{\gamma}{2}N^m & -\gamma M_{mm} \end{pmatrix}, \quad (4.5.3)$$

with

$$\begin{aligned} A^l_{ii} &= \{a(\varphi^{(l)}_i, \varphi^{(l)}_j)\} \qquad x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ A^l_{is} &= \{a(\varphi^{(l)}_i, \varphi_s)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1. \\ A^1_{ss} &= \{a(\varphi_s, \varphi_s)\} \qquad x_s \in \Gamma_1. \\ A^l_{im} &= \{a(\varphi^{(l)}_i, \varphi_m)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2, \\ A^2_{mm} &= \{a(\varphi_m, \varphi_m)\} \qquad x_m \in \Gamma_2, \end{aligned}$$

and

$$F = \begin{pmatrix} F^1_i & F^1_s & F^2_i & F^2_m & 0 \end{pmatrix}^T, \quad (4.5.4)$$

where

$$\begin{aligned} F^l_i &= \{(f^{(l)}_i, \varphi^{(l)}_j)\} \qquad x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ F^l_s &= \{(f^{(l)}_i, \varphi_s)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1, \\ F^l_m &= \{(f^{(l)}_i, \varphi_m)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2, \end{aligned}$$

For fully discrete formulation (4.4.31)-(4.4.32), the matrix formulation associated can be given as:

$$(B + kA)\alpha^n = B\alpha^{n-1} + kF(t_n), \qquad n \ge 1, \quad (4.5.5)$$

with $\alpha = (u^1_i, u^1_s, u^2_i, u^2_m, \lambda_m)^T$.

Here $A$ and $F$ are same as in (4.5.3) and (4.5.4), respectively and $B$ has the following form:

$$B = \begin{pmatrix} B^1_{ii} & B^1_{is} & 0 & 0 & 0 \\ B^1_{si} & B^1_{ss} & 0 & 0 & M_s + \frac{\gamma\sigma}{2}(N^s) + \frac{\gamma\sigma}{2}N^s_n \\ 0 & 0 & B^2_{ii} & B^2_{im} & 0 \\ 0 & 0 & B^2_{mi} & B^2_{mm} & -(M_m)^T + \frac{\gamma\sigma}{2}(N^m)^T + \frac{\gamma\sigma}{2}(N^m_n)^T \\ 0 & M_s + \frac{\gamma}{2}N^s & 0 & -M_m + \frac{\gamma}{2}N^m & -\gamma M_{mm} \end{pmatrix}, \quad (4.5.6)$$

where

$$\begin{aligned} B^l_{ii} &= \{(\varphi^{(l)}_i, \varphi^{(l)}_j)\} \qquad x^l_i, x^l_j \in \mathcal{T}_h(\Omega_l),\ 1 \le l \le 2, \\ B^l_{is} &= \{B(\varphi^{(l)}_i, \varphi_s)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_s \in \Gamma_1, \\ B^1_{ss} &= \{B(\varphi_s, \varphi_s)\} \qquad x_s \in \Gamma_1, \\ B^l_{im} &= \{(\varphi^{(l)}_i, \varphi_m)\} \qquad x^l_i \in \mathcal{T}_h(\Omega_l), x_m \in \Gamma_2. \\ B^2_{mm} &= \{(\varphi_m, \varphi_m)\} \qquad x_m \in \Gamma_2. \end{aligned}$$

### 4.5.1 Numerical Experiments

We choose the following second order elliptic problem on the unit square domain $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary condition and homogeneous jump conditions as follows:

$$\begin{aligned} -\nabla \cdot (\beta_i(x)\nabla u_i) &= f \ \text{ in } \ \Omega_i, \\ u_i &= 0 \ \text{ on } \ \partial\Omega \cap \partial\Omega_i, \\ [[u]]_\Gamma = 0, \quad \left[\left[\beta\frac{\partial u}{\partial n}\right]\right]_\Gamma &= 0 \quad \text{along } \ \Gamma. \end{aligned}$$

The computational domain $\Omega$ is subdivided into two equal subdomains $\Omega_i,\ i = 1,2$. Each subdomain is subdivided into linear triangular elements of different mesh sizes $h_i$. Here, we take $h_1 = 1/14$ and $h_2 = 1/16$. With the penalty parameter $\epsilon = O(h)$, we choose discontinuous coefficients with $\beta_1 = 1$ and $\beta_2 = 10$ in two subdomains. We choose $f$ such that the exact solution is $u(x,y) = \sin\pi x \sin\pi y$.

The order of convergence for the error $e = (u - u_h)$ in $L^2$ norm '$p$' with respect to the space variable parameter $h$ has been computed in Table 4.1. Figure 4.2 shows the computed

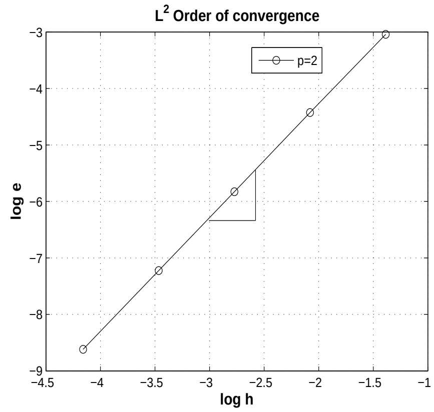



Figure 4.2: Order of Convergence w.r.t. $h$

Table 4.1: $L^2$ Order of Convergence

| $(h_1, h_2)$ | $h = \max_i h_i$ | $\|u - u_h\|_{L^2(\Omega)}$ | Order |
|---|---|---|---|
| $(\frac{1}{4}, \frac{1}{6})$ | 1/4 | 0.047767 | |
| $(\frac{1}{8}, \frac{1}{12})$ | 1/8 | 0.011946 | 1.99948664449448 |
| $(\frac{1}{16}, \frac{1}{24})$ | 1/16 | 0.0029445 | 2.02043304825433 |
| $(\frac{1}{32}, \frac{1}{48})$ | 1/32 | 0.00072768 | 2.01664660906980 |
| $(\frac{1}{64}, \frac{1}{96})$ | 1/64 | 0.00018066 | 2.01002704544642 |

order of convergence with respect to $h$ for $\|u - u_h\|$ in the log-log scale. The computed order of convergence '$p$' matches with the theoretical result (See, Theorem 4.3.2).

For the second order parabolic initial and boundary value problem, consider the unit square domain $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary condition and homogeneous jump conditions. Consider the following parabolic initial-boundary value problem with discontinuous coefficients: Find $u_i = u_i(x,t)$ satisfying

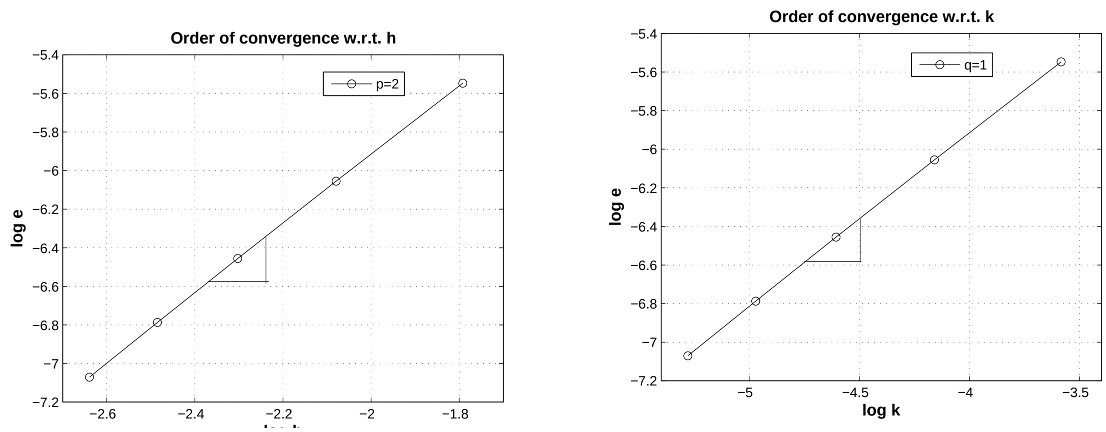
$$u_{i_t} - \nabla \cdot (\beta(x) \nabla u_i) = f \quad \text{in} \quad \Omega_i \times (0,1], \tag{4.5.7}$$

$$u_i(x,t) = 0 \quad \text{on} \quad (\partial\Omega_i \cap \partial\Omega) \times [0,1], \tag{4.5.8}$$

$$u_i(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{4.5.9}$$

$$[[u]]_\Gamma = 0, \quad \left[\left[\beta \frac{\partial u}{\partial n}\right]\right]_\Gamma = 0 \quad \text{along} \quad \Gamma. \tag{4.5.10}$$

The computational domain $\Omega$ is subdivided into two equal subdomains $\Omega_i, \; i = 1, 2$. Each




(a) Order of Convergence w.r.t. discretization parameter $h$

(b) Order of Convergence w.r.t. $k$

Figure 4.3: Order of Convergence with discontinuous coefficients

subdomain is subdivided into linear triangular elements of different mesh sizes $h_i$. With $u_0 = 0$, we consider the discontinuous coefficients along the common interfaces of subdomain. We take the coefficients $(\beta_1, \beta_2) = (1, 10)$. We choose $f$ in such a way that the exact solution is $u(x,t) = e^t \sin \pi x \sin \pi y$. Figure 4.3(a) and 4.3(b) shows the computed order of convergence with respect to $h$ and $k$, respectively, for $\|u - u_h\|$ in the log-log scale, when the penalty parameter $\epsilon$ is taken as $O(h)$. The computed result matches our theoretical result (See, Theorem 4.4.2).

# Chapter 5

# Conclusions

## 5.1 Summary and Some Remarks

In this thesis, we have discussed mortar finite element methods for second order elliptic and parabolic problems. Since mortar finite element methods deal with the independent discretization over each subdomain of the original domain, this process of discretization leads to non-matching grids across the common interfaces of the inter subdomains. In this regard, the mortar finite element method is a locally conforming but globally nonconforming one. In Chapter 1, we have briefly reviewed *mortar finite element methods* for second order elliptic boundary value problems. Here, we have discussed an approximation $Q_h u$ of the solution defined in (1.3.36) and studied the error estimates (Lemma 1.3.6) for the approximation. With the help of this result and the $L^2$-orthogonal projection $\pi_{h_j}$ defined in (1.3.45), we deduce the error estimates (Theorem 1.3.1) for the method (1.3.9). The error estimates in $X$-norm (broken $H^1$-norm) and $L^2$-norm is optimal as in the case of standard finite element method. Next, instead of imposing the constraint (1.3.7) in the finite dimensional space $X_h$, we impose the weak continuity condition across the subdomain interfaces in the variational formulation by means of a Lagrange multiplier. Then optimal error estimates are derived in Theorem 1.3.2. The main contribution of the thesis starts from Chapter 2.

In Chapter 2, we have discussed a *standard mortar finite element method* and a *mortar element method with Lagrange multipliers* for the spatial discretization of a class of *parabolic initial-boundary value problems (2.1.1)-(2.1.3).* A basic approach in deriving error estimates in Chapter 2 is the introduction of a modified elliptic operator $P_h$ from $X$

onto $V_h$, which is defined as :

$$a(u - P_h u, \chi) - \sum_{m=1}^{m_0} \int_{\delta_m \subset \Gamma} a\nabla u \cdot n[[\chi]] d\tau = 0 \qquad \forall \chi \in V_h \tag{5.1.1}$$

where,

$$a(v, w) \;=\; \sum_{l=1}^{K} \int_{\Omega_l} a(x) \nabla v_l \cdot \nabla w_l \; dx.$$

After discussing *a priori* estimates for $u - P_h u$ in broken $H^1$-norm and $L^2$-norm (see, Lemma 2.3.1), optimal order of estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for the semidiscrete method is derived for the parabolic problem. Based on backward Euler method, a completely discrete scheme is analyzed. Further, we have discussed a mortar finite element method with Lagrange multipliers for parabolic problems. In order to find an estimate for the Lagrange multiplier, we have chosen some auxiliary discrete spaces for primal variables and Lagrange multipliers in such a way that they satisfy the LBB condition (See, Proposition 2.6.1). Using the LBB condition, we have derived optimal error estimates (Theorem 2.6.2) in the later part of Chapter 2. The analysis can be easily extended to parabolic problems with discontinuous coefficients and some numerical experiments are conducted to substantiate the theoretical findings.

In order to alleviate the discrete LBB condition, which is crucial for finding the estimates for the Lagrange multipliers discussed in Chapter 2, we have introduced a Nitsche mortaring method in Chapter 3. In Nitsche mortaring method, we have added a penalty term to the discrete formulation in order to establish the stability of the proposed method. We have analyzed the following stabilized problem for elliptic problem with discontinuous coefficients: find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_h \cdot \nabla v_h \; dx + \int_\Gamma \lambda_h [[v_h]] \; d\tau \;=\; \sum_{i=1}^{2} \int_{\Omega_i} f v_h \; dx \quad \forall v_h \in X_h, \tag{5.1.2}$$

$$\int_\Gamma [[u_h]] \mu_h \; d\tau - \epsilon \int_\Gamma \lambda_h \mu_h \; d\tau \;=\; 0 \qquad \forall \mu_h \in W^h(\Gamma), \tag{5.1.3}$$

where $\epsilon$ is a suitably chosen penalty parameter.

We have proved the existence of a unique solution $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ for the problem (5.1.2)-(5.1.3) and derived *a priori* estimates in broken $H^1$ and $L^2$-norms provided $\epsilon = O(h)$

and $u_i \in H^2(\Omega_i),\ i = 1, 2$. Moreover, we have extended this mortaring method to parabolic initial-boundary value problems with discontinuous coefficients. Both semidiscrete and fully discrete schemes have been discussed and error estimates have been derived when $\epsilon = O(h)$. Since the method (5.1.2)-(5.1.3) is inconsistent, in Chapter 3, we have obtained only sub-optimal order of estimates in $H^1$ and $L^2$-norm. This chapter concludes with some computational results.

Finally in Chapter 4, we have proposed a *stabilized Nitsche's mortaring element method*, which is consistent with the original problem. We have discussed the following discrete scheme: find $(u_h, \lambda_h) \in X_h \times W^h(\Gamma)$ such that for all $v_h \in X_h$ and $\mu_h \in W^h(\Gamma)$,

$$\begin{aligned} \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_h \cdot \nabla v_h \, dx &+ \int_\Gamma \lambda_h [[v_h]] \, d\tau + \sigma \int_\Gamma \gamma \lambda_h \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau \\ &- \sigma \int_\Gamma \gamma \left\{ \beta \frac{\partial u_h}{\partial n} \right\} \left\{ \beta \frac{\partial v_h}{\partial n} \right\} d\tau = \sum_{i=1}^{2} \int_{\Omega_i} f v_h \, dx, \quad (5.1.4) \\ \int_\Gamma [[u_h]] \mu_h \, d\tau &+ \int_\Gamma \gamma \left\{ \beta \frac{\partial u_h}{\partial n} \right\} \mu_h \, d\tau - \int_\Gamma \gamma \lambda_h \mu_h d\tau = 0, \quad (5.1.5) \end{aligned}$$

where the penalty parameter $\gamma$ is at our disposal. When $\sigma = 0$, the method (5.1.4)-(5.1.5) is unsymmetric and for $\sigma = 1$, this method is symmetric. We have studied both the symmetric and unsymmetric methods.

Under the assumption on the penalty parameter $\gamma$, that is, $\gamma = O(h)$ (See, Lemma 4.3.3), the method is shown to be stable with respect to the $|||(\cdot, \cdot)|||$-norm defined in (4.3.8). Note that, we have used a natural choice for the discrete space of Lagrange multipliers that is, $W^h(\Gamma)$, the trace space of primal variables $X_h$. After proving the existence of a unique solution to the discrete problem, we have established in this chapter optimal order of estimates with respect to $|||(\cdot, \cdot)|||$-norm and $L^2$-norm for both symmetric and unsymmetric cases (See, Theorem 4.3.1) when $\gamma = O(h)$. We have also analyzed the *Nitsche's mortaring element method* for the parabolic initial-boundary value problems with discontinuous coefficients. Using the elliptic projection (4.3.7), with $\gamma = O(h)$, we have derived optimal order of estimates for both semidiscrete case, (see, Theorem 4.4.1) and fully discrete case (see, Theorem 4.4.2). Moreover, we have derived error estimates for the Lagrange multiplier. This plays a crucial role when we apply domain decomposition method combined with mortar method. Note that, for the parabolic problems, we have only studied the case $\sigma = 1$, that is, when the method (5.1.4)-(5.1.5) is symmetric. For the unsymmetric case, there are certain difficulties in deriving the error estimates. Computational results are discussed at the end of Chapter 4. Compared to Chapter 3, in Chapter 4, we have derived optimal order of convergence for $\|u - u_h\|_{L^2(\Omega)}$, that is of order $O(h^2)$ if the penalty parameter is of order $h$.

## 5.2 Extensions and Future Problems

In this thesis, we have studied mortar element methods for second order elliptic boundary value problems and parabolic initial boundary value problems in two dimensional cases. The results, however, can be extended to problems in three space-dimensions by making appropriate modifications. Mortar element methods for three dimensional problems have been discussed and analyzed in [17, 30]. We can also extend it to non-selfadjoint elliptic boundary value problems. In this thesis, we limit ourselves to $h$-version of finite element methods. For $p$ and $hp$-version of mortar finite element methods it may be of interest to extend our results by following the analysis of [75, 76]. This may be taken up in future.

Since the mortar finite element method deals with the decomposition of computational domain into a finite number subdomains, the matrix system arising from the discretization (2.7.4)-(2.7.5) is a block matrix and is sparse locally, but the global matrix leads to a full matrix. Therefore, it may be possible that the global matrix has a large condition number. Our future plan is to construct suitable preconditioners for the linear system arising from the mortar finite element method.

Except for [66] and references cited there, there is hardly any literature in the direction of mortar finite element method for nonlinear problems. Marcinkowski [66] has analyzed the mortar element method for the following quasilinear strongly monotone elliptic boundary value problems:

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f \quad \text{in } \Omega, \tag{5.2.1}$$

$$u = 0 \quad \text{on } \Gamma, \tag{5.2.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal region with Lipschitz continuous boundary $\partial\Omega$.

Denoting $a_i(x, p_0, p_1, p_2) = a_i(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_2})$ and $p = (p_0, p_1, p_2)$, we assume that $a_i\colon\ \Omega \times \mathbb{R}^3 \to \mathbb{R},\ i = 0, 1, 2$ satisfy the following conditions: For some positive constants $L, \mu_0$,

$$a_i \in C^1(\Omega \times \mathbb{R}^3), \tag{5.2.3}$$

$$max\{|a_i(x, 0, 0, 0)|, |\frac{\partial a_i}{\partial x_k}(x, p)|, |\frac{\partial a_i}{\partial p_j}(x, p)|\} \leq L, \quad for \quad i, j = 0, 1, 2; \quad k = 1, 2; \tag{5.2.4}$$

$$\sum_{i,j=0}^{2} \frac{\partial}{\partial p_j} a_i(x, p)\xi_i \xi_j \geq \mu_0 \sum_{i=1}^{2} |\xi_i|^2 \qquad \text{(Uniform ellipticity)}. \tag{5.2.5}$$

For the error analysis of mortar finite element method applied to (5.2.1)-(5.2.2), Marcinkowski in [66] has used in a crucial way the strongly monotoneness and boundedness property of the bilinear form associated with the elliptic operator.

To the best of our knowledge, there is hardly any result available in mortar element methods for the following quasilinear elliptic boundary value problems of non-monotone type:

$$-\nabla \cdot (a(u)\nabla u) \;=\; f \ \text{ in } \ \Omega, \tag{5.2.6}$$

$$u \;=\; 0 \ \text{ on } \ \partial\Omega, \tag{5.2.7}$$

where $a\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function bounded above and below by positive constants. Here, $a(u) = a(x, u(x))$.

The mortar formulation for the problem (5.2.6)-(5.2.7) is given as: Find $u_h \in V_h$ such that

$$a(u_h; u_h, v_h) \;=\; l(v_h) \qquad \forall\ v_h \in V_h, \tag{5.2.8}$$

where

$$a(u_h; v_h, w_h) = \sum_{l=1}^{K} \int_{\Omega_l} a(u_{h_l}) \nabla v_{l,h} \cdot \nabla w_{l,h}\ dx,$$

and

$$l(v_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h\ dx.$$

It is, therefore, natural to discuss the wellposedness of the discrete problem (5.2.8) and also to derive *a priori* error estimates.

Instead of imposing the mortar condition on the space, it is possible as in case of linear problems to formulate a problem with Lagrange multiplier. The mortar finite element method with Lagrange multiplier for the nonlinear problem is to seek $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$\begin{aligned} a(u_h; u_h, v_h) + b(v_h, \lambda_h) &= l(v_h) \qquad \forall v_h \in X_h && (5.2.9)\\ b(u_h, \mu_h) &= 0 \qquad \forall \mu_h \in M_h, && (5.2.10) \end{aligned}$$

where

$$b(v_h, \mu_h) = \sum_{m=1}^{m_0} \int_{\gamma_m \subset \Gamma} \mu_h [[v_h]]_{|\gamma_m} \, d\tau,$$

and

$$l(v_h) = \sum_{l=1}^{K} \int_{\Omega_l} f v_h \, dx.$$

The main concern is how to prove the wellposedness of the problem (5.2.9)-(5.2.10). Based on the idea of linearization, see [45], it may be possible to use fixed point arguments to prove existence of a discrete solution and we shall take this up in future.

With the help of Taylor series expansion, from (5.2.6)-(5.2.7) and (5.2.8), we obtain

$$\begin{aligned} \mathcal{A}(u; u - u_h, v_h) &= \sum_{l=1}^{K} \int_{\Omega_l} [\tilde{a}_u(u_h)(u - u_h)\nabla(u - u_h) \cdot \nabla v_h \\ &\quad + \tilde{a}_{uu}(u_h)(u - u_h)^2 \nabla u \cdot \nabla v_h] \, dx + \sum_{m=1}^{m_0} \int_{\gamma_m} a(u) \frac{\partial u}{\partial n} [[v_h]] d\tau \quad \forall \ v_h \in V_h, \end{aligned} \tag{5.2.11}$$

where

$$\mathcal{A}(u; v, w) = \sum_{l=1}^{K} \int_{\Omega_l} [a(u_l)\nabla v_l \cdot \nabla w_l + a_u(u_l)\nabla u_l v \cdot \nabla w_l] \, dx,$$

$$a(v) - a(v_h) = \tilde{a}_v(v_h)(v - v_h) = a_v(v)(v - v_h) - \tilde{a}_{vv}(v_h)(v - v_h)^2,$$

and

$$\tilde{a}_v(v_h) = \int_0^1 a_v[v + t(v_h - v)]dt, \qquad \tilde{a}_{vv}(v_h) = \int_0^1 (1-t)a_{vv}[v + t(v_h - v)]dt.$$

Using (5.2.6)-(5.2.7) and (5.2.9)-(5.2.10), we rewrite

$$\begin{aligned} \mathcal{A}(u; u - u_h, v_h) + b(v_h, \lambda - \lambda_h) &= \sum_{l=1}^{K} \int_{\Omega_l} [\tilde{a}_u(u_h)(u - u_h)\nabla(u - u_h) \cdot \nabla v_h \\ &+ \tilde{a}_{uu}(u_h)(u - u_h)^2 \nabla u \cdot \nabla v_h] dx \ \forall v_h \in X_h, \end{aligned} \tag{5.2.12}$$

$$b(u - u_h, \mu_h) = 0 \qquad \forall\ \mu_h \in M_h, \tag{5.2.13}$$

where

$$b(v_h, \mu_h) = -\sum_{m=1}^{m_0} \int_{\gamma_m} \mu_h [[v_h]] d\tau.$$

With appropriate approximation $\tilde{u}_h$ of $u$ in $X_h$, the problem (5.2.12)-(5.2.13) can be written as

$$\begin{aligned} \mathcal{A}(u; \tilde{u}_h - u_h, v_h) + b(v_h, \lambda - \lambda_h) &= A(u_h; \tilde{u}_h - u, v_h) \\ &+ \sum_{l=1}^{K} \int_{\Omega_l} [\tilde{a}_u(u_h)(u - u_h)\nabla(u - u_h) \cdot \nabla v_h \\ &+ \tilde{a}_{uu}(u_h)(u - u_h)^2 \nabla u \cdot \nabla v_h] dx \ \forall v_h \in X_h, \end{aligned} \tag{5.2.14}$$

$$b(\tilde{u}_h - u_h, \mu_h) = b(\tilde{u}_h - u, \mu_h) \qquad \forall\ \mu_h \in M_h. \tag{5.2.15}$$

Now consider a map

$$\Phi \colon X_h \to X_h$$

defined by $\Phi(w) = z$, where $(z, \zeta)$ is the (unique) solution of the system

$$\begin{aligned} \mathcal{A}(u; \tilde{u}_h - z, v_h) + b(v_h, \lambda - \zeta) &= A(u_h; \tilde{u}_h - u, v_h) \\ &+ \sum_{l=1}^{K} \int_{\Omega_l} [\tilde{a}_u(w)(u - w)\nabla(u - w) \cdot \nabla v_h \\ &+ \tilde{a}_{uu}(w)(u - w)^2 \nabla u \cdot \nabla v_h] dx \ \forall v_h \in X_h, \end{aligned} \tag{5.2.16}$$

$$b(\tilde{u}_h - w, \mu_h) = b(\tilde{u}_h - u, \mu_h) \qquad \forall\ \mu_h \in M_h. \tag{5.2.17}$$

For given $w \in X_h$, the problem (5.2.16)-(5.2.17) is linear in $z$, the existence and uniqueness of which follows for small $h$. Note that, here the existence of $(u_h, \lambda_h) \in X_h \times M_h$ satisfying (5.2.9)-(5.2.10) follows, if we can show that $\Phi$ has a fixed point in $X_h$. This can be ensured by applying Brouwer fixed point theorem; that is, by proving that $\Phi$ is continuous and it maps a closed bounded ball of $X_h$ into itself. In future, we would like to pursue this theory and discuss optimal error estimates in broken $H^1$ and $L^2$-norm. Extension of this analysis to parabolic problem is left as a future problem.

In, this thesis, we have not discussed mixed methods combined with mortar element method with and without Lagrange multipliers. Therefore, it may be desirable to explore it further for nonlinear elliptic and parabolic problems.

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# Publication in Journal

- **Ajit Patel, A. K. Pani, N. Nataraj**, Mortar finite element method for parabolic problems, to appear in Numerical Methods for Partial Differential Equations (2008).

# Acknowledgments

I would like to express my deep sense of gratitude and everlasting indebtedness to my thesis supervisor Prof. Amiya Kumar Pani for his enthusiastic supervision, intuitive discussions and kind supports in various way through out my research. He introduced me to the area of research in finite element methods and other related areas of applied mathematics and their various applications in industrial mathematics. His leadership, vast knowledge and skill in many areas, attention to detail and hard work have set an example for me and I hope to match some day. It would not have been possible for me to complete this research without his fruitful suggestions and enlightened lectures. My heartiest thanks to him for showing me the way of solving problems during my Ph.D. I am also very thankful to his wife Mrs. Tapaswani Pani for her affection and care.

I am deeply grateful to my co-supervisor Prof. Neela Nataraj for her invaluable discussion, inspiration and constant encouragement throughout this research. She introduced me the numerical computations for solving the PDEs using finite element methods which helps me a lot to complete my research. Her constructive lectures and discussions on implementations for finite element methods, have helped me a lot in understanding the basics of numerical implementations. I will be ever thankful for her friendly suggestions and moral support which make easier to complete this thesis.

I am greatly thankful to my RPC (Research Progress Committee) members Prof. Rekha P. Kulkarni and Prof. K. Suresh Kumar for their valuable suggestions and cooperation.

I owe my thankfulness to the Head of the Department of Mathematics, IIT Bombay, for providing me with the necessary facilities during my research work.

I am very grateful to Prof. Swadheen Pattanayak, Director, Institute of Mathematics and Applications, Bhubaneswar for his constant encouragement through out my M.Sc. and M.Phil. which was the main source of inspiration to pursue research.

I am very much thankful to my colleagues Debasish, Sarvesh, Dr. Raja Sekhar and Dr. T. Gudi for their valuable suggestions and discussions which help me up to a great extent in my research work. It is also my pleasure to express thanks to all research scholars in

Department of Mathematics, IIT Bombay for their help and cooperation.

Special thanks towards my nearer and dearer Biju, Alok, Dipesh, Benu, Sunita, Suni, Upali, Susmita, Sangram, Jyoti, Bharat, Rukmanda, Akshyada and Garudada for their love, support and presence whenever I need those with me and creating a homely environment during my stay at IIT Bombay.

Thanks are also due to my friends Sanjaya, Subash, Rajendra, Sonalika, Amiya, Sanjeeb, Bikash, Kedar, Brundaban, and elderly Pradeepda and Sarifda for their support and affection towards me during my research.

I would like to express my heartfelt thanks to my parents Shri Tarachand Patel and Smt. Latika Patel for their unconditional love, affection and patience throughout my study. I am also very much thankful to my elder brother Amaresh and all family members for their encouragement and moral support to complete my Ph.D.

Finally, I would like to thank all my well-wishers, whose direct and indirect support helped me in completing my thesis.

**IIT Bombay** **Ajit Patel**