DOMAIN DECOMPOSITION METHODS FOR SECOND ORDER ELLIPTIC AND PARABOLIC PROBLEMS

Thesis

Submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

by

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY
2008

Approval Sheet

The thesis entitled

"DOMAIN DECOMPOSITION METHODS FOR SECOND ORDER ELLIPTIC AND PARABOLIC PROBLEMS"

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is approved for the degree of

DOCTOR OF PHILOSOPHY

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CERTIFICATE OF COURSE WORK

This is to certify that Mr. Debasish Pradhan was admitted to the candidacy of the Ph.D. Degree on 18^{th} July 2002, after successfully completing all the courses required for the Ph.D. Degree programme. The details of the course work done are given below.

Sr. No.	Course No.	Course Name	Credits
1.	MA 534	Modern Theory of Partial Differential Equations	6.00
2.	MA 828	Functional Analysis	6.00
3.	MA 838	Special Topics in Mathematics II	6.00
4.	MA 562	Mathematical Theory of Finite Elements	0.00
5.	MA 530	Nonlinear Analysis	0.00
6.	MA 415	Mathematical Methods	8.00
7.	MA 825	Algebra	6.00
8.	MA 827	Analysis	6.00

I. I. T. Bombay

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Dy. Registrar (Academic)

Abstract

In this dissertation, we have focused on nonoverlapping non-conforming domain decomposition (DD) methods for second order elliptic and parabolic problems using both iterative and non-iterative schemes. We have also analyzed iterative nonoverlapping DD methods for elliptic problems using mixed finite element technique with a scope to apply it to parabolic problems. In Chapter 2 of this thesis, we have discussed a DD method with Lagrange multipliers for elliptic and parabolic problems. The key feature that we have adopted here is the nonconforming Crouzeix-Raviart space for the discretization of the primal variable. The emphasis throughout this study is on the existence and uniqueness of the approximate solutions, and optimal order of estimates in the broken H^1-norm and L^2-norm. Further, we have extended the DD method with Lagrange multipliers to parabolic problems. Optimal error estimates for both semidiscrete and fully discrete schemes are proved. The results of numerical experiments support the theoretical results which are derived in this chapter. Chapter 3 deals with a nonoverlapping iterative DD method for elliptic and parabolic problems. The iterative method has been defined with the help of Robin-type boundary conditions on the artificial interfaces (inter-subdomain boundaries). A convergence analysis is carried out and the convergence of the iterative algorithm is proved for the elliptic problems. In discrete case, the convergence of the iterative scheme is obtained by proving that the spectral radius of the matrix associated with the fixed point iterations is less than one. We have also derived the convergence rate which is shown to be of 1 - $O(h^{1/2}H^{-1/2})$, when the winding number N is not large, H is the maximum diameter of the subdomains and the transmission parameter is of $O(h^{-1/2}H^{-1/2})$. This is the best rate of convergence that can be expected using this iterative procedure. Moreover, we have extended this iterative method to parabolic initial-boundary value problems and demonstrated the convergence of the iteration at each time step. Numerical experiments confirm the theoretical results established in Chapter 3. In Chapter 4, we have analyzed an iterative scheme based on mixed finite element methods using Robin-type boundary condition as transmission data on the artificial interfaces (inter-subdomain boundaries) for nonoverlapping DD method

applied to second order elliptic problems. In this chapter, we have shown the convergence of the iterative scheme for the discrete problem. In the convergence analysis, we have shown that the spectral radius of the matrix associated with the fixed point iterations is less than one. Further, it is shown that the spectral radius has a bound of the form $1 - C\sqrt{h}H_\star$ for quasi-uniform partitions when the coefficients of the lower order term that is b in the elliptic problem $-\Delta u + b\,u = f$ with non-homogeneous boundary condition is positive, where h is the mesh size for triangulations and $H_\star$ is the minimum diameter of the subdomains with appropriate transmission parameter $O(\sqrt{h})$. Finally, the possible extensions with scope for future investigations are discussed in the concluding Chapter.

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Chapter 1

Introduction

1.1 Motivation

With the widespread acceptance of distributed memory multiprocessing as a cost-effective means of solving very large-scale problems in computational fluid dynamics (CFD) and computational structural mechanics (CSM), many engineers and scientists are encouraged with their initial ports of CFD or CSM codes for parallel execution, and are interested in learning whether applied mathematicians and computer scientists have anything to offer as a next step. While parallelization at the level of large linear system of algebraic equations is one option, the Domain decomposition (DD) seems to be a more natural way of parallelizing the algorithms and in this thesis, we explore some of the important roles that remain to be played by DD methods.

DD is a class of methods for solving large linear or nonlinear systems of equations arising from the discretization of partial differential equations by using numerical methods such as finite element methods or finite difference schemes or finite volume methods to obtain fast solutions. These methods are based on decomposing the physical domain into regions, where a problem is modeled by separate partial differential equations (PDEs) with suitable interface conditions between the sub-domains or by the same PDEs with natural transmission conditions on the subdomain interfaces and then obtaining solution by solving smaller problems on these subdomains. Due to the advancement in the high performance computer architectures, these subproblems can be solved in parallel and, thereby, the solution process has a considerable speed-up over traditional methods. Now-a-days, these methods are becoming natural tools for solving problems in parallel specially in CSM and

CFD. Therefore, DD methods turn out to be a subject of intense interest in scientific and engineering computing, see DD Conference Proceedings [29, 73].

The domain can be decomposed into *overlapping* or *nonoverlapping* subregions. Some of the attractive features of these methods include their efficient way of handling complicated geometries in a simple manner, to deal with different type of equations in different parts of the physical domain, and even to take advantage of the parallel processors in computations. After decomposition, the elemental or subdomain problems can be decoupled and solved in each sub-domain independently (to a great extent) except for a matching step, which is necessary for obtaining a smooth global solution from different subdomain solutions. The matching procedure requires communication between the sub-domains. The local interaction is through the exchange of information between neighbouring subregions. DD methods are becoming increasingly popular for solving elliptic and parabolic problems and these methods have been discussed at some length in the existing literature [29, 30, 73, 110, 119, 125]. DD methods can often be viewed as preconditioners for iterative methods like the conjugate gradient (CG) method and generalized minimal residual (GMRES) method.

In this thesis, we first discuss non-iterative, nonoverlapping DD methods and nonconforming finite element methods with Lagrange multipliers for elliptic and parabolic problems. Then, we propose and analyze iterative nonoverlapping DD methods with Robin type transmission conditions on the artificial interfaces between the subdomains.

1.2 Preliminaries

In this section, we discuss the standard Sobolev spaces with some properties which are used in the sequel. Moreover, we appeal to some results which will be useful in the subsequent chapters.

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of non-negative integers. Define a multi-valued index $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_d)$, $\alpha_i \geq 0$, $\alpha_i \in \mathbb{N}$ with $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_d$. Let Ω denote an open bounded convex polygon or polyhedron in $\mathbb{R}^d$, with $d = 2$ or 3, having boundary $\partial\Omega$. For $1 \leq p < \infty$, let $L^p(\Omega)$ denote the real valued measurable functions v on

Ω for which $\displaystyle\int_\Omega |v(x)|^p dx \leq \infty$. The norm on $L^p(\Omega)$ is given by

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v(x)|^p dx\right)^{1/p} \qquad \text{for} \qquad 1 \leq p < \infty.$$

In addition, let $L^\infty(\Omega)$ denote the real valued measurable functions which are essentially bounded in Ω and let its norm be given by

$$\|v\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

With $H^0(\Omega) = L^2(\Omega)$ and for natural numbers $m \geq 1$, let $H^m(\Omega)$ denote the standard Hilbert Sobolev space of order m which is defined by

$$H^m(\Omega) = \left\{v \in L^2(\Omega) : \partial^\alpha v \in L^2(\Omega),\ |\alpha| \leq m\right\}. \tag{1.2.1}$$

$H^m(\Omega)$ is equipped with the seminorm and norm, respectively, defined by

$$|v|_{m,\Omega} = \left(\sum_{|\alpha|=m} ||\partial^\alpha v||^2_{0,\Omega}\right)^{1/2} \qquad \text{for all } m \geq 1, \tag{1.2.2}$$

$$||v||_{m,\Omega} = \left(\sum_{|\alpha|\leq m} ||\partial^\alpha v||^2_{0,\Omega}\right)^{1/2} \qquad \text{for all } m \geq 1, \tag{1.2.3}$$

where $||v||_{0,\Omega} = \left(\displaystyle\int_\Omega v^2(x)\, dx\right)^{1/2}$ denotes the norm in $L^2(\Omega)$. For $d \in \mathbb{N}$ the product space $(\,H^m(\Omega)\,)^d = \left\{\underline{q} = (q_i)_{1\leq i\leq d} :\ q_i \in H^m(\Omega) \ \text{ for all } i = 1, \ldots, d\right\}$ is equipped with the seminorm and norm, respectively, defined by

$$|\underline{q}|_{m,\Omega} = \left(\sum_{i=1}^d |q_i|^2_{m,\Omega}\right)^{1/2} \quad \text{and} \quad ||\underline{q}||_{m,\Omega} = \left(\sum_{i=1}^d ||q_i||^2_{m,\Omega}\right)^{1/2}. \tag{1.2.4}$$

For our subsequent use, we resort to the following notations. Let (a, b) be an interval with $-\infty < a \leq b < \infty$, and let $\mathcal{X}$ be a Banach space with norms $\|.\|_\mathcal{X}$. For $1 \leq p \leq \infty$, we denote by $L^p(a, b; \mathcal{X})$ the space

$$L^p(a, b; \mathcal{X}) := \{\phi : (a, b) \mapsto \mathcal{X} \mid \phi(t) \text{ is measurable in } (a, b) \text{ and } \int_a^b \|\phi(t)\|^p_\mathcal{X} < \infty\}.$$

It is equipped with the following norm for $1 \leq p < \infty$

$$||\phi||_{L^p(a,b;\mathcal{X})} = \left(\int_a^b ||\phi(t)||_{\mathcal{X}}^p \, dt \right)^{1/p}$$

and for $p = \infty$,

$$\|\phi\|_{L^\infty(a,b,\mathcal{X})} := \text{ess} \sup_{t \in (a,b)} \|\phi(t)\|_{\mathcal{X}}.$$

When $-\infty < a \leq b < \infty$, the space

$$C([a,b];\mathcal{X}) := \{\phi : [a,b] \mapsto \mathcal{X} \mid \phi \text{ is continuous in } [a,b]\}$$

is a Banach space equipped with the norm

$$\|\phi\|_{C([a,b];\mathcal{X})} := \max_{t \in [a,b]} \|\phi(t)\|_{\mathcal{X}}.$$

When the interval $[a,b]$ is the time interval $[0,T], T > 0$ fixed, we may conveniently use $L^p(\mathcal{X})$ for $L^p(a,b;\mathcal{X})$ and $C(\mathcal{X})$ for $C(0,T;\mathcal{X})$.
For our future use, we recall the following results.

Theorem 1.2.1 *[22, Theorem 1.6.6, pp. 37] Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$. Then for $1 \leq p \leq \infty$, there exists a constant C depending on Ω such that*

$$||v||_{L^p(\partial\Omega)} \leq C \, ||v||_{L^p(\Omega)}^{1-1/p} \, ||v||_{W^{1,p}(\Omega)}^{1/p} \qquad \forall \, v \in W^{1,p}(\Omega). \tag{1.2.5}$$

We need Sobolev spaces on $\partial\Omega$, or an open subspace of $\partial\Omega$. We have an obvious definition of boundary values, or trace, on $\partial\Omega$, for functions in $C^\infty(\bar{\Omega})$. These maps can be generalized to functions in $H^1(\Omega)$ for a bounded Lipschitz region Ω; see Nečas [103].

Lemma 1.2.1 *[103]* **(Trace and Extension theorem)** *Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$. The trace map $\Upsilon_0 : \; v \to v_{|\partial\Omega}$, defined for $C^\infty(\bar{\Omega})$, has a unique continuous extension from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. This operator has a right continuous inverse.*

As a consequence, we can easily show that the kernel Υ_0 is $H_0^1(\Omega)$, i.e.,

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : \; \Upsilon_0 \, v = 0 \;\; \text{on} \;\; \partial\Omega \right\}.$$

We define the seminorm for the space $H^{1/2}(\partial\Omega)$ by

$$|\mu|_{H^{1/2}(\partial\Omega)} = \inf_{v\in H^1(\Omega),\, \Upsilon_0\, v=\mu} |v|_{H^1(\Omega)}, \tag{1.2.6}$$

and norm for the space $H^{1/2}(\partial\Omega)$ by

$$\|\mu\|^2_{H^{1/2}(\partial\Omega)} = |\mu|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{H}||\mu||^2_{L^2(\partial\Omega)}, \tag{1.2.7}$$

where H is the diameter of Ω. We now introduce spaces that will be used in the mixed formulation of elliptic problems. We denote by $H^{-1/2}(\partial\Omega)$, the dual space of $H^{1/2}(\partial\Omega)$ which is equipped with the norm

$$\|\varphi\|_{H^{-1/2}(\partial\Omega)} = \sup_{\mu\in H^{1/2}(\partial\Omega),\mu\neq 0} \frac{|<\varphi,\mu>_{\partial\Omega}|}{\|\mu\|_{H^{1/2}(\partial\Omega)}}, \tag{1.2.8}$$

where $<\cdot,\cdot>_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. With $\Gamma_0 \subset \partial\Omega$, let $\tilde{v}$ be an extension of $v \in H^{1/2}(\Gamma_0)$ by zero to all of $\partial\Omega$. Then we set $H^{1/2}_{00}(\Gamma_0)$, a subspace of $H^{1/2}(\Gamma_0)$ as

$$H^{1/2}_{00}(\Gamma_0) = \{v \in H^{1/2}(\Gamma_0) :\ \tilde{v} \in H^{1/2}(\partial\Omega)\}.$$

The norm in $H^{1/2}_{00}(\Gamma_0)$ is defined by

$$\|g\|_{H^{1/2}_{00}(\Gamma_0)} = \inf_{v\in H^1_0(\Gamma_0), v_{|_{\Gamma_0}}=g} \|v\|_{H^1(\Omega)}. \tag{1.2.9}$$

Note that $H^{1/2}_{00}(\Gamma_0)$ is strictly contained in $H^{1/2}(\Gamma_0)$ and also continuously embedded in $H^{1/2}(\Gamma_0)$. For a more detailed discussion of trace spaces; cf. Grisvard [75] or Lions and Magenes [93]. The space $H(\text{div};\Omega)$ is defined by

$$H(\text{div};\Omega) = \left\{ \underline{q} = (q_i)_{1\le i\le d} \in (L^2(\Omega))^d :\ \text{div}\underline{q} = \sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} \in L^2(\Omega) \right\}, \tag{1.2.10}$$

and is a Hilbert space with norm

$$||\underline{q}||_{H(\text{div};\Omega)} = \left\{ ||\underline{q}||^2_{0,\Omega} + ||\text{div}\underline{q}||^2_{0,\Omega} \right\}^{1/2}. \tag{1.2.11}$$

Lemma 1.2.2 *[114, Theorem 1.2, pp. 1.05]* **(Trace and Extension theorems for $H(\text{div};\Omega)$)** *The mapping $\underline{q} \to \underline{q}\cdot\nu$ defined from $(H^1(\Omega))^d$ into $L^2(\partial\Omega)$ can be extended to a continuous, linear mapping from $H(\text{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$. Further, we have the following characterization of the norm on $H^{-1/2}(\partial\Omega)$:*

$$||\mu||_{H^{-1/2}(\partial\Omega)} = \inf_{\underline{q}\in H(\text{div};\Omega);\underline{q}\cdot\nu=\mu} ||\underline{q}||_{H(\text{div};\Omega)}. \tag{1.2.12}$$

We also define the space

$$\mathcal{H}(\text{div};\Omega) = \left\{ \underline{q} \in H(\text{div};\Omega) :\ \underline{q}\cdot\nu \in L^2(\partial\Omega) \right\} \tag{1.2.13}$$

which is a Hilbert space with norm

$$||\underline{q}||_{\mathcal{H}(\text{div};\Omega)} = \left\{ ||\underline{q}||^2_{H(\text{div};\Omega)} + ||\underline{q}\cdot\nu||^2_{0,\partial\Omega} \right\}^{1/2}. \tag{1.2.14}$$

We shall make use of the following version of the **Green's formula** : For $v \in H^1(\Omega)$ and $\underline{q} \in H(\text{div};\Omega)$

$$\int_\Omega (v\,\text{div}\underline{q} + \text{grad}\,v\cdot\underline{q})\,dx = \int_{\partial\Omega} v\,\underline{q}\cdot\nu\,ds. \tag{1.2.15}$$

Lemma 1.2.3 *[105]* **(Friedrich's' inequality)** *Let Ω be a bounded domain in $\mathbb{R}^d$. Then there exists a positive constant C depending on Ω such that for $v \in H^1_0(\Omega)$*

$$||v||_{H^1(\Omega)} \le C\,|v|_{H^1(\Omega)}. \tag{1.2.16}$$

Lemma 1.2.4 *[64, 103, 105]* **(Poincaré's inequality)** *Let Ω be a bounded domain in $\mathbb{R}^d$. Then there exists a positive constant C depending on Ω such that for $v \in H^1(\Omega)$*

$$||v||^2_{H^1(\Omega)} \le C\left\{ |v|^2_{H^1(\Omega)} + \frac{1}{H^{2+d}}\left(\int_\Omega v\,dx\right)^2 \right\}. \tag{1.2.17}$$

where H is the diameter of Ω.

Lemma 1.2.5 *[103]* **(Poincaré-Friedrich's inequality)** *Let Γ_0 be an open subset of $\partial\Omega$ with positive measure. Then there exists a positive constant C depending on Ω and Γ_0 such that*

$$||v||^2_{H^1(\Omega)} \le C\left\{ |v|^2_{H^1(\Omega)} + \frac{1}{H}\int_{\Gamma_0} v^2\,ds \right\} \qquad \forall\, v \in H^1(\Omega), \tag{1.2.18}$$

where H is the diameter of Ω.

1.2.1 Triangulation and its properties

Let Ω be a bounded convex polygon or polyhedron in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. Let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$ [34] into triangles for $d = 2$, tetrahedrons for $d = 3$ satisfying

$$T \subset \bar{\Omega}, \quad \forall\, T \in \mathcal{T}_h, \quad \bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

The boundary of T will be denoted by ∂T, T' will be an edge of T when $d = 2$, a triangular face when $d = 3$. We also use the following notations :

$$\begin{aligned}
|T| \quad &= \quad \text{meas } (T), \text{ that is, the Euclidean measure of } T \text{ in } \mathbb{R}^d \\
&\qquad (\text{geometric area if } d = 2, \text{ geometric volume if } d = 3), \\
h_T \quad &= \quad \text{the diameter of } T, \\
\rho_T \quad &= \quad \text{the radius of the circle inscribed in } T \text{ if } d = 2, \text{ of the} \\
&\qquad \text{sphere inscribed in } T \text{ if } d = 3,
\end{aligned} \tag{1.2.19}$$

and

$$h \;=\; \max_{T \in \mathcal{T}_h} h_T.$$

Definition 1.2.1 *[62]* **(Shape-regularity)** *A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be shape regular if there exists σ_0 such that*

$$\sigma_T = \frac{h_T}{\rho_T} \leq \sigma_0 \qquad \forall\, h, \quad \forall\, T \in \mathcal{T}_h.$$

Definition 1.2.2 *[62]* **(Quasi-uniformity)** *A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be quasi uniform if and only if it is shape-regular and there exists σ_1 such that*

$$h_T \geq \sigma_1\, h \qquad \forall\, h, \quad \forall\, T \in \mathcal{T}_h.$$

Remark 1.2.1 (i) *Let T be a triangle and denote by θ_T the smallest of its angles. One readily sees that*

$$\frac{h_T}{\rho_T} \leq \frac{2}{\sin \theta_T}.$$

Therefore, in a shape-regular family of triangulations, the triangles cannot become too flat as $h \to 0$.

(ii) *In dimension* 1, $h_T = \rho_T$, *hence, any mesh family is shape-regular.*

(iii) *A necessary and sufficient condition for quasi-uniformity is that there exists* τ_0 *such that* $\rho_T \geq \tau_0 h$ *for all* h *and* $T \in \mathcal{T}_h$. *Indeed, if* $\{\mathcal{T}_h\}_{h>0}$ *satisfies the above property, then* $\dfrac{h_T}{\rho_T} \leq {\tau_0}^{-1}\dfrac{h_T}{h} \leq {\tau_0}^{-1}$ *for all* h *and* $T \in \mathcal{T}_h$, *thus showing that the family* $\{\mathcal{T}_h\}_{h>0}$ *is shape-regular. Furthermore,* $h_T \geq \rho_T \geq \tau_0 h$ *implies* $h_T \geq \sigma_1 h$. *Conversely, if* $\{\mathcal{T}_h\}_{h>0}$ *is a quasi-uniform mesh family,* $\rho_T \geq \dfrac{1}{\sigma_0} h_T \geq \dfrac{\sigma_1}{\rho_T} h$ *for all* $h > 0$ *and* $T \in \mathcal{T}_h$.

Lemma 1.2.6 *[114, Theorem 1.3, pp. 1.06] Let* $\mathcal{T}_h$ *be such a decomposition of* $\bar{\Omega}$ *with* $\bar{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$. *A function* $v \in L^2(\Omega)$, *whose restriction* $v_{|T}$ *may be identified with a function* $v_T \in H^1(T)$ *for each* $T \in \mathcal{T}_h$, *belongs to* $H^1(\Omega)$ *if and only if for each interface* $T' = T_1 \cap T_2$ *with* $T_1, T_2 \in \mathcal{T}_h$, *the traces of* v_{T_1} *and of* v_{T_2} *on* T' *coincide:*

$$v_{T_1}{}_{|_{T'}} = v_{T_2}{}_{|_{T'}} \qquad \text{for all } T' = T_1 \cap T_2 \text{ with } T_1, T_2 \in \mathcal{T}_h. \tag{1.2.20}$$

Similarly a function $\underline{q} \in (L^2(\Omega))^d$, *whose restriction* $\underline{q}_{|_T}$ *may be identified with a function* $\underline{q} \in H(\text{div}; T)$ *for* $T \in \mathcal{T}_h$, *belongs to* $H(\text{div}; \Omega)$ *if and only if for each interface* $T' = T_1 \cap T_2$ *with* $T_1, T_2 \in \mathcal{T}_h$, *the normal trace of* $\underline{q}_{|_{T_1}}$ *coincides with the negative of that of* $\underline{q}_{|_{T_2}}$*:*

$$\underline{q}_{|_{T_1}} \cdot \nu^{T_1}{}_{|_{T'}} + \underline{q}_{|_{T_2}} \cdot \nu^{T_2}{}_{|_{T'}} = 0 \qquad \text{for all } T' = T_1 \cap T_2 \text{ with } T_1, T_2 \in \mathcal{T}_h, \tag{1.2.21}$$

where ν^T *is the unit exterior normal vector to* ∂T.

Lemma 1.2.7 *[62, Lemma 3.32, pp. 128] Let* $\{\mathcal{T}_h\}_{h>0}$ *be a shape-regular family of geometrically conformal affine meshes. Let* $m \geq 1$ *be a fixed integer. For* $T \in \mathcal{T}_h$, *let* $\psi \in (H^1(T))^m$, *and for a face* $T' \in \partial T$, *set* $\overline{\psi} = \dfrac{1}{meas(T')}\displaystyle\int_{T'} \psi\, dx$. *Then, there exists* C *independent of* h_T *such that for* $v_h \in \bar{X}_h$ *and* $T' \in \partial T$ *with* $T \in \mathcal{T}_h$

$$||\psi - \overline{\psi}||_{0,T'} \leq C\, h_T^{1/2}\, |\psi|_{1,T}, \tag{1.2.22}$$

where $\bar{X}_h$ *is the nonconforming Crouzeix-Raviart space (cf. [39]).*

1.2.2 Some results from functional analysis

We need some well known results from functional analysis, which we state without proof in this subsection.

Definition 1.2.3 *Let $\hat{u}$ be the finite element solution and u^k be the solution at the kth iterative step respectively. If*

$$||u^k - \hat{u}|| \leq CL^k||u^0 - \hat{u}||, \tag{1.2.23}$$

$L \in [0, 1)$, *and C is independent of k, then u^k is said to converge to u with the convergence rate L.*

Lemma 1.2.8 (Hölder Inequality) *Let $1 < p < \infty$ and q satisfy $1/p + 1/q = 1$. If $v \in L^p(\Omega)$, $w \in L^q(\Omega)$, then $vw \in L^1(\Omega)$ and*

$$\int_\Omega |v(x)w(x)| \, dx \leq \|v\|_{L^p(\Omega)} \|w\|_{L^q(\Omega)}. \tag{1.2.24}$$

Lemma 1.2.9 (Young's Inequality) *Let a and b be two positive real numbers, then the following inequality holds for all $\epsilon > 0$*

$$a\,b \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2. \tag{1.2.25}$$

Lemma 1.2.10 (Cauchy-Schwarz Inequality) *Let $1 \leq p, q < \infty$ and $1/p + 1/q = 1$. Suppose that $\{a_i\}$ and $\{b_i\}$ are two sequences of N positive real numbers. Then*

$$\left(\sum_{i=1}^N a_i\, b_i\right) \leq \left(\sum_{i=1}^N a_i^p\right)^{1/p} \left(\sum_{i=1}^N b_i^q\right)^{1/q}. \tag{1.2.26}$$

Now we introduce the spectral radius formula, the complexification of real linear space as well as real linear operators.

Let $\eta_1, \eta_2, \cdots, \eta_s$ be the (real or complex) eigenvalues of a matrix A. Then its spectral radius $\rho(A)$ is defined as:

$$\rho(A) := \max_{1 \leq i \leq s}(|\eta_i|). \tag{1.2.27}$$

Below, we state a lemma without proof which provides a useful upper bound for the spectral radius of a matrix.

Lemma 1.2.11 *Let $A \in \mathbb{C}^{n\times n}$ be a complex-valued matrix and $\rho(A)$ be its spectral radius. For a consistent matrix norm $||\cdot||$ and for $k \in \mathbb{N}$,*

$$\rho(A) \leq \|A^k\|^{1/k} \qquad \forall\, k \in \mathbb{N}. \tag{1.2.28}$$

Theorem 1.2.2 *[35] Let $A \in \mathbb{C}^{n\times n}$ be a complex-valued matrix and $\rho(A)$ be its spectral radius. Then*

$$\lim_{k\to\infty} A^k = 0 \text{ if and only if } \rho(A) < 1. \tag{1.2.29}$$

Moreover, if $\rho(A) > 1$, $\|A^k\|$ is not bounded for increasing k values.

Theorem 1.2.3 *[89, Theorem 12.8, pp. 209]* **(Spectral radius formula)** *Let V be a Banach space over $\mathbb{C}$ and A be a complex linear bounded operator on V to itself. Then*

$$\rho(A) = \inf_{k=1,2,\cdots} ||A^k||^{1/k} = \lim_{k\to\infty} ||A^k||^{1/k}. \tag{1.2.30}$$

Now, we are in a position to construct the complexification of a real linear space. The construction is based on the construction of a complex number field by a real number field.

Definition 1.2.4 *Suppose V is a real n dimensional linear space; we call the tensor product space $\mathbb{C}\otimes V$ the complexification of V, where $\mathbb{C}$ is the complex number field or one dimensional complex linear space. In other words, $\mathbb{C}\otimes V$ is a complex n dimensional space such that*

$$\mathbb{C}\otimes V = \left\{x + \sqrt{(-1)}\, y \mid x, y \in V\right\}.$$

Note that $\mathbb{C}\otimes V$ is equipped with the following addition and scalar multiplication properties:

$$(x_1 + \sqrt{(-1)}y_1) + (x_2 + \sqrt{(-1)}y_2) = (x_1 + x_2) + \sqrt{(-1)}(y_1 + y_2),$$
$$(a + \sqrt{(-1)}b)(x + \sqrt{(-1)}y) = (ax - by) + \sqrt{(-1)}(bx + ay),\ \ a + \sqrt{(-1)}b \in \mathbb{C}.$$

Lemma 1.2.12 *Suppose V is a real linear space equipped with inner product $\langle\cdot,\cdot\rangle$; then we can define an inner product on $\mathbb{C}\otimes V$ as*

$$\langle x_1 + \sqrt{(-1)}y_1, x_2 + \sqrt{(-1)}y_2\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle - \sqrt{(-1)}\langle x_1, y_2\rangle + \sqrt{(-1)}\langle y_1, x_2\rangle.$$

Moreover, if $||\cdot||$ is the norm induced by the inner product, then

$$||x + \sqrt{(-1)}y||^2 = ||x||^2 + ||y||^2. \tag{1.2.31}$$

Definition 1.2.5 *If V is a real linear space and A is a real linear operator of V, we define a complex linear operator $1 \otimes A$ of $\mathbb{C} \otimes V$ by*

$$1 \otimes A(x + \sqrt{(-1)}y) = Ax + \sqrt{(-1)}Ay.$$

We call $1 \otimes A$ the complexification of A. For convenience, we also denote $1 \otimes A$ by $\bar{A}$.

Lemma 1.2.13 *[109] If V is a real linear space and A_1, A_2 are real linear operators of V, then*

$$(1 \otimes A_1)(1 \otimes A_2) = 1 \otimes (A_1 A_2). \tag{1.2.32}$$

In particular,

$$1 \otimes (A^k) = (1 \otimes A)^k, \tag{1.2.33}$$

we denote $1 \otimes (A^k)$ or $(1 \otimes A)^k$ by $\bar{A}^k$.

Proof. Using definition, we observe that

$$\begin{aligned}(1 \otimes A_1)(1 \otimes A_2)(x + \sqrt{(-1)}y) &= (1 \otimes A_1)(A_2 x + \sqrt{(-1)}A_2 y) \\ &= A_1 A_2 x + \sqrt{(-1)}A_1 A_2 y \\ &= 1 \otimes (A_1 A_2)(x + \sqrt{(-1)}y).\end{aligned}$$

This completes the rest of the proof. ■

Lemma 1.2.14 *[109] Let V be a finite dimensional real linear space equipped with an inner-product, and A be a real linear operator on V into itself. Then*

$$||\bar{A}|| = ||A||. \tag{1.2.34}$$

From time to time, we shall use c and C as generic positive constants which do not depend on the discretizing parameters.

1.3 Literature review

Due to the advancement of high speed computers, more attention has been paid to the development of parallel algorithms on massively parallel machines in the last two to three decades. Since DD algorithms help to solve many large scale problems efficiently, which otherwise would be difficult to solve numerically, in the recent past, a large number of articles are devoted to this area. In an early survey article by Chan and Mathew [30], a systematic survey on various DD methods applied to different problems has been presented. In a review article, Xu [128] has discussed the motivation in developing the iterative methods by using the notions of sub-space decomposition and sub-space corrections. Subsequently, a detailed survey article has been written by Xu and Zou [129] on nonoverlapping DD methods which are based on the substructuring-type schemes and the Neumann-Neumann-type methods.

In recent years, DD methods have attracted much attention due to their successful application to many elliptic and parabolic problems. In DD methods, the PDE or its approximation is split into coupled problems on smaller overlapping or non-overlapping sub-domains which form a partition of the original domain. In this thesis, we consider only the case of non-overlapping sub-domains. However, there is a good deal of literature available on overlapping DD methods and we refer the reader to the survey articles [30] and the references, therein.

When the original domain is decomposed into subdomains, the transmission conditions come into picture on the inter-subdomain boundaries. The matching conditions of the solution or the normal derivatives of the solution on the artificial boundary are expressed in terms of Lagrange multipliers. Once the values of the solution or its normal derivatives on the subdomain interfaces are available, then the problem can be solved in parallel in each subdomain. Depending on how we achieve an approximation of the solution or its normal derivatives on the interfaces, the DD methods can be categorized under iterative and non-iterative schemes.

1.3.1 Non-iterative non-overlapping domain decomposition methods

In order to define non-iterative non-overlapping DD methods, we consider the following model problem:

$$\begin{cases} -\Delta u + b(x)\, u &= f \qquad \forall\, x \in \Omega, \\ u &= 0 \qquad \forall\, x \in \partial\Omega, \end{cases} \tag{1.3.1}$$

where Ω is in bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, with sufficiently smooth boundary $\partial\Omega$, f is a given function in $L^2(\Omega)$ and $b(x) \geq 0$. For the multi-domain formulation, let us assume that Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2 with $\Omega = \Omega_1 \cup \Omega_2$ and interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Now we split the original problem (1.3.1) to a problem in the multi-domain framework. Find u_1, u_2 such that:

$$\begin{cases} -\Delta u_i + b\, u_i = f & \text{in} \quad \Omega_i \\ u_i = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ u_1 = u_2 & \text{on} \quad \Gamma \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} & \text{on} \quad \Gamma. \end{cases} \tag{1.3.2}$$

Here $u_i,\ i = 1, 2$ are the restrictions of the solution u of original the problem to $\Omega_i,\ i = 1, 2$ (that is $u_i = u_{|\Omega_i}, i = 1, 2$) and ν^i is the unit outward normal to $\partial\Omega_i \cap \Gamma$ (oriented outward) and $\nu = \nu^1$. The equations $(1.3.2)_3$ and $(1.3.2)_4$ are the transmission conditions for u_1 and u_2 on Γ.

The variational formulation (see, [110, Sect. 1.2]) for the multi-domain problem (1.3.2) is: find $u_1 \in V_1, u_2 \in V_2$ such that

$$\begin{cases} a_i(u_i, v_i) + (b\, u_i, v_i) = (f, v_i) \qquad \forall\ v_i \in V_i^0 \\ u_1 = u_2 \qquad \text{on} \quad \Gamma \\ a_2(u_2, R_2\mu) + (b\, u_2, R_2\mu)_{\Omega_2} = (f, R_2\mu)_{\Omega_2} + (f, R_1\mu)_{\Omega_1} \\ \qquad\qquad -a_1(u_1, R_1\mu) - (b\, u_1, R_1\mu)_{\Omega_1} \quad \forall\ \mu \in \Xi, \end{cases} \tag{1.3.3}$$

where $(w_i, v_i)_{\Omega_i} = \displaystyle\int_{\Omega_i} w_i v_i \, dx$, $a_i(w_i, v_i) = \displaystyle\int_{\Omega_i} \nabla w_i \cdot \nabla v_i \, dx$, $V_i = \{v_i \in H^1(\Omega_i) \mid v_{i|\partial\Omega} = 0\}$, $V_i^0 = H_0^1(\Omega)$, $\Xi = \{\eta \in H^{1/2}(\Gamma) \mid \eta = v_{|\Gamma} \text{ for a suitable } v \in V\}$ and $R_i\,(i = 1, 2)$ denotes any possible extension operator from Ξ to V_i.

We now introduce the following multi-domain finite element approximation of (1.3.2). Let V_h denote a finite dimensional subspace of $H_0^1(\Omega)$ defined by

$$V_h = \left\{ v_h \,|\, v_h \in C^0(\bar{\Omega}),\ v_{h|_T} \in P_r(T)\ \ \forall T \in \mathcal{T}_h,\ r \geq 1 \right\}.$$

Set $V_{i,h} = \{v_{h|_{\Omega_i}} :\ v_h \in V_h\}$, $V_{i,h}^0 = \{v_h \in V_{i,h} :\ v_{h|_\Gamma} = 0\}$ and $\Xi_h = \{v_{h|_\Gamma} :\ v_h \in V_h\}$. The multi-domain finite element approximation to (1.3.3) is to seek $u_{i,h} \in V_{i,h},\ i = 1, 2$ such that

$$a_i(u_{i,h}, v_{i,h}) + (b\, u_{i,h}, v_{i,h})_{\Omega_i} = (f, v_{i,h})_{\Omega_i} \quad \forall\ v_{i,h} \in V_{i,h}^0,\ i = 1, 2, \tag{1.3.4}$$

$$u_{1,h} = u_{2,h} \quad \text{on} \ \ \Gamma, \tag{1.3.5}$$

$$\begin{aligned} a_2(u_{2,h}, R_{2,h}\mu_h) + (b\, u_{2,h}, R_{2,h}\mu_h)_{\Omega_2} &= (f, R_{1,h}\mu_h)_{\Omega_1} + (f, R_{2,h}\mu_h)_{\Omega_2} \\ &\quad - a_1(u_{1,h}, R_{1,h}\mu_h) - (b\, u_{1,h}, R_{1,h}\mu_h)_{\Omega_1} \quad \forall\ \mu_h \in \Xi_h, \end{aligned} \tag{1.3.6}$$

where

$$R_{i,h}\mu_h \ = \ \begin{cases} \mu_h \ \ \textit{on} \ \ \Gamma \\ 0 \ \ \text{at} \ \ \text{other} \ \ \text{nodes of } \Omega_i. \end{cases}$$

To write (1.3.4)-(1.3.6) in **vector matrix form**, let $\{\phi_i\}_{i=1}^{N_1}$ and $\{\chi_i\}_{i=1}^{N_2}$, respectively, be bases for $V_{1,h}^0$ and $V_{2,h}^0$. Further, let $\{\phi_i\}_{i=1}^{N_1} \cup \{\psi_i\}_{i=1}^{N_\Gamma}$ and $\{\chi_i\}_{i=1}^{N_2} \cup \{\psi_i\}_{i=1}^{N_\Gamma}$ be bases for $V_{1,h}$ and $V_{2,h}$, respectively. Here $N_1,\ N_2$ and N_Γ are the dimensions of the spaces $V_{1,h}^0,\ V_{2,h}^0$ and Ξ_h, respectively. Setting

$$u_{1,h} = \ \sum_{i=1}^{N_1} \alpha_i\ \phi_i \ + \ \sum_{j=1}^{N_\Gamma} \lambda_j\ \psi_j, \quad u_{2,h} = \ \sum_{m=1}^{N_2} \beta_m\ \chi_m \ + \ \sum_{j=1}^{N_\Gamma} \lambda_j\ \psi_j,$$

in (1.3.4), (1.3.5), (1.3.6), we arrive at

$$(A_{11})_{N_1\times N_1}(\mathbf{U}_1)_{N_1\times 1} \ + \ (A_{1\Gamma})_{N_1\times N_\Gamma}(\mathbf{U}_\Gamma)_{N_\Gamma\times 1} = (\mathbf{f}_1)_{N_1\times 1}, \tag{1.3.7}$$

$$(A_{22})_{N_2\times N_2}(\mathbf{U}_2)_{N_2\times 1} \ + \ (A_{2\Gamma})_{N_2\times N_\Gamma}(\mathbf{U}_\Gamma)_{N_\Gamma\times 1} = (\mathbf{f}_2)_{N_2\times 1}, \tag{1.3.8}$$

$$(A_{\Gamma 1})_{N_\Gamma\times N_1}(\mathbf{U}_1)_{N_1\times 1} \ + \ (A_{\Gamma 2})_{N_\Gamma\times N_2}(\mathbf{U}_2)_{N_2\times 1} \ + \ (A_{\Gamma\Gamma})_{N_\Gamma\times N_\Gamma}(\mathbf{U}_\Gamma)_{N_\Gamma\times 1} = (\mathbf{f}_\Gamma)_{N_\Gamma\times 1}, \tag{1.3.9}$$

where $(A_{11})_{N_1\times N_1} = (a_1(\phi_i, \phi_j) + (b\,\phi_i, \phi_j)),\ 1 \leq i, j \leq N_1,\ (A_{22})_{N_2\times N_2} = (a_2(\chi_i, \chi_j) + (b\,\chi_i, \chi_j)),\ 1 \leq i, j \leq N_2,\ (A_{\Gamma\Gamma})_{N_\Gamma\times N_\Gamma} = (a_1(\psi_i, \psi_j) + (b\,\psi_i, \psi_j)) + (a_2(\psi_i, \psi_j) + (b\,\psi_i, \psi_j)),\ 1 \leq$

$i, j \leq N_\Gamma$, $(A_{1\Gamma})_{N_1 \times N_\Gamma} = (a_1(\psi_i, \phi_j) + (b\,\psi_i, \phi_j)),\ 1 \leq i \leq N_\Gamma,\ 1 \leq j \leq N_1$, $(A_{2\Gamma})_{N_2 \times N_\Gamma} = (a_2(\psi_i, \chi_j) + (b\,\psi_i, \chi_j)),\ 1 \leq i \leq N_\Gamma,\ 1 \leq j \leq N_2$, while $(A_{\Gamma i})$ denotes the transpose of $(A_{i\Gamma})$, $i = 1, 2$, $a_i(\cdot,\cdot)$ is the restriction of the bilinear form $a(\cdot,\cdot)$ to Ω_i, $(\mathbf{f}_1)_{N_1 \times 1} = (f, \phi_i),\ 1 \leq i \leq N_1$, $(\mathbf{f}_2)_{N_2 \times 1} = (f, \chi_i),\ 1 \leq i \leq N_2$, $(\mathbf{f}_\Gamma)_{N_\Gamma \times 1} = (f, \psi_i),\ 1 \leq i \leq N_\Gamma$. Also,

$$\begin{cases} (A_{\Gamma\Gamma})_{N_\Gamma \times N_\Gamma} = (A^{(1)}_{\Gamma\Gamma})_{N_\Gamma \times N_\Gamma} + (A^{(2)}_{\Gamma\Gamma})_{N_\Gamma \times N_\Gamma}, \\ (\mathbf{f}_\Gamma)_{N_\Gamma \times 1} = (\mathbf{f}^{(1)}_\Gamma)_{N_\Gamma \times 1} + (\mathbf{f}^{(2)}_\Gamma)_{N_\Gamma \times 1}, \quad (\mathbf{U}_\Gamma)_{N_\Gamma \times 1} = (\mathbf{U}^{(1)}_\Gamma)_{N_\Gamma \times 1} + (\mathbf{U}^{(2)}_\Gamma)_{N_\Gamma \times 1}, \end{cases}$$

where $A^{(i)}_{\Gamma\Gamma}$, $\mathbf{U}^{(i)}_\Gamma$ and $\mathbf{f}^{(i)}_\Gamma$ denotes the contribution from the sub-domains $\Omega_i, i = 1, 2$.
From (1.3.7) and (1.3.8),

$$A_{11}\mathbf{U}_1 + A_{1\Gamma}\mathbf{U}_\Gamma = \mathbf{f}_1 \ \Rightarrow\ \mathbf{U}_1 = A_{11}^{-1}\,(\,\mathbf{f}_1 - A_{1\Gamma}\mathbf{U}_\Gamma) \tag{1.3.10}$$

and

$$A_{22}\mathbf{U}_2 + A_{2\Gamma}\mathbf{U}_\Gamma = \mathbf{f}_2 \ \Rightarrow\ \mathbf{U}_2 = A_{22}^{-1}\,(\,\mathbf{f}_2 - A_{2\Gamma}\mathbf{U}_\Gamma). \tag{1.3.11}$$

Substituting $\mathbf{U}_1$ and $\mathbf{U}_2$ from (1.3.10) and (1.3.11) in (1.3.9), we obtain

$$\Sigma_h\ \mathbf{U}_\Gamma = \chi_\Gamma, \tag{1.3.12}$$

where

$$\chi_\Gamma = \mathbf{f}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\mathbf{f}_1 - A_{\Gamma 2}A_{22}^{-1}\mathbf{f}_2 \tag{1.3.13}$$

and

$$\Sigma_h = A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}. \tag{1.3.14}$$

The system (1.3.12) is called the **Schur complement system** and the matrix Σ_h is called the **Schur complement matrix**. However, the matrix Σ_h is a full matrix and is ill-conditioned. Its spectral condition number is of order $1/h$ for triangulations of characteristic mesh size h, see [110, Eqn. 2.3.13, pp. 51]. Compared to the the finite element stiffness matrix A for a second order problem, the condition number of matrix Σ_h is of order $1/h$ where the condition number of the matrix A is of order $1/h^2$. For more detailed analysis and references, we refer to [110, 125]. Therefore, it is a common practice to solve the Schur complement system (1.3.12) iteratively via preconditioned CG methods.

In the next subsection, we are going to introduce iterative substructuring methods for the the elliptic problem (1.3.1). For finding the preconditioner for the matrix Σ_h in the system (1.3.12), we need to define **Steklov-Poincaré operator**, which may also be obtained directly from the interface relationship $(1.3.2)_4$. DD methods depend on the interface equation which is associated with the given problem. This interface problem can be defined in terms of Steklov-Poincaré operator that we are going to introduce below. Let ϑ be the unknown value of u on Γ and we consider the following two Dirichlet problems: For $i = 1,\ 2$, find w_i such that

$$\begin{cases} -\Delta w_i + b\, w_i = f & \text{in} \quad \Omega_i \\ w_i = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ w_i = \vartheta & \text{on} \quad \Gamma. \end{cases} \tag{1.3.15}$$

Since Δ operator is linear, we can split the above problem into two problems as follows. Find u_i^o $(i = 1, 2)$ such that

$$\begin{cases} -\Delta u_i^o + b\, u_i^o = 0 & \text{in} \quad \Omega_i \\ u_i^o = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ u_i^o = \vartheta & \text{on} \quad \Gamma \end{cases} \tag{1.3.16}$$

and find $u_i^\star$ $(i = 1, 2)$ such that

$$\begin{cases} -\Delta u_i^\star + b\, u_i^\star = f & \text{in} \quad \Omega_i \\ u_i^\star = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ u_i^\star = 0 & \text{on} \quad \Gamma \end{cases} \tag{1.3.17}$$

Then $w_i = u_i^o + u_i^\star$ $(i = 1, 2)$. For each $i = 1, 2$, u_i^o is the **harmonic extension** of ϑ into Ω_i and is denoted by $H_i\vartheta$. Since $(-\Delta + b\, I)$ is invertible, we set $u_i^\star = G_i f$, where $G_i = (-\Delta + b\, I)^{-1}$. Now comparing (1.3.2) and (1.3.15), we obtain $w_i = u_i$, $i = 1, 2$, if and only if $\dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu}$ on Γ. Since $\dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu}$ on Γ, using the definition of w_i, we find that $\dfrac{\partial u_1^o}{\partial \nu} - \dfrac{\partial u_2^o}{\partial \nu} = \dfrac{\partial u_2^\star}{\partial \nu} - \dfrac{\partial u_1^\star}{\partial \nu}$ on Γ. As u_i^o is the harmonic extension of ϑ into Ω_i, we obtain

$$\frac{\partial (H_1\vartheta)}{\partial \nu} - \frac{\partial (H_2\vartheta)}{\partial \nu} = \frac{\partial (G_2 f)}{\partial \nu} - \frac{\partial (G_1 f)}{\partial \nu} \text{ on } \Gamma.$$

Setting the **Steklov-Poincaré operator** as

$$S\eta = \frac{\partial(H_1\vartheta)}{\partial\nu} - \frac{\partial(H_2\vartheta)}{\partial\nu} = \sum_{i=1}^{2}\frac{\partial(H_i\eta)}{\partial\nu^i},$$

we now arrive at

$$S\vartheta = \chi \quad \text{on} \quad \Gamma, \tag{1.3.18}$$

where

$$\chi = \frac{\partial(G_2 f)}{\partial\nu} - \frac{\partial(G_1 f)}{\partial\nu} = -\sum_{i=1}^{2}\frac{\partial(G_i f)}{\partial\nu^i}.$$

The equation (1.3.18) is the **Steklov-Poincaré interface equation**. In particular, we split S as

$$S = S_1 + S_2, \quad \text{where} \quad S_i\eta = \frac{\partial(H_i\eta)}{\partial\nu^i},\ i = 1, 2.$$

The variational formulation corresponding to (1.3.18) is given as follows: Find $\vartheta \in \Xi$ such that

$$\langle S\vartheta, \mu\rangle = \langle\chi, \mu\rangle \quad \forall\mu \in \Xi. \tag{1.3.19}$$

The functions $u_i^o = H_i\vartheta$ $(i = 1, 2)$ and $u_i^* = G_i f$ $(i = 1, 2)$ introduced in (1.3.16) and (1.3.17) are, respectively, the solutions to the following variational problems:

$$\begin{cases} \text{Find } \ H_i\vartheta \in V_i \text{ such that} \\ a_i(H_i\vartheta, v_i) + (b\, H_i\vartheta, v_i)_{\Omega_i} = 0 \quad \forall v_i \in V_i^0, \\ H_i\vartheta = \vartheta \quad \text{on} \quad \Gamma \end{cases} \tag{1.3.20}$$

and

$$\begin{cases} \text{find } \ G_i f \in V_i^0 \text{ such that} \\ a_i(G_i f, v_i) + (b\, G_i f, v_i)_{\Omega_i} = (f, v_i)_{\Omega_i} \quad \forall v_i \in V_i^0. \end{cases} \tag{1.3.21}$$

Note that the variational form of the Steklov-Poincaré equation can be obtained directly from the interface relation $(1.3.3)_3$. The corresponding finite element approximation of the the Steklov-Poincaré operator can be stated as follows:

$$\begin{cases} \text{Find } H_{i,h}\eta_h \in V_{i,h} \text{ such that} \\ a_i(H_{i,h}\eta_h, v_{i,h}) + (H_{i,h}\eta_h, v_{i,h}) = 0 \quad \forall\ v_{i,h} \in V_{i,h}^0, \\ H_{i,h}\eta_{h|_\Gamma} = \eta_h \quad \text{on} \quad \Gamma \end{cases} \tag{1.3.22}$$

and

$$\begin{cases} \text{find } G_{i,h}f \in V_{i,h}^0 \text{ such that} \\ a_i(G_{i,h}f, v_{i,h}) + (G_{i,h}f, v_{i,h}) = (f, v_{i,h}) \quad \forall\ v_{i,h} \in V_{i,h}^0. \end{cases} \tag{1.3.23}$$

Then find $\vartheta_h \in \Xi_h$ an approximation of ϑ such that

$$S_h\vartheta_h \ = \ \chi_h \ \text{ on } \ \Gamma, \tag{1.3.24}$$

where

$$\chi_h = \ -\sum_{i=1}^{2} \frac{\partial(G_{i,h}f)}{\partial\nu^i}, \quad S_h\eta_h = \sum_{i=1}^{2} \ S_{i,h}\eta_h, \quad S_{i,h}\eta_h = \frac{\partial(H_{i,h}\eta_h)}{\partial\nu^i} \quad \forall\ \eta_h \in \Xi_h. \tag{1.3.25}$$

In variational form, we rewrite (1.3.24) as

$$\langle S_h\eta_h, \mu_h\rangle = \langle \chi_h, \mu_h\rangle \ \forall\ \mu_h \in \Xi_h, \tag{1.3.26}$$

where

$$\begin{aligned} \langle S_h\eta_h, \mu_h\rangle \ &= \ \sum_{i=1}^{2} \{a_i(H_{i,h}\eta_h, R_{i,h}\mu_h) + (b\, H_{i,h}\eta_h, R_{i,h}\mu_h)\} \\ &= \ \sum_{i=1}^{2} \{a_i(H_{i,h}\eta_h, H_{i,h}\mu_h) + (b\, H_{i,h}\eta_h, H_{i,h}\mu_h)\} = \sum_{i=1}^{2}\langle S_{i,h}\eta_h, \mu_h\rangle \end{aligned}$$

and

$$\langle \chi_h\eta_h, \mu_h\rangle = \sum_{i=1}^{2} \ [(f, R_{i,h}\mu_h) - \{a_i(G_{i,h}f, R_{i,h}\mu_h) + (b\, G_{i,h}f, R_{i,h}\mu_h)\}] \ \forall\ \eta_h,\ \mu_h \in \Xi_h.$$

Here $R_{i,h}$, $i = 1, 2$, is any extension operator from Ξ_h into $V_{i,h}$. Similarly, we obtain a matrix Σ_h which is precisely the algebraic counterpart of the discrete Steklov-Poincaré operator S_h as

$$[\ \Sigma_h\eta_h, \mu_h\ ] = \langle S_h\eta_h, \mu_h\rangle \ \ \forall\eta_h, \mu_h \in \Xi_h, \tag{1.3.27}$$

where $[\cdot, \cdot]$ is the Euclidean scalar product in $\Re^{N_\Gamma}$ and for each $\mu_h \in \Xi_h$, μ_h denotes the set of its values at the nodes on Γ. For $i = 1, 2$, we define $\Sigma_{i,h}$ as

$$[\ \Sigma_{i,h}\eta_h, \mu_h\ ] = \langle S_{i,h}\eta_h, \mu_h\rangle \ \ \forall\eta_h, \mu_h \in \Xi_h. \tag{1.3.28}$$

The above results are discussed in [110].

Another approach called Lagrange multiplier based approach is also used in the literature [48, 123]. In Lagrange multiplier approach, we obtain solution as well as its normal derivate on the subdomain interfaces. Through this approach it is possible to relax the continuity conditions at the interfaces of the subdomains. Lagrange multiplier based framework can be defined in terms of Steklov-Poincaré operator that we are going to introduce below.

Let γ_i be the trace operator mapping functions in $H^1_\Gamma(\Omega_i) = V_i,\ i = 1, 2$ to their traces in Γ. Let $H^{1/2}_{00}(\Gamma)$ be the fractional order Sobolev space on Γ consisting of traces of functions in $H^1_\Gamma(\Omega_i)$ and let $(H^{1/2}_{00}(\Gamma))'$ denote its dual. Using the continuity of fluxes, we will split the problem into two subproblems for $i = 1, 2$ such that

$$\begin{cases} -\Delta u_i + b\, u_i &= f \qquad \text{in} \quad \Omega_i \\ u_i &= 0 \qquad \text{on} \quad \partial\Omega_i \cap \partial\Omega_i \\ \nabla u_i \cdot \nu &= (-1)^{i+1}\lambda \quad \text{on} \quad \Gamma. \end{cases} \tag{1.3.29}$$

The vector ν is the outward normal to Γ oriented, from Ω_1 to Ω_2. The weak formulation corresponding to the problems (1.3.29) is to find $u_i \in H^1_\Gamma(\Omega_i),\ i = 1, 2$ such that

$$a_i(u_i, v_i) + (b\, u_i, v_i)_{\Omega_i} = \langle \lambda, v_i \rangle_\Gamma + (f_i, v_i)_{\Omega_i}, \quad \forall\, v_i \in H^1_\Gamma(\Omega_i). \tag{1.3.30}$$

Here, we first reduce the problem to a problem on the subdomain interface using Steklov-Poincaré operators. For the unknown Neumann data λ on Γ, we define the Steklov-Poincaré operators $S_i^\star\ :\ (H^{1/2}_{00}(\Gamma))' \to H^{1/2}_{00}(\Gamma)$, $i = 1, 2$ by

$$S_i^\star \lambda = \gamma_i u_i, \tag{1.3.31}$$

where $\lambda \in (H^{1/2}_{00}(\Gamma))'$ and u_i is the solution of (1.3.30) with $f_i = 0$. Here u_i is the harmonic function satisfying the Neumann condition given by λ. In other words, the Steklov-Poincaré operator maps the Neumann boundary condition into the corresponding Dirichlet boundary condition as :

$$S_i^\star\ :\ \frac{\partial u_i}{\partial \nu} \to \gamma_i u_i. \tag{1.3.32}$$

Furthermore, we define $G_i^\star\ :\ (H^1_\Gamma(\Omega_i))' \to H^{1/2}_{00}(\Gamma)$, $i = 1, 2$ by the equation

$$G_i^\star f_i = \gamma_i u_i, \tag{1.3.33}$$

where $f_i \in L^2(\Omega_i)$, u_i is the solution of (1.3.30) with $\lambda = 0$. In terms of the Steklov-Poincaré operators, the problem is to find the solution λ such that

$$(S_1^\star + S_2^\star)\lambda = G_2^\star f_2 - G_1^\star f_1, \tag{1.3.34}$$

that is to find the Neumann data λ on Γ such that the traces of the solutions $u_i,\ i = 1, 2$ of (1.3.30) coincide on Γ.

The standard finite element method with Lagrange multipliers was first introduced by Babuška in [6] for second order elliptic problems with Dirichlet boundary conditions. He further showed that an application of Lagrange multipliers would avoid the difficulty in fulfilling essential boundary conditions on the finite element spaces. In the primal hybrid finite element method of Raviart and Thomas [112] the usefulness of Lagrange multipliers which approximate normal derivatives on the boundary of each finite element is shown. Subsequently, Bramble [17] has reformulated the Lagrange multiplier method of Babuška [6], and discussed estimates for the solution and the boundary flux.

The Lagrange multiplier approach to enforce the continuity of the solution is linked to interface formulation using Poincaré-Steklov operators in the DD context by Dorr [48]. This Lagrange multiplier technique consists in relaxing the continuity conditions at the corners of the subdomains and gives a saddle-point problem without Lagrange multipliers associated with vertices, where the normal derivative may not be well defined as the normal vector field is discontinuous at these points. He has used the Lagrange formulation to introduce finite element spaces of smaller dimension on the interfaces for regular meshes. This can reduce the size of the problem substantially, but it is restricted to regular meshes. Swann [123] has used cell discretization method in his analysis. In his approach, the domain of a problem is partitioned into cells; approximations are made on each cell, and the approximations are forced to be weakly continuous across the boundaries of each cell by using Lagrange multipliers. The only requirement for convergence of this method, which is referred to as moment collocation is that the basis functions on each cell constitute a Schauder basis in an appropriate space. The finite element tearing and interconnecting (FETI) method is an iterative substructuring method using Lagrange multipliers to enforce the continuity of the finite element solution across the subdomain interface, see [63, 96]. Exploiting the structure of the Lagrange multipliers, Belgacem [11] has analyzed the mortar element method with

Lagrange multiplier by setting it under the frame work of a primal hybrid formulation. A basic requirement for the Lagrange multiplier method is to construct multiplier spaces which satisfy certain criteria known as the inf-sup properties for the scheme to be stable. To achieve stability of the corresponding Lagrange multiplier scheme, we need to choose the multiplier space appropriately so that the discrete spaces for the primal variable and the multiplier satisfy the inf-sup condition, also known as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition. When the Lagrange multiplier is used to relax the mortaring condition on the finite element spaces, the corresponding discrete formulation gives rise to an indefinite system. The mortar element method using dual spaces for the Lagrange multipliers has been studied in [126]. The Lagrange multiplier space is replaced by a dual space without losing the optimality of the method. The advantage of this approach is that all the basis functions are locally supported. Compared to the standard mortar method where a linear system of equations for the mortar projection must be solved; in this case the matrix associated with mortar is represented by a diagonal matrix. In [88], Lamichhane and Wohlmuth extended the mortar finite elements with Lagrange multipliers to elliptic interface problems. Many natural and convenient choices of these spaces are ruled out as these spaces do not satisfy the inf-sup condition. In order to alleviate this problem, stabilized multiplier techniques or Nitsche's method [120] is used. In this method, the original bilinear forms of the problem are modified by adding suitable stabilized terms in order to improve stability without compromising on the consistency of the method. We refer to [7, 9, 10] for the various penalty methods applied to elliptic problems and discuss how to circumvent the inf-sup condition in order to achieve the consistency and stability of the methods. The drawback of most of the stabilized methods is that they use jump in the primal variables as one of stabilized term across the subdomain interfaces. To mitigate this problem, Hansbo et al. [80] have proposed a stabilization method which avoids the cumbersome integration of products of unrelated mesh functions.

Another approach based on the balancing DD algorithm uses solution of local problems on the subdomains in each iteration coupled with a coarse problem that is used to propagate the error globally and to guarantee that the possibly singular local problems are consistent. The abstract theory introduced in [94] is used to develop bound on the condition numbers for conforming linear elements in two and three dimensions. It is to be observed that the balancing DD algorithm is known as the Neumann-Neumann algorithm for non-overlapping DD methods. For related results on the balancing DD algorithms, we refer to [95, 97, 98].

From an engineering point of view, the mixed finite element methods for approximating flux for elliptic problems with discontinuous and rapidly varying coefficients provide efficient and accurate solutions. Glowinski and Wheeler [74] have proposed and analyzed DD techniques combined with mixed finite element methods for elliptic problems. However, their approach requires that the resulting discrete systems should be solved exactly by a fast direct method on the subdomains. Other DD methods with nonoverlapping partitions for mixed finite element methods are discussed by Cowsar and Wheeler [37], Rusten and Winther [116], and Cowsar, Mandel, and Wheeler [38]. In [116], Rusten and Winther have derived DD preconditioners for the linear systems arising from mixed finite element discretizations of second-order elliptic boundary value problems. The preconditioners are based on subproblems with either Neumann or Dirichlet boundary conditions on the interior boundary. In [32], Chen has shown that the mixed finite element formulation can be algebraically condensed to a symmetric and positive definite system for Lagrange multipliers using the features of the existing mixed finite element spaces for elliptic problems. Subsequently, Chen et al. [33] have discussed the DD algorithms for mixed finite element methods based on the approach described in [32] for second order elliptic problems.

Most of the above methods are designed for elliptic partial differential equations (PDEs). In principle, DD methods can be applied to the resulting elliptic problem at each time level when implicit time discretization applied to parabolic problems. In the context of parabolic problems, explicit schemes are parallel and also easy to implement, but they usually require small time steps because of stability constraints. On the other hand, implicit schemes are necessary for finding the steady state solutions or computing slowly unsteady problems where one needs to march with large time steps. However, the implicit schemes are not inherently parallel because at each time step essentially an elliptic type of problem needs to be solved.

DD methods for time dependent problems have been discussed in [40, 41, 42, 58, 59, 60, 87, 110, 130] and the references, therein. In [40, 60, 87], the authors have discussed the DD method in the frame work of finite difference schemes. Kuznetsov [87] has proposed an explicit-implicit scheme to solve parabolic problems based on a partition of Ω into non-overlapping regions. The boundary value of u^{n+1} on the interface Γ is first computed using an explicit method (or even an implicit scheme) in a small neighborhood of Γ. Using these boundary values, Dirichlet problems can be solved on each sub-domain to provide the solution u^{n+1} on the whole domain Ω. This idea is particularly appealing on the grids containing regions of refinement, see [87]. Another alternate direct approach was proposed by Dawson, Du and Dupont [40] by finite difference methods in the context of finite difference methods. In this procedure, interface values between subdomains are found by an explicit difference formula. Dawson and Du [41] has extended earlier work by Dawson et al. [40] based on finite element methods. In this procedure, subdomain interface data are updated using an explicit procedure in one dimension, and an "implicit in y, explicit in x" procedure in two dimensions. Dawson and Dupont [42] has discussed explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems. In this procedure, the domain is partitioned into many non-overlapping sub-domains with interface Γ and special basis functions are constructed having support in a small 'tube' of width $O(H)$ containing the interface Γ. In the first step approximate flux using explicit procedure on Γ using these special basis functions. Finally, using these boundary values, the solution u^{n+1} is determined at the interior of the sub-domains, see [42]. The explicit nature of the flux calculation induces a time step limitation necessary to preserve stability, although this constraint is not necessary sharp which comes with a fully explicit method.

In contrast, a second approach based on the discretization of the parabolic problems which leads to a DD algorithm as a direct method as given by Dryja [58] and corresponds to a domain decomposed matrix splitting (fractional step method) involving two non-overlapping subregions. The resulting scheme can be shown to be unconditionally stable. Unfortunately, the discretization error of splitting scheme becomes the square root of the discretization error of the original scheme. In the two-dimensional finite element case Dryja [58] has proved $\hat{\Sigma}_{2,h}$ is the preconditioner for $\hat{\Sigma}_h$, where the condition number, $\kappa(\hat{\Sigma}_{2,h}^{-1}\hat{\Sigma}_h)$ is bounded by $C(1+\log\frac{H}{h})^2$, $C > 0$ is a constant independent of h, H and Δt, $\hat{\Sigma}_h$ being the Schur complement matrix and H being the diameter of the sub-domain.

Dryja [59] used Crank-Nicolson scheme for time discretization of parabolic problems, but this algorithm is stable and convergent with an error bound $O(\Delta t + h)$ in an appropriate norm. The error bound obtained for the method is same as for the backward Euler scheme. Zheng et al. [132] have discussed nonoverlapping DD method for parabolic problems based on stabilized explicit Lagrange multipliers. First they formulate the problem into a differential algebraic equations and then solve them using Runge-Kutta-Chebyshev projection method [131]. To develop a stabilized explicit DD finite element method, they use the mass lumping technique [127]. In [106], Pradhan et al. have discussed the application of DD methods to a parabolic integro-differential equations.

Another approach was proposed by Girault, Glowinski and Lopez [72], in which the domain is partitioned into many non-overlapping sub-domains, where the sub-domain meshes need not be quasi-uniform. They are composed of triangles or quadrilaterals that do not match at interfaces. For the case of computation, this lack of continuity is compensated by a mortar technique based on piecewise constant (discontinuous) multipliers on the interfaces, thus making the implementation simpler. But the price to pay is asymptotically a half-order loss in accuracy compared with mortar methods, see [72].

1.3.2 Iterative non-overlapping domain decomposition methods

In this subsection, we discuss iterative procedures to solve the multi-domain problem (1.3.2). Under the iterative schemes assuming either the value of the solution or its normal derivative or a combination of both the solution and its normal derivative on the intersubdomain interfaces, the problem can be solved in parallel in each subdomain and then an iterative technique is invoked to update the values of the solution or its normal derivative on the interfaces. To motivate the iterative schemes, we now introduce a sequence of subproblems in Ω_1 and Ω_2 for which the conditions $(1.3.2)_3$ and $(1.3.2)_4$ provide the Dirichlet and Neumann data, respectively, on the interface Γ. In general, we expect that the two sequences of functions $\{u_1^k\}$ and $\{u_2^k\}$ starting from initial guesses u_1^0, u_2^0 will converge to u_1 and u_2 respectively.

Dirichlet-Neumann iterative scheme. Given ϑ^0, find u_1^{k+1}, u_2^{k+1} and ϑ^{k+1} for each

$k \geq 0$ such that

$$\begin{cases} -\Delta u_1^{k+1} + b\, u_1^{k+1} = f & \text{in} \quad \Omega_1, \\ u_1^{k+1} = 0 & \text{on} \quad \partial\Omega_1 \cap \partial\Omega, \\ u_1^{k+1} = \vartheta^k & \text{on} \quad \Gamma, \end{cases} \tag{1.3.35}$$

$$\begin{cases} -\Delta u_2^{k+1} + b\, u_1^{k+1} = f & \text{in} \quad \Omega_2, \\ u_2^{k+1} = 0 & \text{on} \quad \partial\Omega_2 \cap \partial\Omega, \\ \dfrac{\partial u_2^{k+1}}{\partial \nu} = \dfrac{\partial u_1^{k+1}}{\partial \nu} & \text{on} \quad \Gamma, \end{cases} \tag{1.3.36}$$

and

$$\vartheta^{k+1} \;=\; \theta\, u_2^{k+1}{}_{|\Gamma} \;+\; (1-\theta)\, \vartheta^k, \tag{1.3.37}$$

where θ is an acceleration parameter with $0 \leq \theta < 1$. This method was considered by Bjorstad and Widlund [13], Funaro et al. [65] and Marini and Quarteroni [99]. It is shown in [110] that the Dirichlet-Neumann iterative scheme is convergent and the rate of convergence is independent of h, where h is the mesh size for triangulations. It is to be noted that the Dirichlet-Neumann iterative scheme is algorithmically sequential. Next, we define Neumann-Neumann iterative procedures to solve the multi-domain problem (1.3.2).

Neumann-Neumann iterative scheme. Given ϑ^0, find $u_i^{k+1},\ \psi_i^{k+1} \in V_i,\ i = 1,\ 2$ for each $k \geq 0$ such that

$$\begin{cases} -\triangle u_i^{k+1} + b\, u_i^{k+1} &= f \quad \text{on} \ \Omega_i, \\ u_i^{k+1} &= 0 \quad \text{on} \ \partial\Omega_i \cap \partial\Omega, \\ u_i^{k+1} &= \vartheta^k \quad \text{on} \ \Gamma \end{cases} \tag{1.3.38}$$

and then

$$\begin{cases} -\triangle \psi_i^{k+1} + b\, \psi_i^{k+1} &= 0 \quad \text{on} \ \Omega_i, \\ \psi_i^{k+1} &= 0 \quad \text{on} \ \partial\Omega_i \cap \partial\Omega, \\ \dfrac{\partial \psi_i^{k+1}}{\partial \nu} &= \dfrac{\partial u_1^{k+1}}{\partial n} - \dfrac{\partial u_2^{k+1}}{\partial \nu} \quad \text{on} \ \Gamma, \end{cases} \tag{1.3.39}$$

with $\vartheta^{k+1} = \vartheta^k - \theta\left(\sigma_1\, \psi_1^{k+1}{}_{|\Gamma} - \sigma_2\, \psi_2^{k+1}{}_{|\Gamma}\right)$, $\theta > 0$ and σ_1 and σ_2 are two positive averaging coefficients. It is observed that in [14] that the Neumann-Neumann iterative scheme is convergence and the rate of convergence is shown to be independent of the grid-size h. Further, we note that the Neumann-Neumann iterative scheme is algorithmically parallel.

Now, we define Robin iterative procedures to solve the multi-domain problem (1.3.2).

Robin iterative scheme. Given u_2^0, find u_1^{k+1} and u_2^{k+1} for each $k \geq 0$ such that

$$\begin{cases} -\Delta u_1^{k+1} + b\, u_1^{k+1} = f & \text{in} \quad \Omega_1, \\ u_1^{k+1} = 0 & \text{on} \quad \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_1^{k+1}}{\partial \nu} + \gamma_1\, u_1^{k+1} = \dfrac{\partial u_2^k}{\partial \nu} + \gamma_1\, u_2^k & \text{on} \quad \Gamma, \end{cases} \tag{1.3.40}$$

$$\begin{cases} -\Delta u_2^{k+1} + b\, u_1^{k+1} = f & \text{in} \quad \Omega_2, \\ u_2^{k+1} = 0 & \text{on} \quad \partial\Omega_2 \cap \partial\Omega, \\ \dfrac{\partial u_2^{k+1}}{\partial \nu} - \gamma_2\, u_2^{k+1} = \dfrac{\partial u_1^{k+1}}{\partial \nu} - \gamma_2\, u_1^{k+1} & \text{on} \quad \Gamma, \end{cases} \tag{1.3.41}$$

where γ_1 and γ_2 are non-negative acceleration parameters satisfying $\gamma_1 + \gamma_2 > 0$. For the sake of parallelisation, in (1.3.41) we could also consider u_1^k instead of u_1^{k+1} and assigning in that case also u_1^0. The Robin-type boundary conditions as interface conditions was proposed by Lions in [92] as a tool for the domain decomposition iterative methods. This method is now referred to as Lions nonoverlapping DD method (**Lions method**). In [92] only the convergence of the Lions method in the multi-domain case has been proved when $b(x) \geq 0$, that is, there are no estimates of error reduction factor at each iteration, nor any information about the rate of convergence. We refer the reader to Agoshkov [1] for a similar formulation at the algebraic level. Later on, Despres [45, 46] has applied Lions idea to the Helmholtz problems. In 1993, Douglas et al. [49] have discussed parallel iterative procedure to approximate the solution of (1.3.1) by using mixed finite element methods and obtained the rate of convergence through a spectral radius estimation of the iterative solution. Note that each triangle is considered as a subdomain. Further, it is shown that the spectral radius has a bound of the form $1 - Ch$ for quasiregular partitions when $b(x) \geq b_0 > 0$, where h is the mesh size for triangulations. Subsequently, Douglas et al. [52] have established the convergence rate as $1 - Ch$ for nonconforming finite element methods by again using the spectral radius estimation of the iterative solution for the elliptic problems (1.3.1) on quasiregular partitions when $b(x) \geq b_0 > 0$.

Later, Deng [43, 44] has developed and analyzed another non-overlapping DD iterative procedure for elliptic problems (1.3.1), which are based on the following subproblems: Given $g_{ij}^0,\ 1 \leq j \neq i \leq 2$ arbitrarily, find $u_i^k,\ i = 1, 2$ for each $k \geq 0$ such that

$$\begin{cases} -\Delta u_i^k + b\, u_i^k = f & \text{in} \quad \Omega_i, \\ u_i^k = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega, \\ \dfrac{\partial u_i^k}{\partial \nu^i} + \beta\, u_i^k = g_{ij}^k & \text{on} \quad \Gamma, \quad \forall 1 \leq j \leq 2, \quad j \neq i, \end{cases} \tag{1.3.42}$$

and then update the Robin data of the transmission condition as

$$g_{ij}^{k+1} = 2\beta\, u_j^k - g_{ji}^k \quad \text{on} \quad \Gamma, \quad \forall 1 \leq j \leq 2, \quad j \neq i, \tag{1.3.43}$$

where $\beta > 0$ is the transmission coefficient. Note that the updation technique of Robin data g in (1.3.43) is different from Lions method [92]. Deng has analyzed the convergence when $b(x) = 0$ in [44] for (1.3.1) and obtained the convergence rate by a spectral radius estimation of the iterative solution when $b(x) \geq b_0 > 0$. He has shown that the spectral radius has a bound of the form $1 - Ch$ for quasiregular partitions, provided $b(x) \geq b_0 > 0$. In [44, 49, 52], the iterative method is shown to be convergent but the rate of convergence is not established, when $b(x) = 0$. Recently, Gou and Hou [79] have analyzed a one-parameter generalization of Lions nonoverlapping method [92] for solutions of (1.3.1). They have established the convergence and acceleration properties of the finite element versions of the proposed method, when $b(x) = 0$. But there are no estimates of the error reduction factor at each iteration, nor any information about the rate of convergence of the proposed method. Due to lack of coercivity of the associated bilinear form in the inner-subdomains, particular attention is needed when $b(x) = 0$ to achieve the convergence rate of the iterative method. Based on the method proposed in [44], Qin and Xu [109] have derived the convergence rate, in general, when the lower term vanishes, i.e., $b(x) = 0$ and the convergence rate is shown to be of order 1 - $O(h^{1/2}H^{-1/2})$, when the winding number N (see, the definition 3.2.1 given in chapter 3) is not large and H is the maximum diameter of the subdomains. In [84], Kim et al. have discussed iterative DD method to approximate the solution of a nonlinear parabolic problems based on fully discrete mixed finite element method. In this paper, they have used Robin type boundary conditions for inter-subdomains boundaries and demonstrated the convergence of the iteration at each time step.

1.3.3 Various other domain decomposition methods

There are other classes of *direct* and *iterative* methods, which are quite popular amongst the DD community. Since in this dissertation, we have not touched upon these classes of methods, we only briefly present some earlier results. In the earliest 19th century, Schwarz [118] proposed an iterative method for the solution of classical boundary value problems for harmonic functions. It consists of solving successively a similar problem in subdomains, going alternatively from one to other. The convergence of this process was proved using of the maximum principle. This is called as iterative Schwarz alternating procedure for overlapping DD method. In 1953, Kron [86] introduced the set of principles and a systematic procedure to establish the exact solutions of very large and complicated physical systems, without solving a large number of simultaneous equations. The procedure consists of dividing the system into several smaller sub-systems. To obtain a solution of the original system, Kron has interconnected sub-system solutions through a set of transformations and this method is subsequently known as fast direct DD solvers (substructuring or tearing methods) in literature. Subsequently, in 1963, Przemieniecki [108] discussed a matrix method of linear structural analysis for the calculation of stresses and deflections in an aircraft structure divided into a number of structural components. This direct matrix method is called substructuring. In 1982, Dryja [53] has described algorithms for the solution of the system of linear equations arising from the application of finite element method to the Dirichlet problem on a polygonal region based on the capacitance matrix technique. Exploiting the capacitance matrix technique, Dryja [54] has applied it to the symmetric elliptic problem with the Dirichlet condition on an arbitrary region. In 1984, Dryja [55] has again employed the same method to a general elliptic problems. In DD terminology, this is a "Schur complement matrix" system, see [29, 36, 73, 110, 119, 125]. A good approximation to the Schur complement of a linear system can be constructed algebraically by investigating its numerical structure. This idea is introduced by Dryja [53] and further developed in a paper by Golub and Mayers [77] that refered to the symmetric two dimensional case. The subdomain structuring of the Schur complement matrix or capacitance matrix can lead to block direct methods. It can lead to block iterative methods via preconditioners, see [36]. Gropp and Keyes [76], Langer et al. [82] have discussed preconditioners for DD methods.

The Schur complement system can be extended by iterative coupling of the subregions. There are two approaches widely followed for the construction of DD preconditioner. One is a (modified) Schur complement preconditioner that has been studied by the DD community very intensively, see [18, 19, 53]. Another is a preconditioner for the local problems with homogeneous Dirichlet boundary conditions arising in each sub-domain. The most sensitive part is the transformation operator transforming the nodal finite element basis on the interfaces into the approximate discrete harmonic basis. However, we provide here the results from some articles which play crucial role in developing DD methods, see [110]. See [13, 18, 65] for the Dirichlet-Neumann algorithm for non-overlapping DD methods. Often, as in preconditioner conjugate gradient (PCG), the objective is to produce an iterative method in which the matrix is symmetric positive definite. Meyer [102] has proposed a parallelization and preconditioning of the conjugate gradient (CG) method on the basis of a non-overlapping DD approach. A survey of preconditioners for DD is given by Chan and Resasco [28]; see also Meurant [101].

In [61], Ehrlich has discussed the iterative Schwarz alternating procedure for overlapping DD method. For Schwarz alternating algorithm in a variational framework, see Dryja and Widlund [56], Matsokin and Nepomnyaschikh [100] and Lions [90]. The original two-subdomain Schwarz method is now called the multiplicative Schwarz method, see [12]. First one subdomain is solved with pseudo-boundary conditions, then the information is transfered to the pseudo-boundary conditions for the other subdomain. This method is algorithmically effective. Subsequently, Haase and Langer [81] have discussed a multiplicative Schwarz method for non-overlapping DD procedure. Although the Schwarz alternating method is straightforward and intuitive, it is, in fact, a very effective procedure, see the reference [90, 91]. We now conclude this section with a quotation of P . L. Lions [91] ”In some sense, even if many interesting and important variants have been introduced recently, the Schwarz algorithm remains the prototype of such methods and also presents some properties (like robustness, or indifference to the type of equations considered...) which do not seem to be enjoyed by other methods".

1.4 Outline of the Thesis

The organization of thesis is as follows. Chapter 1, which is introductory in nature consists of some definitions, inequalities and some results to be used in subsequent chapters. Further, it deals with a brief survey on DD methods.

In Chapter 2, an effort has been made to apply non-iterative non-overlapping DD methods combined with non-conforming finite element methods with Lagrange multipliers for elliptic problems. When the original domain is decomposed into subdomains, the transmission conditions come into picture on the inter-subdomain boundaries. The matching conditions are expressed in terms of Lagrange multipliers for the Neumann boundary condition on the artificial boundary, which produce good approximation of the normal derivatives of the exact solution across the interfaces. The key feature that we have adopted here is the nonconforming Crouzeix-Raviart space for the discretization of the primal variable.

For parabolic equations a completely discrete scheme based on backward Euler scheme is discussed. Optimal error estimates in L^2 and H^1-norms are demonstrated. The results of numerical experiments support the theoretical results which are derived in this chapter.

Chapter 3 is concerned with the analysis of an iterative non-overlapping DD method with Robin-type boundary conditions on the artificial interfaces, that is, on the inter subdomain boundaries of the elliptic problems. The rate of convergence is derived to be of $1 - O(h^{1/2}H^{-1/2})$, where h is the finite element mesh parameter and H is the maximum diameter of the subdomains. This chapter is concluded with an application to parabolic equations. Finally, some numerical experiments are conducted to illustrate the theoretical results.

In Chapter 4, we propose and analyze an iterative non-overlapping DD method for elliptic problems based on mixed finite element methods. We have used Robin-type boundary conditions to obtain the transmission data on the inter-subdomain boundaries. The convergence analysis of the parallel iterative procedure is discussed in details. The rate of convergence is estimated as $1 - O(h^{1/2}H_\star)$, where h is the finite element mesh parameter and $H_\star$ is the minimum diameter of the subdomains.

Finally, we present, in Chapter 5, we first present a summary of the results with some observations. Further, we conclude this Chapter with a discussion of some possible extensions and future problems.

Chapter 2

A Non-Conforming Finite Element Method with Lagrange Multipliers

2.1 Introduction

In this chapter, we discuss a non-overlapping domain decomposition procedure for approximating the solution of second order elliptic and parabolic equations using non-conforming finite element methods. When the original domain is decomposed into subdomains, the transmission conditions come into play on the inter-subdomain boundaries. The matching conditions are expressed in terms of the Lagrange multiplier for the Neumann boundary condition on the artificial boundary, which produces good approximation of the normal derivatives of the exact solution across the interfaces. Lagrange multiplier technique helps in relaxing the continuity conditions at the interfaces of the subdomains. A basic requirement for the Lagrange multiplier method is to construct multiplier spaces which satisfy certain criteria known as the inf-sup properties for the scheme to be stable. To achieve stability of the corresponding Lagrange multiplier scheme, we need to choose the multiplier space appropriately so that the discrete spaces for the primal variable and the multiplier satisfy the inf-sup condition, also known as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition.

Earlier, a finite element method with Lagrange multipliers was first introduced by Babuška in [6] for second order elliptic problems with Dirichlet boundary condition. In his paper, he showed that an application of Lagrange multipliers would avoid the difficulty in fulfilling essential boundary conditions on the finite element spaces. Subsequently, Bramble

[17] reformulated the Lagrange multiplier method of Babuška [6], and discussed estimates for the solution and the boundary flux. The Lagrange multiplier approach to enforcing solution continuity is related to interface formulations using Poincaré-Steklov operators on the regular mesh by Dorr [48]. Exploiting the structure of the Lagrange multipliers, Belgacem [11] has applied it to the mortar finite element method. Further, he has discussed the construction of the discrete Lagrange multiplier space, which is compatible to the discrete trace space, so that the Babuška-Brezzi condition (inf-sup condition) is satisfied. In [126], Wohlmuth has analyzed the mortar finite element method with Lagrange multipliers using dual Lagrange multiplier spaces. In [88], Lamichhane and Wohlmuth have extended the mortar finite elements with Lagrange multipliers to elliptic interface problems. Subsequently, Hansbo et al. [80] has analyzed the Lagrange multiplier method for the finite element solution of the multi-domain elliptic PDEs using non-matching meshes. Moreover, they introduced a penalty term as a stabilizer and derived a priori error bounds.

DD methods for time dependent problems have been discussed in [40, 41, 42, 58, 59, 60, 87, 110, 130] and the references, therein. In [40, 60, 87], the authors have discussed the DD method in the frame work of finite difference schemes. Kuznetsov [87] has proposed a modified approximation scheme of mixed type, where the standard second order implicit scheme is used inside each subdomain, while the explicit Euler scheme is applied to update the interface values on the new time level. Once the interface values are available, the global problem is fully decoupled and can, thus, be computed in parallel. A similar scheme was proposed in [40, 41, 42], where instead of using the same spacing h as for the interior points where the implicit scheme is applied, a larger spacing H is used at each interface point where the explicit scheme is applied. Due to stability and accuracy requirements, both methods do not lead to satisfactory computational results. In the two-dimensional finite element case Dryja [58] has proved $\hat{\Sigma}_{2,h}$ is the preconditioner for $\hat{\Sigma}_h$, where the condition number, $\kappa(\hat{\Sigma}_{2,h}^{-1}\hat{\Sigma}_h)$ is bounded by $C(1+\log\frac{H}{h})^2$, $C>0$ is a constant independent of h, H and Δt, $\hat{\Sigma}_h$ being the Schur complement matrix and H being the diameter of the subdomain. Dryja [59] used Crank-Nicolson scheme for the time discretization of parabolic problems, and this algorithm is stable and convergent with an error bound $O(\Delta t + h)$ in an appropriate norm. But the error bound obtained for the method is same as for the backward Euler scheme. Zheng et al. [132] have discussed nonoverlapping DD method for parabolic problems based on stabilized explicit Lagrange multipliers. First they formulate the problem into a differential algebraic equations and then solve them using Runge-Kutta-Chebyshev projection method [131]. To develop a stabilized explicit DD finite element method, they use the mass lumping technique [127].

A brief outline of this chapter is as follows. In Section 2.2, we formulate the elliptic multidomain problem and we introduce Lagrange multipliers on inter-element subdomain boundaries. The key feature that we have adopted here is nonconforming Crouzeix-Raviart space for the discretization of the primal variable. In Section 2.3, we have discussed both L^2 and H^1 error estimates. In Section 2.5-2.7, we extend the method to parabolic initial and boundary value problems and analyze the error estimates for both semidiscrete and fully discrete schemes. Finally, Section 2.4 and Section 2.8 deals with some numerical experiments to support our theoretical results.

2.2 The elliptic problem

We consider the following second order problem:

$$\begin{cases} -\Delta u &= f \qquad \forall x \in \Omega, \\ u &= 0 \qquad \forall x \in \partial\Omega, \end{cases} \tag{2.2.1}$$

where Ω is a bounded convex polygon or polyhedron in $\mathbb{R}^d$, $d = 2$ or 3 and $f \in L^2(\Omega)$. The weak formulation of (2.2.1) is to find $\bar{u} \in H_0^1(\Omega)$ such that

$$a_\Omega(\bar{u}, v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{2.2.2}$$

where

$$a_\Omega(v, w) = \int_\Omega \nabla v \cdot \nabla w \ \ dx. \tag{2.2.3}$$

To describe finite element approximations for (2.2.2), we begin with a regular triangulation of $\bar{\Omega}$. Let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$ into triangles for $d = 2$, tetrahedrons for $d = 3$. Let the boundary of T be denoted by ∂T and let T' denote an edge of T when $d = 2$,

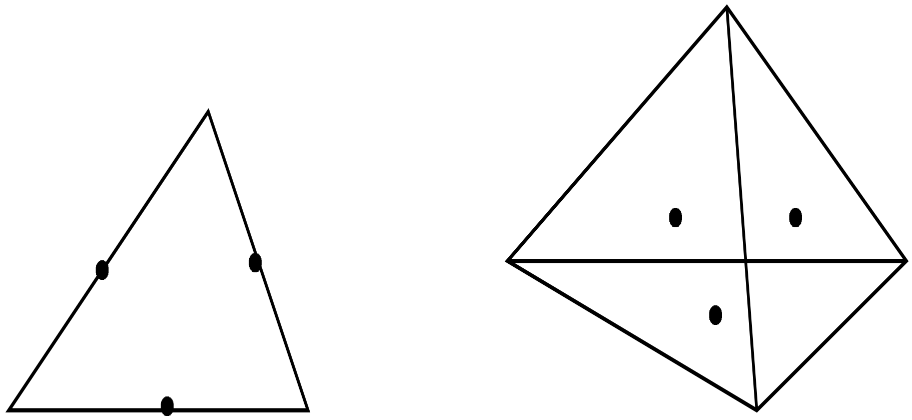

Figure 2.1: Nonconforming finite elements

a triangular face when $d = 3$ (see details in Chapter 1). Let $P_r(T)$ denote the space of polynomials of degree less than or equal to r in two variables defined on the triangle T. Now, we define the nonconforming Crouzeix-Raviart space (cf. [39]) associated with the triangulation $\mathcal{T}_h$. Let

$$\bar{X}_h = \left\{ v \in L^2(\Omega) \mid v_{|T} \in P_1(T),\ T \in \mathcal{T}_h,\ v \text{ is continuous at } p \in N_h \right.$$
$$\left. \text{and vanishes at } p \in \Gamma_h \right\}, \quad (2.2.4)$$

where N_h is the set of all face barycenters of elements of $\mathcal{T}_h$ in the interior of Ω and Γ_h is the set of all face barycenters of elements of $\mathcal{T}_h$ on the boundary of $\partial\Omega$. A function in $\bar{X}_h$ is completely determined by its values at centers of the sides of the triangle ($d = 2$) or tetrahedron ($d = 3$) in $\mathcal{T}_h$ (cf. Figure 2.1). Then, the nonconforming Galerkin approximation of (2.2.2) is defined as the solution $\bar{u}_h \in \bar{X}_h$ of

$$a^h_\Omega(\bar{u}_h, v_h) = (f, v_h) \qquad \forall v_h \in \bar{X}_h, \quad (2.2.5)$$

where

$$a^h_\Omega(v_h, w_h) = \int_\Omega \nabla v_h \cdot \nabla w_h \, dx. \quad (2.2.6)$$

Lemma 2.2.1 *The problem (2.2.5) has a unique solution.*

Proof: Since (2.2.5) leads to a system of linear algebraic equations, it is enough to prove uniqueness. Setting $f = 0$ and $v_h = u_h$ in (2.2.5), we obtain $a^h_\Omega(\bar{u}_h, u_h) = 0$. Hence on each

$T \in \mathcal{T}_h$, $\dfrac{\partial \bar{u}_h}{\partial x_i} = 0$, where $i = 1,\ 2$, when $d = 2$ or $i = 1,\ 2,\ 3$, when $d = 3$. Thus, $\bar{u}_h$ is constant on each element $T \in \mathcal{T}_h$. Since $\bar{u}_h \in X_h$, $\bar{u}_h$ is continuous at $p \in N_h \cap \Gamma_h$ and $\bar{u}_h(p)$ vanishes at $p \in \partial\Omega$. Therefore, $\bar{u}_h$ vanishes for all elements T if at least one face belongs to $\partial\Omega$. We can continue the argument for elements T in the interior of Ω not necessarily having boundary ∂T a part of $\partial\Omega$ and obtain $\bar{u}_h = 0$. Hence, the problem (2.2.5) has a unique solution and this completes the proof. ■

Lemma 2.2.2 *[62, Lemma 3.31, pp. 127] (Extended Poincaré inequality). There exists $C(\Omega)$ depending only on Ω such that, for all $h \leq 1$,*

$$||v_h||_{0,\Omega} \leq C(\Omega) \left(\sum_{T \in \mathcal{T}_h} ||\nabla v_h||_{0,T}^2 \right)^{1/2} \qquad \forall\, v_h \in \bar{X}_h. \tag{2.2.7}$$

The next theorem follows from [15, Theorem 1.5, pp. 106].

Theorem 2.2.1 *Suppose Ω is a convex and bounded domain. Then, there exists a constant $C > 0$ independent of h such that*

$$||\bar{u} - \bar{u}_h||_{0,\Omega} + h \left(\sum_{T \in \mathcal{T}_h} ||\nabla(\bar{u} - \bar{u}_h)||_{0,T}^2 \right)^{1/2} \leq C\, h^2 \left(\sum_{T \in \mathcal{T}_h} ||\bar{u}||_{2,T}^2 \right)^{1/2}, \tag{2.2.8}$$

where $\bar{u}$ and $\bar{u}_h$ are the solution of (2.2.2) and (2.2.5), respectively.

Lemma 2.2.3 *[107, Lemma A.3, pp. 39] Let T be a triangle or a quadrilateral in a shape regular triangulation $\mathcal{T}_h$. Then, there exists a constant $C > 0$ such that for $v \in H^1(T)$*

$$||v||_{0,\partial T}^2 \leq C \left(\frac{1}{h_T} ||v||_{0,T}^2 + ||v||_{0,T}\, ||\nabla v||_{0,T} \right). \tag{2.2.9}$$

2.2.1 Lagrange multiplier on inter subdomain interfaces

In this subsection, we discuss the variational formulation for the multi-domain problem and introduce Lagrange multipliers on inter-element subdomain interfaces.

For the domain decomposition method, let the domain Ω be partitioned into a finite number of non-overlapping sub-domains Ω_i $(i = 1, 2, \cdots, M)$ with $\bar{\Omega} = \bigcup_{i=1}^{M} \Omega_i$, and let

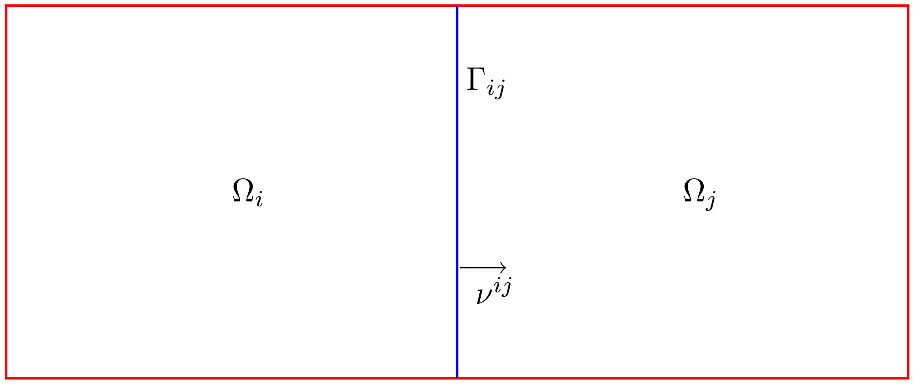

Figure 2.2: Normal vector ν^{ij} outward to Ω_i

$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j = \Gamma_{ji}$ with $|\Gamma_{ij}|$ as the measure of Γ_{ij}. Further let $\Gamma = \bigcup_{i=1,\ i<j\in N(i)}^{M} \Gamma_{ij}$ and $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ denote the interior interfaces, where $N(i) = \{j \neq i \,|\, |\Gamma_{ij}| > 0\}$. Now we are in a position to write the multi-domain problems. Find $u_i,\ i = 1, 2, \cdots, M$ satisfying the following subproblems:

$$\begin{cases} -\Delta u_i &= f_i & \text{in } \ \Omega_i, \\ u_i &= 0 & \text{on } \ \partial\Omega_i \cap \partial\Omega, \\ u_i &= u_j & \text{on } \ \Gamma_{ij}, \quad j \in N(i), \\ \dfrac{\partial u_i}{\partial \nu} &= \dfrac{\partial u_j}{\partial \nu} & \text{on } \ \Gamma_{ij}, \quad j \in N(i), \end{cases} \tag{2.2.10}$$

where $u_i = u_{|\Omega_i}$, $f_i = f_{|\Omega_i}$, $i = 1,\ 2, \cdots, M$, and $\nu = \nu^{ij} = -\nu^{ji}$ on Γ_{ij} and ν^{ij} and ν^{ji} are unit outward normals to $\partial\Omega_i$ and $\partial\Omega_j$, respectively. Note that last two conditions $(2.2.10)_2$-$(2.2.10)_3$ are called the transmission conditions on the artificial interface Γ.

Let $X_i = \{v \in H^1(\Omega_i) \mid v_{|\partial\Omega_i\cap\partial\Omega} = 0\}$, $i = 1, 2, \cdots, M$ and $X = \prod_{i=1}^{M} X_i$. The space X endowed with the norm

$$||v||_X^2 = \sum_{i=1}^{M} ||v_i||_{1,\Omega_i}^2 \tag{2.2.11}$$

is a Hilbert space. Note that $|v|_X^2 = \sum_{i=1}^{M} |v_i|_{1,\Omega_i}^2$ is a semi norm.

Now we are looking for the variational formulation of the multi-domain problem. Multiply both sides of $(2.2.10)_1$ by a test function $v_i \in X_i$ and integrate over Ω_i to obtain

$$\int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx - \sum_{j \in N(i)} \left\langle \frac{\partial u_i}{\partial \nu^{ij}}, v_i \right\rangle = \int_{\Omega_i} f_i v_i \, dx,$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ and ν^{ij} is unit outward normal to Ω_i. Finally, sum over $1 \leq i \leq M$ to find that

$$\sum_{i=1}^{M} \left(a_{\Omega_i}(u_i, v_i) - \sum_{j \in N(i)} \left\langle \frac{\partial u_i}{\partial \nu^{ij}}, v_i \right\rangle \right) = \sum_{i=1}^{M} (f_i, v_i)_{\Omega_i} \qquad \forall v_i \in X, \tag{2.2.12}$$

where

$$a_{\Omega_i}(v, w) = \int_{\Omega_i} \nabla v \cdot \nabla w \, dx, \qquad (v, w)_{\Omega_i} = \int_{\Omega_i} v \, w \, dx. \tag{2.2.13}$$

Define the space $Y_{ij} = H^{-\frac{1}{2}}(\Gamma_{ij})$ and $Y = \prod_{i=1}^{M} \prod_{i<j \in N(i)} Y_{ij}$. Define

$$||\mu||_Y = \sup_{v \in H^{\frac{1}{2}}(\Gamma) \setminus \{0\}} \frac{\langle v, \mu \rangle}{||v||_{\frac{1}{2}, \Gamma}}. \tag{2.2.14}$$

We are in a position to introduce Lagrange multipliers on interface. Set the Lagrange multipliers as

$$\lambda_{ij} = \nabla u_i \cdot \nu^{ij} = -\nabla u_j \cdot \nu^{ji} \text{ on } \Gamma_{ij} \qquad \text{and} \qquad \lambda_{ij} = -\lambda_{ji} \text{ on } \Gamma_{ij}, \tag{2.2.15}$$

where ν^{ij} is the normal vector oriented from Ω_i to Ω_j (see Figure 2.2). Using (2.2.15) in (2.2.12) at the interface, we derive the following equations: Find $u = (u_1, u_2, \cdots, u_M) \in X = \prod_{i=1}^{M} X_i$ and $\lambda \in Y = \prod_{i=1}^{M} \prod_{i<j \in N(i)} Y_{ij}$ such that

$$a(u, v) - b(v, \lambda) = (f, v) \qquad \forall v \in X, \tag{2.2.16}$$

$$b(u, \mu) = 0 \qquad \forall \mu \in Y, \tag{2.2.17}$$

where the bilinear form $a : X \times X \to \mathbb{R}$ is given by

$$a(w, v) = \sum_{i=1}^{M} a_{\Omega_i}(w_i, v_i), \tag{2.2.18}$$

the bilinear form $b : X \times Y \to \mathbb{R}$ is defined by

$$b(v, \mu) = \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left\langle v_i - v_j,\, \mu_{|\Gamma_{ij}} \right\rangle, \tag{2.2.19}$$

and $(f, v) = \sum_{i=1}^{M} (f_i, v_i)_{\Omega_i}$. We now define a space Z by

$$Z = \{v \in X \;:\; b(v, \mu) = 0 \quad \forall \mu \in Y\}. \tag{2.2.20}$$

The space Z may be identified with $H_0^1(\Omega)$ (see, [112, pp. 394]).

Lemma 2.2.4 *The variational formulation of a single domain problem (2.2.2) and multi-domain problem (2.2.16)-(2.2.17) are equivalent under the following conditions: the test function* $(v_1, v_2, \cdots, v_M) \in X = \prod_{i=1}^{M} X_i$ *belongs to* $H_0^1(\Omega)$ *and* $\lambda_{ij} = \nabla u_i \cdot \nu^{ij} = -\nabla u_j \cdot \nu^{ji}$ on $\Gamma_{ij}, \quad 1 \le i \le M,\ j \in N(i)$.

Proof. Let $\bar{u} \in H_0^1(\Omega)$ is a solution of a single domain problem (2.2.2). Setting $u_i = \bar{u}_{|\Omega_i}$, we obtain (2.2.16)-(2.2.17). Let $(u, \lambda) \in X \times Y$ be a solution of problem (2.2.16)-(2.2.17). Then $u \in Z$ and hence $u \in H_0^1(\Omega)$. Choosing $v \in H_0^1(\Omega)$ in (2.2.16), we arrive at

$$\sum_{i=1}^{M} a_{\Omega_i}(u_i, v_i) = \sum_{i=1}^{M} (f, v_i), \tag{2.2.21}$$

where $\bar{u}_{|\Omega_i} = u_i$ and $\bar{v}_{|\Omega_i} = v_i$. Therefore, (2.2.21) can be written as

$$a_{\Omega}(\bar{u}, v) = (f, v) \qquad \forall v \in H_0^1(\Omega). \tag{2.2.22}$$

This completes the rest of the proof. ■

Theorem 2.2.2 *[112, Theorem 1, pp. 395] Problem (2.2.16)-(2.2.17) has a unique solution* $(u, \lambda) \in X \times Y$*. Moreover if* $\bar{u} \in H_0^1(\Omega)$ *is a solution of the problem (2.2.2) with* $u_i = \bar{u}_{|\Omega_i}$ *and we have* $\lambda_{ij} = \nabla u_i \cdot \nu^{ij} = -\nabla u_j \cdot \nu^{ji}$ on Γ_{ij}, $1 \le i \le M,\ j \in N(i)$.

Below, we state a Lemma on the inf-sup condition satisfied by $b(\cdot, \cdot)$ without proof. For a proof, see [8, Lemma 3.1(c), pp. 614]. We need Lemma 2.2.5 in our future analysis.

Lemma 2.2.5 *There exists a constant $K_0 > 0$ such that*

$$\inf_{0\neq\mu\in Y} \sup_{0\neq v\in X} \frac{b(v,\mu)}{||v||_X||\mu||_Y} \geq K_0. \tag{2.2.23}$$

Discrete multidomain formulation. Now we focus our attention on the discretization of the problem (2.2.16)-(2.2.17) based on the Crouzeix-Raviart element. For the triangulation $\mathcal{T}_h$, we now assume that the triangles (resp. rectangles) T should not cross the interface Γ_{ij}, and thus, each element is either contained in $\bar{\Omega}_i$ or in $\bar{\Omega}_j$ and they share the same edges of Γ_{ij}. For multi-domain problem, let $X_{i,h} = \bar{X}_{h|_{\Omega_i}}$, where $\bar{X}_h$ is defined in (2.2.4). Define $X^0_{i,h} = \{v_h | v_h \in X_{i,h}$ and $v_h(p) = 0$ at $p \in \partial\Omega_{i,h}\}$. We now define two discrete spaces $Y_{i,h}$ and $Y_{ij,h}$ on $\partial\Omega_i$ and Γ_{ji}, respectively, as follows. Let $Y_{i,h}$ consist of piecewise constant elements on triangulation $\mathcal{T}_{h,i|_{\partial\Omega_i}}$, where $\mathcal{T}_{h,i|_{\partial\Omega_i}}$ is the triangulation of $\partial\Omega_i \setminus \partial\Omega$ inherited from $\mathcal{T}_h$, i.e., $\mathcal{T}_{h,i|_{\partial\Omega_i}} = \mathcal{T}_{h|_{\partial\Omega_i\setminus\partial\Omega}}$. Furthermore, let $Y_{ij,h} = Y_{i,h|_{\Gamma_{ij}}}$. The spaces are nonconforming, since $X_{i,h}$ is not subspace of X_i. For $v \in X_{i,h}$, set the discrete H^1 semi-norm as

$$|v|^2_{1,h,\Omega_i} = \sum_{T\in\mathcal{T}_{h,i}} \int_T |\nabla v|^2 dx, \tag{2.2.24}$$

and defines the H^1 norm by

$$||v||^2_{1,h,\Omega_i} = |v|^2_{1,h,\Omega_i} + ||v||^2_{0,\Omega_i}. \tag{2.2.25}$$

Note that

$$||v||^2_{1,h} = \sum_{i=1}^{M} ||v||^2_{1,h,\Omega_i} \tag{2.2.26}$$

defines a norm on X_h. Given the finite dimensional spaces $X_{i,h}$, $Y_{i,h}$ and $Y_{ij,h}$, we now introduce linear operators:

$$\pi_i : X_{i,h} \to Y_{i,h} \qquad \text{and} \qquad \pi_{ij} : X_{i,h} \to Y_{ij,h}, \tag{2.2.27}$$

respectively, as

$$\pi_i v_{i|_e} \equiv v_i(p) \quad \forall e \in \mathcal{T}_{h,i|_{\partial\Omega_i}} \qquad \text{and} \qquad \pi_{ij} v_i = \pi_i v_{i|_{\Gamma_{ij}}}, \tag{2.2.28}$$

where $e \in \partial T \cap \partial\Omega_i$ is edge of the triangle $T \in \mathcal{T}_{h,i|_{\partial\Omega_i}}$ and p is the face barycenter of T. Similarly, we define the linear operators

$$S_i : Y_{i,h} \rightarrow X_{i,h} \qquad \text{and} \qquad S_{ij} : Y_{ij,h} \rightarrow X_{i,h} \tag{2.2.29}$$

as

$$S_i w_i = \begin{cases} w_i & \text{freedom on } \partial\Omega_i, \\ 0 & \text{other freedom,} \end{cases} \qquad \text{and} \qquad S_{ij} w_{ij} = \begin{cases} w_{ij} & \text{freedom on } \Gamma_{ij}, \\ 0 & \text{other freedom.} \end{cases} \tag{2.2.30}$$

From (2.2.29) and (2.2.30), we note that in general $\pi_i v_i \neq v_{i|_{\partial\Omega_i}}$ and $S_i w_{i|_{\partial\Omega_i}} \neq w_i$. Further, we observe that

$$v_i - S_i \pi_i v_i \in X^0_{i,h}, \tag{2.2.31}$$

and

$$\pi_i S_i = Id_i, \qquad \pi_{ij} S_{ij} = Id_{ij}, \tag{2.2.32}$$

where Id_i and Id_{ij} are identity operators on $Y_{i,h}$ and $Y_{ij,h}$, respectively.

Lemma 2.2.6 *[109, Lemma 2.1, pp. 2542] There exists a positive constant C independent of h such that*

$$||\pi_{ij} v_i||_{0,\Gamma_{ij}} \;\; \leq \;\; C\, ||v_{i|_{\Gamma_{ij}}}||_{0,\Gamma_{ij}}, \qquad \forall\, v \in X_{i,h}. \tag{2.2.33}$$

Also, for $w_{ij} \in Y_{ij,h}$,

$$||S_{ij} w_{ij}||_{0,\Omega_i} \;\; \leq \;\; Ch^{1/2} ||w_{ij}||_{0,\Gamma_{ij}}, \tag{2.2.34}$$

and

$$|S_{ij} w_{ij}|_{1,h,\Omega_i} \;\; \leq \;\; Ch^{-1/2} ||w_{ij}||_{0,\Gamma_{ij}}. \tag{2.2.35}$$

Now we are in a position to state the nonconforming multidomain approximation of (2.2.16)-(2.2.17). Given $f \in L^2(\Omega)$, find $u_h = (u_{1,h}, \cdots, u_{M,h}) \in X_h = \prod_{i=1}^{M} X_{i,h}$ and $\lambda_h \in Y_h =$

$\prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{ij,h}$ such that

$$a^h(u_h, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} [\pi\, v_h]\, ds = \sum_{i=1}^{M} (f, v_h)_{\Omega_i} \qquad \forall v_h \in X_h, \quad (2.2.36)$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\, u_h]\, \mu_h\, ds = 0 \qquad \forall \mu_h \in Y_h, \quad (2.2.37)$$

where

$$a^h(v_h, w_h) = \sum_{i=1}^{M} a^h_{\Omega_i}(v_{i,h}, w_{i,h}) = \sum_{i=1}^{M} \int_{\Omega_i} \nabla v_{i,h} \cdot \nabla w_{i,h}\, dx \quad (2.2.38)$$

and

$$[\pi\, v_h] = \pi_{ij}\, v_{i,h} - \pi_{ji}\, v_{j,h} \ \text{ on } \ \Gamma_{ij}. \quad (2.2.39)$$

Since $\mu_h \in Y_h$ and $\pi_{ij} v_h \in Y_h$ are constants on Γ_{ij}, using mid-point rule we obtain

$$\int_{\Gamma_{ij}} \pi_{ij} v_h\, \mu_h\, ds = \sum_{p\in\Gamma_{ij}\cap N_h} v_h(p)\mu_h(p)|s_p| \qquad \forall\ v_h \in X_h,\ \mu_h \in Y_h, \quad (2.2.40)$$

where s_p is the element face with p as its barycenter and $|s_p|$ is the measure of s_p.

Lemma 2.2.7 *Let* $u_h = (u_{1,h}, \cdots, u_{M,h}) \in X_h = \prod_{i=1}^{M} X_{i,h}$. *Then* $\bar{u}_h \in \bar{X}_h$ *if and only if*

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\, u_h]\, \mu_h = 0 \qquad \forall \mu_h \in Y_h, \quad (2.2.41)$$

where $u_h = (u_{1,h}, \cdots, u_{M,h})$ *and* $\bar{u}_h$ *are the discrete solutions of (2.2.36)-(2.2.37) and (2.2.5), respectively.*

Proof. Here $X_{i,h} = \bar{X}_{h|_{\Omega_i}}$, $i = 1, 2, \cdots, M$, i.e., localize the nonconforming Galerkin space $\bar{X}_h$ by removing the midpoint continuity constraints on the interfaces between two adjacent subdomains. Let us consider first $\bar{u}_h \in \bar{X}_h$, i.e., $\bar{u}_{i,h}(p) - \bar{u}_{j,h}(p) = 0$ on Γ_{ij}, where p denotes the midpoints of the triangle edges. Hence, (2.2.41) is satisfied, where $u_{i,h}(p) = \bar{u}_h(p)_{|_{\Omega_i}}$. Similarly, from (2.2.41), we obtain $u_{i,h}(p) - u_{j,h}(p) = 0$ on Γ_{ij}, that is, the midpoint continuity condition on the interfaces between two adjacent subdomains is satisfied. Thus, $\bar{u}_h \in \bar{X}_h$ and this completes the proof. ■

Lemma 2.2.8 *Let (u_h, λ_h) be the solution of (2.2.36)-(2.2.37). Then there is a positive constant C independent of h such that*

$$||\lambda_{ij,h}||_{0,\Gamma_{ij}} \leq C\left(h^{-1/2}|u_{i,h}|_{1,h,\Omega_i} + h^{1/2}||f||_{0,\Omega_i}\right), \quad i = 1, 2, \cdots, M, \quad \forall j \in N(i), \tag{2.2.42}$$

where M is the number of subdomains.

Proof. Choose $v_h = (0, 0, \cdots, S_{ij}\lambda_{ij,h}, \cdots, 0)$ in (2.2.36). Using Lemma 2.2.6, we obtain

$$\begin{aligned} ||\lambda_{ij,h}||^2_{0,\Gamma_{ij}} &= a^h_{\Omega_i}(u_{i,h}, S_{ij}\lambda_{ij,h}) - (f, S_{ij}\lambda_{ij,h}) \\ &\leq |u_{i,h}|_{1,h,\Omega_i}|S_{ij}\lambda_{ij,h}|_{1,h,\Omega_i} + ||f||_{0,\Omega_i}||S_{ij}\lambda_{ij,h}||_{0,\Omega_i} \\ &\leq Ch^{-1/2}|u_{i,h}|_{1,h,\Omega_i}||\lambda_{ij,h}||_{0,\Gamma_{ij}} + Ch^{1/2}||f||_{0,\Omega_i}||\lambda_{ij,h}||_{0,\Gamma_{ij}}. \end{aligned}$$

This completes the rest of the proof. ■

Theorem 2.2.3 *Problem (2.2.36)-(2.2.37) has a unique solution.*

Proof. Since the problem problem (2.2.36)-(2.2.37) leads to a square system of linear algebraic equations, it is enough to prove uniqueness. Setting $f = 0$, $v_h = (0, 0, \cdots, u_{i,h}, \cdots, 0)$ in (2.2.36) and $\mu_h = (0, 0, \cdots, \lambda_{ij,h}, \cdots, 0)$ in (2.2.37), we obtain

$$\sum_{i=1}^{M} a^h_{\Omega_i}(u_{i,h}, u_{i,h}) = 0. \tag{2.2.43}$$

From (2.2.43), we can conclude that $u_{i,h}$ is constant on each Ω_i. Now, we consider the subdomains Ω_i, having at least one face belonging to $\partial\Omega$. We know that $u_{i,h}(p) = 0$ on $\partial\Omega_i \cap \partial\Omega$, where p is the face barycenters of the triangulation on $\partial\Omega_i \cap \partial\Omega$ inherited from $\mathcal{T}_h$. Hence, we obtain $u_{i,h} = 0$ in Ω_i, where Ω_i belongs to boundary subdomain. In the next step, we consider the subdomains Ω_j adjacent to Ω_i. Then the continuity of $u_{i,h}$ at the midpoint of Γ_{ij} shows the $u_{j,h} = 0$ in Ω_j. Similarly, we continue the analysis further and obtain $u_{i,h} = 0$ for all subdomains. Next we wish to show that $\lambda_{ij,h} = 0$ for each Γ_{ij}. Setting $f = 0$ in (2.2.36), using Lemma 2.2.8, we obtain $\lambda_{ij,h} = 0$ for each Γ_{ij} and this completes the rest of the proof. ■

2.3 Convergence analysis

In this section, we derive the error estimate of the discrete multidomain problem. Below we discuss an interpolation operator for our future use. Given $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$, let $I_h\phi \in \bar{X}_h \cap C^0(\bar{\Omega})$ be the continuous piecewise linear function which interpolates ϕ at the vertices of the triangulation. Define $I_h : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow \bar{X}_h \cap C^0(\bar{\Omega})$ with

$$(I_h\phi)(p) = \frac{1}{2}(\phi(v_1) + \phi(v_2)) \qquad \forall\, \phi \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.3.1}$$

where p denotes the midpoints of the triangle edges, and v_1 and v_2 being the endpoints of the edge. Note that the interpolation operator $I_h\psi \in X_h$ $(I_h\psi \in \bar{X}_h)$ satisfies

$$||\psi - I_h\psi||_{0,\Omega} + h\,||\psi - I_h\psi||_{1,h} \leq C\,h^2 \sum_{i=1}^{M} ||\psi_i||_{2,\Omega_i}. \tag{2.3.2}$$

2.3.1 Consistency error

Since X_h is not a subspace of X, we, therefore, consider the consistency error for the proposed nonconforming finite element discretization using Strang's second lemma [34, 121, 122]. Furthermore, we prove below that the discretization error is bounded by the best approximation error and the consistency error [15].

Lemma 2.3.1 *Let $(u_h, \lambda_h) \in X_h \times Y_h$ be the solution of (2.2.36)-(2.2.37) and let $(u, \lambda) \in X \times Y$ be the solution of (2.2.16)-(2.2.17). Then there exists a constant C independent of h such that*

$$||u - u_h||_{1,h} \leq C \left\{ \inf_{v_h \in X_h} ||u - v_h||_{1,h} + \sup_{w_h \in X_h} \frac{\left| F(w_h) + \sum_{i=1}^{M} \sum_{i<j \in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}\,[\pi w_h]\,ds - a^h(u, w_h) \right|}{||w_h||_{1,h}} \right\}, \tag{2.3.3}$$

where $F(w_h) = \sum_{i=1}^{M}(f, w_{i,h})_{\Omega_i}$ and $a^h(v, w_h) = \sum_{i=1}^{M} a^h_{\Omega_i}(v_i, w_{i,h})$.

Proof. Using Lemma 2.2.2 we find that for all $v_h \in X_h$,

$$a^h(v_h, v_h) \geq \alpha ||v_h||^2_{1,h}. \tag{2.3.4}$$

For $z_h \in X_h$,

$$\begin{aligned}
\alpha ||u_h - z_h||_{1,h}^2 &\leq \sum_{i=1}^{M} a_{\Omega_i}^h (u_{i,h} - z_{i,h}, u_{i,h} - z_{i,h}) \\
&= \sum_{i=1}^{M} \left[a_{\Omega_i}^h (u_i - z_{i,h}, u_{i,h} - z_{i,h}) + a_{\Omega_i}^h (u_{i,h} - u_i, u_{i,h} - z_{i,h}) \right] \\
&\leq C \sum_{i=1}^{M} ||u - z_h||_{1,h} ||u_h - z_h||_{1,h} + \sum_{i=1}^{M} a_{\Omega_i}^h (u_{i,h} - u_i, u_{i,h} - z_{i,h}). \quad (2.3.5)
\end{aligned}$$

To estimate the last term on the right hand side of (2.3.5), we note from (2.2.36) with $w_h \in X_h$ that

$$\begin{aligned}
a^h(u_h - u, w_h) = \sum_{i=1}^{M} a_{\Omega_i}^h (u_{i,h} - u_i, w_{i,h}) = \sum_{i=1}^{M} (f, w_{i,h})_{\Omega_i} \\
+ \sum_{i=1}^{M} \left[\sum_{i<j \in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} \left[\pi \, w_h \right] ds - a_{\Omega_i}^h (u_i, w_{i,h}) \right]. \qquad (2.3.6)
\end{aligned}$$

The proof of the lemma follows from (2.3.6) with $w_{i,h} = u_{i,h} - z_{i,h}$, (2.3.5) and the triangle inequality and this completes the rest of the proof. ■

For finding the consistency error, we need to introduce a projection operator $Q_h : L^2(\Gamma_{ij}) \rightarrow Y_{ij,h}$, which is defined as

$$\int_{\Gamma_{ij}} (Q_h \mu) \, \pi_{ij} v_h \, ds = \int_{\Gamma_{ij}} \mu \, (\pi_{ij} v_h) \, ds \qquad \forall \, v_h \in X_{i,h}. \qquad (2.3.7)$$

The operator Q_h given by (2.3.7) is well-defined and continuous. It is easy to see that Q_h is identity

$$Q_h \mu = \mu \qquad \forall \, \mu \in Y_{ij,h}. \qquad (2.3.8)$$

Furthermore, the operator Q_h is L^2-stable in the sense that

$$||Q_h \mu||_{0,\Gamma_{ij}} \leq C \, ||\mu||_{0,\Gamma_{ij}}. \qquad (2.3.9)$$

Using (2.3.8) and (2.3.9), it is easy to establish the following approximation result.

Lemma 2.3.2 *There exists a positive constant C independent of h such that for $\mu \in H^{1/2}(\Gamma_{ij})$*

$$||\mu - Q_h\mu||_{0,\Gamma_{ij}} \le C\, h^{1/2} ||\mu||_{1/2,\Gamma_{ij}}. \qquad (2.3.10)$$

Proof. For $T \in \mathcal{T}_{h,i}$, $\mu \in L^2(\Gamma_{ij})$, and each edge $T' \in \partial T \cap \Gamma_{ij}$, we define the average value $\bar{\mu}$ on T' as

$$\bar{\mu} = \frac{1}{\text{meas}(T')} \int_{T'} \mu\, ds. \qquad (2.3.11)$$

From (2.3.8), we note that $Q_h\bar{\mu} = \bar{\mu}$. Hence, using the triangle inequality, (2.3.9) and Lemma 1.2.7, we find that

$$\begin{aligned} ||\mu - Q_h\mu||_{0,T'} &\le ||\mu - \bar{\mu}||_{0,T'} + ||Q_h(\mu - \bar{\mu})||_{0,T'} \\ &\le C\, ||\mu - \bar{\mu}||_{0,T'} \le C\, h^{1/2} ||\mu||_{1/2,T'}. \end{aligned} \qquad (2.3.12)$$

The global estimate (2.3.10) is obtained by summing over all local contributions and this completes the rest of the proof. ■

Lemma 2.3.3 *(Asymptotic consistency) Given $f \in L^2(\Omega)$, let $(u, \lambda) \in X \times Y$ be the solution of (2.2.16)-(2.2.17). Assume that $u = (u_1, \cdots, u_M) \in \prod_{i=1}^{M} H^2(\Omega_i)$. Then, there exists a constant C independent of h such that*

$$\frac{\left|F(w_h) + \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}\, [\pi w_h]\, ds - a^h(u, w_h)\right|}{||w_h||_{1,h}} \le C\, h \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}\ \forall\, w_h \in X_h. \quad (2.3.13)$$

Proof. Since $f_{|\Omega_i} = -\Delta u_i$ and $w_h \in X_h$, using integration by parts we obtain

$$\begin{aligned} & a^h(u, w_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}\, [\pi w_h]\, ds - F(w_h) \\ &= \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\int_T \nabla u_i \cdot \nabla w_{i,h}\, dx - \sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\ne\phi} \int_{\partial T} \lambda_{ij,h}\, [\pi w_h]\, ds - \int_T f_{|\Omega_i} w_{i,h}\, dx \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T} w_{i,h}\, ds \\
&\quad + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\neq\phi} \left(\int_{\partial T} \lambda_{ij}\,[w_h]\, ds - \int_{\partial T} \lambda_{ij,h}\,[\pi w_h]\, ds \right) \right] \\
&= I_1 \ + \ I_2, \qquad (2.3.14)
\end{aligned}
$$

where

$$
\partial T = \begin{cases} \partial T_{int}, \ \text{each edge/face of an element } T\in\mathcal{T}_{h,i} \text{ located inside } \Omega_i \\ \qquad\qquad \text{and neither in } \partial T\cap\Gamma_{ij} \text{ nor in } \partial T\cap\partial\Omega, \\ \partial T_{ext}, \ \text{other freedom, that is, } T\in\mathcal{T}_{h,i}, j\in N(i) \text{ and } \partial T\cap\Gamma_{ij}\neq\phi. \end{cases} \qquad (2.3.15)
$$

We now estimate each term of the right hand side of (2.3.14). For the first term of (2.3.14), we obtain

$$
I_1 = \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T} w_{i,h}\, ds = \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \sum_{e\in\partial T_{int}} \int_e \nabla u_i \cdot \nu^T\, w_{i,h}\, ds. \qquad (2.3.16)
$$

Since each face e of an element T located inside Ω_i appears twice in the above sum (2.3.16), we can subtract from $w_{i,h}$ its mean-value $\overline{w}_{i,h}$ on the face e. If e is on $\partial\Omega$, it is clear that $\overline{w}_{i,h} = 0$. Therefore, the equation (2.3.16) can be written as

$$
I_1 = \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \sum_{e\in\partial T_{int}} \int_e \nabla u_i \cdot \nu^T\, (w_{i,h} - \overline{w}_{i,h})\, ds. \qquad (2.3.17)
$$

It follows from the definition of $\overline{w}_{i,h}$ that $\int_e (w_{i,h} - \overline{w}_{i,h})\, ds = 0$. The values of the integrals also do not change if we subtract a constant multiple of $\nabla u_i \cdot \nu^T$ on each face e. We can also subtract from ∇u_i its mean-value $\overline{\nabla u_i}$ on e and obtain

$$
I_1 = \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \sum_{e\in\partial T_{int}} \int_e (\nabla u_i - \overline{\nabla u_i}) \cdot \nu^T\, (w_{i,h} - \overline{w}_{i,h})\, ds. \qquad (2.3.18)
$$

An application of Cauchy-Schwarz inequality with Lemma 1.2.7 yields

$$
\begin{aligned}
I_1 \ &\le \ \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \sum_{e\in\partial T_{int}} ||\nabla u_i - \overline{\nabla u_i})||_{0,e}\, ||w_{i,h} - \overline{w}_{i,h}||_{0,e} \\
&\le \ C \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} h_T^{1/2}\, |u_i|_{2,T}\, h_T^{1/2}\, |w_{i,h}|_{1,T}
\end{aligned}
$$

$$\leq \; C\,h \sum_{i=1}^{M} \left(\sum_{T \in \mathcal{T}_{h,i}} |u_i|^2_{2,T} \right)^{1/2} \left(\sum_{T \in \mathcal{T}_{h,i}} |w_{i,h}|^2_{1,T} \right)^{1/2}$$

$$\leq \; C\,h \sum_{i=1}^{M} |u_i|_{2,\Omega_i} \, ||w_{i,h}||_{1,h,\Omega_i}. \tag{2.3.19}$$

For the second term on the right hand side of (2.3.14), we note that

$$I_2 = \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[\sum_{T \in \mathcal{T}_{h,i}} \sum_{\partial T \cap \Gamma_{ij} \neq \phi} \left(\int_{\partial T} \lambda_{ij} \, [w_h] \, ds - \int_{\partial T} \lambda_{ij,h} \, [\pi w_h] \, ds \right) \right] \tag{2.3.20}$$

$$\begin{aligned} &= \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[\sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} \left(\int_e \lambda_{ij} \, [w_h] \, ds - \int_e Q_h \lambda_{ij} \, [\pi w_h] \, ds \right) \right] \\ &\quad + \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[\sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} \left(\int_e Q_h \lambda_{ij} \, [\pi w_h] \, ds - \int_e \lambda_{ij,h} \, [\pi w_h] \, ds \right) \right] \\ &= I_{2,1} \, + \, I_{2,2}, \end{aligned} \tag{2.3.21}$$

Next, we need to estimate $I_{2,1}$ and $I_{2,2}$. Observe that

$$\begin{aligned} \sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} \int_e [w_h] \, ds \;&=\; \sum_{p \in N_{h,i} \cap \Gamma_{ij}} [w_h(p)] \, |e| \\ &=\; \sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} \int_e [\pi w_h] \, ds, \end{aligned} \tag{2.3.22}$$

where e is the element face with p as its barycenter, $|e|$ is the measure of e and $N_{h,i}$ is the set of all barycenters of $\mathcal{T}_{h,i}$. Using (2.3.22), we obtain

$$\begin{aligned} I_{2,1} \;&=\; \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[\sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} \int_e (\lambda_{ij} - Q_h \lambda_{ij}) \; [w_h] \, ds \right] \\ &\leq\; \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[\sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T \cap \Gamma_{ij} \neq \phi} ||\lambda_{ij} - Q_h \lambda_{ij}||_{0,e} \left(||w_{i,h_{|\Gamma_{ij}}}||_{0,e} + ||w_{j,h_{|\Gamma_{ij}}}||_{0,e} \right) \right] \\ &\leq\; C \sum_{i=1}^{M} \sum_{i<j \in N(i)} \left[||\lambda_{ij} - Q_h \lambda_{ij}||_{0,\Gamma_{ij}} \left(||w_{i,h_{|\Gamma_{ij}}}||_{0,\Gamma_{ij}} + ||w_{j,h_{|\Gamma_{ij}}}||_{0,\Gamma_{ij}} \right) \right]. \end{aligned} \tag{2.3.23}$$

For the midpoint rule, it is easy to see that

$$\int_{\Gamma_{ij}} w_h^2 ds \leq C\, h \int_{\Omega_i} |\nabla w_h|^2 dx. \tag{2.3.24}$$

Using Lemma 2.3.2 and (2.3.24) in (2.3.23), we arrive at

$$\begin{aligned} I_{2,1} &\leq C\, h^{1/2} \sum_{i=1}^{M} \left(\sum_{i<j\in N(i)} ||\lambda_{ij} - Q_h \lambda_{ij}||_{0,\Gamma_{ij}} \right) |w_{i,h}|_{1,h,\Omega_i} \\ &\leq C\, h \sum_{i=1}^{M} \left(\sum_{i<j\in N(i)} ||\lambda_{ij}||_{H^{1/2}(\Gamma_{ij})} \right) |w_{i,h}|_{1,h,\Omega_i} \\ &\leq C\, h \sum_{i=1}^{M} ||u||_{2,\Omega_i} \, ||w_h||_{1,h,\Omega_i}. \end{aligned} \tag{2.3.25}$$

Using Lemma 2.2.6, Lemma 2.3.4, (2.3.24) and (2.3.31), we estimate $I_{2,2}$ as

$$\begin{aligned} I_{2,2} &\leq \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left[\sum_{T\in\mathcal{T}_{h,i}} \sum_{e\in\partial T\cap\Gamma_{ij}\neq\phi} ||\lambda_{ij,h} - Q_h\lambda_{ij}||_{0,e} \left(||\pi_{ij} w_{i,h}||_{0,e} + ||\pi_{ji} w_{j,h}||_{0,e} \right) \right] \\ &\leq C \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left[||\lambda_{ij,h} - Q_h\lambda_{ij}||_{0,\Gamma_{ij}} \left(||\pi_{ij} w_{i,h}||_{0,\Gamma_{ij}} + ||\pi_{ji} w_{j,h}||_{0,\Gamma_{ij}} \right) \right] \\ &\leq C \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left[||\lambda_{ij,h} - Q_h\lambda_{ij}||_{0,\Gamma_{ij}} \left(||w_{i,h|_{\Gamma_{ij}}}||_{0,\Gamma_{ij}} + ||w_{j,h|_{\Gamma_{ij}}}||_{0,\Gamma_{ij}} \right) \right] \\ &\leq C\, h^{1/2} \sum_{i=1}^{M} \left(\sum_{i<j\in N(i)} ||\lambda_{ij,h} - Q_h\lambda_{ij}||_{0,\Gamma_{ij}} \right) |w_{i,h}|_{1,h,\Omega_i} \\ &\leq C\, h \sum_{i=1}^{M} ||u||_{2,\Omega_i} \, ||w_h||_{1,h,\Omega_i}. \end{aligned} \tag{2.3.26}$$

Employing (2.3.19), (2.3.21), (2.3.25) and (2.3.26) in (2.3.14), we arrive at (2.3.13) and this completes the rest of the proof. ∎

Lemma 2.3.4 *Let* $(u_h, \lambda_h) \in X_h \times Y_h$ *be the solution of (2.2.36)-(2.2.37) and let* $(u, \lambda) \in X \times Y$ *be the solution of (2.2.16)-(2.2.17) with given data* $f \in L^2(\Omega)$. *Assume that* $u =$

$(u_1, \cdots, u_M) \in \prod_{i=1}^{M} H^2(\Omega_i)$. *Then, there exists a constant C independent of h such that*

$$||\lambda - \lambda_h||_{0,\Gamma} \leq C\, h^{1/2} \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}. \tag{2.3.27}$$

Proof. From (2.2.36), we obtain using interpolant I_h in (2.3.1)

$$\begin{aligned}
\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}\, [\pi\, v_h]\, ds &= \sum_{i=1}^{M} \left[\, a^h_{\Omega_i}(u_{i,h}, v_{i,h}) - (f, v_{i,h})_{\Omega_i} \,\right] \\
&= \sum_{i=1}^{M} \left[\, a^h_{\Omega_i}(I_h u_i - u_i, v_{i,h}) + a^h_{\Omega_i}(u_{i,h} - I_h u_i, v_{i,h}) \,\right] + \sum_{i=1}^{M} \left[\, a^h_{\Omega_i}(u_i, v_{i,h}) - (f, v_{i,h})_{\Omega_i} \,\right] \\
&= \sum_{i=1}^{M} \left[\, a^h_{\Omega_i}(I_h u_i - u_i, v_{i,h}) + a^h_{\Omega_i}(u_{i,h} - I_h u_i, v_{i,h}) \,\right] + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T} \frac{\partial u_i}{\partial \nu^T} v_{i,h}\, ds. \qquad (2.3.28)
\end{aligned}$$

Using the operator Q_h in (2.3.28), we can rewrite it as

$$\begin{aligned}
\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} (\lambda_{ij,h} - Q_h\lambda_{ij})\, [\pi\, v_h]\, ds &= \sum_{i=1}^{M} \left[\, a^h_{\Omega_i}(I_h u_i - u_i, v_h) + a^h_{\Omega_i}(u_{i,h} - I_h u_i, v_{i,h}) \,\right] \\
&\quad + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T} \frac{\partial u_i}{\partial \nu^T}\, v_{i,h}\, ds - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} Q_h\lambda_{ij}\, [\pi\, v_h]\, ds \\
&= \sum_{i=1}^{M} a^h_{\Omega_i}(I_h u_i - u_i, v_{i,h}) + \sum_{i=1}^{M} a^h_{\Omega_i}(u_{i,h} - I_h u_i, v_{i,h}) + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T}\, v_{i,h}\, ds \\
&\quad + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\neq\phi} \left(\int_{\partial T} \lambda_{ij}\, [v_h]\, ds - \int_{\partial T} Q_h\lambda_{ij}\, [\pi v_h]\, ds \right) \right]. \qquad (2.3.29)
\end{aligned}$$

Using Cauchy-Schwarz inequality for the first and second terms, (2.3.19) for the third term, and (2.3.25) for the fourth term on the right hand side of (2.3.29), we arrive at

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} (\lambda_{ij,h} - Q_h\lambda_{ij})\, [\pi\, v_h]\, ds \leq C\, h \sum_{i=1}^{M} ||u_i||_{2,\Omega_i} |v_h|_{1,h,\Omega_i}. \tag{2.3.30}$$

Choose $v_h = S_{ij}(\lambda_{ij,h} - Q_h\lambda_{ij})$ in (2.3.30) and using Lemma 2.2.6, we obtain

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} ||\lambda_{ij,h} - Q_h\lambda_{ij}||_{0,\Gamma_{ij}} \leq C\, h^{1/2} \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}. \tag{2.3.31}$$

Using triangle inequality, we arrive at (2.3.27) and this completes the rest of the proof. ■

Combining the Lemma 2.3.1, Lemma 2.3.3 and Lemma 2.3.4, we obtain the following estimates.

Theorem 2.3.1 *Let $(u_h, \lambda_h) \in X_h \times Y_h$ be the solution of (2.2.36)-(2.2.37) and let $(u, \lambda) \in X \times Y$ be the solution of (2.2.16)-(2.2.17). Then, there exists a positive constant C independent of h such that*

$$||u - u_h||_{1,h} + h^{1/2}\, ||\lambda - \lambda_h||_{0,\Gamma} \leq C\, h \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}. \tag{2.3.32}$$

2.3.2 A priori estimates in L^2-norm

For L^2-error estimates, we appeal to Aubin and Nitsche duality argument (see, [4, 22, 34, 15]).

Theorem 2.3.2 *Let $(u_h, \lambda_h) \in X_h \times Y_h$ be the solution of (2.2.36)-(2.2.37) and let $(u, \lambda) \in X \times Y$ be the solution of (2.2.16)-(2.2.17). Assume that $u = (u_1, \cdots, u_M) \in \prod_{i=1}^{M} H^2(\Omega_i)$. Then, there exists a positive constant C independent of h such that*

$$||u - u_h||_{0,\Omega} \leq C\, h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}. \tag{2.3.33}$$

Proof. For $i = 1, 2, \cdots, M$, let $\zeta_i = u_i - u_{i,h}$, $\zeta = (\zeta_1, \cdots, \zeta_M)$ and let $\psi_i = \psi_{|\Omega_i} \in H^2(\Omega_i) \cap H_0^1(\Omega)$ be a solution of the transmission problem :

$$\begin{cases} -\Delta\psi_i &= \zeta_i \quad \text{in } \Omega_i, \\ \psi_i &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ \psi_i &= \psi_j \quad \text{on } \Gamma_{ij}, \quad j \in N(i), \\ \dfrac{\partial\psi_i}{\partial\nu} &= \dfrac{\partial\psi_j}{\partial\nu} \quad \text{on } \Gamma_{ij}, \quad j \in N(i), \end{cases} \tag{2.3.34}$$

which satisfies the regularity condition

$$\sum_{i=1}^{M} ||\psi||_{2,\Omega_i} \leq C\, ||\zeta||_{0,\Omega}. \tag{2.3.35}$$

Since $\zeta = (\zeta_1, \cdots, \zeta_M) \in X_h$, we multiply both the sides of (2.3.34) by ζ_i and integrate over Ω_i. Now integration by parts yields

$$
\begin{aligned}
||\zeta||^2_{0,\Omega} &= \sum_{i=1}^{M} ||\zeta_i||^2_{0,\Omega_i} = \sum_{i=1}^{M}(-\Delta\psi_i, \zeta_i) = \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left(\int_T \nabla\psi_i \cdot \nabla\zeta_i \, dx - \int_{\partial T} \frac{\partial \psi_i}{\partial \nu^T} \zeta_i \, ds \right) \\
&= \sum_{i=1}^{M} a^h_{\Omega_i}(\psi_i, \zeta_i) - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial \psi_i}{\partial \nu^T} \zeta_i \, ds \\
&\qquad - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\neq\phi} \int_{\partial T} \frac{\partial \psi}{\partial \nu} [\zeta] \, ds \right].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
||\zeta||^2_{0,\Omega} &= \sum_{i=1}^{M} a^h_{\Omega_i}(\zeta_i, \psi_i - I_h\psi_i) + \sum_{i=1}^{M} a^h_{\Omega_i}(\zeta_i, I_h\psi_i) - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial \psi_i}{\partial \nu^T} \zeta_i \, ds \\
&\qquad - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\neq\phi} \int_{\partial T} \frac{\partial \psi}{\partial \nu} [\zeta] \, ds \right]. \qquad (2.3.36)
\end{aligned}
$$

Since $I_h\psi_i \in X_h$, and using (2.3.6) and (2.3.14), we obtain

$$
\sum_{i=1}^{M} a^h_{\Omega_i}(\zeta_i, I_h\psi_i) = \sum_{i=1}^{M} \left[\sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T} \frac{\partial u_i}{\partial \nu^T} I_h\psi_i \, ds \; - \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} \, [\pi I_h\psi] \, ds \right]. \qquad (2.3.37)
$$

Substituting (2.3.37) in (2.3.36), we find that

$$
\begin{aligned}
||\zeta||^2_{0,\Omega} &= \sum_{i=1}^{M} a^h_{\Omega_i}(\zeta_i, \psi_i - I_h\psi_i) - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial \psi_i}{\partial \nu^T} \zeta_i \, ds \\
&\quad - \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left[\sum_{i<j\in N(i)} \sum_{\partial T\cap\Gamma_{ij}\neq\phi} \int_{\partial T} \frac{\partial \psi}{\partial \nu} [\zeta] \, ds \right] + \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \left(\int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T} I_h\psi_i \, ds \right) \\
&\qquad + \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left[\int_{\Gamma_{ij}} \lambda_{ij} \, [I_h\psi] \, ds - \int_{\Gamma_{ij}} \lambda_{ij,h} \, [\pi I_h\psi] \, ds \right] \\
&= I_3 \, + \, I_4 \, + \, I_5 \, + \, I_6 \, + \, I_7. \qquad (2.3.38)
\end{aligned}
$$

Now, we have to estimate each of the term on the right-hand side of (2.3.38). For I_3, using Cauchy-Schwartz inequality, (2.3.32) and (2.3.2), we arrive at

$$I_3 \leq \sum_{i=1}^{M} ||\zeta_i||_{1,h,\Omega_i} ||\psi_i - I_h\psi_i||_{1,h,\Omega_i} \leq C\,h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i} ||\psi_i||_{2,\Omega_i} \leq C\,h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i} ||\zeta_i||_{0,\Omega}. \quad (2.3.39)$$

For obtaining the estimates of I_4 and I_5, we proceed similarly as in the estimate of I_1 in the previous subsection and obtain

$$|I_4| \leq C\,h \sum_{i=1}^{M} ||\psi_i||_{2,\Omega_i} ||\zeta_i||_{1,h,\Omega_i} \leq C\,h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i} ||\zeta_i||_{0,\Omega_i}, \quad (2.3.40)$$

and

$$|I_5| \leq C\,h \sum_{i=1}^{M} ||\psi_i||_{2,\Omega_i} ||\zeta_i||_{1,h,\Omega_i} \leq C\,h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i} ||\zeta_i||_{0,\Omega_i}. \quad (2.3.41)$$

Since $\psi_i = \psi_{|T'} = \psi_j$, $\psi_i = \psi_{|T_1}$ and $\psi_j = \psi_{|T_2}$, where T_1 and T_2 are two triangles in $\mathcal{T}_{h,i}$ with T' as the common edge, we find that

$$\int_{T' \in \partial T_1 \cap \partial T_2} \frac{\partial u_i}{\partial \nu^{T_1}} \psi_i \, ds + \int_{T' \in \partial T_1 \cap \partial T_2} \frac{\partial u_j}{\partial \nu^{T_2}} \psi_j \, ds = 0. \quad (2.3.42)$$

Using (2.3.42) in I_6, we obtain

$$I_6 = \sum_{i=1}^{M} \sum_{T \in \mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T} (I_h\psi_i - \psi_i)\, ds = \sum_{i=1}^{M} \sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T_{int}} \int_e \frac{\partial u}{\partial \nu_e} [I_h\psi - \psi]\, ds, \quad (2.3.43)$$

where $[v_h] = v_{h|T_1} - v_{h|T_2}$, and let e denote the common face of two triangles. Note that

$$\int_e [v_h]\, ds = 0, \quad (2.3.44)$$

since $[v_h]$ is linear and vanishes at the midpoint of e. Using (2.3.44) in (2.3.43), we obtain

$$I_6 = \sum_{i=1}^{M} \sum_{T \in \mathcal{T}_{h,i}} \sum_{e \in \partial T_{int}} \int_e (\nabla u - \overline{\nabla u}) \cdot \nu_e \, [I_h\psi - \psi]\, ds, \quad (2.3.45)$$

where $\overline{\nabla u}$ is the mean value of ∇u on e. An application of Cauchy-Schwarz inequality with Lemma 1.2.7, Lemma 2.2.3, and (2.3.2) yields

$$
\begin{aligned}
I_6 &\leq Ch^{1/2}\sum_{i=1}^{M}||u_i||_{2,\Omega_i}\sum_{T\in\mathcal{T}_{h,i}}\Big(h^{-1/2}||\psi_i - I_h\psi_i||_{0,T} + ||\psi_i - I_h\psi_i||_{0,T}^{1/2}\,|\psi_i - I_h\psi_i|_{1,h,T}^{1/2}\Big)\\
&\leq Ch^{1/2}\sum_{i=1}^{M}||u_i||_{2,\Omega_i}\left\{\sum_{T\in\mathcal{T}_{h,i}}h^{-1/2}||\psi_i - I_h\psi_i||_{0,T} + \left(\sum_{T\in\mathcal{T}_{h,i}}||\psi_i - I_h\psi_i||_{0,T}\right)^{1/2}\right.\\
&\qquad\left.\times\left(\sum_{T\in\mathcal{T}_{h,i}}||\psi_i - I_h\psi_i||_{1,h,T}\right)^{1/2}\right\}\\
&\leq C\,h^{1/2}\sum_{i=1}^{M}||u_i||_{2,\Omega_i}\,\Big\{h^{3/2}\,||\psi_i||_{2,\Omega_i} + \big(h^2\,||\psi_i||_{2,\Omega_i}\,h\,||\psi_i||_{2,\Omega_i}\big)^{1/2}\Big\}\\
&\leq C\,h^2\sum_{i=1}^{M}||u_i||_{2,\Omega_i}\,||\psi_i||_{2,\Omega_i} \leq C\,h^2\sum_{i=1}^{M}||u_i||_{2,\Omega_i}\,||\zeta_i||_{0,\Omega_i}.
\end{aligned}
\tag{2.3.46}
$$

For I_7, we rewrite it as

$$
\begin{aligned}
I_7 = \sum_{i=1}^{M}\sum_{i<j\in N(i)}\Bigg\{&\left[\int_{\Gamma_{ij}}\lambda_{ij}\,[I_h\psi] - \int_{\Gamma_{ij}}Q_h\lambda_{ij}\,[\pi I_h\psi]\,ds\right]\\
&+\int_{\Gamma_{ij}}(Q_h\lambda_{ij} - \lambda_{ij,h})\,[\pi I_h\psi]\,ds\Bigg\}.
\end{aligned}
\tag{2.3.47}
$$

Using (2.3.22) in (2.3.47), we obtain

$$
\begin{aligned}
I_7 \;&=\; \sum_{i=1}^{M}\sum_{i<j\in N(i)}\left[\sum_{T\in\mathcal{T}_{h,i}}\;\sum_{e\in\partial T\cap\Gamma_{ij}\neq\phi}\int_e(\lambda_{ij} - Q_h\lambda_{ij})\,[I_h\psi]\,ds\right]\\
&+\;\sum_{i=1}^{M}\sum_{i<j\in N(i)}\left[\sum_{T\in\mathcal{T}_{h,i}}\;\sum_{e\in\partial T\cap\Gamma_{ij}\neq\phi}\int_e(Q_h\lambda_{ij} - \lambda_{ij,h})\,[I_h\psi]\,ds\right].
\end{aligned}
\tag{2.3.48}
$$

Since $\psi_i = \psi_j$ on Γ_{ij}, we therefore, arrive at

$$
\begin{aligned}
I_7 \;&=\; \sum_{i=1}^{M}\sum_{i<j\in N(i)}\left[\sum_{T\in\mathcal{T}_{h,i}}\;\sum_{e\in\partial T\cap\Gamma_{ij}\neq\phi}\int_e(\lambda_{ij} - Q_h\lambda_{ij})\,[I_h\psi - \psi]\,ds\right]\\
&+\;\sum_{i=1}^{M}\sum_{i<j\in N(i)}\left[\sum_{T\in\mathcal{T}_{h,i}}\;\sum_{e\in\partial T\cap\Gamma_{ij}\neq\phi}\int_e(Q_h\lambda_{ij} - \lambda_{ij,h})\,[I_h\psi - \psi]\,ds\right].
\end{aligned}
\tag{2.3.49}
$$

We proceed similarly as in the estimates of $I_{2,1}$ and $I_{2,2}$ in the previous subsection and using (2.3.2), we find that

$$|I_7| \leq C\, h^2 \sum_{i=1}^{M} ||u_i||_{2,\Omega_i}\, ||\zeta_i||_{0,\Omega_i}. \tag{2.3.50}$$

Substituting (2.3.39), (2.3.41), (2.3.46) and (2.3.50) into (2.3.38) and using the triangle inequality, we obtain (2.3.33). This completes the proof of the theorem. ■

2.4 Numerical experiments

In this section, we have applied the discrete scheme to a model problem.

The numerical implementation scheme has been performed in a sequential machine using MATLAB.

Consider the problem (2.2.1) with $f = 2[x(1-x) + y(1-y)]$. The exact solution of the problem (2.2.1) is given by $u = x(1-x)y(1-y)$. Here we consider $\Omega = (0,1) \times (0,1)$. We decompose the square into $[0,3/4] \times [0,1]$ and $[3/4,1] \times [0,1]$, with interface $\Gamma = \{3/4\} \times (0,1)$.

h	D.O.F. in Ω_1	D.O.F. in Ω_2	$e_h = \|u - u_h\|_{0,\Omega}$	Rate
1/8	138	46	$2.13638547 \times 10^{-4}$	-
1/16	564	188	$5.55749496 \times 10^{-5}$	1.9427
1/24	1278	426	$2.48861646 \times 10^{-5}$	1.9818
1/32	2280	760	$1.40354724 \times 10^{-5}$	1.9908
1/40	3570	1190	$8.99370414 \times 10^{-6}$	1.9945
1/48	5148	1716	$6.24978544 \times 10^{-6}$	1.9964

Table 2.1: L^2 error and order of convergence for the 2-domain case

In Figure 2.3, the graph of the L^2 error $\|u - u_h\|$ is plotted as a function of the discretization step $'h'$ in the $log - log$ scale. The slope of the graph gives the computed order of convergence as approximately 2.0. These results match with the theoretical results obtained in Theorem 2.3.2.

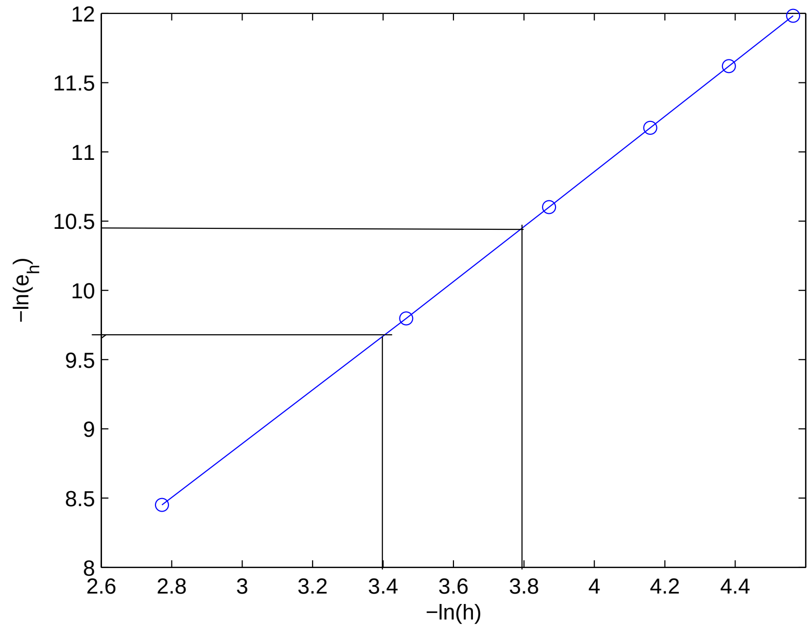

Figure 2.3: The order of convergence

In Table 2.1, the L^2 error $e_h = \|u - u_h\|$ for $h = 1/8$, $h = 1/16$, $h = 1/24$, $h = 1/32$, $h = 1/40$ and $h = 1/48$ are given.

2.5 The parabolic problem

In the remaining part of this chapter, we consider the following parabolic initial and boundary value problem. Given $f \in L^2(\Omega)$ and $u_0(x) \in L^2(\Omega)$, find $u = u(x,t)$ such that

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \ Q_T = (0,T] \times \Omega, \\ u(x,t) = 0 & \text{on } \ \partial\Omega, \ t \in (0,T], \\ u(x,0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{2.5.1}$$

where Ω is a bounded convex polygon or polyhedron in $\mathbb{R}^d$, $d = 2$ or 3 with a Lipschitz continuous, piecewise C^1 boundary $\partial\Omega$.

The weak formulation corresponding to the problem (2.5.1) may be stated as follows:

given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, find $\bar{u} : (0, T] \to H_0^1(\Omega)$ such that

$$\begin{cases} (\bar{u}_t, v) + a_\Omega(\bar{u}, v) &= (f, v) \qquad \forall\, v \in H_0^1(\Omega), \\ u(0) &= u_0, \end{cases} \tag{2.5.2}$$

where

$$a_\Omega(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx, \quad \text{and} \quad (v, w) = \int_\Omega v\, w \, dx. \tag{2.5.3}$$

Theorem 2.5.1 *Assume that the bilinear form $a(\cdot, \cdot)$ is both continuous and coercive in $H_0^1(\Omega) \times H_0^1(\Omega)$. Then, given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, there exits a unique solution $\bar{u} : [0, T] \to H_0^1(\Omega)$ to (2.5.2). Moreover, $\bar{u}$ depends continuously on the data; i.e., there exists a constant C such that*

$$\max_{t \in [0,T]} ||\bar{u}||^2_{0,\Omega} + \int_0^T ||\bar{u}||^2_{1,\Omega} \leq C \left(||u_0||^2_{0,\Omega} + \int_0^T ||f||^2_{0,\Omega} \right). \tag{2.5.4}$$

For a proof of this theorem, we refer to [93].

Now we are in a position to write the multi-domain problems. Find $u_i, \ i = 1, \ 2, \cdots, M$ satisfying the following subproblems:

$$\begin{cases} u_{it} - \Delta u_i = f_i(x, t) & \text{in } \ \Omega_i, \ t \in (0, T], \\ u_i = 0 & \text{on } \ \partial\Omega_i \cap \partial\Omega, \ t \in (0, T], \\ u_i = u_j & \text{on } \ \Gamma_{ij}, \ \ j \in N(i), \ t \in (0, T], \\ \dfrac{\partial u_i}{\partial \nu} = \dfrac{\partial u_j}{\partial \nu} & \text{on } \ \Gamma_{ij}, \ \ j \in N(i), \ t \in (0, T], \\ u_i(0) = u_{0|_{\Omega_i}} & \text{in } \ \Omega_i, \end{cases} \tag{2.5.5}$$

where $u_i = \bar{u}_{|\Omega_i}$, $u_{ti} = \bar{u}_{t|\Omega_i}$, $f_i = f_{|\Omega_i}$, $i = 1, \ 2, \cdots, M$ and $\nu = \nu^{ij} = \nu^{ji}$ on Γ_{ij} and ν^{ij} and ν^{ji} are unit outward normals to Ω_i and Ω_j, respectively. Note that $(2.5.5)_3$ - $(2.5.5)_4$ are called the consistency conditions on the artificial interface Γ_{ij}. Now we are looking for the variational formulation for the multi-domain problems (2.5.5). Multiply both sides of $(2.5.5)_1$ by a test function $v_i \in X_i$ and integrate over Ω_i to obtain

$$\int_{\Omega_i} u_{it} v_i dx + \int_{\Omega_i} \nabla u_i \cdot \nabla v_i dx - \sum_{j \in N(i)} \left\langle \frac{\partial u_i}{\partial \nu^{ij}}, v_i \right\rangle = \int_{\Omega_i} f_i v_i dx,$$

where $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ and ν_{ij} are unit outward normals to $\partial\Omega_i$. Finally, sum over $1 \le i \le M$ to find that

$$\sum_{i=1}^{M}\left[(u_{ti}, v_i)_{\Omega_i} + a_{\Omega_i}(u_i, v_i) - \sum_{j\in N(i)}\left\langle \frac{\partial u_i}{\partial \nu^{ij}}, v_i\right\rangle\right] = \sum_{i=1}^{M}(f_i, v_i)_{\Omega_i} \qquad \forall v_i \in X, \quad (2.5.6)$$

where

$$a_{\Omega_i}(v, w) = \int_{\Omega_i} \nabla v \cdot \nabla w \; dx, \qquad (v, w)_{\Omega_i} = \int_{\Omega_i} v\, w \; dx. \quad (2.5.7)$$

Below, we discuss the Lagrange multipliers method on interface Γ_{ij}. Find $u = (u_1, u_2, \cdots, u_M)$: $(0, T] \in X = \prod_{i=1}^{M} X_i$ and $\lambda : (0, T] \in Y = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{ij}$ such that

$$(u_t, v) + a(u, v) - b(v, \lambda) = (f, v) \qquad \forall v \in X, \quad (2.5.8)$$

$$b(u, \mu) = 0 \qquad \forall \mu \in Y, \quad (2.5.9)$$

where the bilinear form $a : X \times X \to \mathbb{R}$ is given by

$$a(w, v) = \sum_{i=1}^{M} a_{\Omega_i}(w_i, v_i), \quad (2.5.10)$$

the bilinear form $b : X \times Y \to \mathbb{R}$ is defined as

$$b(v, \mu) = \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left\langle \mu_{|\Gamma_{ij}},\; v_i - v_j \right\rangle \quad (2.5.11)$$

and $(\cdot,\cdot)$ denotes L^2 inner product.

Below, we state a Lemma and Theorem without proof.

Lemma 2.5.1 *The variational formulation of a single domain problem (2.2.2) and multi-domain problem (2.5.8)-(2.5.8) are equivalent under the following conditions: the test function* $(v_1, v_2, \cdots, v_M) \in X = \prod_{i=1}^{M} X_i$ *belongs to* $H_0^1(\Omega)$ *and* $\lambda_{ij}(t) = \nabla u_i \cdot \nu^{ij} = -\nabla u_j \cdot \nu^{ji}$ on Γ_{ij}, $\quad 1 \le i \le M,\; j \in N(i)$.

Theorem 2.5.2 *Problem (2.5.8)-(2.5.8) has a unique solution* $(u, \lambda) \in C^0([0, T] :\in X \times Y)$. *Moreover if* $\bar{u} : [0, T] \in H_0^1(\Omega)$ *is a solution of problem (2.5.2) with* $u_i = \bar{u}_{|\Omega_i}$ *and we have* $\lambda_{ij}(t) = \nabla u_i \cdot \nu^{ij} = -\nabla u_j \cdot \nu^{ji}$ on Γ_{ij}, $1 \le i \le M,\; j \in N(i)$.

The proof of Lemma 2.5.1 and Theorem 2.5.2 follow in the same way as those of proof of Lemma 2.2.4 and Theorem 2.2.2, respectively.

2.6 Semi-discrete approximation

In this section, we focus our attention on the spatial discretization of the problem (2.5.8)-(2.5.9). We state the variational formulation for the semi-discrete problem. Given $f \in L^2(Q_T)$, find $u_h = (u_{1,h}, \cdots, u_{M,h}) : (0,T] \to X_h = \prod_{i=1}^{M} X_{i,h}$ and $\lambda_h : [0,T] \to Y_h = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{i,h}$ such that

$$(u_{h,t}, v_h) + a^h(u_h, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} \, [\pi \, v_h] \, ds = \sum_{i=1}^{M} (f, v_h)_{\Omega_i} \quad \forall v \in X_h, \tag{2.6.1}$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi \, u_h] \, \mu_h \, ds = 0 \qquad\qquad \forall \mu_h \in Y_h, \tag{2.6.2}$$

and initial condition

$$u_h(0) = u_{0,h}, \tag{2.6.3}$$

where

$$a^h(v_h, w_h) = \sum_{i=1}^{M} a^h_{\Omega_i}(v_{i,h}, w_{i,h}) = \sum_{i=1}^{M} \int_{\Omega_i} \nabla v_{i,h} \cdot \nabla w_{i,h} \, dx, \tag{2.6.4}$$

$$[\pi \, v_h] = \pi_{ij} \, v_{i,h} - \pi_{ji} \, v_{j,h} \;\text{ on }\; \Gamma_{ij} \tag{2.6.5}$$

and $u_{0,h}$ is an approximation of u_0 onto X_h to be defined later.

Theorem 2.6.1 *Problem (2.6.1) - (2.6.2) has a unique solution* $u_h = (u_{1,h}, \cdots, u_{M,h}) : [0,T] \to X_h = \prod_{i=1}^{M} X_{i,h}$ *and* $\lambda_h : [0,T] \to Y_h = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{i,h}$. *Moreover, there exist two constant* C *and* α *independent of* h *such that*

$$||u_h||_{0,\Omega} \leq C \left(||u_{0h}||_{0,\Omega} + ||f||_{L^2([0,T],L^2(\Omega))} \right). \tag{2.6.6}$$

Proof. For simplicity, we prove the above theorem for the two fixed subdomains, i.e., $M = 2$. Since X_h and Λ_h are finite dimensional, the semidiscrete problem (2.6.1)-(2.6.2) leads to

$$M_{11}\frac{d\alpha_h^1}{dt} + A_{11}\alpha_h^1 - B_{1\Gamma}\beta_h = F_1, \tag{2.6.7}$$

$$M_{22}\frac{d\alpha_h^2}{dt} + A_{22}\alpha_h^2 + B_{2\Gamma}\beta_h = F_2, \tag{2.6.8}$$

$$B_{\Gamma 1}\alpha_h^1 - B_{\Gamma 2}\alpha_h^2 = 0, \tag{2.6.9}$$

where $M_{ii} = [m^i_{jk}]$ with $m^i_{jk} = (\phi_j, \phi_k)$, $A_{ii} = [a^i_{jk}]$ with $a^i_{jk} = a_i(\phi_j, \phi_k)$, $F_i = (F^i_j)$ with $F^i_j = (f_i, \phi_j)$, $j, k = 1, 2 \cdots N_i$, $i = 1, 2$, and $B_{i\Gamma} = [b^i_{js}]$ with $b^i_{js} = b(\phi_j, \psi_s)$, $j = 1, 2 \cdots N_i$, $s = 1, 2 \cdots N_\Gamma$, $B_{\Gamma i} = B^T_{i\Gamma}$, $i = 1, 2$. Here N_i is the number of unknowns in the Ω_i including the interface Γ and N_Γ denotes the number of unknowns on the interface Γ. Since the mass matrix $M_{ii},\ i = 1, 2$ is invertible, we obtain

$$\frac{d\alpha_h^1}{dt} = M_{11}^{-1}F_1 - M_{11}^{-1}A_{11}\alpha_h^1 + M_{11}^{-1}B_{1\Gamma}\beta_h, \tag{2.6.10}$$

$$\frac{d\alpha_h^2}{dt} = M_{22}^{-1}F_2 - M_{22}^{-1}A_{22}\alpha_h^2 - M_{22}^{-1}B_{2\Gamma}\beta_h. \tag{2.6.11}$$

Differentiate (2.6.9) with respect to time, and find that

$$B_{\Gamma 1}\frac{d\alpha_h^1}{dt} - B_{\Gamma 2}\frac{d\alpha_h^1}{dt} = 0. \tag{2.6.12}$$

Substituting (2.6.10)-(2.6.11) into (2.6.12), we arrive at

$$\begin{aligned}\left(B_{\Gamma 1}M_{11}^{-1}B_{1\Gamma} + B_{\Gamma 2}M_{22}^{-1}B_{2\Gamma}\right)\beta_h &= \left(-B_{\Gamma 1}M_{11}^{-1}F_1 + B_{\Gamma 1}M_{11}^{-1}A_{11}\alpha_h^1\right) \\ &+ \left(B_{\Gamma 2}M_{22}^{-1}F_2 - B_{\Gamma 2}M_{22}^{-1}A_{22}\alpha_h^2\right).\end{aligned} \tag{2.6.13}$$

Since $\left(B_{\Gamma 1}M_{11}^{-1}B_{1\Gamma} + B_{\Gamma 2}M_{22}^{-1}B_{2\Gamma}\right)$ is positive definite, we, therefore, obtain

$$\begin{aligned}\beta_h = \left(B_{\Gamma 1}M_{11}^{-1}B_{1\Gamma} + B_{\Gamma 2}M_{22}^{-1}B_{2\Gamma}\right)^{-1}&\left(-B_{\Gamma 1}M_{11}^{-1}F_1 + B_{\Gamma 2}M_{22}^{-1}F_2\right. \\ &\left.+ B_{\Gamma 1}M_{11}^{-1}A_{11}\alpha_h^1 - B_{\Gamma 2}M_{22}^{-1}A_{22}\alpha_h^2\right).\end{aligned} \tag{2.6.14}$$

Setting $\Sigma = B_{\Gamma 1}M_{11}^{-1}B_{1\Gamma} + B_{\Gamma 2}M_{22}^{-1}B_{2\Gamma}$ and substituting (2.6.14) into (2.6.10)-(2.6.11), we now arrive at a system of linear ordinary differential equations

$$\begin{aligned}\frac{d\alpha_h^1}{dt} + \left(I + M_{11}^{-1}B_{1\Gamma}(\Sigma)^{-1}B_{\Gamma 1}\right)M_{11}^{-1}A_{11}\alpha_h^1 - M_{11}^{-1}B_{1\Gamma}(\Sigma)^{-1}B_{\Gamma 2}M_{22}^{-1}A_{22}\alpha_h^2 \\ = M_{11}^{-1}B_{1\Gamma}(\Sigma)^{-1}B_{\Gamma 1}M_{11}^{-1}F_1 - M_{11}^{-1}B_{1\Gamma}(\Sigma)^{-1}B_{\Gamma 2}M_{22}^{-1}F_2\end{aligned} \tag{2.6.15}$$

and

$$\frac{d\alpha_h^2}{dt} + \left(I + M_{22}^{-1}B_{2\Gamma}(\Sigma)^{-1}B_{\Gamma 2}\right)M_{22}^{-1}A_{22}\alpha_h^2 - M_{22}^{-1}B_{2\Gamma}(\Sigma)^{-1}B_{\Gamma 1}M_{11}^{-1}A_{11}\alpha_h^1$$
$$= M_{22}^{-1}B_{2\Gamma}(\Sigma)^{-1}B_{\Gamma 2}M_{22}^{-1}F_2 - M_{22}^{-1}B_{2\Gamma}(\Sigma)^{-1}B_{\Gamma 1}M_{11}^{-1}F_1 \quad (2.6.16)$$

with given $\alpha_h(0)$. An appeal to Picard's theorem yields the existence of a unique solution $\alpha_h = (\alpha_h^1, \alpha_h^2)$ of (2.6.15)-(2.6.16) on $[0, T]$. Substituting the value of α_h in (2.6.14), we obtain a unique β_h. This completes the proof of existence and uniqueness of (u_h, λ_h) on (2.6.1)-(2.6.2).

Suppose (u_h, λ_h) is a solution of (2.6.1) and (2.6.2). Choose $v_h = u_h$ in (2.6.1) and $\mu_h = \lambda_h$ in (2.6.2), then we arrive at

$$\frac{1}{2}\frac{d}{dt}||u_h||_{0,\Omega}^2 + a^h(u_h, u_h) = (f, u_h). \quad (2.6.17)$$

Using Cauchy-Schwarz inequality, coercive property of a^h, $|(f, v_h)| \leq ||f||_{0,\Omega}\,||v_h||_{0,\Omega}$ and $||v_h||_{0,\Omega} \leq C||v_h||_{1,h}$ in (2.6.17), we obtain

$$\frac{d}{dt}||u_h||_{0,\Omega}^2 + \alpha\,||u_h||_{1,h}^2 \leq C(\alpha)||f||_{0,\Omega}^2. \quad (2.6.18)$$

Here we have used $||f||_{0,\Omega}\,||v_h||_{0,\Omega} \leq C\,||f||_{0,\Omega}\,||v_h||_{1,\Omega} \leq C(\alpha)\,||f||_{0,\Omega}^2 + \frac{\alpha}{2}||v_h||_{1,\Omega}^2$. Now integrate (2.6.18) over 0 to T to obtain (2.6.6). Similarly we can proceed for more than two subdomains. This completes the rest of the proof. ■

2.6.1 Error estimates

In this subsection, we discuss error estimates for the semi-discrete scheme.

For given u and λ, define $R_h u \in X_h$ and $G_h\lambda \in Y_h$ by

$$a^h(u - R_h u, v_h) - \sum_{i=1}^{M}\sum_{i<j\in N(i)}\left[\int_{\Gamma_{ij}}\lambda_{ij}\,[v_h]\,ds - G_h\lambda_{ij}\,[\pi\, v_h]\,ds\right]$$
$$= \sum_{i=1}^{M}\sum_{T\in\mathcal{T}_{h,i}}\int_{\partial T_{int}}\frac{\partial u_i}{\partial \nu^T}\,v_{i,h}\,ds \qquad \forall v_h \in X_h, \quad (2.6.19)$$

$$\sum_{i=1}^{M}\sum_{i<j\in N(i)}\int_{\Gamma_{ij}}[u - \pi\, R_h u]\,\mu_h\,ds = 0 \qquad \forall \mu_h \in Y_h. \quad (2.6.20)$$

Lemma 2.6.1 *Let $R_h u$ and $G_h\lambda$ be satisfy (2.6.19) and (2.6.20). Assume that $\{u,\ u_t, u_{tt},\ u_{ttt}\} \in \prod_{i=1}^{M} H^2(\Omega_i)$. Then there exists a constant C independent of h such that*

$$\left\|\frac{\partial^m}{\partial t^m}(u - R_h u)\right\|_{1,h} + h^{1/2}\left\|\frac{\partial^m}{\partial t^m}(\lambda - G_h\lambda)\right\|_{0,\Gamma} \le C\,h\sum_{i=1}^{M}\sum_{l=0}^{m}\left\|\frac{\partial^l}{\partial t^l}\,u_i\right\|_{2,\Omega_i}, \quad m = 0,\ 1, \tag{2.6.21}$$

and

$$\left\|\frac{\partial^m}{\partial t^m}(u - R_h u)\right\|_{0,\Omega} \le C\,h^2\sum_{i=1}^{M}\sum_{l=0}^{m}\left\|\frac{\partial^l}{\partial t^l}\,u_i\right\|_{2,\Omega_i}, \qquad m = 0,\ 1,\ 2,\ 3. \tag{2.6.22}$$

The proof follows easily from Theorem 2.3.1 and Theorem 2.3.2.

Theorem 2.6.2 *Let (u, λ) and (u_h, λ_h) be the solutions of the equations (2.5.8)-(2.5.9) and (2.6.1)-(2.6.2), respectively. Assume that $u_0 \in \prod_{i=1}^{M} H^2(\Omega_i)$ and $u_t \in \prod_{i=1}^{M} L^2(0,T;H^1(\Omega_i))$. Then there exists a positive constant C independent of h such that for $(0,T]$,*

$$||u-u_h||_{1,h} \le C\left\{||u(0)-u_{0,h}||_{1,h} + h\sum_{i=1}^{M}||u_0||_{H^2(\Omega_i)} + h\sum_{i=1}^{M}||u_t||_{L^2(0,T;H^1(\Omega_i))}\right\}. \tag{2.6.23}$$

In addition, if $u_t \in \prod_{i=1}^{M} L^2(0,T;H^2(\Omega_i))$, then

$$||u-u_h||_{0,\Omega} \le C\left\{||u(0)-u_{0,h}||_{0,\Omega} + h^2\sum_{i=1}^{M}\left(||u_0||_{H^2(\Omega_i)} + ||u_t||_{L^2(0,T;H^2(\Omega_i))}\right)\right\}. \tag{2.6.24}$$

Proof. Setting

$$u - u_h = \underbrace{(u - R_h u)}_{\eta} - \underbrace{(u_h - R_h u)}_{\theta} \quad \text{and} \quad \lambda - \lambda_h = \underbrace{(\lambda - G_h\lambda)}_{\Phi} - \underbrace{(\lambda_h - G_h\lambda)}_{\Psi}, \tag{2.6.25}$$

we now rewrite

$$\begin{aligned}\sum_{i=1}^{M} a_{\Omega_i}^h(\theta_i, v_{i,h}) &= \sum_{i=1}^{M}\left[a_{\Omega_i}^h(u_{i,h}, v_{i,h}) - a_{\Omega_i}^h(u_i, v_{i,h}) + a_{\Omega_i}^h(u_i - R_h u_i, v_{i,h})\right]\\ &= -(u_{h,t}, v_h) + \sum_{i=1}^{M}\sum_{i<j\in N(i)}\int_{\Gamma_{ij}}\lambda_{ij,h}\,[\pi v_h]\,ds + (f, v_h)\\ &\quad - \sum_{i=1}^{M} a_{\Omega_i}^h(u_i, v_{i,h}) + \sum_{i=1}^{M} a_{\Omega_i}^h(u_i - R_h u_i, v_{i,h}).\end{aligned} \tag{2.6.26}$$

Using (2.6.19) in (2.6.26) and subtracting (2.6.20) from (2.6.2), we arrive at

$$(\theta_t, v_h) + a^h(\theta, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi\,[\pi v_h]\, ds = (\eta_t, v_h) \qquad \forall v_h \in X_h, \tag{2.6.27}$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\,\theta]\,\mu_h\, ds = 0 \qquad \mu_h \in Y_h. \tag{2.6.28}$$

Substituting $v_h = \theta$ in (2.6.27) and $\mu_h = \Psi$ in (2.6.28) and using Cauchy-Schwarz inequality, extended Poincaré inequality and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}||\theta||^2_{0,\Omega} + \alpha||\theta||^2_{1,h} \le C(\alpha)||\eta_t||^2_{0,\Omega} + \frac{\alpha}{2}||\theta||^2_{1,h}. \tag{2.6.29}$$

Integrating from 0 to T with respect to time, we find that

$$||\theta(t)||^2_{0,\Omega} + \alpha \int_0^T ||\theta||^2_{1,h} \le ||\theta(0)||^2_{0,\Omega} + C(\alpha)\int_0^T ||\eta_t||^2_{0,\Omega} ds. \tag{2.6.30}$$

Using (2.6.22),

$$\begin{aligned} ||\theta(0)||_{0,\Omega} = ||R_h u(0) - u_h(0)||_{0,\Omega} &\le ||u(0) - u_h(0)||_{0,\Omega} + ||R_h u(0) - u(0)||_{0,\Omega} \\ &\le ||u(0) - u_{0,h}||_{0,\Omega} + C\,h^2 \sum_{i=1}^{M} ||u_0||_{H^2(\Omega_i)}. \end{aligned} \tag{2.6.31}$$

Using (2.6.22) and (2.6.31), we derive the estimate (2.6.24).

Differentiate (2.6.28) with respect to t. Choose $\mu_h = \Psi$ to obtain

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\,\theta_t]\,\Psi\, ds = 0. \tag{2.6.32}$$

Substituting $v_h = \theta_t$ in (2.6.27) and using (2.6.32), we arrive at

$$||\theta_t||^2_{0,\Omega} + \frac{1}{2}\frac{d}{dt}a^h(\theta,\theta) = (\eta_t, \theta_t). \tag{2.6.33}$$

Using Cauchy-Schwarz inequality, extended Poincaré inequality, Young's inequality and integrating with respect to time, we obtain

$$\int_0^T ||\theta_t||^2_{0,\Omega}\, ds + \alpha||\theta(t)||^2_{1,h} \le ||\theta(0)||^2_{1,h} + C\int_0^T ||\eta_t||^2_{0,\Omega}. \tag{2.6.34}$$

Using (2.6.21)

$$||\theta(0)||_{1,h} = ||R_h u(0) - u_h(0)||_{1,h} \leq ||u(0) - u_h(0)||_{1,h} + ||R_h u(0) - u(0)||_{1,h}$$
$$\leq ||u(0) - u_{0,h}||_{1,h} + C\, h \sum_{i=1}^{M} ||u_0||_{H^2(\Omega_i)}. \quad (2.6.35)$$

Using (2.6.34) and (2.6.35), we derive the estimate (2.6.23). This completes the proof of the theorem. ■

Theorem 2.6.3 *Let (u, λ) and (u_h, λ_h) be the solutions of the equations (2.5.8)-(2.5.9) and (2.6.1)-(2.6.2), respectively. Assume that $u_0 \in \prod_{i=1}^{M} H^2(\Omega_i)$, $\{u_t, u_{tt}\} \in \prod_{i=1}^{M} L^2(0,T;H^1(\Omega_i))$. Then there exists a positive constant C independent of h such that for $(0,T]$,*

$$h^{1/2}\, ||\lambda - \lambda_h||_{0,\Gamma} \leq C \Bigg\{ ||u(0) - u_{0,h}||_{1,h} + h \sum_{i=1}^{M} ||u_0||_{H^2(\Omega_i)}$$
$$+ \, h \sum_{i=1}^{M} \left(||u_t||_{L^2(0,T;H^1(\Omega_i))} + ||u_{tt}||_{L^2(0,T;H^1(\Omega_i))} \right) \Bigg\}. \quad (2.6.36)$$

Proof. From (2.6.27), we obtain

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi\, [\pi\, v_h]\, ds = (\theta_t, v_h) + a^h(\theta, v_h) - (\eta_t, v_h) \qquad \forall v_h \in X_h. \quad (2.6.37)$$

Now choose $v_h = S_{ij}\Psi_{ij}$ in (2.6.37), using Lemma 2.2.6, extended Poincaré inequality and Cauchy-Schwarz inequality, we find that

$$||\Psi||_{0,\Gamma} \leq C\, h^{-1/2} \left(||\eta_t||_{0,\Omega} + ||\theta||_{1,h} + ||\theta_t||_{0,\Omega} \right). \quad (2.6.38)$$

To estimate (2.6.38), we need to an estimation of $||\theta_t||_{0,\Omega}$. Now differentiate (2.6.27) and (2.6.28) with respect to the time to obtain

$$(\theta_{tt}, v_h) + a^h(\theta_t, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi_t\, [\pi\, v_h]\, ds = (\eta_{tt}, v_h), \quad (2.6.39)$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\, \theta_t]\, \mu_h\, ds = 0. \quad (2.6.40)$$

Substituting $v_h = t\,\theta_t$ in (2.6.39) and $\mu_h = t\,\Psi_t$ in (2.6.40), we arrive at

$$t\,(\theta_{tt}, \theta_t) + t\,a^h(\theta_t, \theta_t) = t\,(\eta_{tt}, \theta_t), \tag{2.6.41}$$

and hence,

$$\frac{d}{dt}\left(t\,||\theta_t||^2_{0,\Omega}\right) + t\,||\theta_t||^2_{1,h} \leq C\,t\,||\eta_{tt}||^2_{0,\Omega} + ||\theta_t||^2_{0,\Omega}. \tag{2.6.42}$$

Now integrating with respect to time from 0 to T, we find that

$$t||\theta_t||^2_{0,\Omega} \leq \int_0^T s\,||\eta_{tt}||^2_{0,\Omega} ds + \int_0^T ||\theta_t||^2_{0,\Omega} ds \leq C\,\int_0^T s\,||\eta_{tt}||^2_{0,\Omega} ds + \int_0^T ||\theta_t||^2_{0,\Omega} ds. \tag{2.6.43}$$

From (2.6.43), we obtain

$$||\theta_t||^2_{0,\Omega} \leq \frac{1}{t}\int_0^T s\,||\eta_{tt}||^2_{0,\Omega} ds + \frac{1}{t}\int_0^T ||\theta_t||^2_{0,\Omega} ds. \tag{2.6.44}$$

Using (2.6.21) and substituting (2.6.34) and (2.6.35) in (2.6.44), and applying (2.6.34), we arrive at

$$||\theta_t||^2_{0,\Omega} \leq \frac{C}{t}\Bigg\{||u_0 - u_{0,h}||^2_{1,h} + h^2\sum_{i=1}^{M}\Big[||u_0||^2_{H^2(\Omega_i)} + ||u_t||^2_{L^2(0,T;H^1(\Omega_i))} + ||u_{tt}||^2_{L^2(0,T;H^1(\Omega_i))}\Big]\Bigg\}. \tag{2.6.45}$$

An application of triangle inequality completes the rest of the proof. ■

2.7 Fully discrete approximation

In this section, we discuss a completely discrete scheme which is based on backward Euler method for the problem (2.5.8)-(2.5.9). Let $0 < t_1 < t_2 < \cdots < t_N$ be a partition of $[0, T]$ into N subintervals with $T = N\Delta t$, $\Delta t = t_n - t_{n-1}$ being the time step and $t_n = n\Delta t$. For a continuous function Θ on $[0, T]$, define

$$\bar{\partial}_t\Theta^n = \frac{\Theta^n - \Theta^{n-1}}{\Delta t}, \tag{2.7.1}$$

where $\Theta^n = \Theta(t_n),\ n = 1, 2, 3, \cdots, N$.

Given $f \in L^2(Q_T)$ and $U^{n-1} \in X_h$, find $U^n = (U_1^n, \cdots, U_M^n) \in X_h = \prod_{i=1}^{M} X_{i,h}$ and $\lambda_h^n \in$

$Y_h = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{i,h}$ for $n = 1, 2, 3, \cdots, N$, such that

$$(\bar{\partial}_t U^n, v_h) + a^h(U^n, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}^n \, [\pi \, v_h] \, ds = (f^n, v_h) \qquad \forall v_h \in X_h, \quad (2.7.2)$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi \, U^n] \, \mu_h \, ds = 0 \qquad \forall \mu_h \in Y_h, \quad (2.7.3)$$

and

$$U^0 = u_{0,h}, \tag{2.7.4}$$

where $u_{0,h}$ is an approximation of u_0 onto X_h to be defined later. We now rewrite (2.7.2)-(2.7.3) as

$$(U^n, v_h) + \Delta t \, a^h(U^n, v_h) - \Delta t \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}^n \, [\pi \, v_h] \, ds = (U^{n-1}, v_h) + \Delta t \, (f^n, v_h)$$
$$\forall v_h \in X_h, \quad (2.7.5)$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi \, U^n] \, \mu_h \, ds = 0 \qquad \forall \mu_h \in Y_h. \quad (2.7.6)$$

Theorem 2.7.1 *Given $(U^{n-1}, \lambda_h^{n-1})$, there exists a unique solution (U^n, λ_h^n) to problem (2.7.5) and (2.7.6).*

Proof. For simplicity, we prove the result for two fixed subdomains, i.e., $M = 2$. Since X_h and Y_h are finite dimensional, the problem (2.7.5)- (2.7.6) leads to

$$\begin{pmatrix} \hat{A}_{11} & 0 & \hat{B}_{1\Gamma} \\ 0 & \hat{A}_{22} & -\hat{B}_{2\Gamma} \\ \hat{B}_{\Gamma 1} & -\hat{B}_{\Gamma 2} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1^n \\ \mathbf{U}_2^n \\ \lambda_h^n \end{pmatrix} = \begin{pmatrix} \mathbf{G}_1^n \\ \mathbf{G}_2^n \\ 0 \end{pmatrix}, \tag{2.7.7}$$

where $\hat{A}_{ii} = M_{ii} + \Delta t A_{ii}$, $M_{ii} = [m_{jk}^i]$ with $m_{jk}^i = (\phi_j, \phi_k)$, $A_{ii} = [a_{jk}^i]$ with $a_{jk}^i = a_i(\phi_j, \phi_k)$, $\hat{B}_{i\Gamma} = \Delta t B_{i\Gamma}$, $B_{i\Gamma} = [b_{js}^i]$ with $b_{js}^i = b(\phi_j, \psi_s)$, $\hat{B}_{\Gamma i} = \hat{B}_{i\Gamma}^T$, $G_i^n = M_{ii}\mathbf{U}_1^{n-1} + \Delta t F_i$, $F_i = (F_j^i)$ with $(F_j^i) = (f_i, \phi_j)$, $j = 1, 2 \cdots N_i$, $k = 1, 2 \cdots N_i$, $s = 1, 2 \cdots N_\Gamma$. Here N_i is the number of unknowns in the Ω_i including the interface Γ and N_Γ denotes the number of unknowns on the interface Γ for all $i = 1, 2,$. Since $\hat{A}_{ii}$ is invertible,

$$\mathbf{U}_1^n = \hat{A}_{11}^{-1} \, (\, \mathbf{G}_1^n - \hat{B}_{1\Gamma} \lambda_h^n) \tag{2.7.8}$$

and

$$\mathbf{U}_2^n \;=\; \hat{A}_{22}^{-1} \left(\mathbf{G}_2^n \;+\; \hat{B}_{2\Gamma}\lambda_h^n \right). \tag{2.7.9}$$

Substituting $\mathbf{U}_1^n$ and $\mathbf{U}_2^n$ from (2.7.8) and (2.7.9) in (2.7.7), we obtain

$$\hat{\Sigma}_h \; \lambda_h^n \;=\; \chi_\Gamma^n, \tag{2.7.10}$$

where

$$\chi_\Gamma^n \;=\; \hat{B}_{\Gamma 1}\hat{A}_{11}^{-1}\mathbf{G}_1^n \;-\; \hat{B}_{\Gamma 2}\hat{A}_{22}^{-1}\mathbf{G}_2^n, \tag{2.7.11}$$

and

$$\hat{\Sigma}_h \;=\; \hat{B}_{\Gamma 1}\hat{A}_{11}^{-1}\hat{B}_{1\Gamma} \;+\; \hat{B}_{\Gamma 2}\hat{A}_{22}^{-1}\hat{B}_{2\Gamma}. \tag{2.7.12}$$

The system (2.7.10) is called the **Schur complement system** and the the matrix $\hat{\Sigma}_h$ is called the **Schur complement matrix**. Rewrite $\hat{\Sigma}_h$ as

$$\hat{\Sigma}_h = \hat{\Sigma}_{1,h} + \hat{\Sigma}_{2,h}, \quad \text{with} \;\; \hat{\Sigma}_{i,h} = \hat{B}_{\Gamma i}\hat{A}_{ii}^{-1}\hat{B}_{i\Gamma}. \tag{2.7.13}$$

Since $\hat{\Sigma}_h$ is positive definite, $\hat{\Sigma}_h$ is invertible, and, hence, we obtain from (2.7.10) a unique λ_h^n. Substituting λ_h^n in (2.7.8)-(2.7.9), we obtain a unique $U^n = (U_1^n, U_2^n)$, for $n = 1, 2, \cdots, N$. Similarly, we can proceed for more than two subdomains and this completes the rest of the proof. ■

2.7.1 Error estimates

In this subsection, we discuss error estimates for the completely discrete scheme (2.7.2)-(2.7.3).

Theorem 2.7.2 *Let (u^n, λ^n) and (U^n, λ_h^n) be the solutions of (2.5.8)-(2.5.9) and (2.7.2)-(2.7.3) respectively. Assume that $u(0) \in \prod_{i=1}^{2} H^2(\Omega_i)$, $u \in \prod_{i=1}^{2} H^2(\Omega_i)$, $u_{tt} \in L^2(0,T;L^2(\Omega))$ and $u_t \in \prod_{i=1}^{2} L^2(0,T;H^1(\Omega_i))$. Then there exists a positive constant C independent of h*

such that for $(0,T]$,

$$\max_{0\le n\le N} ||u^n - U^n||_{1,h} \le C\big\{||u(0) - U^0||_{1,h} + \Delta t\, ||u_{tt}||_{L^2(0,T;L^2(\Omega))}$$
$$+ h \sum_{i=1}^{M} \big(||u(0)||_{H^2(\Omega_i)} + ||u||_{H^2(\Omega_i)} + ||u_t||_{L^2(0,T;H^1(\Omega_i))}\big)\ \big\}. \quad (2.7.14)$$

In addition, if $u_t \in \prod_{i=1}^{2} L^2(0,T;H^2(\Omega_i))$, *then*

$$\max_{0\le n\le N} ||u^n - U^n||_{0,\Omega} \le C\big\{||u(0) - U^0||_{0,\Omega} + \Delta t\, ||u_{tt}||_{L^2(0,T;L^2(\Omega))}$$
$$+h^2 \sum_{i=1}^{M} \big(||u(0)||_{H^2(\Omega_i)} + ||u||_{H^2(\Omega_i)} + ||u_t||_{L^2(0,T;H^2(\Omega_i))}\big)\ \big\}. \quad (2.7.15)$$

Proof. Set

$$u(t_n) - U^n = \underbrace{(u(t_n) - R_h u(t_n))}_{\eta^n} - \underbrace{(U^n - R_h u(t_n))}_{\theta^n} \quad (2.7.16)$$

and

$$\lambda(t_n) - \lambda_h^n = \underbrace{(\lambda(t_n) - G_h\lambda(t_n))}_{\Phi^n} - \underbrace{(\lambda_h^n - G_h\lambda(t_n))}_{\Psi^n}. \quad (2.7.17)$$

Since the estimates for η^n and Φ^n are known, it is enough to estimate the error θ^n and Ψ^n. From (2.7.2), we now rewrite

$$\sum_{i=1}^{M} a^h_{\Omega_i}(\theta_i^n, v_{i,h}) = \sum_{i=1}^{M} \big[a^h_{\Omega_i}(U_i^n, v_{i,h}) - a^h_{\Omega_i}(u_i(t_n), v_{i,h}) + a^h_{\Omega_i}(u_i(t_n) - R_h u_i(t_n), v_{i,h})\big]$$

$$= -(\bar{\partial}_t U^n, v_h) + \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda^n_{ij,h}\,[\pi v_h]\, ds + (f^n, v_h)$$
$$- \sum_{i=1}^{M} a^h_{\Omega_i}(u_i(t_n), v_{i,h}) + \sum_{i=1}^{M} a^h_{\Omega_i}(u_i(t_n) - R_h u_i(t_n), v_{i,h}). \quad (2.7.18)$$

Using (2.6.19) in (2.7.18) at $t = t_n$ and subtracting (2.6.20) from (2.6.28) at $t = t_n$, we arrive at

$$(\bar{\partial}_t \theta^n, v_h) + a^h(\theta^n, v_h) - \sum_{i=1}^{M} \sum_{i<j \in N(i)} \int_{\Gamma_{ij}} \Psi^n \, [\pi v_h] \, ds = (\rho^n, v_h) + (\bar{\partial}_t \eta^n, v_h) \quad \forall v_h \in X_h, \tag{2.7.19}$$

$$\sum_{i=1}^{M} \sum_{i<j \in N(i)} \int_{\Gamma_{ij}} [\pi \, \theta^n] \, \mu_h \, ds = 0 \qquad \forall \mu_h \in Y_h, \tag{2.7.20}$$

where

$$\rho^n = u_t(t_n) - \bar{\partial}_t u(t_n) \qquad \text{and} \qquad \bar{\partial}_t \eta^n = \bar{\partial}_t \left(u(t_n) - R_h u(t_n) \right). \tag{2.7.21}$$

We note that

$$(\bar{\partial}_t \theta^n, \theta^n) = \frac{1}{2} \bar{\partial}_t \left(||\theta^n||^2_{0,\Omega} \right) + \frac{\Delta t}{2} ||\bar{\partial}_t \theta^n||^2_{0,\Omega}. \tag{2.7.22}$$

Choosing $v_h = \theta^n$ in (2.7.19), $\mu_h = \Psi^n$ in (2.7.20) and using (2.7.22), Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\bar{\partial}_t \left(||\theta^n||^2_{0,\Omega} \right) + \Delta t ||\bar{\partial}_t \theta^n||^2_{0,\Omega} + \alpha ||\theta^n||^2_{1,h} \le C_1(\alpha) ||\rho^n||^2_{0,\Omega} + C_2(\alpha) ||\bar{\partial}_t \eta^n||^2_{0,\Omega}. \tag{2.7.23}$$

Multiplying (2.7.23) by Δt and summing over n, we arrive at

$$||\theta^n||^2_{0,\Omega} + \alpha \Delta t \sum_{k=1}^{n} ||\theta^k||^2_{1,h} \le ||\theta^0||^2_{0,\Omega} + C_1(\alpha) \Delta t \sum_{k=1}^{n} ||\rho^k||^2_{0,\Omega} + C_2(\alpha) \Delta t \sum_{k=1}^{n} ||\bar{\partial}_t \eta^k||^2_{0,\Omega}. \tag{2.7.24}$$

We now estimate each term of the right hand side of (2.7.24). The first term of (2.7.24), we obtain

$$\begin{aligned} ||\theta^0||_{0,\Omega} = ||U^0 - R_h u(0)||_{0,\Omega} &\le ||U^0 - u(0)||_{0,\Omega} + ||u(0) - R_h u(0)||_{0,\Omega} \\ &\le ||U^0 - u(0)||_{0,\Omega} + C \, h^2 \sum_{i=1}^{M} ||u(0)||_{H^2(\Omega_i)}. \end{aligned} \tag{2.7.25}$$

Using Taylor's expansion, write

$$\rho^k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (s - t_{k-1}) u_{tt} ds \tag{2.7.26}$$

and hence

$$||\rho^k||^2_{0,\Omega} \leq \left(\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}(s-t_{k-1})||u_{tt}||ds\right)^2 \leq C\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}(s-t_{k-1})^2||u_{tt}||^2ds \leq C\Delta t||u_{tt}||^2_{L^2(t_{k-1},t_k;L^2(\Omega))}. \qquad (2.7.27)$$

The third term of (2.7.24) is estimated as

$$\begin{aligned} ||\bar{\partial}_t\eta^k||^2_{0,\Omega} &= \sum_{i=1}^M\int_{\Omega_i}|\bar{\partial}_t u_i(t_k) - \bar{\partial}_t R_h u_i(t_k)|^2dx \\ &\leq \sum_{i=1}^M(\Delta t)^{-1}\int_{t_{k-1}}^{t_k}\int_{\Omega_i}|u_{ti}(t_k) - R_h u_{ti}(t_k)|^2dxdt \\ &\leq C(\Delta t)^{-1}h^4\sum_{i=1}^M||u_t||^2_{L^2(t_{k-1},t_k;H^2(\Omega_i))}. \qquad (2.7.28)\end{aligned}$$

Substituting (2.7.25), (2.7.27) and (2.7.28) into (2.7.24) and using the triangle inequality, we obtain (2.7.15).

Choosing $v_h = \bar{\partial}_t\theta^n$ in (2.7.19), $\mu_h = \Psi^n$ in (2.7.20) and using Cauchy-Schwarz inequality and Young's inequality, we obtain

$$||\bar{\partial}_t\theta^n||^2_{0,\Omega} + a^h(\theta^n, \bar{\partial}_t\theta^n) \leq \frac{1}{4}||\rho^n||^2_{0,\Omega} + \frac{1}{2}||\bar{\partial}_t\theta^n||^2_{0,\Omega} + \frac{1}{4}||\bar{\partial}_t\eta^n||^2_{0,\Omega} \qquad (2.7.29)$$

with $\sum_{i=1}^M\sum_{i<j\in N(i)}\int_{\Gamma_{ij}}[\pi\,\theta^n]\,\Psi^n\,ds = 0$. Multiplying in (2.7.29) by Δt and summing over n, the error bound shows

$$\frac{\Delta t}{2}\sum_{j=1}^n||\bar{\partial}_t\theta^j||^2_{0,\Omega} + \frac{\alpha}{2}||\theta^n||^2_{1,h} \leq C(\alpha)||\theta^0||^2_{1,h} + \frac{\Delta t}{4}\sum_{j=1}^n||\rho^j||^2_{0,\Omega} + \frac{\Delta t}{4}\sum_{j=1}^n||\bar{\partial}_t\eta^j||^2_{0,\Omega}. \qquad (2.7.30)$$

We now estimate each term of the right hand side of (2.7.30). The first term of (2.7.30), we obtain

$$\begin{aligned} ||\theta^0||_{1,h} = ||U^0 - R_h u(0)||_{1,h} &\leq ||U^0 - u(0)||_{1,h} + ||u(0) - R_h u(0)||_{1,h} \\ &\leq ||U^0 - u(0)||_{1,h} + Ch\sum_{i=1}^M||u(0)||_{H^2(\Omega_i)}. \qquad (2.7.31)\end{aligned}$$

The third term of (2.7.30) is estimated as

$$\begin{aligned} ||\bar{\partial}_t \eta^k||^2_{0,\Omega} &= \sum_{i=1}^{M} \int_{\Omega_i} |\bar{\partial}_t u_i(t_k) - \bar{\partial}_t R_h u_i(t_k)|^2 dx \\ &\leq \sum_{i=1}^{M} (\Delta t)^{-1} \int_{t_{k-1}}^{t_k} \int_{\Omega_i} |u_{ti}(t_k) - R_h u_{ti}(t_k)|^2 dx dt \\ &\leq C(\Delta t)^{-1} h^2 \sum_{i=1}^{M} ||u_t||^2_{L^2(t_{k-1},t_k;H^1(\Omega_i))}. \end{aligned} \quad (2.7.32)$$

Substituting (2.7.31), (2.7.27) and (2.7.32) into (2.7.30) and using the triangle inequality, we obtain (2.7.14). This completes the rest of the proof. ■

Theorem 2.7.3 *Let (u^n, λ^n) and (U^n, λ_h^n) be the solutions of the equations (2.5.8)-(2.5.9) and (2.7.2)-(2.7.3), respectively. Assume that $u(0) \in \prod_{i=1}^{M} H^2(\Omega_i)$, $u \in \prod_{i=1}^{M} H^2(\Omega_i)$, $u_t \in L^\infty(H^1(\Omega_i))$ $u_t \in \prod_{i=1}^{M} L^2(0,T;H^1(\Omega_i))$, $u_{tt} \in \prod_{i=1}^{M} L^2(0,T;H^1(\Omega_i))$, $u_{tt} \in L^2(0,T;L^2(\Omega))$, $u_{ttt} \in L^2(0,T;L^2(\Omega))$, and $u_{tt} \in L^\infty(L^2(\Omega))$. And also assume that $U^0 - R_h u(0) = 0$. Then there exists a positive constant C independent of h such that for $(0,T]$*

$$\begin{aligned} \max_{0\leq n\leq N} h^{1/2}\, ||\lambda^n - \lambda_h^n||_{0,\Gamma} \leq C\big\{ \Delta t \ \big[||u_{tt}||_{L^2(0,T;L^2(\Omega))} + ||u_{tt}||_{L^\infty(L^2(\Omega))} \\ + ||u_{ttt}||_{L^2(0,T;L^2(\Omega))} \big] + h \sum_{i=1}^{M} \big(||u(0)||_{H^2(\Omega_i)} + ||u||_{H^2(\Omega_i)} + ||u_t||_{L^2(0,T;H^1(\Omega_i))} \\ + ||u_t||_{L^\infty(H^1(\Omega_i))} + ||u_{tt}||_{L^2(0,T;H^1(\Omega_i))} \big) \big\}. \end{aligned} \quad (2.7.33)$$

Proof. Now Choose $v_h = \theta^n$ in (2.7.19), we obtain

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi^n \, [\pi v_h] \, ds = (\bar{\partial}_t \theta^n, v_h) + a^h(\theta^n, v_h) - (\rho^n, v_h) - (\bar{\partial}_t \eta^n, v_h). \quad (2.7.34)$$

Now choose $v_h = S_{ij} \Psi^n_{ij}$ in (2.7.34), using Lemma 2.2.6, extended Poincaré inequality and Cauchy-Schwarz inequality, we find that

$$||\Psi^n||_{0,\Gamma} \leq C\, h^{-1/2} \, \big(||\bar{\partial}_t \theta^n||_{0,\Omega_i} + ||\theta^n_i||_{1,h,\Omega_i} + ||\rho^n||_{0,\Omega_i} + ||\bar{\partial}_t \eta^n||_{0,\Omega_i} \big). \quad (2.7.35)$$

We now estimate each term of the right hand side of (2.7.35). Estimates of second, third and fourth terms of (2.7.35) are known. Only the first term of (2.7.35) has to be estimated.

The equation (2.7.19) is true for every n. Then we can write $n \in \{1, 2, \cdots, N\}$ such that

$$(\bar{\partial}_t \theta^n, v_h) + a^h(\theta^n, v_h) = \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi^n [\pi v_h] \, ds + (\rho^n, v_h) + (\bar{\partial}_t \eta^n, v_h). \quad (2.7.36)$$

Also, for $n \in \{2, \cdots, N\}$, such that

$$(\bar{\partial}_t \theta^{n-1}, v_h) + a^h(\theta^{n-1}, v_h) = \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \Psi^{n-1} [\pi v_h] \, ds + (\rho^{n-1}, v_h) + (\bar{\partial}_t \eta^{n-1}, v_h). \quad (2.7.37)$$

For $n \in \{2, \cdots, N\}$, subtracting (2.7.37) from (2.7.36), then we obtain

$$(\bar{\partial}_{tt} \theta^n, v_h) + a^h(\bar{\partial}_t \theta^n, v_h) = \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \bar{\partial}_t \Psi^n [\pi v_h] \, ds + (\bar{\partial}_t \rho^n, v_h) + (\bar{\partial}_{tt} \eta^n, v_h). \quad (2.7.38)$$

We note that

$$(\bar{\partial}_{tt} \theta^n, \bar{\partial}_t \theta^n) = \frac{1}{2} \bar{\partial}_t \left(||\bar{\partial}_t \theta^n||^2_{0,\Omega} \right) + \frac{\Delta t}{2} ||\bar{\partial}_{tt} \theta^n||^2_{0,\Omega}. \quad (2.7.39)$$

Choosing $v_h = \bar{\partial}_t \theta^n$ in (2.7.38), then apply Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2} \bar{\partial}_t \left(||\bar{\partial}_t \theta^n||^2_{0,\Omega} \right) + \frac{\Delta t}{2} ||\bar{\partial}_{tt} \theta^n||^2_{0,\Omega} + \frac{\alpha}{2} ||\bar{\partial}_t \theta^n||^2_{1,h} \leq C_1(\alpha) ||\bar{\partial}_t \rho^n||^2_{0,\Omega} + C_2(\alpha) ||\bar{\partial}_{tt} \eta^n||^2_{0,\Omega}, \quad (2.7.40)$$

with $\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \bar{\partial}_t \Psi^n [\pi \bar{\partial}_t \theta^n] \, ds = 0$. Now we have to estimate each term of the right hand side of (2.7.35). From Taylor's series expansion, we know

$$u(t_n) = u(t_{n-1}) + \Delta t u_t(t_{n-1}) + \frac{(\Delta t)^2}{2!} u_{tt}(t_{n-1}) + \frac{1}{2!} \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 u_{ttt}(s) \, ds, \quad (2.7.41)$$

$$u(t_{n-2}) = u(t_{n-1}) - \Delta t u_t(t_{n-1}) + \frac{(\Delta t)^2}{2!} u_{tt}(t_{n-1}) - \frac{1}{2!} \int_{t_{n-1}}^{t_n} (t_{n-1} - s)^2 u_{ttt}(s) \, ds, \quad (2.7.42)$$

$$u(t_n) = u(t_{n-1}) + \Delta t u_t(t_{n-1}) + \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt}(s) \, ds, \quad (2.7.43)$$

$$u(t_{n-2}) = u(t_{n-1}) - \Delta t u_t(t_{n-1}) + \int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s) u_{tt}(s) \, ds, \quad (2.7.44)$$

$$u_t(t_n) = u_t(t_{n-1}) + \Delta t u_{tt}(t_{n-1}) + \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{ttt}(s) \, ds. \quad (2.7.45)$$

The first term of (2.7.40) is estimated as

$$\begin{aligned}\bar{\partial}_t \rho^n &= \frac{1}{\Delta t}\left[\left(u_t(t_n) - \bar{\partial}_t u(t_n)\right) - \left(u_t(t_{n-1}) - \bar{\partial}_t u(t_{n-1})\right)\right] \\ &= \frac{1}{(\Delta t)^2}\left[\Delta t\left(u_t(t_n) - u_t(t_{n-1})\right) - \left(u(t_n) - 2u(t_{n-1}) + u(t_{n-2})\right)\right]. \quad (2.7.46)\end{aligned}$$

Substituting (2.7.41), (2.7.42) and (2.7.45) into (2.7.46)

$$\begin{aligned}\bar{\partial}_t \rho^n = \frac{1}{(\Delta t)^2}\Bigg[\Delta t \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{ttt}(s)\, ds - \frac{1}{2!}\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 u_{ttt}(s)\, ds \\ - \frac{1}{2!}\int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)^2 u_{ttt}(s)\, ds\Bigg]. \quad (2.7.47)\end{aligned}$$

$$\begin{aligned}||\bar{\partial}_t \rho^n||_{0,\Omega}^2 &\leq \frac{C}{(\Delta t)^4}\Bigg[(\Delta t)^3 \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 |||u_{ttt}(s)|||^2\, ds \\ &\quad + \frac{\Delta t}{4}\int_{t_{n-2}}^{t_{n-1}} (s - t_{n-1})^4 |||u_{ttt}(s)|||^2\, ds + \frac{\Delta t}{4}\int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)^4 |||u_{ttt}(s)|||\, ds\Bigg] \\ &\leq C\Delta t\left[\int_{t_{n-1}}^{t_n} |||u_{ttt}(s)|||^2\, ds + \int_{t_{n-2}}^{t_{n-1}} |||u_{ttt}(s)|||^2\, ds\right]. \quad (2.7.48)\end{aligned}$$

The second term of (2.7.40) is estimated as

$$\begin{aligned}\bar{\partial}_{tt} \eta^n &= \frac{1}{(\Delta t)^2}\left[\left(u(t_n) - R_h u(t_n)\right) - 2\left(u(t_{n-1}) - R_h u(t_{n-1})\right) + \left(u(t_{n-2}) - R_h u(t_{n-2})\right)\right] \\ &= \frac{1}{(\Delta t)^2}\left[\left(u(t_n) - 2u(t_{n-1}) + u(t_{n-2})\right) + R_h\left(u(t_n) - 2u(t_{n-1}) + u(t_{n-2})\right)\right]. \quad (2.7.49)\end{aligned}$$

Substituting (2.7.43) and (2.7.44) into (2.7.49), we obtain

$$\bar{\partial}_{tt} \eta^n = \frac{1}{(\Delta t)^2}\left[\int_{t_{n-1}}^{t_n} (s - t_{n-1})(u_{tt}(s) - R_h u_{tt}(s))\, ds - \int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)(u_{tt}(s) - R_h u_{tt}(s))\, ds\right].$$

$$\begin{aligned}||\bar{\partial}_{tt} \eta^n||_{0,\Omega}^2 &\leq \frac{2}{(\Delta t)^3}\Bigg[\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 ||u_{tt} - R_h u_{tt}||_{0,\Omega}^2 ds \\ &\qquad + \int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)^2 ||u_{tt} - R_h u_{tt}||_{0,\Omega}^2 ds\Bigg] \\ &\leq C(\Delta t)^{-1} h^2 \sum_{i=1}^{M}\left[\int_{t_{n-1}}^{t_n} ||u_{tt}(s)||_{H^1(\Omega_i)}^2\, ds + \int_{t_{n-2}}^{t_{n-1}} ||u_{tt}(s)||_{H^1(\Omega_i)}^2\, ds\right]. \quad (2.7.50)\end{aligned}$$

Substituting (2.7.50) and (2.7.48) into (2.7.40), multiplying Δt and summing over $n \in \{2, 3, \cdots, N\}$, we obtain

$$||\bar{\partial}_t \theta^n||^2_{0,\Omega} \leq ||\bar{\partial}_t \theta^1||^2_{0,\Omega} + C\left\{(\Delta t)^2 ||u_{ttt}||^2_{L^2(0,T;L^\infty)} + h^2 \sum_{i=1}^{2} ||u_{tt}||^2_{L^2(0,T;H^1(\Omega_i))}\right\}. \quad (2.7.51)$$

From (2.7.40) with $n = 1$, we obtain

$$\begin{aligned} \Delta t||\bar{\partial}_t \theta^1||^2_{0,\Omega} + \alpha||\theta^1||^2_{1,h} &\leq ||\theta^0||^2_{1,h} + C\Delta t\{||\rho^1||^2_{0,\Omega} + ||\bar{\partial}_t \eta^1||^2_{0,\Omega}\} \\ &\leq ||\theta^0||^2_{1,h} + C\Delta t\left\{(\Delta t)^2 ||u_{tt}||^2_{L^\infty(L^\infty)} + h^2 \sum_{i=1}^{M} ||u_t||^2_{L^\infty(H^1(\Omega_i))}\right\}. \quad (2.7.52) \end{aligned}$$

Substitute (2.7.52) in (2.7.51) and an application of triangle inequality completes the rest of the proof. ■

2.8 Numerical Experiments

In this section, we have applied the fully discrete scheme to a model problem. The numerical implementation scheme has been performed in a sequential machine using MATLAB.

h	D.O.F. in Ω_1	D.O.F. in Ω_2	$e_h = \|u(\cdot, t^N) - U^N\|_{0,\Omega}$	Rate
1/8	138	46	5.84592952×10^{-4}	-
1/12	315	105	2.69221264×10^{-4}	1.9123
1/16	564	188	1.53057306×10^{-4}	1.9630
1/20	885	295	9.84439608×10^{-5}	1.9778
1/24	1278	426	6.85490517×10^{-5}	1.9852
1/28	1743	581	5.04449543×10^{-5}	1.9894

Table 2.2: L^2 error and order of convergence for the 2-domain case

Consider the problem (2.5.1) with $f(x, y, t) = e^t[x(1-x)+y(1-y)+2x(1-x)+2y(1-y)]$ and $u(x, y, 0) = u_0(x, y)$. The exact solution of the problem (2.5.1) is given by $u(x, y, t) = e^t x(1 - x)y(1 - y)$.

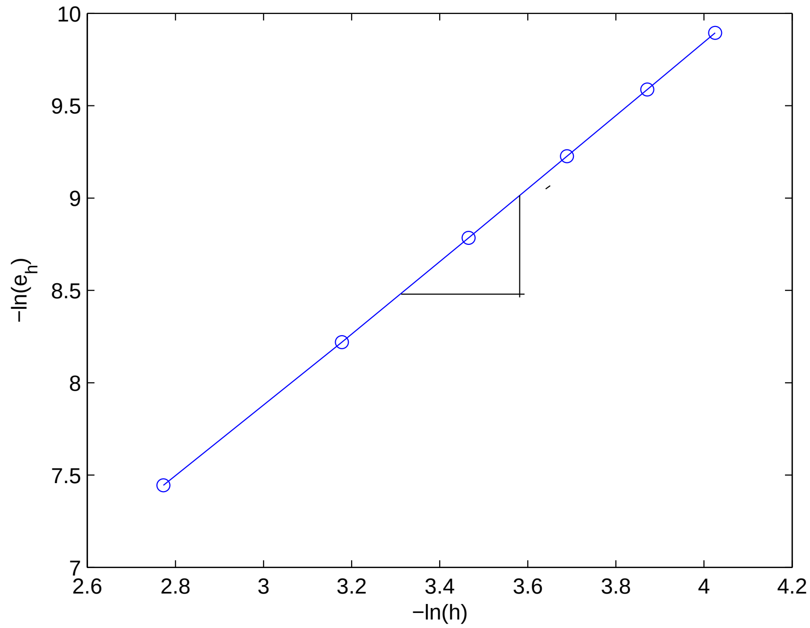

Figure 2.4: The order of convergence

Here we take $\Omega = (0,1) \times (0,1)$. We decompose the square into $[0,3/4] \times [0,1]$ and $[3/4,1] \times [0,1]$, with interface $\Gamma = \{3/4\} \times (0,1)$.

In Figure 2.4, the graph of the L^2 error $\|u - u_h\|$ is plotted as a function of the discretization step $'h'$ in the $log - log$ scale. The slope of the graph gives the computed order of convergence as approximately 2.0. These results match with the theoretical results obtained in Theorem 2.7.2.

In Table 2.2, the L^2 error $e_h = \|u(\cdot, t^N) - U^N\|$ for $h = 1/8$, $h = 1/12$, $h = 1/16$, $h = 1/20$, $h = 1/24$ and $h = 1/28$, and $\Delta t = h^2$ at time $t = 1$ are given.

Chapter 3

A Robin-Type Non-Overlapping Domain Decomposition Procedure for Second Order Elliptic Problems

3.1 Introduction

In this chapter, we discuss the analysis of an iterative nonoverlapping DD method for second order elliptic and parabolic problems using Robin-type transmission condition on the artificial interfaces, that is, on the inter subdomain boundaries. The nonoverlapping DD method using Robin-type boundary condition as transmission condition on the artificial interface (inter subdomain boundary) is becoming an an important tool for solving the following second order elliptic problems:

$$\left\{ \begin{array}{rcll} -\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x) \frac{\partial u}{\partial x_j}\right) + b(x)\, u & = & f & \forall\, x \in \Omega, \\ u & = & 0 & \forall\, x \in \partial\Omega, \end{array} \right. \tag{3.1.1}$$

where the coefficients $a_{i,j}(x)$ and $b(x)$ are in $L^{\infty}(\Omega)$ and the coefficients $a_{i,j}(x)$ satisfies ellipticity condition

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^d,\ \forall\, x \in \Omega,$$

for a suitable constant $\alpha_0 > 0$ and $b(x) \geq 0$. The Robin-type boundary conditions as interface conditions was proposed by P. L. Lions in [92] as a tool for domain decomposition iterative methods and the convergence properties by taking a suitable pseudo energy was also investigated in [92]. This idea has been applied to a more difficult Helmholtz problem by Despres [45, 46]. Exploiting the structure of the mixed finite elements, Douglas et al. [49] have obtained a more precise convergence rate by a spectral radius estimation of the iterative solution and the spectral radius has a bound of the form $1 - Ch$ for quasiregular partitions when $b(x) \geq b_0 > 0$. Subsequently in [52], Douglas et al. have discussed the convergence rate as $1 - Ch$ for nonconforming finite element methods by again using the spectral radius estimation of the iterative solution for the elliptic problems (3.1.1) on quasiregular partitions when $b(x) \geq b_0 > 0$. An improved variant of Lions method is proposed by Q. Deng and its convergence rate is analyzed in [43, 44]. Deng obtained the convergence rate by a spectral radius estimation of the iterative solution and the spectral radius has a bound of the form $1 - Ch$ for quasiregular partitions when $b(x) \geq b_0 > 0$. In [49, 52, 44], the iterative method is shown to be convergent but without the rate of convergence, when $b(x) = 0$. Based on the method proposed in [44], L. Qin and X. Xu [109] have derived the convergence rate, in general, when the lower term vanishes, i.e., $b(x) = 0$ and the convergence rate is shown to be of 1 - $O(h^{1/2}H^{-1/2})$, when the winding number N (see, the Definition 3.2.1 given in section 3) is not large.

A brief outline of this chapter is as follows. In Section 3.2, we introduce an iterative method for the elliptic multidomain problem. The key feature that we have adopted here is the introduction of the nonconforming Crouzeix-Raviart space for the discretization of the primal variable. In Section 3.3, we have discussed discrete iterative multidomain formulation. In Section 3.4, we have shown the discrete iterative multidomain problem is convergent. In Section 3.5, we have calculated the rate of convergence for iterative scheme. In Section 3.7, we extend the iterative method to a parabolic initial and boundary value problems and analyze the convergence, spectral radius and rate of convergence for fully discrete schemes. Finally, Section 3.6 and Section 3.8 deals with some numerical experiments to support our theoretical results.

3.2 Problem formulation.

We consider the following second order elliptic problem:

$$\begin{cases} -\Delta u &= f \qquad \forall x \in \Omega, \\ u &= 0 \qquad \forall x \in \partial\Omega, \end{cases} \tag{3.2.1}$$

where Ω is a bounded domain in $\mathbb{R}^d(d = 2, 3)$ and $f \in L^2(\Omega)$. The weak formulation of (3.2.1) is to find $\bar{u} \in H_0^1(\Omega)$ such that

$$a_\Omega(\bar{u}, v) = (f, v) \qquad \forall\, v \in H_0^1(\Omega), \tag{3.2.2}$$

where

$$a_\Omega(v, w) = \int_\Omega \nabla v \cdot \nabla w \; dx. \tag{3.2.3}$$

To describe finite element approximations for (3.2.2), we begin with a triangulation of $\bar{\Omega}$. Let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$ into triangles (resp. rectangles) satisfying

$$T \subset \bar{\Omega}, \quad \forall T \in \mathcal{T}_h, \quad \bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T. \tag{3.2.4}$$

Let h be the length of the greatest side of the $T \in \mathcal{T}_h$. Let $P_r(T)$ denote the space of polynomials of degree less than or equal to r in two variables defined on the triangle T. Now we define the nonconforming Crouzeix-Raviart space (cf. [39]) associated with the triangulation $\mathcal{T}_h$. Let

$$\bar{X}_h = \big\{ v \in L^2(\Omega) \mid v_{|T} \in P_1(T),\ T \in \mathcal{T}_h,\ v \text{ continuous at } p \in N_h \text{ and vanishes at } p \in \Gamma_h \big\}, \tag{3.2.5}$$

where N_h is the set of all face barycenters of elements of $\mathcal{T}_h$ in the interior of Ω and Γ_h is the set of all face barycenters of elements of $\mathcal{T}_h$ on the boundary of $\partial\Omega$. A function in X_h is completely determined by its nodal values at centers of the sides of the triangles ($d = 2$) or tetrahedra ($d = 3$) in $\mathcal{T}_h$ (cf. Figure 2.1). Then, the nonconforming Galerkin approximation of (3.2.2) is defined as the solution $u_h \in X_h$ of the equations

$$a_\Omega^h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in X_h, \tag{3.2.6}$$

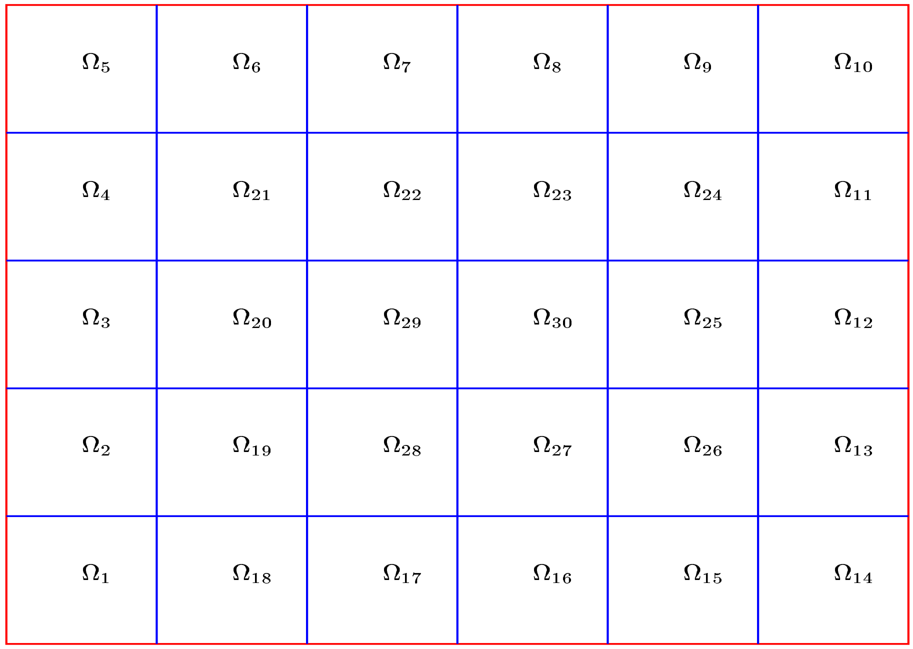

Figure 3.1: Non-overlapping decompositions of the domain into 30 disjoint sub-domains

where

$$a_{\Omega}^{h}(v_h, w_h) = \int_{\Omega} \nabla v_h \cdot \nabla w_h \, dx. \tag{3.2.7}$$

Lemma 3.2.1 *The problem (3.2.6) has a unique solution.*

For a proof, see, the Lemma 2.2.1 given in Chapter 2.

For the domain decomposition method, the domain $\bar{\Omega}$ is partitioned into a finite number of sub-domains. We define a sequence of sets D_i whose elements are subdomains by induction:

$$\begin{aligned} D_1 &= \{\Omega_i \mid \text{at least one face of } \Omega_i \text{ belongs to } \partial\Omega\}, \\ D_{r+1} &= \{\Omega_i \mid \Omega_i \notin D_r, \Omega_i \text{ share one face with atleast some } \Omega_j \in D_r\}. \end{aligned}$$

Definition 3.2.1 *[109] There exists an integer N called the winding number of the domain decomposition such that $\bigcup_{i=1}^{N} D_i$ contains all subdomains of Ω.*

For example (see Figure 3.1), the integer i in each subdomain means that this subdomain is Ω_i. So

$$\begin{aligned} D_1 &= \{\Omega_i \mid i = 1, 2, \cdots, 18\}, \\ D_2 &= \{\Omega_i \mid i = 19, 20, \cdots, 28\}, \\ D_3 &= \{\Omega_{29}, \Omega_{30}\}, \end{aligned}$$

and the winding number $N = 3$. For notational convenience, we denote a subdomain belonging to D_r by D_{i^r}. For example

$$\begin{aligned} D_1 &= (D_{i^1})_{1\le i\le 18} = \{\Omega_1,\ \Omega_2, \cdots, \Omega_{18}\}, \\ D_2 &= (D_{i^2})_{19\le i\le 28} = \{\Omega_{19},\ \Omega_{20}, \cdots, \Omega_{28}\}, \\ D_3 &= (D_{i^3})_{\{i=29,\ 30\}} = \{\Omega_{29}, \Omega_{30}\}. \end{aligned}$$

3.2.1 Iterative Method for the Multidomain Problem

In this subsection, a nonoverlapping DD procedure is developed and analyzed. Since the domain Ω is partitioned into a finite number of non-overlapping sub-domains Ω_i ($i = 1, 2, \cdots, M$), we define an iterative procedure as:

$$\left\{ \begin{aligned} -\Delta u_i^k &= f \qquad \text{in} \ \ \Omega_i, \\ \frac{\partial u_i^k}{\partial \nu_{ij}} &= \lambda_{ij}^k \qquad \text{on} \ \ \Gamma_{ij},\ j \in N(i), \\ u_i^k &= 0 \qquad \text{on} \ \ \partial\Omega_i \cap \partial\Omega, \end{aligned} \right. \tag{3.2.8}$$

$$\lambda_{ij}^k = -(\ \beta_{ij} u_i^k - \beta_{ji} u_j^{k-1}\ ) - \lambda_{ji}^{k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{3.2.9}$$

where $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ with $|\Gamma_{ij}|$ as the measure of Γ_{ij}, $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ denotes the interior interfaces, $\beta_{ij} = \beta_{ji} > 0$ are parameters and

$$N(i) = \{j \neq i |\ |\Gamma_{ij}| > 0\}. \tag{3.2.10}$$

Let $H^1_{\Gamma_i}(\Omega_i) = \{u_i | u_i \in H^1(\Omega_i)$ and $u_i = 0$ on $\partial\Omega_i \cap \partial\Omega\}$. The weak formulation corresponding to the problem (3.2.8) may be stated as follows: Given $\{u_i^0, \lambda_{ij}^0, \lambda_{ji}^0\} \in \{H^1_{\Gamma_i}(\Omega_i),$

$L^2(\Gamma_{ij}),\ L^2(\Gamma_{ji})\}$ and $f \in L^2(\Omega_i)$, find $u_i^k \in H^1_{\Gamma_i}(\Omega_i),\ i = 1, \cdots, M$ such that

$$a_{\Omega_i}(u_i^k, v) + \sum_{j\in N(i)} \beta_{ij} \int_{\Gamma_{ij}} u_i^k v ds = (f, v)_{\Omega_i} + \sum_{j\in N(i)} \beta_{ji} \int_{\Gamma_{ij}} u_j^{k-1} v ds - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ji}^{k-1} v ds \qquad \forall v \in H^1_{\Gamma_i}(\Omega_i), \tag{3.2.11}$$

and

$$\lambda_{ij}^k = -(\ \beta_{ij} u_i^k - \beta_{ji} u_j^{k-1}\ ) - \lambda_{ji}^{k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.2.12}$$

Let u be the solution of (3.2.1) and $u_i^k (1 \le i \le M)$ be the solutions of (3.2.11)-(3.2.12). For $1 \le i \le M$,

$$u_i = u_{|\Omega_i}, \qquad u = (u_i)_{1\le i\le M} \in \prod_{i=1}^M H^1_{\Gamma_i}(\Omega_i), \tag{3.2.13}$$

$$u^k = (u_i^k)_{1\le i\le M} \in \prod_{i=1}^M H^1_{\Gamma_i}(\Omega_i), \tag{3.2.14}$$

$$e_i^k = u_i^k - u_i, \qquad e^k = (e_i^k)_{1\le i\le M} \in \prod_{i=1}^M H^1_{\Gamma_i}(\Omega_i), \tag{3.2.15}$$

and

$$\mu_{ij}^k = \lambda_{ij}^k - \lambda_{ij}, \qquad \mu_{ji}^k = \lambda_{ji}^k - \lambda_{ji}, \qquad \mu^k = (\mu_{ij}^k) \prod_{i=1,\ j\in N(i)}^M L^2(\Gamma_{ij}), \tag{3.2.16}$$

where λ_{ij}, λ_{ji} are defined in the (2.2.15), and e^k and μ^k are the errors at iterative step k. Assume that $u \in H^1_0(\Omega) \cap H^{3/2}(\Omega)$, $\dfrac{\partial u_i}{\partial \nu_{ij}} \in L^2(\Gamma_{ij}),\ j \in N(i)$. Due to linearity of (3.2.1) and (3.2.8)-(3.2.9), the equations in e_i^k and μ_{ij}^k satisfy

$$\left\{ \begin{array}{rcll} -\Delta e_i^k &=& 0 & \text{in } \Omega_i, \\ \dfrac{\partial e_i^k}{\partial \nu_{ij}} &=& \mu_{ij}^k & \text{on } \Gamma_{ij},\ j \in N(i), \\ e_i^k &=& 0 & \text{on } \partial\Omega_i \cap \partial\Omega, \end{array} \right. \tag{3.2.17}$$

$$\mu_{ij}^k = -\beta\,(\ e_i^k - e_j^{k-1}\ ) - \mu_{ji}^{k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{3.2.18}$$

where $\beta = \beta_{ij} = \beta_{ji}$. The weak formulation corresponding to the problem (3.2.17) may be stated as follows:

$$a_{\Omega_i}(e_i^k, v) - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \mu_{ij}^k \, v \, ds = 0 \qquad \forall v \in H^1_{\Gamma_i}(\Omega_i). \tag{3.2.19}$$

Setting $v = e_i^k$ in (3.2.19), we arrive at the following equation :

$$a_{\Omega_i}(e_i^k, e_i^k) = \sum_{j \in N(i)} \int_{\Gamma_{ij}} \mu_{ij}^k e_i^k ds. \tag{3.2.20}$$

Define

$$E_i^k = E_i(e_i^k, \mu_{ij}^k) = \sum_{j \in N(i)} ||\mu_{ij}^k + \beta e_i^k||^2_{0,\Gamma_{ij}}, \tag{3.2.21}$$

and

$$E^k = E(e^k, \mu^k) = \sum_{i=1}^{M} E_i^k = \sum_{i=1}^{M} E_i(e_i^k, \mu_{ij}^k). \tag{3.2.22}$$

Lemma 3.2.2 *Let E_i^k and E^k be defined, respectively, by (3.2.21) and (3.2.22). Then, the following identity*

$$E^k = E^{k-1} - 4\beta \sum_{i=1}^{M} a_{\Omega_i}(e_i^{k-1}, e_i^{k-1}) \tag{3.2.23}$$

holds true.

Proof. From (3.2.20) and (3.2.21), we obtain

$$\begin{aligned} E_i^k &= \sum_{j \in N(i)} \left(||\mu_{ij}^k||^2_{0,\Gamma_{ij}} + \beta^2 ||e_i^k||^2_{0,\Gamma_{ij}} \right) + 2\beta \sum_{j \in N(i)} \int_{\Gamma_{ij}} \mu_{ij}^k e_i^k ds \\ &= \sum_{j \in N(i)} \left(||\mu_{ij}^k||^2_{0,\Gamma_{ij}} + \beta^2 ||e_i^k||^2_{0,\Gamma_{ij}} \right) + 2\beta \; a_{\Omega_i}(e_i^k, e_i^k). \end{aligned} \tag{3.2.24}$$

Then, from (3.2.18), (3.2.21) and (3.2.24), we arrive at

$$\begin{aligned} E_i^k &= \sum_{j \in N(i)} ||\mu_{ij}^k + \beta e_i^k||^2_{0,\Gamma_{ij}} = \sum_{j \in N(i)} || - \mu_{ji}^{k-1} + \beta e_j^{k-1}||^2_{0,\Gamma_{ij}} \\ &= \sum_{j \in N(i)} \left(||\mu_{ij}^{k-1}||^2_{0,\Gamma_{ij}} + \beta^2 ||e_i^{k-1}||^2_{0,\Gamma_{ij}} \right) - 2\beta \sum_{j \in N(i)} \int_{\Gamma_{ij}} \mu_{ij}^{k-1} e_i^{k-1} ds \\ &= \sum_{j \in N(i)} \left(||\mu_{ij}^{k-1}||^2_{0,\Gamma_{ij}} + \beta^2 ||e_i^{k-1}||^2_{0,\Gamma_{ij}} \right) - 2\beta \; a_{\Omega_i}(e_i^{k-1}, e_i^{k-1}) \\ &= E_i^{k-1} - 4\beta \; a_{\Omega_i}(e_i^{k-1}, e_i^{k-1}), \end{aligned} \tag{3.2.25}$$

and this completes the proof. ■

Theorem 3.2.1 *Let $u \in H_0^1(\Omega)$ be the solution of (3.2.2) which also belongs to $H^2(\Omega)$; $u_i = u_{|\Omega_i}$, and $\lambda_{ij} = \dfrac{\partial u_i}{\partial \nu_{ij}}$ on Γ_{ij}, $j \in N(i)$, with $\nu = \nu_{ij} = -\nu_{ji}$. Let $u_i^k \in H^1_{\Gamma_i}(\Omega_i)$ $(i = 1, 2, \cdots, M)$ be the solution of (3.2.11). Then for any initial guess $\{u_i^0, \lambda_{ij}^0, \lambda_{ji}^0\} \in \{H^1_{\Gamma_i}(\Omega_i), L^2(\Gamma_{ij}), L^2(\Gamma_{ji})\}$, $\forall j \in N(i)$, the following convergence result holds true :*

$$||u^k - u||_{1,\Omega} = \left(\sum_{i=1}^{M} ||u_i^k - u_i||^2_{1,\Omega_i}\right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty \tag{3.2.26}$$

and

$$||\lambda^k - \lambda||_{H^{-1/2}(\Gamma)} = \left(\sum_{i=1}^{M} \sum_{j \in N(i)} ||\lambda_{ij}^k - \lambda_{ij}||^2_{H^{-1/2}(\Gamma_{ij})}\right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty. \tag{3.2.27}$$

Proof. Since $e_i^k = u_i^k - u_i$ and $\mu_{ij}^k = \lambda_{ij}^k - \lambda_{ij}$, it is enough to show that for each i

$$||e_i^k||^2_{1,\Omega_i} \to 0, \quad \text{as} \quad k \to \infty, \tag{3.2.28}$$

and

$$||\mu_{ij}^k||^2_{H^{-1/2}(\Gamma_{ij})} \to 0, \quad \text{as} \quad k \to \infty, \ \forall j \in N(i). \tag{3.2.29}$$

From Lemma 3.2.2 and (3.2.21)-(3.2.22), we note that each $E^k \geq 0$ and

$$E^k + 4\,\beta \sum_{i=1}^{M} a_{\Omega_i}(e_i^{k-1}, e_i^{k-1}) = E^{k-1}. \tag{3.2.30}$$

The second term on the left hand side of (3.2.30) is non-negative, $0 \leq E^k \leq E^{k-1}$ and hence, $\{E^k\}$ is a decreasing sequence of non-negative terms which is bounded above by E^0. Therefore, $\{E^k\}$ converges. Moreover,

$$4\,\beta \sum_{i=1}^{M} a_{\Omega_i}(e_i^{k-1}, e_i^{k-1}) = E^{k-1} - E^k. \tag{3.2.31}$$

On summing from $k = 1$ to N_1, where N_1 is a large number, we obtain

$$4\,\beta \sum_{k=1}^{N_1} \sum_{i=1}^{M} a_{\Omega_i}(e_i^k, e_i^k) = E^0 - E^{N_1} \leq 2\,E^0, \tag{3.2.32}$$

and hence, as $N_1 \to \infty$, we find that

$$0 \le \sum_{k=1}^{N_1} \sum_{i=1}^{M} a_{\Omega_i}(e_i^k, e_i^k) < \infty. \tag{3.2.33}$$

Thus,

$$a_{\Omega_i}(e_i^k, e_i^k) \to 0 \;\text{ as }\; k \to \infty, \qquad i = 1, 2, \cdots, M. \tag{3.2.34}$$

Therefore,

$$||\nabla e_i^k||_{0,\Omega_i} \to 0 \;\text{ as }\; k \to \infty, \qquad i = 1, 2, \cdots, M. \tag{3.2.35}$$

First we consider the subdomains $\Omega_i \in D_1$, that is one face of the subdomains Ω_i, belongs to the boundary $\partial\Omega$. From $(3.2.17)_{(iii)}$, for all i, $\Omega_i \in D_1$,

$$e_i^k = 0 \qquad \text{on} \qquad \partial\Omega_i \cap \partial\Omega. \tag{3.2.36}$$

Therefore, it follows from (3.2.35)-(3.2.36) and the Poincaré-Friedrich's inequality (Lemma 1.2.5) that

$$||e_i^k||_{1,\Omega_i} \le C||\nabla e_i^k||_{0,\Omega_i} \to 0 \;\text{ as }\; k \to \infty,\; \forall i, \;\; \Omega_i \in D_1. \tag{3.2.37}$$

Hence, an use of the trace theorem (Theorem 1.2.1) yields for all i, $\Omega_i \in D_1$

$$||e_i^k||_{L^2(\Gamma_{ij})} \to 0 \;\text{ as }\; k \to \infty,\; \forall\, j \in N(i). \tag{3.2.38}$$

From (3.2.19), (3.2.35), (3.2.37)-(3.2.38), and using Lemma 2.2.5 in (3.2.19), we obtain for all i, $\Omega_i \in D_1$

$$||\mu_{ij}^k||_{H^{-1/2}(\Gamma_{ij})} \to 0 \;\text{ as }\; k \to \infty,\; \forall\, j \in N(i). \tag{3.2.39}$$

Now we consider the domains $\Omega_i \in D_2$. Using (3.2.18) in (3.2.19) with $\beta = \beta_{ij} = \beta_{ji}$, we arrive at

$$a_{\Omega_i}(e_i^k, v) + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} e_i^k v ds = \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} e_j^{k-1} v ds - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \mu_{ij}^{k-1} v ds \qquad \forall v \in H^1_{\Gamma_i}(\Omega_i). \tag{3.2.40}$$

Now, choose $v \in H^1_{\Gamma_i}(\Omega_i)$ such that

$$v = \begin{cases} e_i^k \quad \text{on} \quad \Gamma_{ij}, \ \forall \ j \in N(i), \ \Omega_j \in D_1 \\ 0 \quad \text{elsewhere} \quad \text{on} \quad \partial\Omega_i. \end{cases} \tag{3.2.41}$$

Substituting (3.2.41) into (3.2.40), we find that

$$\beta \sum_{j \in N(i)} ||e_i^k||^2_{L^2(\Gamma_{ij})} \leq ||\nabla e_i^k||_{0,\Omega_i} ||\nabla v||_{0,\Omega_i} + \beta \sum_{j \in N(i)} ||e_j^{k-1}||_{L^2(\Gamma_{ij})} ||e_i^k||_{L^2(\Gamma_{ij})} + \sum_{j \in N(i)} ||\mu_{ji}^{k-1}||_{L^2(\Gamma_{ij})} ||e_i^k||_{L^2(\Gamma_{ij})}. \tag{3.2.42}$$

Using (3.2.35), (3.2.38) and (3.2.39) in (3.2.42), we obtain for all i, $\Omega_i \in D_2$

$$||e_i^k||_{L^2(\Gamma_{ij})} \to 0 \quad \text{as} \quad k \to \infty, \ \forall \ j \in N(i), \quad \Omega_j \in D_1. \tag{3.2.43}$$

From the definition of D_r, for all i, $\Omega_i \in D_2$, there exists at least one j such that $\Omega_j \in D_1$, with meas$(\Gamma_{ij}) > 0$. Therefore, it follows from (3.2.35), (3.2.43), and the Poincaré inequality that

$$||e_i^k||_{1,\Omega_i} \leq C \left(||\nabla e_i^k||_{0,\Omega_i} + \sum_{j \in N(i), \ \Omega_j \in D_1} ||e_i^k||_{L^2(\Gamma_{ij})} \right) \to 0 \quad \text{as} \quad k \to \infty, \ \forall \ i, \quad \Omega_i \in D_2. \tag{3.2.44}$$

Similarly, we can continue the argument until the domain is exhausted and this completes the rest of the proof. ■

3.3 Discrete multidomain formulation

In this subsection, we discuss iterative method based on the nonconforming finite element problem (3.2.6).

For the triangulation $\mathcal{T}_h$, we now assume that the triangles (resp. rectangles) T should not cross the interface Γ_{ij}, and thus, each element is either contained in $\bar{\Omega}_i$ or in $\bar{\Omega}_j$ and they share the same edges of Γ_{ij}. For the multi-domain problem, let $X_{i,h} = X_{h|_{\Omega_i}}$. Define $X^0_{i,h} = \{v_h | v_h \in X_{i,h} \text{ and } v_h(p) = 0 \text{ at } p \in \partial\Omega_{i,h}\}$. We now define two discrete spaces $Y_{i,h}$ and

$Y_{ij,h}$ on $\partial\Omega_i$ and Γ_{ij}, respectively, as follows. Let $Y_{i,h}$ consist of piecewise constant elements on triangulation $\mathcal{T}_{h,i|_{\partial\Omega_i}}$, where $\mathcal{T}_{h,i|_{\partial\Omega_i}}$ is the triangulation of $\partial\Omega_i \setminus \partial\Omega$ inherited from $\mathcal{T}_h$, i.e., $\mathcal{T}_{h,i|_{\partial\Omega_i}} = \mathcal{T}_{h|_{\partial\Omega_i\setminus\partial\Omega}}$. Furthermore, let $Y_{ij,h} = Y_{i,h|_{\Gamma_{ij}}}$. The spaces are nonconforming, since $X_{i,h}$ is not subspace of $H^1_{\Gamma_i}(\Omega_i)$. For $v \in X_{i,h}$, set the discrete H^1 semi-norm as

$$|v|^2_{1,h,\Omega_i} = \sum_{T\in\mathcal{T}_{h,i}} \int_T |\nabla v|^2 dx. \tag{3.3.1}$$

We define the weighted H^1 energy norm for $v \in X_{i,h}$ by

$$||v||^2_{1,h,\Omega_i} = |v|^2_{1,h,\Omega_i} + \frac{1}{H^2}||v||^2_{0,\Omega_i}, \tag{3.3.2}$$

and

$$||v||^2_{1,h} = \sum_{i=1}^{M} ||v||^2_{1,h,\Omega_i}, \tag{3.3.3}$$

where H is the diameter of the subdomain. Given the finite element spaces $X_{i,h}$, $Y_{i,h}$ and $Y_{ij,h}$, we now introduce the linear operators:

$$\pi_i : X_{i,h} \to Y_{i,h} \qquad \text{and} \qquad \pi_{ij} : X_{i,h} \to Y_{ij,h} \tag{3.3.4}$$

as

$$\pi_i v_{i|_\tau} \equiv v_i(p) \quad \forall \tau \in \mathcal{T}_{h,i|_{\partial\Omega_i}} \qquad \text{and} \qquad \pi_{ij} v_i = \pi_i v_{i|_{\Gamma_{ij}}}. \tag{3.3.5}$$

Similarly, we define the linear operators

$$S_i : Y_{i,h} \to X_{i,h} \qquad \text{and} \qquad S_{ij} : Y_{ij,h} \to X_{i,h} \tag{3.3.6}$$

as

$$S_i w_i = \begin{cases} w_i & \text{freedom on } \partial\Omega_i, \\ 0 & \text{other freedom,} \end{cases} \qquad \text{and} \qquad S_{ij} w_{ij} = \begin{cases} w_{ij} & \text{freedom on } \Gamma_{ij}, \\ 0 & \text{other freedom.} \end{cases} \tag{3.3.7}$$

From the equation (3.3.6) and (3.3.7), we note that in general $\pi_i v_i \neq v_{i|_{\partial\Omega_i}}$ and $S_i w_{i|_{\partial\Omega_i}} \neq w_i$. Furthermore, we observe that

$$v_i - S_i\pi_i v_i \in X^0_{i,h}, \tag{3.3.8}$$

and

$$\pi_i S_i = Id_i, \qquad \pi_{ij} S_{ij} = Id_{ij}, \tag{3.3.9}$$

where Id_i and Id_{ij} are identity operators on $Y_{i,h}$ and $Y_{ij,h}$, respectively.

Lemma 3.3.1 *[109, Lemma 2.1, pp. 2542] There exists a positive constant C independent of h such that*

$$||\pi_{ij} v_i||_{0,\Gamma_{ij}} \leq C||v_{i|_{\Gamma_{ij}}}||_{0,\Gamma_{ij}} \qquad \forall\, v_i \in X_{i,h}, \tag{3.3.10}$$

$$||S_{ij} w_{ij}||_{0,\Omega_i} \leq Ch^{1/2}||w_{ij}||_{0,\Gamma_{ij}}. \tag{3.3.11}$$

Also, $\forall w_{ij} \in Y_{ij,h}$,

$$|S_{ij} w_{ij}|_{1,h,\Omega_i} \leq Ch^{-1/2}||w_{ij}||_{0,\Gamma_{ij}}. \tag{3.3.12}$$

The next lemma is a Poincaré Friedrich's inequality (cf. [20, (1.1)] and [117, Lemma 5]) for nonconforming P_1 elements.

Lemma 3.3.2 *(Poincaré-Friedrich's inequality). Let $H = \max_{1\leq i\leq M} diam(\Omega_i)$ and let Γ_{ij} be a face of Ω_i. Then, there exists a constant C constant independent of Ω_i such that for $v \in X_{i,h}$ we have*

$$||v||^2_{0,\Omega_i} \leq CH^2|v|^2_{1,\Omega_i} + CH^{2-d}\left(\int_{\Gamma_{ij}} v(s)ds\right)^2, \tag{3.3.13}$$

where $d = 2,3$ is the dimension of Ω_i. Further, if $\int_{\Gamma_{ij}} v(s)ds = 0$, the following version of Poincaré inequality holds :

$$||v||_{0,\Omega_i} \leq CH|v|_{1,\Omega_i}. \tag{3.3.14}$$

The next lemma is a the special trace theorem for Crouzeix-Raviart element space. For a proof, see [109, pp. 2544].

Lemma 3.3.3 *[109]* (Special trace theorem) Let the diameter of each subdomain Ω_i $(i = 1, 2, \cdots, M)$ be $O(H)$, and let Γ_{ij}, Γ_{il} be two faces of Ω_i. Then, there exists a positive constant C independent of Ω_i such that for $v_i \in X_{i,h}$, $1 \le l, j \le M$, $l \ne j$,

$$||\pi_{il} v_i||^2_{0,\Gamma_{il}} \le CH|v_i|^2_{1,h,\Omega_i} + C||\pi_{ij} v_i||^2_{0,\Gamma_{ij}}. \tag{3.3.15}$$

Now we are in a position to state the nonconforming Galerkin multidomain approximation corresponding to (3.2.11) and (3.2.12). Given $\{u^0_{i,h}, \lambda^0_{ij,h}, \lambda^0_{ji,h}\} \in \{X_{i,h}, Y_{ij,h}, Y_{ji,h}\}$ and $f \in L^2(\Omega)$, find $u^k_{i,h} \in X_{i,h}$, $\lambda^k_{ij,h} \in Y_{ij,h}$ and $\lambda^k_{ji,h} \in Y_{ji,h}$ such that

$$a^h_{\Omega_i}(u^k_{i,h}, v_h) + \sum_{j\in N(i)} \beta_{ij} \int_{\Gamma_{ij}} \pi_{ij} u^k_{i,h}\, \pi_{ij} v_h\, ds = (f, v_h)_{\Omega_i} + \sum_{j\in N(i)} \beta_{ji} \int_{\Gamma_{ij}} \pi_{ji} u^{k-1}_{j,h}\, \pi_{ij} v_h\, ds$$
$$- \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda^{k-1}_{ji,h} \pi_{ij} v_h\, ds \qquad \forall v_h \in X_{i,h}, \tag{3.3.16}$$

$$\lambda^k_{ij,h} = -(\ \beta_{ij}\pi_{ij} u^k_{i,h}(p) - \beta_{ji}\pi_{ji} u^{k-1}_{j,h}(p)\ ) - \lambda^{k-1}_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{3.3.17}$$

where

$$a^h_{\Omega_i}(v_{i,h}, w_{i,h}) = \int_{\Omega_i} \nabla v_{i,h} \cdot \nabla w_{i,h}\, dx. \tag{3.3.18}$$

Remark 3.3.1 *(3.3.16)-(3.3.17) is well posed can be proved similar as the proof of Theorem 2.2.3.*

Since $v_h, w_h \in X_h$ are linear polynomials on Γ_{ij}, using midpoint rule we obtain

$$\int_{\Gamma_{ij}} \pi_{ij} v_h\, \pi_{ij} w_h\, ds = \sum_{p\in\Gamma_{ij}\cap N_h} v_h(p) w_h(p)|s_p| \qquad \forall\, v_h, w_h \in X_h, \tag{3.3.19}$$

where s_p is the element face with p as its barycenter and $|s_p|$ is the measure of s_p.

3.4 Convergence Analysis

For convergence analysis, we now state the discrete nonconforming multidomain variational formulation based on Lagrange multipliers as (see, Chapter 2, (2.2.36)-(2.2.37)) : Given

$f \in L^2(\Omega)$, find $u_h = (u_{1,h}, \cdots, u_{M,h}) \in X_h = \prod_{i=1}^{M} X_{i,h}$ and $\lambda_h \in Y_h = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{ij,h}$ such that

$$a^h(u_h, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}\,[\pi\, v_h]\, ds = \sum_{i=1}^{M} (f, v_h)_{\Omega_i} \qquad \forall v \in X_h, \tag{3.4.1}$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [\pi\, u_h]\, \mu_h\, ds = 0 \qquad \forall \mu_h \in Y_h, \tag{3.4.2}$$

where

$$a^h(v_h, w_h) = \sum_{i=1}^{M} a^h_{\Omega_i}(v_{i,h}, w_{i,h}) = \sum_{i=1}^{M} \int_{\Omega_i} \nabla v_{i,h} \cdot \nabla w_{i,h}\, dx. \tag{3.4.3}$$

Lemma 3.4.1 *Let u_h and λ_h be the solution of (3.4.1)-(3.4.2). Then*

$$||\lambda_{ij,h}||_{0,\Gamma_{ij}} \leq C\left(h^{-1/2}|u_{i,h}|_{1,h,\Omega_i} + h^{1/2}||f||_{0,\Omega_i}\right), \quad i = 1, 2, \cdots, M, \quad \forall j \in N(i), \tag{3.4.4}$$

where C is a positive constant independent of h and M is the number of subdomains.
The proof of Lemma 3.4.1 is similar to that of the proof of Lemma 2.2.8.

From (3.4.1), we note that in each subdomain Ω_i

$$a^h_{\Omega_i}(u_{i,h}, v_h) - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} \pi_{ij} v_h\, ds = (f, v_h) \qquad \forall\, v_h \in X_{i,h}. \tag{3.4.5}$$

Since $\lambda_{ij,h} = -\lambda_{ji,h}$, then from (3.4.2) we obtain

$$\lambda_{ij,h} = -\lambda_{ji,h} - \beta(\pi_{ij} u_{i,h}(p) - \pi_{ji} u_{j,h}(p)). \tag{3.4.6}$$

Set

$$e^k_{i,h} = u^k_{i,h} - u_{i,h},\ \mu^k_{ij,h} = \lambda^k_{ij,h} - \lambda_{ij,h} \text{ and } \mu^k_{ji,h} = \lambda^k_{ji,h} - \lambda_{ji,h}. \tag{3.4.7}$$

Then, subtracting (3.4.5) from (3.3.16) and (3.4.6) from (3.3.17) with $\beta = \beta_{ij} = \beta_{ji}$, for $1 \leq i \leq M$, we obtain the error equations

$$a^h_{\Omega_i}(e^k_{i,h}, v_h) - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu^k_{ij,h} \pi_{ij} v_h ds = 0 \qquad \forall\, v_h \in X_{i,h}, \tag{3.4.8}$$

$$\mu^k_{ij,h} = -\left(\ \beta_{ij} \pi_{ij} e^k_{i,h}(p) - \beta_{ji} \pi_{ji} e^{k-1}_{j,h}(p)\ \right) - \mu^{k-1}_{ji,h} \qquad \forall\, x \in \Gamma_{ij},\ j \in N(i). \tag{3.4.9}$$

Setting $v_h = (0, \cdots, e^k_{i,h}, \cdots, 0)$ in (3.4.8), we arrive at the following equality

$$a^h_{\Omega_i}(e^k_{i,h}, e^k_{i,h}) = \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu^k_{ij,h}\pi_{ij}e^k_{i,h}ds. \tag{3.4.10}$$

Define

$$E^k_{i,h} = E_{i,h}(e^k_{i,h}, \mu^k_{ij,h}) = \sum_{j\in N(i)} ||\mu^k_{ij,h} + \beta\pi_{ij}e^k_{i,h}||^2_{0,\Gamma_{ij}}, \tag{3.4.11}$$

and

$$E^k_h = E_h(e^k_h, \mu^k_h) = \sum_{i=1}^{M} E^k_{i,h} = \sum_{i=1}^{M} E_{i,h}(e^k_{i,h}, \mu^k_{ij,h}). \tag{3.4.12}$$

Lemma 3.4.2 *Let E^k_h and $E^k_{i,h}$ be defined, respectively, by (3.4.12) and (3.4.11). Then following identity*

$$E^k_h = E^{k-1}_h - 4\beta \sum_{i=1}^{M} a^h_{\Omega_i}(e^{k-1}_{i,h}, e^{k-1}_{i,h}) \tag{3.4.13}$$

holds true.

Proof. From (3.4.11) and (3.4.10), we obtain

$$\begin{aligned} E^k_{i,h} &= \sum_{j\in N(i)} \Big(||\mu^k_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}e^k_{i,h}||^2_{0,\Gamma_{ij}}\Big) + 2\beta \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu^k_{ij,h}\pi_{ij}e^k_{i,h}ds \\ &= \sum_{j\in N(i)} \Big(||\mu_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}e^k_{i,h}||^2_{0,\Gamma_{ij}}\Big) + 2\beta\ a^h_{\Omega_i}(e^k_{i,h}, e^k_{i,h}). \end{aligned} \tag{3.4.14}$$

Then, from (3.4.9), (3.4.11) and (3.4.14), we arrive at

$$\begin{aligned} E^k_{i,h} &= \sum_{j\in N(i)} ||\mu^k_{ij,h} + \beta\pi_{ij}e^k_{i,h}||^2_{0,\Gamma_{ij}} = \sum_{j\in N(i)} ||-\mu^{k-1}_{ji,h} + \beta\pi_{ji}e^{k-1}_{j,h}||^2_{0,\Gamma_{ij}} \\ &= \sum_{j\in N(i)} \Big(||\mu^{k-1}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}e^{k-1}_{i,h}||^2_{0,\Gamma_{ij}}\Big) - 2\beta \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu^{k-1}_{ij,h}\pi_{ij}e^{k-1}_{i,h}ds \\ &= \sum_{j\in N(i)} \Big(||\mu^{k-1}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}e^{k-1}_{i,h}||^2_{0,\Gamma_{ij}}\Big) - 2\beta\ a^h_{\Omega_i}(e^{k-1}_{i,h}, e^{k-1}_{i,h}) \\ &= E^{k-1}_{i,h} - 4\beta\ a^h_{\Omega_i}(e^{k-1}_{i,h}, e^{k-1}_{i,h}), \end{aligned} \tag{3.4.15}$$

and this completes the proof. ■

Theorem 3.4.1 *Let* $(u_{i,h}, \lambda_{ij,h})$, $i = 1, 2, \cdots, M$, *be the solutions of the problem (3.4.5)-(3.4.6) and let* $(u^k_{i,h}, \lambda^k_{ij,h})$ *be the solutions of the discrete iterative problem (3.3.16) and (3.3.17) at iterative step* k. *Then, for any initial guess* $\{u^0_{i,h}, \lambda^0_{ij,h}, \lambda^0_{ji,h}\} \in \{X_{i,h}, Y_{ij,h}, Y_{ji,h}\}$ $\forall j \in N(i)$, *the iterative method converges in the sense that*

$$||u^k_h - u_h||_{1,h} = \left(\sum_{i=1}^{M} ||u^k_{i,h} - u_{i,h}||^2_{1,h,\Omega_i} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty, \tag{3.4.16}$$

and

$$||\lambda^k_h - \lambda_h||_0 = \left(\sum_{i=1}^{M} \sum_{j \in N(i)} ||\lambda^k_{ij,h} - \lambda_{ij,h}||^2_{0,\Gamma_{ij}} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty. \tag{3.4.17}$$

Proof. Since $e^k_{i,h} = u^k_{i,h} - u_{i,h}$ and $\mu^k_{ij,h} = \lambda^k_{ij,h} - \lambda_{ij,h}$, it is enough to show that for each i,

$$||e^k_{i,h}||^2_{1,h,\Omega_i} \quad \to \quad 0, \quad \text{as} \quad k \to \infty, \tag{3.4.18}$$

$$||\mu^k_{ij,h}||^2_{0,\Gamma_{ij}} \quad \to \quad 0, \quad \text{as} \quad k \to \infty, \ \forall j \in N(i). \tag{3.4.19}$$

From (3.4.10) and (3.4.11)-(3.4.12), we note that each $E^k_{i,h} \geq 0$ and

$$E^k_h + 4\,\beta \sum_{i=1}^{M} a^h_{\Omega_i}(e^{k-1}_{i,h}, e^{k-1}_{i,h}) = E^{k-1}_h. \tag{3.4.20}$$

The second term on the left hand side of (3.4.20) is non-negative, $0 \leq E^k_h \leq E^{k-1}_h$ and hence, $\{E^k_h\}$ is a decreasing sequence of non-negative terms which is bounded above by E^0_h. Therefore, $\{E^k_h\}$ converges. Moreover,

$$4\,\beta \sum_{i=1}^{M} a^h_{\Omega_i}(e^{k-1}_{i,h}, e^{k-1}_{i,h}) = E^{k-1}_h - E^k_h. \tag{3.4.21}$$

On summing from $k = 1$ to N_1, where N_1 is a large number, we obtain

$$4\,\beta \sum_{k=1}^{N_1} \sum_{i=1}^{M} a_{\Omega_i}(e^k_{i,h}, e^k_{i,h}) = E^0_h - E^{N_1}_h \leq 2\, E^0_h, \tag{3.4.22}$$

and hence, as $N_1 \to \infty$, we find that

$$0 \leq \sum_{k=1}^{N_1} \sum_{i=1}^{M} a^h_{\Omega_i}(e^k_{i,h}, e^k_{i,h}) < \infty. \tag{3.4.23}$$

Thus,

$$a^h_{\Omega_i}(e^k_{i,h}, e^k_{i,h}) \to 0 \ \text{ as } \ k \to \infty, \qquad i = 1, 2, \cdots, M. \tag{3.4.24}$$

Therefore,

$$||\nabla e^k_{i,h}||_{0,\Omega_i} \to 0 \ \text{ as } \ k \to \infty, \qquad i = 1, 2, \cdots, M. \tag{3.4.25}$$

Setting $\lambda_{ij,h} = \mu^k_{ij,h}$, $u_{i,h} = e^k_{i,h}$ and $f = 0$ in Lemma 3.4.1, and (3.4.25), then for all i, we obtain

$$||\mu^k_{ij,h}||_{0,\Gamma_{ij}} \to 0 \ \text{ as } \ k \to \infty, \ \forall\, j \in N(i). \tag{3.4.26}$$

First we consider the subdomains $\Omega_i \in D_1$, that is, one face of the subdomains Ω_i, belongs to the boundary $\partial\Omega$. Since, for all i, $\Omega_i \in D_1$,

$$e^k_{i,h}(p) = 0 \qquad \text{on} \qquad \partial\Omega_i \cap \partial\Omega, \tag{3.4.27}$$

where p denote any nodal point on Γ_i. Therefore, it follows from (3.3.2), (3.4.25) and the Poincaré inequality (3.3.14) that

$$||e^k_{i,h}||_{1,h,\Omega_i} \le C||\nabla e^k_{i,h}||_{0,\Omega_i} \to 0 \ \text{ as } \ k \to \infty, \ \forall i, \ \ \Omega_i \in D_1. \tag{3.4.28}$$

Hence, by the special trace theorem (Lemma 3.3.3), (3.4.27) and (3.4.28) implies that for all i, $\Omega_i \in D_1$

$$||\pi_{ij} e^k_{i,h}||_{0,\Gamma_{ij}} \to 0 \ \text{ as } \ k \to \infty, \ \forall\, j \in N(i). \tag{3.4.29}$$

From (3.4.9) with $\beta = \beta_{ij} = \beta_{ji}$, it follows that

$$\beta\pi_{ij} e^k_{i,h}(p) = -\mu^k_{ij,h} + \beta\pi_{ji} e^{k-1}_{j,h}(p) - \mu^{k-1}_{ji,h} \qquad \forall x \in \Gamma_{ij}, \ j \in N(i). \tag{3.4.30}$$

Using (3.4.29) and (3.4.26) in (3.4.30), we obtain for $\Omega_i \in D_2$, $\forall\, i$

$$||\pi_{ij} e^k_{i,h}||_{0,\Gamma_{ij}} \to 0, \ \ \text{as} \ \ k \to \infty, \ \forall\, j \in N(i), \ \ \Omega_j \in D_1. \tag{3.4.31}$$

From the definition of the D_r, for all i, $\Omega_i \in D_2$, there exists at least one j such that $\Omega_j \in D_1$, with $\text{meas}(\Gamma_{ij}) > 0$. Therefore, it follows from (3.4.25), (3.4.31), and the Poincaré

Friedrich's inequality that

$$||e^k_{i,h}||_{1,h,\Omega_i} \leq C\left(H||\nabla e^k_{i,h}||_{0,\Omega_i} + \sum_{j\in N(i),\ \Omega_j\in D_1} ||\pi_{ij}e^k_{i,h}||_{0,\Gamma_{ij}}\right) \rightarrow 0$$
$$\text{as}\ \ k\rightarrow\infty,\ \forall\ i,\ \ \Omega_i\in D_2. \qquad (3.4.32)$$

Similarly, we can continue the argument until the domain is exhausted and this completes the proof. ■

3.5 Convergence Rate

Let

$$\tilde{X}_h = \prod_{i=1}^{M} X_{i,h}, \qquad \tilde{Y}_h = \prod_{i=1}^{M} Y_{ij,h}, \qquad \forall j\in N(i). \qquad (3.5.1)$$

Also, let $T_f : \tilde{X}_h \times \tilde{Y}_h \rightarrow \tilde{X}_h \times \tilde{Y}_h$ be a mapping such that for any $(w_h,\theta_h)\in\tilde{X}_h\times\tilde{Y}_h$, $(z_h,\eta_h)\equiv T_f(w_h,\theta_h)$ is the solution, for all i, of

$$\begin{aligned} a^h_{\Omega_i}(z_{i,h},v_h) + \sum_{j\in N(i)}\beta\int_{\Gamma_{ij}}\pi_{ij}z_{i,h}\pi_{ij}v_h\,ds &= (f,v_h)_{\Omega_i} + \sum_{j\in N(i)}\beta\int_{\Gamma_{ij}}\pi_{ji}w_{j,h}\pi_{ij}v_h\,ds \\ &- \sum_{j\in N(i)}\int_{\Gamma_{ij}}\theta_{ji,h}\pi_{ij}v_h\,ds\ \ \forall v_h\in X_{i,h}, \quad (3.5.2)\end{aligned}$$

$$\eta_{ij,h} = -\beta(\ \pi_{ij}z_{i,h}(p) - \pi_{ji}w_{j,h}(p)\ ) - \theta_{ji,h} \qquad \forall x\in\Gamma_{ij},\ j\in N(i), \qquad (3.5.3)$$

where $z_{i,h} = z_{h|_{\Omega_i}}$, $w_{i,h} = w_{h|_{\Omega_i}}$, $\eta_{ij,h} = \eta_{h|_{\Gamma_{ij}}}$ and $\theta_{ji,h} = \theta_{h|_{\Gamma_{ij}}}$. Since the operator T_f is linear, we can now split the operator T_f as $T_f(w_h,\theta_h) = T_0(w_h,\theta_h) + T_f(0,0)$, where the operators T_0 and T_f are defined as follows : Given (w_h,θ_h), $(z^\star_h,\eta^\star_h) = T_0(w_h,\theta_h)$ satisfies for all i,

$$\begin{aligned} a^h_{\Omega_i}(z^\star_{i,h},v_h) + \sum_{j\in N(i)}\beta\int_{\Gamma_{ij}}\pi_{ij}z^\star_{i,h}\pi_{ij}v_h\,ds &= \sum_{j\in N(i)}\beta\int_{\Gamma_{ij}}\pi_{ji}w_{j,h}\pi_{ij}v_h\,ds \\ &- \sum_{j\in N(i)}\int_{\Gamma_{ij}}\theta_{ji,h}\pi_{ij}v_h\,ds\ \ \forall v_h\in X_{i,h}, \quad (3.5.4)\end{aligned}$$

$$\eta^{\star}_{ij,h} = -\beta(\ \pi_{ij} z^{\star}_{i,h}(p) - \pi_{ji} w_{j,h}(p)\ ) - \theta_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{3.5.5}$$

and $(z^o_h, \eta^o_h) = T_f(0,0)$ satisfies, for all i,

$$a^h_{\Omega_i}(z^o_{i,h}, v_h) + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z^o_{i,h} \pi_{ij} v_h \, ds = (f, v_h)_{\Omega_i} \qquad \forall v_h \in X_{i,h}, \tag{3.5.6}$$

$$\eta^o_{ij,h} = -\beta\ \pi_{ij} z^o_{i,h}(p) \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.5.7}$$

Then $(z_h, \eta_h) = (z^{\star}_h, \eta^{\star}_h) + (z^o_h, \eta^o_h)$.

Lemma 3.5.1 *The pair $(z_h, \eta_h) \in \tilde{X}_h \times \tilde{Y}_h$ is a solution, for all i, of*

$$\begin{aligned} a^h_{\Omega_i}(z_{i,h}, v_h) + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z_{i,h} \pi_{ij} v_h ds &= (f, v_h)_{\Omega_i} + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ji} z_{j,h} \pi_{ij} v_h ds \\ &- \sum_{j \in N(i)} \int_{\Gamma_{ij}} \eta_{ji,h} \pi_{ij} v_h ds \qquad \forall v_h \in X_{i,h}, \end{aligned} \tag{3.5.8}$$

$$\eta_{ij,h} = -\beta(\ \pi_{ij} z_{i,h}(p) - \pi_{ji} z_{j,h}(p)\ ) - \eta_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{3.5.9}$$

where $\eta_{ij,h} = -\eta_{ji,h}$ if and only if it is a fixed point of the operator T_f.

It is easy to check that for each i any solution of (3.4.5)-(3.4.6) is a fixed point of T_f and conversely a fixed point of T_f is a solution of (3.4.5)-(3.4.6).

Lemma 3.5.2 *Let (u_h, λ_h) be a fixed point of T_f. Then $\pi_{ij} u_{i,h}(p) = \pi_{ji} u_{j,h}(p)$ and $\lambda_{ij,h} = -\lambda_{ji,h}$ for all Γ_{ij}. Furthermore, $\bar{u}_h \in \bar{X}_h$ is the solution of (3.2.6).*

Proof. Let (u_h, λ_h) be a fixed point of T_f. Then, substituting (3.5.3) into (3.5.2) yields

$$a^h_{\Omega_i}(u_{i,h}, v_h) - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h} \pi_{ij} v_h \, ds = (f, v_h)_{\Omega_i} \qquad \forall v_h \in X_{i,h}, \tag{3.5.10}$$

and, hence, for each i, $(u_{i,h}, \lambda_{ij,h})$ satisfies (3.4.5). From (3.5.3), we obtain

$$\lambda_{ij,h} = -(\ \beta \pi_{ij} u_{i,h}(p) - \beta \pi_{ji} u_{j,h}(p)\ ) - \lambda_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i).$$

Thus, (3.4.6) is also satisfied. From (3.5.3), $\lambda_{ij,h} = -\beta(\ \pi_{ij} u_{i,h}(p) - \pi_{ji} u_{j,h}(p)\ ) - \lambda_{ji,h}$, it is clear that $\pi_{ij} u_{i,h}(p) = \pi_{ji} u_{j,h}(p)$ since $\lambda_{ij,h} = -\lambda_{ji,h}$. Also from Lemma 2.2.7, $\bar{u}_h$ is the solution of (3.2.6). This completes the rest of the proof. ■

Since

$$(z_h, \eta_h) = T_f(w_h, \theta_h) = T_0(w_h, \theta_h) + T_f(0, 0), \tag{3.5.11}$$

the fixed point (z_h, η_h) of T_f that is $T_f(z_h, \eta_h) = (z_h, \eta_h)$ is indeed a solution of

$$(I - T_0)(z_h, \eta_h) = T_f(0, 0). \tag{3.5.12}$$

Note that, from (3.7.7)-(3.7.11), we conclude that

$$(e_h^k, \mu_h^k) = T_0(e_h^{k-1}, \mu_h^{k-1}). \tag{3.5.13}$$

If (z_h, η_h) is a fixed point of T_0, then from (3.5.4)-(3.5.5), we write the operator T_0 satisfies the following problem

$$a^h_{\Omega_i}(z_{i,h}, v_h) - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \eta_{ij,h} \pi_{ij} v_h ds = 0 \qquad \forall v_h \in X_{i,h}, \tag{3.5.14}$$

$$\eta_{ij,h} = -\beta(\ \pi_{ij} z_{i,h}(p) - \pi_{ji} z_{j,h}(p)\ ) - \eta_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.5.15}$$

Lemma 3.5.3 *Let $(z_h, \eta_h) \in \tilde{X}_h \times \tilde{Y}_h$ be the solution of (3.5.14) and (3.5.15). Then*

$$||\eta_{ij,h}||^2_{0,\Gamma_{ij}} \leq Ch^{-1} |z_{i,h}|^2_{1,h,\Omega_i} \quad \forall j \in N(i). \tag{3.5.16}$$

Proof. Now choosing $v_h = (0, \cdots, S_{ij}\eta_{ij,h}, \cdots, 0)$ in (3.5.14), and using (3.3.9) and Lemma 3.3.1, we obtain

$$\begin{aligned} ||\eta_{ij,h}||^2_{0,\Gamma_{ij}} &= \int_{\Gamma_{ij}} \eta_{ij,h}.\pi_{ij} S_{ij} \eta_{ij,h}\, ds = a^h_{\Omega_i}(z_{i,h}, S_{ij}\eta_{ij,h}) \\ &\leq |z_{i,h}|_{1,h,\Omega_i} |S_{ij}\eta_{ij,h}|_{1,h,\Omega_i} \\ &\leq Ch^{-1/2} |z_{i,h}|_{1,h,\Omega_i} ||\eta_{ij,h}||_{0,\Gamma_{ij}} \quad \forall j \in N(i). \end{aligned} \tag{3.5.17}$$

This completes the rest of the proof. ■

Since the errors e_h^k, μ_h^k satisfy (3.5.13). Our next aim to find the spectral radius of T_0.

Remark 3.5.1 *Here $\tilde{X}_h \times \tilde{Y}_h$ is a real linear space and T_0 is a real linear operator. In general, the spectral radius formula does not hold for the real case. So the complexification of the real linear space and the real linear operator is necessary.*

Now, we recall the linear operator T_0 defined in (3.5.13) and the linear space $\tilde{X}_h \times \tilde{Y}_h$ defined in (3.5.1). Using Lemmas given in the Chapter 1, how $\tilde{X}_h$, $\tilde{Y}_h$ are defined and also $\bar{T}_0$. The next lemma shows that the relation between $||T_0^k||$ and $\rho(\bar{T}_0)$.

Lemma 3.5.4 *Let $\tilde{X}_h \times \tilde{Y}_h$ be equipped with an inner-product and*

$$\rho(\bar{T}_0) \leq 1 - R, \qquad R \in (0,1). \tag{3.5.18}$$

Then for every positive integer k, there is a constant C independent of k such that

$$||T_0^k|| \leq C(1 - R/2)^k. \tag{3.5.19}$$

Although, the proof of Lemma 3.5.4 is available in [109, Lemma 3.6, pp. 2547], but for making the thesis self content, we sketch briefly below a proof.

Proof. From Lemmas 1.2.13 and 1.2.14 we find that

$$||\bar{T}_0^k|| = ||T_0^k||. \tag{3.5.20}$$

Since $\bar{T}_0$ is a complex linear operator on the complex linear space $\mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$, then by the spectral radius formula (see, Chapter 1, Theorem 1.2.3)

$$\rho(\bar{T}_0) = \lim_{k \to \infty} ||\bar{T}_0^k||^{1/k}, \tag{3.5.21}$$

for $\epsilon > 0$, there exists a natural number N such that for $k > N$, we have

$$||\bar{T}_0^k||^{1/k} \leq \rho(\bar{T}_0) + \epsilon,$$

and hence

$$||\bar{T}_0^k|| \leq (\rho(\bar{T}_0) + \epsilon)^k.$$

Choose a constant $C > 1$ such that

$$||\bar{T}_0^k|| \leq C(\rho(\bar{T}_0) + \epsilon)^k$$

for $k = 1, 2, \cdots, N$. Then $\forall\, k$

$$||T_0^k|| = ||\bar{T}_0^k|| \leq C(\rho(\bar{T}_0) + \epsilon)^k. \tag{3.5.22}$$

With $\epsilon = R/2$ in (3.5.22), we complete the rest of the proof. ■

3.5.1 Spectral radius without quasi-uniformity assumptions

Let $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$, i.e.,

$$(\bar{z}_h, \bar{\eta}_h) = (\tilde{z}_h, \tilde{\eta}_h) + \sqrt{(-1)}(\hat{z}_h, \hat{\eta}_h), \tag{3.5.23}$$

where $(\tilde{z}_h, \tilde{\eta}_h), (\hat{z}_h, \hat{\eta}_h) \in \tilde{X}_h \times \tilde{Y}_h$. Using Lemma 1.2.12, we obtain the following identity.

Lemma 3.5.5 *Let $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$, and $(\tilde{z}_h, \tilde{\eta}_h), (\hat{z}_h, \hat{\eta}_h) \in \tilde{X}_h \times \tilde{Y}_h$ satisfy (3.5.23). Then*

$$|\bar{z}_{i,h}|^2_{1,h,\Omega_i} = |\tilde{z}_{i,h}|^2_{1,h,\Omega_i} + |\hat{z}_{i,h}|^2_{1,h,\Omega_i} \tag{3.5.24}$$

$$||\bar{\eta}_{ij,h}||^2_{0,ij} = ||\tilde{\eta}_{ij,h}||^2_{0,ij} + ||\hat{\eta}_{ij,h}||^2_{0,ij}, \tag{3.5.25}$$

and

$$||\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,ij} = ||\pi_{ij}\tilde{z}_{i,h}||^2_{0,ij} + ||\pi_{ij}\hat{z}_{i,h}||^2_{0,ij}, \tag{3.5.26}$$

where $\bar{\pi}_{ij}$ is the complexification of π_{ij}. For the sake of convenience, let us define another notation $G_{i,h}$ similar to $E^k_{i,h}$, but both having the same property, where each $G_{i,h}$ acts on complex values and each $E^k_{i,h}$ acts on real values:

$$G_{i,h} = G_{i,h}(\bar{z}_{i,h}, \bar{\eta}_{ij,h}) = \sum_{j \in N(i)} ||\bar{\eta}_{ij,h} + \beta \bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}}, \tag{3.5.27}$$

and

$$G_h = G_h(\bar{z}_h, \bar{\eta}_h) = \sum_{i=1}^{M} G_{i,h} = \sum_{i=1}^{M} G_{i,h}(\bar{z}_{i,h}, \bar{\eta}_{ij,h}). \tag{3.5.28}$$

Lemma 3.5.6 *Let G_h and $G_{i,h}$ be defined, respectively, by (3.5.27) and (3.5.28). Then the following identity holds true :*

$$G_h(\bar{z}_h, \bar{\eta}_h) = \sum_{i=1}^{M} \sum_{j \in N(i)} \left(||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2 ||\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \right) + 2\beta \sum_{i=1}^{M} a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}). \tag{3.5.29}$$

Proof. Setting $v_h = z_{i,h} \in X_{i,h}$ in (3.5.14), we arrive at the following equality

$$a^h_{\Omega_i}(z_{i,h}, z_{i,h}) = \sum_{j \in N(i)} \int_{\Gamma_{ij}} \eta_{ij,h}.\pi_{ij} z_{i,h} ds. \tag{3.5.30}$$

From (3.5.27) and (3.5.28), and using Lemma 1.2.12, we obtain

$$\begin{aligned} G_{i,h} &= \sum_{j\in N(i)} ||\bar{\eta}_{ij,h} + \beta\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \\ &= \sum_{j\in N(i)} ||\tilde{\eta}_{ij,h} + \beta\pi_{ij}\tilde{z}_{i,h}||^2_{0,\Gamma_{ij}} + \sum_{j\in N(i)} ||\hat{\eta}_{ij,h} + \beta\pi_{ij}\hat{z}_{i,h}||^2_{0,\Gamma_{ij}} \\ &= I_1 + I_2. \end{aligned} \tag{3.5.31}$$

Since $(z_{i,h}, \eta_{ij,h}) \in X_{i,h} \times Y_{ij,h}$, by (3.5.30),

$$\begin{aligned} I_1 &= \sum_{j\in N(i)} \left(||\tilde{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}\tilde{z}_{i,h}||^2_{0,\Gamma_{ij}}\right) + 2\beta \sum_{j\in N(i)} \langle\tilde{\eta}_{ij,h}, \pi_{ij}\tilde{z}_{i,h}\rangle_{|\Gamma_{ij}} \\ &= \sum_{j\in N(i)} \left(||\tilde{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}\tilde{z}_{i,h}||^2_{0,\Gamma_{ij}}\right) + 2\beta\ a^h_{\Omega_i}(\tilde{z}_{i,h}, \tilde{z}_{i,h}). \end{aligned} \tag{3.5.32}$$

Similarly, we find that

$$\begin{aligned} I_2 &= \sum_{j\in N(i)} \left(||\hat{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}\hat{z}_{i,h}||^2_{0,\Gamma_{ij}}\right) + 2\beta \sum_{j\in N(i)} \langle\hat{\eta}_{ij,h}, \pi_{ij}\hat{z}_{i,h}\rangle_{|\Gamma_{ij}} \\ &= \sum_{j\in N(i)} \left(||\hat{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\pi_{ij}\hat{z}_{i,h}||^2_{0,\Gamma_{ij}}\right) + 2\beta\ a^h_{\Omega_i}(\hat{z}_{i,h}, \hat{z}_{i,h}). \end{aligned} \tag{3.5.33}$$

Using (3.5.32), (3.5.33) and Lemma 1.2.12 in (3.5.31), we arrive at

$$G_{i,h} = \sum_{j\in N(i)} \left(||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}}\right) + 2\beta\ a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}), \tag{3.5.34}$$

where

$$a^h_{\Omega_i}(\tilde{z}_{i,h}, \tilde{z}_{i,h}) + a^h_{\Omega_i}(\hat{z}_{i,h}, \hat{z}_{i,h}) = ||\nabla\tilde{z}_{i,h}||^2_{0,\Omega_i} + ||\nabla\hat{z}_{i,h}||^2_{0,\Omega_i} = ||\nabla\bar{z}_{i,h}||^2_{0,\Omega_i} = a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}).$$

This completes the rest of the proof. ■

Theorem 3.5.1 *Let $\rho(\bar{T}_0)$ be the spectral radius of $\bar{T}_0$. Then*

$$\rho(\bar{T}_0) < 1. \tag{3.5.35}$$

Proof. Let γ be an eigenvalue of $\bar{T}_0$ and let $(\bar{z}_h, \bar{\eta}_h) \neq (0, 0)$ be the corresponding eigenvector. Then

$$\bar{T}_0(\bar{z}_h, \bar{\eta}_h) = \gamma\,(\bar{z}_h, \bar{\eta}_h). \tag{3.5.36}$$

It follows from (3.5.27) and (3.5.29) that

$$G_h(\bar{T}_0(\bar{z}_h, \bar{\eta}_h)) = |\gamma|^2 G_h(\bar{z}_h, \bar{\eta}_h). \tag{3.5.37}$$

On the other hand,

$$\begin{aligned} G_{i,h}(\bar{T}_0(\bar{z}_{i,h}, \bar{\eta}_{i,h})) &= \sum_{j\in N(i)} ||\gamma\bar{\eta}_{ij,h} + \beta\gamma\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \\ &= \sum_{j\in N(i)} ||\gamma\tilde{\eta}_{ij,h} + \beta\gamma\pi_{ij}\tilde{z}_{i,h}||^2_{0,\Gamma_{ij}} + \sum_{j\in N(i)} ||\gamma\hat{\eta}_{ij,h} + \beta\gamma\pi_{ij}\hat{z}_{i,h}||^2_{0,\Gamma_{ij}} \\ &= \sum_{j\in N(i)} ||-\tilde{\eta}_{ji,h} + \beta\pi_{ji}\tilde{z}_{j,h}||^2_{0,\Gamma_{ij}} + \sum_{j\in N(i)} ||-\hat{\eta}_{ji,h} + \beta\pi_{ji}\hat{z}_{j,h}||^2_{0,\Gamma_{ij}} \\ &= I_3 + I_4. \end{aligned} \tag{3.5.38}$$

To find the estimates of I_3 and I_4, we proceed in the same way of finding the estimates of I_1 and I_2 in (3.5.32) and (3.5.33), respectively. Then using (3.5.34) and (3.5.29), we obtain

$$G_h(\bar{T}_0(\bar{z}_h, \bar{\eta}_h)) = G_h(\bar{z}_h, \bar{\eta}_h) - 4\beta \sum_{i=1}^{M} a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}) \tag{3.5.39}$$

and hence,

$$|\gamma|^2 = 1 - \frac{4\beta}{G_h(\bar{z}_h, \bar{\eta}_h)} \sum_{i=1}^{M} a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}). \tag{3.5.40}$$

From (3.5.40), we conclude that $|\gamma| \leq 1$. Note that $|\gamma| = 1$ if and only if

$$a^h_{\Omega_i}(\tilde{z}_{i,h}, \tilde{z}_{i,h}) = 0 \qquad \text{and} \qquad a^h_{\Omega_i}(\hat{z}_{i,h}, \hat{z}_{i,h}) = 0 \qquad \forall i = 1, 2, \cdots, M. \tag{3.5.41}$$

Then proceeding as in the proof of Theorem 2.2.3, it is easy to show that $(\bar{z}_h, \bar{\eta}_h)$ is trivial, i.e., $(\bar{z}_h, \bar{\eta}_h) = (0, 0)$ and this leads to a contradiction as $(\bar{z}_h, \bar{\eta}_h)$ is an eigenvector of T_0. Hence, $|\gamma| < 1$ and this completes the rest of the proof. ■

3.5.2 Rate of convergence with quasi-uniformity assumption on the mesh

From (3.5.40), we obtain

$$|\gamma|^2 \leq 1 - \frac{1}{Q}, \tag{3.5.42}$$

where $Q > 1$ is such that

$$G_h(\bar{z}_h, \bar{\eta}_h) \leq 4\, Q\, \beta \sum_{i=1}^{M} a^h_{\Omega_i}(\bar{z}_{i,h}, \bar{z}_{i,h}). \tag{3.5.43}$$

Note that the estimation of Q yields the convergence rate for the iterative procedure (3.3.16)-(3.3.17).

Lemma 3.5.7 *If $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h),\ j \in N(i)$, then*

$$||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} \leq C h^{-1} |\bar{z}_{i,h}|^2_{1,h,\Omega_i}, \tag{3.5.44}$$

where C is independent of h.

Proof. Using (3.5.23), Lemma 3.5.5 and Lemma 3.5.3, we obtain (3.5.44). This completes the proof. ■

Lemma 3.5.8 *For every $\bar{v}_h \in \mathbb{C} \otimes \tilde{X}_h,\ \forall j, l \in N(i)$, then*

$$||\bar{\pi}_{il} \bar{v}_i||^2_{0,\Gamma_{il}} \leq C H |\bar{v}_i|^2_{1,h,\Omega_i} + C ||\bar{\pi}_{ij} \bar{v}_i||^2_{0,\Gamma_{ij}}, \tag{3.5.45}$$

where $\bar{\pi}_{ij}$ and $\bar{\pi}_{il}$ are the complexifications of π_{ij} and π_{il}, respectively, and the positive constant C is independent of H.

Proof. Using (3.5.23), Lemma 3.5.5 and Lemma 3.3.3, we obtain (3.5.45). This completes the proof. ■

Lemma 3.5.9 *Let $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}_0$ such that $\bar{T}_0(\bar{z}_h, \bar{\eta}_h) = \gamma(\bar{z}_h, \bar{\eta}_h)$. Then*

$$\gamma \bar{\eta}_{ij,h} = -\beta(\ \gamma \bar{\pi}_{ij} \bar{z}_{i,h}(p) - \bar{\pi}_{ji} \bar{z}_{j,h}(p)\ ) - \bar{\eta}_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.5.46}$$

Lemma 3.5.10 *Let $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}_0$ such that $\bar{T}_0(\bar{z}_h, \bar{\eta}_h) = \gamma(\bar{z}_h, \bar{\eta}_h)$. Then there is a positive constant C independent of Γ_{ij} and β such that*

$$||\bar{\pi}_{ij} \bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \leq C \beta^{-2} \left(||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + ||\bar{\eta}_{ji,h}||^2_{0,\Gamma_{ij}} \right) + C ||\bar{\pi}_{ji} \bar{z}_{j,h}||^2_{0,\Gamma_{ij}} \quad \forall j \in N(i). \tag{3.5.47}$$

Proof. From (3.5.46), we note that

$$\beta \bar{\pi}_{ij} \bar{z}_{i,h} = \bar{\eta}_{ij,h} + \gamma \bar{\eta}_{ji,h} + \beta \gamma \bar{\pi}_{ji} \bar{z}_{j,h} \quad \forall j \in N(i). \tag{3.5.48}$$

Using (3.5.23), Lemma 3.5.5, we obtain

$$\beta^2||\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \leq ||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \gamma^2||\bar{\eta}_{ji,h}||^2_{0,\Gamma_{ij}} + \beta^2\gamma^2||\bar{\pi}_{ji}\bar{z}_{j,h}||^2_{0,\Gamma_{ij}}. \tag{3.5.49}$$

We know from Theorem 3.5.1 that $|\gamma| < 1$ and this completes the rest of the proof. ■

It follows from (3.5.27), (3.5.23) and Lemma 3.5.5 that

$$\begin{aligned} G_{i,h}(\bar{z}_{i,h}, \bar{\eta}_{ij,h}) &= \sum_{j\in N(i)} ||\bar{\eta}_{ij,h} + \beta\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}} \\ &\leq 2\sum_{j\in N(i)} \left(||\bar{\eta}_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\bar{\pi}_{ij}\bar{z}_{i,h}||^2_{0,\Gamma_{ij}}\right). \end{aligned} \tag{3.5.50}$$

Below, we discuss a bound for the terms in the bracket which appear on the right hand side of (3.5.50).

Theorem 3.5.2 *Let Ω_{i^r} be the sets of subdomain in D_r, and $\beta = O(h^{-1/2}H^{-1/2})$. Further, let $(\bar{z}_h, \bar{\eta}_h) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}_0$ such that $\bar{T}_0(\bar{z}_h, \bar{\eta}_h) = \gamma(\bar{z}_h, \bar{\eta}_h)$, then*

$$\begin{aligned} ||\bar{\eta}_{i^rj,h}||^2_{0,\Gamma_{i^rj}} + \beta^2||\bar{\pi}_{i^rj}\bar{z}_{i^r,h}||^2_{0,\Gamma_{i^rj}} &\leq C_1^3h^{-1/2}H^{1/2}\beta|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}} \\ &+ C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} \\ +\cdots &+ C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}} \forall j \in N(i^r), \end{aligned} \tag{3.5.51}$$

where $C_1 \geq 1.5$ is independent of h and H, and N is the winding number.

Proof. First we consider when $r = 1$, i.e., $\Omega_{i^1} \in D_1$. Note that there is at least one face of Ω_{i^1} belonging to $\partial\Omega$ and $\bar{\pi}_{i^1l}\bar{z}_{i^1,h}$ vanishes on this face. Then using Lemma 3.5.8, we find that

$$||\bar{\pi}_{i^1j}\bar{z}_{i^1,h}||^2_{0,\Gamma_{i^1j}} \leq CH|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \quad \forall j \in N(i^1). \tag{3.5.52}$$

From Lemma 3.5.7 and (3.5.52), we arrive at

$$||\bar{\eta}_{i^1j,h}||^2_{0,\Gamma_{i^1j}} + \beta^2||\bar{\pi}_{i^1j}\bar{z}_{i^1,h}||^2_{0,\Gamma_{i^1j}} \leq C_1h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \quad \forall j \in N(i^1), \tag{3.5.53}$$

and hence (3.5.51) holds for $r = 1$. Next, we consider when $r = 2$, i.e., $\Omega_{i^2} \in D_2$. In this case, at least one face of Ω_{i^2} is common to some $\Omega_{i^1} \in D_1$. From Lemma 3.5.10, we find that

$$\begin{aligned} \beta^2||\bar{\pi}_{i^2i^1}\bar{z}_{i^2,h}||^2_{0,\Gamma_{i^2i^1}} &\leq C_1||\bar{\eta}_{i^2i^1,h}||^2_{0,\Gamma_{i^2i^1}} \\ &+ C_1\left(||\bar{\eta}_{i^1i^2,h}||^2_{0,\Gamma_{i^2i^1}} + \beta^2||\bar{\pi}_{i^1i^2}\bar{z}_{i^1,h}||^2_{0,\Gamma_{i^2i^1}}\right) \quad \forall i^1 \in N(i^2). \end{aligned} \tag{3.5.54}$$

Using Lemma 3.5.7 and substituting (3.5.53) in (3.5.54), we arrive at

$$\beta^2||\bar{\pi}_{i^2i^1}\bar{z}_{i^2,h}||^2_{0,\Gamma_{i^2i^1}} \leq C_1^2 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} + C_1^2 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}. \tag{3.5.55}$$

Substituting (3.5.55) in Lemma 3.5.8, we obtain $\forall j \in N(i^2)$

$$\begin{aligned}\beta^2||\bar{\pi}_{i^2j}\bar{z}_{i^2,h}||^2_{0,\Gamma_{i^2j}} \leq C_1 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} + C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} \\ + C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}.\end{aligned} \tag{3.5.56}$$

From Lemma 3.5.7 and (3.5.56), we arrive at

$$\begin{aligned}||\bar{\eta}_{i^2j,h}||^2_{0,\Gamma_{i^2j}} + \beta^2||\bar{\pi}_{i^2j}\bar{z}_{i^2,h}||^2_{0,\Gamma_{i^2j}} &\leq C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} \\ &+ C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}} \quad \forall j \in N(i^2),\end{aligned} \tag{3.5.57}$$

where $C_1 \geq 1.5$. Next, we consider when $r = 3$, i.e., $\Omega_{i^3} \in D_3$. That means at least one face of Ω_{i^3} is common to one of $\Omega_{i^2} \in D_2$. From Lemma 3.5.10, we find that

$$\begin{aligned}\beta^2||\bar{\pi}_{i^3i^2}\bar{z}_{i^3,h}||^2_{0,\Gamma_{i^3i^2}} &\leq C_1||\bar{\eta}_{i^3i^2,h}||^2_{0,\Gamma_{i^3i^2}} \\ &+ C_1\left(||\bar{\eta}_{i^2i^3,h}||^2_{0,\Gamma_{i^3i^2}} + \beta^2||\bar{\pi}_{i^2i^3}\bar{z}_{i^2,h}||^2_{0,\Gamma_{i^3i^2}}\right) \quad \forall i^2 \in N(i^3).\end{aligned} \tag{3.5.58}$$

Using Lemma 3.5.7 and (3.5.57) in (3.5.58), we arrive at

$$\begin{aligned}\beta^2||\bar{\pi}_{i^3i^2}\bar{z}_{i^3,h}||^2_{0,\Gamma_{i^3i^2}} \leq C_1^2 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^3,h}|^2_{1,h,\Omega_{i^3}} + C_1^4 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} \\ + C_1^4 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}.\end{aligned} \tag{3.5.59}$$

Substituting (3.5.59) in Lemma 3.5.8, we obtain $\forall j \in N(i^3)$

$$\begin{aligned}\beta^2||\bar{\pi}_{i^3j}\bar{z}_{i^3,h}||^2_{0,\Gamma_{i^3j}} \leq C_1 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^3,h}|^2_{1,h,\Omega_{i^3}} + C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^3,h}|^2_{1,h,\Omega_{i^3}} \\ + C_1^5 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} + C_1^5 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}.\end{aligned} \tag{3.5.60}$$

From Lemma 3.5.7 and 3.5.60, we arrive at $\forall j \in N(i^3)$

$$\begin{aligned}||\bar{\eta}_{i^3j,h}||^2_{0,\Gamma_{i^3j}} &+ \beta^2||\bar{\pi}_{i^3j}\bar{z}_{i^3,h}||^2_{0,\Gamma_{i^3j}} \leq C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^3,h}|^2_{1,h,\Omega_{i^3}} \\ &+ C_1^5 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^2,h}|^2_{1,h,\Omega_{i^2}} + C_1^5 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}},\end{aligned} \tag{3.5.61}$$

where $C_1 \geq 1.5$. Similarly, we can continue the argument until the entire domain is exhausted. In general, we obtain $\forall j \in N(i^r)$

$$\begin{aligned}
\beta^2||\bar{\pi}_{i^r j}\bar{z}_{i^r,h}||^2_{0,\Gamma_{i^r j}} &\leq C_1 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}} + C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}} \\
&+ C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} \\
&+ \cdots + C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \qquad (3.5.62)
\end{aligned}$$

and

$$\begin{aligned}
||\bar{\eta}_{i^r j,h}||^2_{0,\Gamma_{i^r j}} + \beta^2||\bar{\pi}_{i^r j}\bar{z}_{i^r,h}||^2_{0,\Gamma_{i^r j}} &\leq C_1^3 h^{-1/2}H^{1/2}\beta|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}} \\
&+ C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} \\
&+ \cdots + C_1^{2r-1}h^{-1/2}H^{1/2}\beta|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \qquad (3.5.63)
\end{aligned}$$

where $C_1 \geq 1.5$. This completes the rest of the proof. ■

From Theorem 3.5.2 and (3.5.50), we find that

$$\begin{aligned}
\sum_{\Omega_{i^r}\in D_r} G_{i,h}(\bar{z}_{i,h},\bar{\eta}_{ij,h}) &\leq 2\sum_{\Omega_{i^r}\in D_r}\sum_{j\in N(i^r)}\left(||\bar{\eta}_{i^r j,h}||^2_{0,\Gamma_{i^r j}} + \beta^2||\bar{\pi}_{i^r j}\bar{z}_{i^r,h}||^2_{0,\Gamma_{i^r j}}\right) \\
&\leq RC_1^3 h^{-1/2}H^{1/2}\beta\sum_{\Omega_{i^r}\in D_r}|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}} \\
&+ RC_1^{2r-1}h^{-1/2}H^{1/2}\beta\sum_{\Omega_{i^{r-1}}\in D_{r-1}}|\bar{z}_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} \\
&+ \cdots + RC_1^{2r-1}h^{-1/2}H^{1/2}\beta\sum_{\Omega_{i^1}\in D_1}|\bar{z}_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \qquad (3.5.64)
\end{aligned}$$

where R is the total number of interfaces. Now we sum up all the subdomains using (3.5.64), and arrive at

$$\begin{aligned}
G_h(\bar{z}_h,\bar{\eta}_h) = \sum_{r=1}^{N}\sum_{\Omega_{i^r}\in D_r} G_{i,h}(\bar{z}_{i,h},\bar{\eta}_{ij,h}) &\leq Rh^{-1/2}H^{1/2}\beta\sum_{r=1}^{N}\left(C_1^{2r-1}\sum_{\Omega_{i^r}\in D_r}|\bar{z}_{i^r,h}|^2_{1,h,\Omega_{i^r}}\right) \\
&\leq RC_1^{2N}h^{-1/2}H^{1/2}\beta\sum_{i=1}^{M} a^h_{\Omega_i}(\bar{z}_{i,h},\bar{z}_{i,h}). \qquad (3.5.65)
\end{aligned}$$

From the estimate (3.5.65), we obtain that (3.5.43), i.e., $4Q = RC_1^{2N}h^{-1/2}H^{1/2}$.

Theorem 3.5.3 *Assume that the parameter* $\beta = \beta_{ij} = \beta_{ji}$ *in the iterative procedure (3.3.16)-(3.3.17) satisfies* $\beta = O(h^{-1/2}H^{-1/2})$. *Then, the spectral radius* $\rho(\bar{T}_0)$ *of the operator is bounded as follows:*

$$\rho(\bar{T}_0) \leq 1 - Ch^{1/2}H^{-1/2} \equiv \gamma_0, \tag{3.5.66}$$

where $C = \dfrac{4}{RC_1^{2N}}$ *and the iteration (3.3.16)-(3.3.17) converges with an error at the* k^{th} *iteration bounded asymptotically by* $O(\gamma_0^k)$.

3.6 Numerical experiments

In this section, we have applied the present results to a model problem. The numerical implementation scheme has been performed in a sequential machine using MATLAB.

Consider the problem (3.2.1) with $f = 2[x(1-x) + y(1-y)]$. The exact solution of the problem (3.2.1) problem is given by $u = x(1-x)y(1-y)$.

Here we take $\Omega = (0,1) \times (0,1)$. We decompose the square into $[0, 3/4] \times [0,1]$ and $[3/4, 1] \times [0,1]$, with interface $\Gamma = \{3/4\} \times (0,1)$.

We triangulate the domain uniformly and mesh size is h. Here, we consider the winding number $N = 1$. We choose the initial guess $\{u_{i,h}^0, \lambda_{ij,h}^0\} = \{0, 0\}$. The stop criterion is $\|u_h^k - u_h\|_\infty \leq 10^{-4}$, where iteration number is k. We choose the relaxation parameter $\beta = O(h^{-1/2}H^{-1/2})$.

h	H	D.O.F. in Ω_1	D.O.F. in Ω_2	k = No. of Iter.	$e_h = \|u - u_h\|_{0,\Omega}$	Rate
1/8	1	138	46	6	2.13200154×10^{-4}	-
1/16	1	564	188	10	5.53207760×10^{-5}	1.9463
1/24	1	1278	426	12	2.44792188×10^{-5}	2.0108
1/32	1	2280	760	14	1.36365473×10^{-5}	2.0337
1/40	1	3570	1190	16	8.66312732×10^{-6}	2.0331
1/48	1	5148	1716	17	5.82667301×10^{-6}	2.1754

Table 3.1: L^2 error and the rate of convergence for the 2-domain case

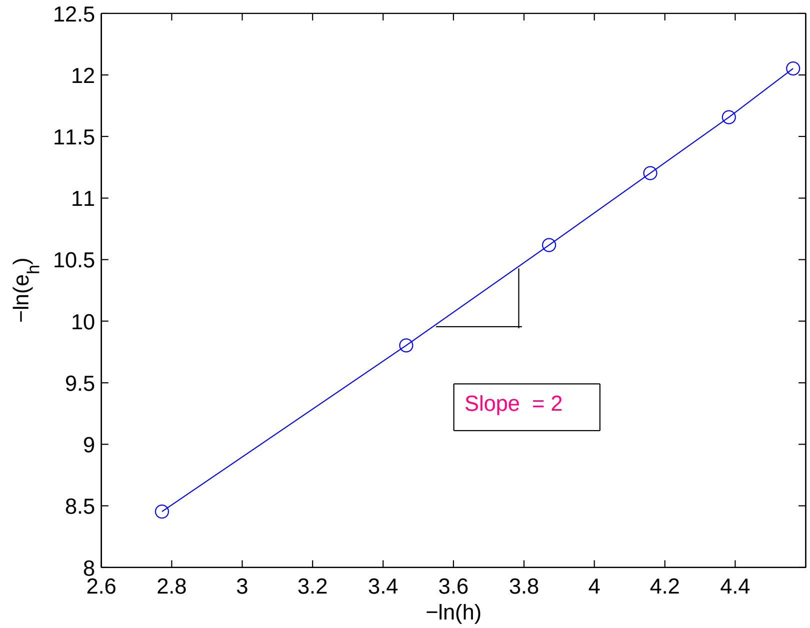

Figure 3.2: The order of convergence

In Figure 3.2, the graph of the L^2 error $\|u - u_h\|$ is plotted as a function of the discretization step $'h'$ in the $log - log$ scale. The slope of the graph provides the computed order of convergence as approximately 2.0.

In Table 3.1, the iteration number, order of convergence and L^2 error $e_h = \|u - u_h\|$ for $h = 1/8$, $h = 1/16$, $h = 1/24$, $h = 1/32$, $h = 1/40$ and $h = 1/48$ are given. The numerical result confirms our theoretical result.

3.7 The parabolic problem

In this section, we discuss the fully discrete non-conforming finite element method combined with nonoverlapping DD method using Robin-type boundary conditions across the inter-subdomains boundary at each time step for the following linear second order parabolic initial and boundary value problem. Find $u = u(x,t)$ such that

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \Omega,\ t \in (0,T], \\ u(x,t) = 0 & \text{on } \partial\Omega,\ t \in (0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.7.1}$$

where Ω is a bounded convex polygon or polyhedron in $\mathbb{R}^d$, $d = 2$ or 3 with a Lipschitz continuous, piecewise C^1 boundary $\partial\Omega$. Here the non-homogeneous term $f = f(x,t)$ and $u_0(x)$ are given functions.

In section 2.7, we have stated a completely discrete scheme which is based on backward Euler method for the multi-domain problem. The weak formulation corresponding to the multi-domain problem stated as follows (see, chapter 2, problem (2.7.2)-(2.7.4)): Given $f \in L^2(Q_T)$ and $U^{n-1} \in X_h$, find $U^n = (U_1^n, \cdots, U_M^n) \in X_h = \prod_{i=1}^{M} X_{i,h}$ and $\lambda_h^n \in Y_h = \prod_{i=1}^{M} \prod_{i<j\in N(i)} Y_{i,h}$ for $n = 1, 2, 3, \cdots, N$, such that

$$\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}, v_h\right) + a^h(U^n, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}^n \left[\pi\, v_h\right] ds = (f^n, v_h) \ \forall v_h \in X_h, \tag{3.7.2}$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} \left[\pi\, U^n\right] \mu_h \, ds = 0 \qquad \forall \mu_h \in Y_h, \tag{3.7.3}$$

and

$$U^0 = u_{0,h}. \tag{3.7.4}$$

Let us formulate an iterative version of (3.7.2)-(3.7.3). Consider the Lagrange multiplier to be $\lambda_{ij,h}^n$ as seen from Ω_i and $\lambda_{ji,h}^n$ as seen from Ω_j. Then, the iterative procedure is to compute $\{U_i^{n,k}, \lambda_{ij,h}^{n,k}\} \in X_{i,h} \times Y_{ij,h}$ recursively as the solution of

$$\begin{aligned} \left(\frac{U_i^{n,k} - U_i^{n-1}}{\Delta t}, v_h\right) + a_{\Omega_i}^h(U_i^{n,k}, v_h) + \sum_{j\in N(i)} \beta_{ij} \int_{\Gamma_{ij}} \pi_{ij} U_i^{n,k}\, \pi_{ij} v_h \, ds = (f^n, v_h)_{\Omega_i} \\ + \sum_{j\in N(i)} \beta_{ji} \int_{\Gamma_{ij}} \pi_{ji} U_{j,h}^{n,k-1}\, \pi_{ij} v_h \, ds - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ji,h}^{n,k-1} \pi_{ij} v_h \, ds \qquad \forall v_h \in X_{i,h}, \end{aligned} \tag{3.7.5}$$

and

$$\lambda_{ij,h}^{n,k} = -(\ \beta_{ij}\pi_{ij}U_i^{n,k}(p) - \beta_{ji}\pi_{ji}U_j^{n,k-1}(p)\ ) - \lambda_{ji,h}^{n,k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \quad (3.7.6)$$

with initial guess $\{U_i^{n,0}, \lambda_{ij,h}^{n,0}, \lambda_{ji,h}^{n,0}\} \in \{X_{i,h}, Y_{ij,h}, Y_{ji,h}\}$ as given in the time level t_{n-1}.

3.7.1 Convergence of iterative scheme

In this subsection, we discuss the convergence of the iteration defined by (3.7.5)-(3.7.6). From (3.7.3), we note that in each subdomain Ω_i,

$$\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}, v_h\right) + a_{\Omega_i}^h(U_i^n, v_h) - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}^n\, \pi_{ij}\, v_h\, ds = (f^n, v_h)\ \forall v_h \in X_{i,h}. \quad (3.7.7)$$

Since $\lambda_{ij,h}^n = -\lambda_{ji,h}^n$, then form (3.7.3), we obtain

$$\lambda_{ij,h}^n = -\lambda_{ji,h}^n - \beta(\pi_{ij}\, U_i^n(p) - \pi_{ji}\, U_j^n(p)). \quad (3.7.8)$$

Set

$$e_{i,h}^{n,k} = U_i^{n,k} - U_i^n,\ \mu_{ij,h}^{n,k} = \lambda_{ij,h}^{n,k} - \lambda_{ij,h}^n \text{ and } \mu_{ji,h}^{n,k} = \lambda_{ji,h}^{n,k} - \lambda_{ji,h}^n. \quad (3.7.9)$$

Then, subtracting (3.7.7) from (3.7.5) and (3.7.8) from (3.7.6) with $\beta = \beta_{ij} = \beta_{ji}$, lead to the following equations:

$$\left(\frac{e_{i,h}^{n,k}}{\Delta t}, v_h\right) + a_{\Omega_i}^h(e_{i,h}^{n,k}, v_h) - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda_{ij,h}^{n,k}\, \pi_{ij}\, v_h\, ds = 0 \qquad \forall v_h \in X_{i,h} \quad (3.7.10)$$

and

$$\lambda_{ij,h}^{n,k} = -\beta\left(\pi_{ij}\, e_{i,h}^{n,k}(p) - \pi_{ji}\, e_{j,h}^{n,k-1}(p)\right) - \lambda_{ji,h}^{n,k-1}. \quad (3.7.11)$$

Setting $v_h = e_{i,h}^{n,k}$ in (3.7.10), we arrive at

$$\frac{1}{\Delta t}||e_{i,h}^{n,k}||_{0,\Omega_i}^2 + a_{\Omega_i}^h(e_{i,h}^{n,k}, e_{i,h}^{n,k}) = \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu_{ij,h}^{n,k}\pi_{ij}e_{i,h}^{n,k} ds. \quad (3.7.12)$$

For analyzing convergence, we now define

$$E_{i,h}^{n,k} = E_{i,h}(e_{i,h}^{n,k}, \mu_{ij,h}^{n,k}) = \sum_{j\in N(i)} ||\mu_{ij,h}^{n,k} + \beta\pi_{ij}e_{i,h}^{n,k}||_{0,\Gamma_{ij}}^2, \quad (3.7.13)$$

and

$$E_h^{n,k} = E_h(e_h^{n,k}, \mu_h^{n,k}) = \sum_{i=1}^{M} E_{i,h}^{n,k} = \sum_{i=1}^{M} E_{i,h}(e_{i,h}^{n,k}, \mu_{ij,h}^{n,k}). \tag{3.7.14}$$

Lemma 3.7.1 *Let $E_h^{n,k}$ and $E_{i,h}^{n,k}$ be defined, respectively, by (3.7.14) and (3.7.13). Then following identity*

$$E_h^{n,k} = E_h^{n,k-1} - 4\beta \sum_{i=1}^{M} \left(a_{\Omega_i}^h(e_{i,h}^{n,k-1}, e_{i,h}^{n,k-1}) + \frac{1}{\Delta t}||e_{i,h}^{n,k-1}||_{0,\Omega_i}^2 \right) \tag{3.7.15}$$

holds true.

Proof. From (3.7.13) and (3.7.12), we obtain

$$\begin{aligned} E_{i,h}^{n,k} &= \sum_{j\in N(i)} \left(||\mu_{ij,h}^{n,k}||_{0,\Gamma_{ij}}^2 + \beta^2 ||\pi_{ij} e_{i,h}^{n,k}||_{0,\Gamma_{ij}}^2 \right) + 2\beta \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu_{ij,h}^{n,k} \pi_{ij} e_{i,h}^{n,k} ds \\ &= \sum_{j\in N(i)} \left(||\mu_{ij,h}^{n,k}||_{0,\Gamma_{ij}}^2 + \beta^2 ||\pi_{ij} e_{i,h}^{n,k}||_{0,\Gamma_{ij}}^2 \right) + 2\beta \left(a_{\Omega_i}^h(e_{i,h}^{n,k}, e_{i,h}^{n,k}) + \frac{1}{\Delta t}||e_{i,h}^{n,k}||_{0,\Omega_i}^2 \right). \end{aligned} \tag{3.7.16}$$

Then, from (3.7.11), (3.7.13) and (3.7.16), we arrive at

$$\begin{aligned} E_{i,h}^{n,k} &= \sum_{j\in N(i)} ||\mu_{ij,h}^{n,k} + \beta\pi_{ij} e_{i,h}^{n,k}||_{0,\Gamma_{ij}}^2 = \sum_{j\in N(i)} ||-\mu_{ji,h}^{n,k-1} + \beta\pi_{ji} e_{j,h}^{n,k-1}||_{0,\Gamma_{ij}}^2 \\ &= \sum_{j\in N(i)} \left(||\mu_{ij,h}^{n,k-1}||_{0,\Gamma_{ij}}^2 + \beta^2 ||\pi_{ij} e_{i,h}^{n,k-1}||_{0,\Gamma_{ij}}^2 \right) - 2\beta \sum_{j\in N(i)} \int_{\Gamma_{ij}} \mu_{ij,h}^{n,k-1} \pi_{ij} e_{i,h}^{n,k-1} ds \\ &= \sum_{j\in N(i)} \left(||\mu_{ij,h}^{n,k-1}||_{0,\Gamma_{ij}}^2 + \beta^2 ||\pi_{ij} e_{i,h}^{n,k-1}||_{0,\Gamma_{ij}}^2 \right) \\ &\qquad - 2\beta \left(a_{\Omega_i}^h(e_{i,h}^{n,k-1}, e_{i,h}^{n,k-1}) + \frac{1}{\Delta t}||e_{i,h}^{n,k-1}||_{0,\Omega_i}^2 \right) \\ &= E_{i,h}^{n,k-1} - 4\beta \left(a_{\Omega_i}^h(e_{i,h}^{n,k-1}, e_{i,h}^{n,k-1}) + \frac{1}{\Delta t}||e_{i,h}^{n,k-1}||_{0,\Omega_i}^2 \right), \end{aligned}$$

and this completes the proof. ■

Theorem 3.7.1 *Let $(U_i^n, \lambda_{ij,h}^n)$, $i = 1, 2, \cdots, M$, be the solutions of the problem (3.7.7)-(3.7.8) and let $(U_i^{n,k}, \lambda_{ij,h}^{n,k})$ be the solutions of the discrete iterative problem (3.7.5) and*

(3.7.6) at iterative step k. Then, for any initial guess $\{U_i^{n,0}, \lambda_{ij,h}^{n,0}, \lambda_{ji,h}^{n,0}\} \in \{X_{i,h}, Y_{ij,h}, Y_{ji,h}\}$ $\forall j \in N(i)$*, the iterative method converges in the sense that*

$$||U^{n,k} - U^n||_{1,h} = \left(\sum_{i=1}^{M} ||U_i^{n,k} - U_i^n||^2_{1,h,\Omega_i} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty, \tag{3.7.17}$$

and

$$||\lambda_h^{n,k} - \lambda_h^n||_0 = \left(\sum_{i=1}^{M} \sum_{j \in N(i)} ||\lambda_{ij,h}^{n,k} - \lambda_{ij,h}^n||^2_{0,\Gamma_{ij}} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty. \tag{3.7.18}$$

For a proof of Theorem 3.7.1, we refer to Theorem 3.4.1.

3.7.2 Spectral radius

Let $T_f^n : \tilde{X}_h \times \tilde{Y}_h \to \tilde{X}_h \times \tilde{Y}_h$ be an mapping such that for any $(w_h^n, \theta_h^n) \in \tilde{X}_h \times \tilde{Y}_h$, $(z_h^n, \eta_h^n) \equiv T_f^n(w_h^n, \theta_h^n)$ is the solution, for all i, of

$$\begin{aligned} \frac{1}{\Delta t}(z_{i,h}^n, v_h) + a_{\Omega_i}^h(z_{i,h}^n, v_h) + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z_{i,h}^n \pi_{ij} v_h \, ds = (f^n, v_h)_{\Omega_i} \\ + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ji} w_{j,h}^n \pi_{ij} v_h \, ds - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \theta_{ji,h}^n \pi_{ij} v_h \, ds \qquad \forall v_h \in X_{i,h}, \end{aligned} \tag{3.7.19}$$

$$\eta_{ij,h}^n = -\beta(\, \pi_{ij} z_{i,h}^n(p) - \pi_{ji} w_{j,h}^n(p) \,) - \theta_{ji,h}^n \qquad \forall x \in \Gamma_{ij}, \; j \in N(i), \tag{3.7.20}$$

where $z_{i,h}^n = z^n_{h|_{\Omega_i}}$, $w_{i,h}^n = w^n_{h|_{\Omega_i}}$, $\eta_{ij,h}^n = \eta^n_{h|_{\Gamma_{ij}}}$ and $\theta_{ij,h}^n = \theta^n_{h|_{\Gamma_{ij}}}$. Since the operator T_f^n is linear, we can now split the operator T_f^n as $T_f^n(w_h^n, \theta_h^n) = T_0^n(w_h^n, \theta_h^n) + T_f^n(0,0)$ where the operators T_0^n and T_f^n are defined as follows : Given (w_h^n, θ_h^n), $(z_h^{n,\star}, \eta_h^{n,\star}) = T_0^n(w_h^n, \theta_h^n)$ satisfies for all i,

$$\begin{aligned} \frac{1}{\Delta t}(z_{i,h}^{n,\star}, v_h) + a_{\Omega_i}^h(z_{i,h}^{n,\star}, v_h) + \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z_{i,h}^{n,\star} \pi_{ij} v_h \, ds = \sum_{j \in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ji} w_{j,h}^n \pi_{ij} v_h \, ds \\ - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \theta_{ji,h}^n \pi_{ij} v_h \, ds \qquad \forall v_h \in X_{i,h}, \end{aligned} \tag{3.7.21}$$

$$\eta_{ij,h}^{n,\star} = -\beta(\, \pi_{ij} z_{i,h}^{n,\star}(p) - \pi_{ji}^n w_{j,h}(p) \,) - \theta_{ji,h}^n \qquad \forall x \in \Gamma_{ij}, \; j \in N(i), \tag{3.7.22}$$

and given $(z_h^{n,o}, \eta_h^{n,o}) = T_f^n(0,0)$ satisfies for all i,

$$\frac{1}{\Delta t}(z_{i,h}^{n,o}, v_h) + a_{\Omega_i}^h(z_{i,h}^{n,o}, v_h) + \sum_{j\in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z_{i,h}^{n,o} \pi_{ij} v_h \, ds = (f^n, v_h)_{\Omega_i} \qquad \forall v_h \in X_{i,h}, \tag{3.7.23}$$

$$\eta_{ij,h}^{n,o} = -\beta \; \pi_{ij} z_{i,h}^{n,o}(p) \qquad \forall x \in \Gamma_{ij}, \; j \in N(i). \tag{3.7.24}$$

Then $(z_h^n, \eta_h^n) = (z_h^{n,\star}, \eta_h^{n,\star}) + (z_h^{n,o}, \eta_h^{n,o})$.

Lemma 3.7.2 *The pair $(z_h^n, \eta_h^n) \in \tilde{X}_h \times \tilde{Y}_h$ is a solution, for all i, of*

$$\frac{1}{\Delta t}(z_{i,h}^{n}, v_h) + a_{\Omega_i}^h(z_{i,h}^{n}, v_h) + \sum_{j\in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ij} z_{i,h}^{n} \pi_{ij} v_h ds = (f^n, v)_{\Omega_i}$$
$$+ \sum_{j\in N(i)} \beta \int_{\Gamma_{ij}} \pi_{ji} z_{j,h}^{n} \pi_{ij} v_h ds - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \eta_{ji,h}^n \pi_{ij} v_h ds \qquad \forall v_h \in X_{i,h}, \tag{3.7.25}$$

$$\eta_{ij,h}^n = -\beta(\; \pi_{ij} z_{i,h}^n(p) - \pi_{ji} z_{j,h}^n(p) \;) - \eta_{ji,h}^n \qquad \forall x \in \Gamma_{ij}, \; j \in N(i), \tag{3.7.26}$$

where $\eta_{ij,h}^n = -\eta_{ji,h}^n$ if and only if it is a fixed point of the operator T_f^n.

It is easy to check that for each i any solution of (3.7.7)-(3.7.8) is a fixed point of T_f^n and conversely a fixed point of T_f^n is a solution of (3.7.7)-(3.7.8).

Lemma 3.7.3 *Let (u_h^n, λ_h^n) be a fixed point of T_f^n. Then $\pi_{ij} u_{i,h}^n(p) = \pi_{ji} u_{j,h}^n(p)$ and $\lambda_{ij,h}^n = -\lambda_{ji,h}^n$ for all Γ_{ij}.*

Note that the operator $T_f^n(z_h^n, \eta_h^n)$ can be decomposed into a sum of two operators $T_0^n(z_h^n, \eta_h^n)$ and $T_f^n(0,0)$. Then then

$$(z_h^n, \eta_h^n) = T_f^n(w_h^n, \theta_h^n) = T_0^n(w_h^n, \theta_h^n) + T_f^n(0,0). \tag{3.7.27}$$

The fixed point (z_h^n, η_h^n) of T_f^n that is $T_f^n(z_h^n, \eta_h^n) = (z_h^n, \eta_h^n)$ is a solution of

$$(I - T_0^n)(z_h^n, \eta_h^n) = T_f^n(0,0). \tag{3.7.28}$$

Lemma 3.7.4 *Let (u_h^n, λ_h^n) be a fixed point of T_f^n. Then from (3.7.27), we write*

$$(u_h^n, \lambda_h^n) = T_f^n(u_h^n, \lambda_h^n) = T_0^n(u_h^n, \lambda_h^n) + T_f^n(0,0). \tag{3.7.29}$$

Moreover,

$$(e_h^{n,k}, \mu_h^{n,k}) = T_0^n(e_h^{n,k-1}, \mu_h^{n,k-1}). \tag{3.7.30}$$

If (z_h^n, η_h^n) is a fixed point of T_0^n, then from (3.7.27), the operator T_0^n satisfies

$$\frac{1}{\Delta t}(z_{i,h}^n, v_h) + a_{\Omega_i}^h(z_{i,h}^n, v_h) - \sum_{j \in N(i)} \int_{\Gamma_{ij}} \eta_{ij,h}^n \pi_{ij} v_h ds = 0 \qquad \forall v_h \in X_{i,h}, \tag{3.7.31}$$

$$\eta_{ij,h}^n = -\beta(\ \pi_{ij} z_{i,h}^n(p) - \pi_{ji} z_{j,h}^n(p)\ ) - \eta_{ji,h}^n \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.7.32}$$

Lemma 3.7.5 *Let $(z_h^n, \eta_h^n) \in \tilde{X}_h \times \tilde{Y}_h$ be the solution of (3.7.31) and (3.7.32). Then*

$$||\eta_{ij,h}^n||_{0,\Gamma_{ij}} \leq C h^{-1/2} |z_{i,h}^n|_{1,h,\Omega_i} + C\frac{h^{1/2}}{\Delta t}||z_{i,h}^n||_{0,\Omega_i} \quad \forall j \in N(i). \tag{3.7.33}$$

Proof. Now choose $v_h = S_{ij}\eta_{ij,h}^n$ in (3.7.31), and using Lemma 2.2.6, we obtain

$$\begin{aligned} ||\eta_{ij,h}^n||_{0,\Gamma_{ij}}^2 &= a_{\Omega_i}^h(z_{i,h}^n, S_{ij}\eta_{ij,h}^n) + \frac{1}{\Delta t}(z_{i,h}^n, S_{ij}\eta_{ij,h}^n) \\ &\leq |z_{i,h}^n|_{1,h,\Omega_i} |S_{ij}\eta_{ij,h}^n|_{1,h,\Omega_i} + \frac{1}{\Delta t}||z_{i,h}^n||_{0,\Omega_i} |S_{ij}\eta_{ij,h}^n|_{0,\Omega_i} \\ &\leq C h^{-1/2} |z_{i,h}^n|_{1,h,\Omega_i} ||\eta_{ij,h}^n||_{0,\Gamma_{ij}} + C\frac{h^{1/2}}{\Delta t}||z_{i,h}^n||_{0,\Omega_i} ||\eta_{ij,h}^n||_{0,\Gamma_{ij}}. \end{aligned} \tag{3.7.34}$$

This completes the rest of the proof. ■

Now next aim to find the spectral radius of T_0^n.

Here $\tilde{X}_h \times \tilde{Y}_h$ is a real linear space and T_0^n is a real linear operator. In general, the spectral radius formula does not hold in the real case. So the complexification of a real linear space and a real linear operator is necessary. Now, we recall the linear operator T_0^n defined in (3.5.13) and the linear space $\tilde{X}_h \times \tilde{Y}_h$ defined in (3.5.1). Our main idea to find $||T_0^{n,k}||$, which is dominated by $\rho(\bar{T}_0^n)$, where $\rho(\bar{T}_0^n)$ is the spectral radius of $\bar{T}_0^n$. The next lemma shows that the relation between $||T_0^{n,k}||$ and $\rho(\bar{T}_0^n)$.

Lemma 3.7.6 *Let $\tilde{X}_h \times \tilde{Y}_h$ be equipped with an inner-product and*

$$\rho(\bar{T}_0^n) \leq 1 - R, \qquad R \in (0,1). \tag{3.7.35}$$

Then for all positive integer number k, there is a constant C independent of k such that

$$||T_0^{n,k}|| \leq C(1 - R/2)^k. \tag{3.7.36}$$

Let $(\bar{z}_h^n, \bar{\eta}_h^n) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$, i.e.,

$$(\bar{z}_h^n, \bar{\eta}_h^n) = (\tilde{z}_h^n, \tilde{\eta}_h^n) + \sqrt{(-1)}(\hat{z}_h^n, \hat{\eta}_h^n), \tag{3.7.37}$$

where $(\tilde{z}_h^n, \tilde{\eta}_h^n), (\hat{z}_h^n, \hat{\eta}_h^n) \in \tilde{X}_h \times \tilde{Y}_h$. Using the Lemma 1.2.12, we obtain the following identity.

Lemma 3.7.7 *Let $(\bar{z}_h^n, \bar{\eta}_h^n) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$, and $(\tilde{z}_h^n, \tilde{\eta}_h^n), (\hat{z}_h^n, \hat{\eta}_h^n) \in \tilde{X}_h \times \tilde{Y}_h$. Then*

$$|\bar{z}_{i,h}^n|_{1,h,\Omega_i}^2 = |\tilde{z}_{i,h}^n|_{1,h,\Omega_i}^2 + |\hat{z}_{i,h}^n|_{1,h,\Omega_i}^2 \tag{3.7.38}$$

$$||\bar{\eta}_{ij,h}^n||_{0,ij}^2 = ||\tilde{\eta}_{ij,h}^n||_{0,ij}^2 + ||\hat{\eta}_{ij,h}^n||_{0,ij}^2, \tag{3.7.39}$$

and

$$||\bar{\pi}_{ij}\bar{z}_{i,h}^n||_{0,ij}^2 = ||\pi_{ij}\tilde{z}_{i,h}^n||_{0,ij}^2 + ||\pi_{ij}\hat{z}_{i,h}^n||_{0,ij}^2 \tag{3.7.40}$$

where $\bar{\pi}_{ij}$ is the complexification of π_{ij}.

For the sake of convenience, let us define another notation $G_{i,h}^n$ similar to $E_{i,h}^n$, but both having the same property, where each $G_{i,h}^n$ acts on complex values and each $E_{i,h}^n$ acts on real values.

$$G_{i,h}^n = G_{i,h}(\bar{z}_{i,h}^n, \bar{\eta}_{ij,h}^n) = \sum_{j \in N(i)} ||\bar{\eta}_{ij,h}^n + \beta\bar{\pi}_{ij}\bar{z}_{i,h}^n||_{0,\Gamma_{ij}}^2, \tag{3.7.41}$$

and

$$G_h^n = G_h(\bar{z}_h^n, \bar{\eta}_h^n) = \sum_{i=1}^{M} G_{i,h}^n = \sum_{i=1}^{M} G_{i,h}(\bar{z}_{i,h}^n, \bar{\eta}_{ij,h}^n). \tag{3.7.42}$$

Lemma 3.7.8 *Let G_h^n and $G_{i,h}^n$ be defined, respectively, by (3.7.41) and (3.7.42). Then the following identity holds :*

$$G_h(\bar{z}_h^n, \bar{\eta}_h^n) = \sum_{i=1}^{M} \sum_{j \in N(i)} \left(||\bar{\eta}_{ij,h}^n||_{0,\Gamma_{ij}}^2 + \beta^2 ||\bar{\pi}_{ij}\bar{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 \right)$$
$$+2\beta \sum_{i=1}^{M} \left(a_{\Omega_i}^h(\bar{z}_{i,h}^n, \bar{z}_{i,h}^n) + \frac{1}{\Delta t}||\bar{z}_{i,h}^n||_{0,\Omega_i}^2 \right). \tag{3.7.43}$$

Theorem 3.7.2 *Let $\rho(\bar{T}_0^n)$ be the spectral radius of $\bar{T}_0^n$. Then*

$$\rho(\bar{T}_0^n) < 1. \tag{3.7.44}$$

Proof. Let γ be an eigenvalue of $\bar{T}_0^n$ and let $(\bar{z}_h^n, \bar{\eta}_h^n) \neq (0,0)$ be its corresponding eigenvector. Then

$$\bar{T}_0^n(\bar{z}_h^n, \bar{\eta}_h^n) = \gamma\,(\bar{z}_h^n, \bar{\eta}_h^n). \tag{3.7.45}$$

It follows from (3.7.41) and (3.7.43) that

$$G_h(\bar{T}_0^n(\bar{z}_h^n, \bar{\eta}_h^n)) = |\gamma|^2 G_h(\bar{z}_h^n, \bar{\eta}_h^n). \tag{3.7.46}$$

In the other hand,

$$\begin{aligned} G_{i,h}(\bar{T}_0^n(\bar{z}_{i,h}^n, \bar{\eta}_{i,h}^n)) &= \sum_{j\in N(i)} ||\gamma\bar{\eta}_{ij,h}^n + \beta\gamma\bar{\pi}_{ij}\bar{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 \\ &= \sum_{j\in N(i)} ||\gamma\tilde{\eta}_{ij,h}^n + \beta\gamma\pi_{ij}\tilde{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 + \sum_{j\in N(i)} ||\gamma\hat{\eta}_{ij,h}^n + \beta\gamma\pi_{ij}\hat{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 \\ &= \sum_{j\in N(i)} ||-\tilde{\eta}_{ji,h}^n + \beta\pi_{ji}\tilde{z}_{j,h}^n||_{0,\Gamma_{ij}}^2 + \sum_{j\in N(i)} ||-\hat{\eta}_{ji,h}^n + \beta\pi_{ji}\hat{z}_{j,h}^n||_{0,\Gamma_{ij}}^2 \\ &= I_5 + I_6. \end{aligned} \tag{3.7.47}$$

To find the estimates of I_5 and I_6, we proceed in the same way as finding the estimates of I_1 and I_2 in (3.5.32) and (3.5.32), respectively. By the simple calculation, we obtain

$$G_h(\bar{T}_0^n(\bar{z}_h^n, \bar{\eta}_h^n)) = G_h(\bar{z}_h^n, \bar{\eta}_h^n) - 4\beta\sum_{i=1}^{M}\left(a_{\Omega_i}^h(\bar{z}_{i,h}^n, \bar{z}_{i,h}^n) + \frac{1}{\Delta t}||\bar{z}_{i,h}^n||_{0,\Omega_i}^2\right). \tag{3.7.48}$$

Therefore,

$$|\gamma|^2 = 1 - \frac{4\beta}{G_h(\bar{z}_h^n, \bar{\eta}_h^n)}\sum_{i=1}^{M}\left(a_{\Omega_i}^h(\bar{z}_{i,h}, \bar{z}_{i,h}) + \frac{1}{\Delta t}||\bar{z}_{i,h}^n||_{0,\Omega_i}^2\right). \tag{3.7.49}$$

From (3.7.49), we concluded that $|\gamma| \leq 1$. Note that $|\gamma| = 1$ if and only if $\forall i = 1, 2, \cdots, M$

$$a_{\Omega_i}^h(\tilde{z}_{i,h}^n, \tilde{z}_{i,h}^n) + \frac{1}{\Delta t}||\tilde{z}_{i,h}^n||_{0,\Omega_i}^2 = 0 \qquad \text{and} \qquad a_{\Omega_i}^h(\hat{z}_{i,h}^n, \hat{z}_{i,h}^n) + \frac{1}{\Delta t}||\hat{z}_{i,h}^n||_{0,\Omega_i}^2 = 0. \tag{3.7.50}$$

Then using the argument of proof of Theorem 2.2.3, it is easy to show that $(\bar{z}_h^n, \bar{\eta}_h^n)$ is trivial, i.e., $(\bar{z}_h^n, \bar{\eta}_h^n) = (0,0)$ and this leads to a contradiction as $(\bar{z}_h^n, \bar{\eta}_h^n)$ is an eigenvector of T_0^n. Hence, $|\gamma| < 1$ and this completes the rest of the proof. ■

3.7.3 Rate of convergence

From (3.7.49), we obtain

$$|\gamma|^2 \leq 1 - \frac{1}{Q_1}, \tag{3.7.51}$$

where $Q_1 > 1$ is such that

$$G_h(\bar{z}_h^n, \bar{\eta}_h^n) \leq 4\, Q_1\, \beta \sum_{i=1}^{M} \left(a_{\Omega_i}^h(\bar{z}_{i,h}^n, \bar{z}_{i,h}^n) + \frac{1}{\Delta t} ||\bar{z}_{i,h}^n||_{0,\Omega_i}^2 \right). \tag{3.7.52}$$

Note that estimation of Q_1 with yields the convergence rate for the iterative procedure (3.7.5) and (3.7.6).

Lemma 3.7.9 *If $(\bar{z}_h^n, \bar{\eta}_h^n) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h),\ j \in N(i)$, then*

$$||\bar{\eta}_{ij,h}^n||_{0,\Gamma_{ij}} \leq Ch^{-1/2} |\bar{z}_{i,h}^n|_{1,h,\Omega_i} + C \frac{h^{1/2}}{\Delta t} ||\bar{z}_{i,h}^n||_{0,\Omega_i}, \tag{3.7.53}$$

where C is independent of h.

Proof. Using (3.7.37), Lemma 3.7.7 and Lemma 3.7.5, we obtain (3.7.53). This completes the proof. ■

Lemma 3.7.10 *Let $(\bar{z}_h^n, \bar{\eta}_h^n) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}_0^n$ such that $\bar{T}_0^n(\bar{z}_h^n, \bar{\eta}_h^n) = \gamma(\bar{z}_h^n, \bar{\eta}_h^n)$. Then*

$$\gamma \bar{\eta}_{ij,h}^n = -\beta(\ \gamma \bar{\pi}_{ij} \bar{z}_{i,h}^n(p) - \bar{\pi}_{ji} \bar{z}_{j,h}^n(p)\ ) - \bar{\eta}_{ji,h}^n \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{3.7.54}$$

Lemma 3.7.11 *Let $(\bar{z}_h^n, \bar{\eta}_h^n) \in \mathbb{C} \otimes (\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}_0^n$ such that $\bar{T}_0^n(\bar{z}_h^n, \bar{\eta}_h^n) = \gamma(\bar{z}_h^n, \bar{\eta}_h^n)$. Then there is a positive constant C independent of Γ_{ij} and β such that*

$$||\bar{\pi}_{ij} \bar{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 \leq C\beta^{-2} \left(||\bar{\eta}_{ij,h}^n||_{0,\Gamma_{ij}}^2 + ||\bar{\eta}_{ji,h}^n||_{0,\Gamma_{ij}}^2 \right) + C ||\bar{\pi}_{ji} \bar{z}_{j,h}^n||_{0,\Gamma_{ij}}^2 \quad \forall j \in N(i). \tag{3.7.55}$$

Proof. From (3.7.54), we note that

$$\beta \bar{\pi}_{ij} \bar{z}_{i,h}^n = \bar{\eta}_{ij,h}^n + \gamma \bar{\eta}_{ji,h}^n + \beta \gamma \bar{\pi}_{ji} \bar{z}_{j,h}^n \quad \forall j \in N(i). \tag{3.7.56}$$

Using (3.7.37), Lemma 3.7.7, we obtain

$$\beta^2 ||\bar{\pi}_{ij} \bar{z}_{i,h}^n||_{0,\Gamma_{ij}}^2 \leq ||\bar{\eta}_{ij,h}^n||_{0,\Gamma_{ij}}^2 + \gamma^2 ||\bar{\eta}_{ji,h}^n||_{0,\Gamma_{ij}}^2 + \beta^2 \gamma^2 ||\bar{\pi}_{ji} \bar{z}_{j,h}^n||_{0,\Gamma_{ij}}^2. \tag{3.7.57}$$

We know from Theorem 3.7.2 that $|\gamma| < 1$ and this completes the rest of the proof. ■

It follows from (3.7.41), (3.7.37) and Lemma 3.7.7 that

$$\begin{aligned} G_{i,h}(\bar{z}^n_{i,h}, \bar{\eta}_{ij,h}) &= \sum_{j\in N(i)} ||\bar{\eta}^n_{ij,h} + \beta\bar{\pi}_{ij}\bar{z}^n_{i,h}||^2_{0,\Gamma_{ij}} \\ &\le 2\sum_{j\in N(i)} \left(||\bar{\eta}^n_{ij,h}||^2_{0,\Gamma_{ij}} + \beta^2||\bar{\pi}_{ij}\bar{z}^n_{i,h}||^2_{0,\Gamma_{ij}}\right). \end{aligned} \tag{3.7.58}$$

Below, we discuss a bound for the terms in the bracket which appear in the right-hand side of (3.7.58).

Theorem 3.7.3 *Let Ω_{i^r} be the sets of subdomain in D_r, and $\beta = O(h^{-1/2}H^{-1/2})$. Further, let $(\bar{z}^n_h, \bar{\eta}^n_h) \in \mathbb{C}\otimes(\tilde{X}_h \times \tilde{Y}_h)$ be an eigenvector of $\bar{T}^n_0$ such that $\bar{T}^n_0(\bar{z}^n_h, \bar{\eta}^n_h) = \gamma(\bar{z}^n_h, \bar{\eta}^n_h)$, then*

$$\begin{aligned} &||\bar{\eta}^n_{i^rj,h}||^2_{0,\Gamma_{i^rj}} + \beta^2||\bar{\pi}_{i^rj}\bar{z}^n_{i^r,h}||^2_{0,\Gamma_{i^rj}} \\ &\qquad \le C_1^3 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}} + \frac{1}{\Delta t}||\bar{z}^n_{i^r,h}||^2_{0,\Omega_{i^r}}\right) \\ &\qquad +C_1^{2r-1}h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} + \frac{1}{\Delta t}||\bar{z}^n_{i^{r-1},h}||^2_{0,\Omega_{i^{r-1}}}\right) \\ &\qquad +\cdots + C_1^{2r-1}h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right)\forall j\in N(i^r), \end{aligned} \tag{3.7.59}$$

where $C_1 \ge 1.5$ is independent of h and H, and N is the winding number.

Proof. First we consider when $r = 1$, i.e., $\Omega_{i^1} \in D_1$. Note that there is at least one face of Ω_{i^1} belonging to $\partial\Omega$ and $\bar{\pi}_{i^1l}\bar{z}^n_{i^1,h}$ vanishes on this face. Then using Lemma 3.5.8, we find that

$$||\bar{\pi}_{i^1j}\bar{z}^n_{i^1,h}||^2_{0,\Gamma_{i^1j}} \le CH|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}}, \quad \forall j\in N(i^1). \tag{3.7.60}$$

From Lemma 3.7.9 and (3.7.60), we arrive at

$$\begin{aligned} &||\bar{\eta}^n_{i^1j,h}||^2_{0,\Gamma_{i^1j}} + \beta^2||\bar{\pi}_{i^1j}\bar{z}^n_{i^1,h}||^2_{0,\Gamma_{i^1j}} \\ &\quad \le C_1h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right), \quad \forall j\in N(i^1), \end{aligned} \tag{3.7.61}$$

and hence (3.7.59) holds for $r = 1$. Next, we consider when $r = 2$, i.e., $\Omega_{i^2} \in D_2$. In this case, at least one face of Ω_{i^2} is common to some $\Omega_{i^1} \in D_1$. From Lemma 3.7.11, we find that

$$\begin{aligned}\beta^2||\bar{\pi}_{i^2i^1}\bar{z}^n_{i^2,h}||^2_{0,\Gamma_{i^2i^1}} &\le C_1||\bar{\eta}^n_{i^2i^1,h}||^2_{0,\Gamma_{i^2i^1}} \\ &+ C_1\left(||\bar{\eta}^n_{i^1i^2,h}||^2_{0,\Gamma_{i^2i^1}} + \beta^2||\bar{\pi}_{i^1i^2}\bar{z}^n_{i^1,h}||^2_{0,\Gamma_{i^2i^1}}\right) \quad \forall i^1 \in N(i^2). \quad (3.7.62)\end{aligned}$$

Using Lemma 3.7.9 and substituting (3.7.61) in (3.7.62), we arrive at

$$\begin{aligned}\beta^2||\bar{\pi}_{i^2i^1}\bar{z}^n_{i^2,h}||^2_{0,\Gamma_{i^2i^1}} &\le C_1^2h^{-1/2}H^{1/2}\left(1+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}} + \frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\ &+C_1^2h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right). \quad (3.7.63)\end{aligned}$$

Substituting (3.7.63) in Lemma 3.5.8, we obtain $\forall j \in N(i^2)$

$$\begin{aligned}\beta^2||\bar{\pi}_{i^2j}\bar{z}^n_{i^2,h}||^2_{0,\Gamma_{i^2j}} &\le C_1h^{-1/2}H^{1/2}\beta|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}} \\ &+C_1^3h^{-1/2}H^{1/2}\left(1+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}} + \frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\ &+C_1^3h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right). \quad (3.7.64)\end{aligned}$$

From Lemma 3.7.9 and (3.7.64), we arrive at

$$\begin{aligned}&||\bar{\eta}^n_{i^2j,h}||^2_{0,\Gamma_{i^2j}} + \beta^2||\bar{\pi}_{i^2j}\bar{z}^n_{i^2,h}||^2_{0,\Gamma_{i^2j}} \\ &\le C_1^3h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}} + \frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\ &+C_1^3h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right) \quad \forall j \in N(i^2), \quad (3.7.65)\end{aligned}$$

where $C_1 \ge 1.5$. Next, we consider when $r = 3$, i.e., $\Omega_{i^3} \in D_3$. That means at least one face of Ω_{i^3} is common to one of $\Omega_{i^2} \in D_2$. From Lemma 3.7.11, we find that

$$\begin{aligned}\beta^2||\bar{\pi}_{i^3i^2}\bar{z}^n_{i^3,h}||^2_{0,\Gamma_{i^3i^2}} &\le C_1||\bar{\eta}^n_{i^3i^2,h}||^2_{0,\Gamma_{i^3i^2}} \\ &+ C_1\left(||\bar{\eta}^n_{i^2i^3,h}||^2_{0,\Gamma_{i^3i^2}} + \beta^2||\bar{\pi}_{i^2i^3}\bar{z}^n_{i^2,h}||^2_{0,\Gamma_{i^3i^2}}\right) \quad \forall i^2 \in N(i^3). \quad (3.7.66)\end{aligned}$$

Using Lemma 3.7.9 and (3.7.65) in (3.7.66), we arrive at

$$\begin{aligned}\beta^2||\bar{\pi}_{i^3i^2}\bar{z}^n_{i^3,h}||^2_{0,\Gamma_{i^3i^2}} &\le C_1^2h^{-1/2}H^{1/2}\left(1+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^3,h}|^2_{1,h,\Omega_{i^3}} + \frac{1}{\Delta t}||\bar{z}^n_{i^3,h}||^2_{0,\Omega_{i^3}}\right) \\ &+C_1^4h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}} + \frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\ &+C_1^4h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right). \quad (3.7.67)\end{aligned}$$

Substituting (3.7.67) in Lemma 3.5.8, we obtain $\forall j \in N(i^3)$

$$\begin{aligned}
\beta^2||\bar{\pi}_{i^3j}\bar{z}^n_{i^3,h}||^2_{0,\Gamma_{i^3j}} &\leq C_1 h^{-1/2}H^{1/2}\beta|\bar{z}^n_{i^3,h}|^2_{1,h,\Omega_{i^3}} \\
&+C_1^3 h^{-1/2}H^{1/2}\left(1+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^3,h}|^2_{1,h,\Omega_{i^3}}+\frac{1}{\Delta t}||\bar{z}^n_{i^3,h}||^2_{0,\Omega_{i^3}}\right) \\
&+C_1^5 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}}+\frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\
&+C_1^5 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}}+\frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right). \qquad (3.7.68)
\end{aligned}$$

From Lemma 3.7.9 and 3.7.68, we arrive at $\forall j \in N(i^3)$

$$\begin{aligned}
||\bar{\eta}^n_{i^3j,h}||^2_{0,\Gamma_{i^3j}} &+ \beta^2||\bar{\pi}_{i^3j}\bar{z}^n_{i^3,h}||^2_{0,\Gamma_{i^3j}} \\
&\leq C_1^3 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^3,h}|^2_{1,h,\Omega_{i^3}}+\frac{1}{\Delta t}||\bar{z}^n_{i^3,h}||^2_{0,\Omega_{i^3}}\right) \\
&+C_1^5 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^2,h}|^2_{1,h,\Omega_{i^2}}+\frac{1}{\Delta t}||\bar{z}^n_{i^2,h}||^2_{0,\Omega_{i^2}}\right) \\
&+C_1^5 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}}+\frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right), \qquad (3.7.69)
\end{aligned}$$

where $C_1 \geq 1.5$. Similarly, we can continue the argument until the entire domain is exhausted. In general,we obtain $\forall j \in N(i^r)$

$$\begin{aligned}
\beta^2||\bar{\pi}_{i^rj}\bar{z}^n_{i^r,h}||^2_{0,\Gamma_{i^rj}} &\leq C_1 h^{-1/2}H^{1/2}\beta|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}} \\
&+C_1^3 h^{-1/2}H^{1/2}\left(1+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}}+\frac{1}{\Delta t}||\bar{z}^n_{i^r,h}||^2_{0,\Omega_{i^r}}\right) \\
&+C_1^{2r-1} h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}}+\frac{1}{\Delta t}||\bar{z}^n_{i^{r-1},h}||^2_{0,\Omega_{i^{r-1}}}\right) \\
&+\cdots+C_1^{2r-1} h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}}+\frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right). \qquad (3.7.70)
\end{aligned}$$

and

$$\begin{aligned}
||\bar{\eta}^n_{i^rj,h}||^2_{0,\Gamma_{i^rj}} &+ \beta^2||\bar{\pi}_{i^rj}\bar{z}^n_{i^r,h}||^2_{0,\Gamma_{i^rj}} \\
&\leq C_1^3 h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}}+\frac{1}{\Delta t}||\bar{z}^n_{i^r,h}||^2_{0,\Omega_{i^r}}\right) \\
&+C_1^{2r-1} h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}}+\frac{1}{\Delta t}||\bar{z}^n_{i^{r-1},h}||^2_{0,\Omega_{i^{r-1}}}\right) \\
&+\cdots+C_1^{2r-1} h^{-1/2}H^{1/2}\left(2+\frac{h^2}{\Delta t}\right)\beta\left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}}+\frac{1}{\Delta t}||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}}\right), \qquad (3.7.71)
\end{aligned}$$

where $C_1 \geq 1.5$. This completes the rest of the proof. ∎

From Theorem 3.7.3 and (3.7.58), we find that

$$\sum_{\Omega_{i^r}\in D_r} G_{i,h}(\bar{z}^n_{i,h}, \bar{\eta}^n_{ij,h}) \leq 2 \sum_{\Omega_{i^r}\in D_r} \sum_{j\in N(i^r)} \left(||\bar{\eta}^n_{i^r j,h}||^2_{0,\Gamma_{i^r j}} + \beta^2 ||\bar{\pi}_{i^r j} \bar{z}^n_{i^r,h}||^2_{0,\Gamma_{i^r j}} \right)$$

$$\leq R_1 C_1^3 h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right) \beta \sum_{\Omega_{i^r}\in D_r} \left(|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}} + \frac{1}{\Delta t} ||\bar{z}^n_{i^r,h}||^2_{0,\Omega_{i^r}} \right)$$
$$+ R_1 C_1^{2r-1} h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right) \beta \sum_{\Omega_{i^{r-1}}\in D_{r-1}} \left(|\bar{z}^n_{i^{r-1},h}|^2_{1,h,\Omega_{i^{r-1}}} + \frac{1}{\Delta t} ||\bar{z}^n_{i^{r-1},h}||^2_{0,\Omega_{i^{r-1}}} \right)$$
$$+ \cdots + R_1 C_1^{2r-1} h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right) \beta \sum_{\Omega_{i^1}\in D_1} \left(|\bar{z}^n_{i^1,h}|^2_{1,h,\Omega_{i^1}} + \frac{1}{\Delta t} ||\bar{z}^n_{i^1,h}||^2_{0,\Omega_{i^1}} \right), \quad (3.7.72)$$

where R_1 is the total number of interfaces. Now we sum up all the subdomains using (3.7.72), we arrive at

$$G_h(\bar{z}^n_h, \bar{\eta}^n_h) = \sum_{r=1}^{N} \sum_{\Omega_{i^r}\in D_r} G_{i,h}(\bar{z}^n_{i,h}, \bar{\eta}^n_{ij,h})$$
$$\leq R_1 h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right) \beta \sum_{r=1}^{N} \left(C_1^{2r-1} \sum_{\Omega_{i^r}\in D_r} \left(|\bar{z}^n_{i^r,h}|^2_{1,h,\Omega_{i^r}} + \frac{1}{\Delta t} ||\bar{z}^n_{i^r,h}||^2_{0,\Omega_{i^r}} \right) \right)$$
$$\leq R_1 C_1^{2N} h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right) \beta \sum_{i=1}^{M} \left(a^h_{\Omega_i}(\bar{z}^n_{i,h}, \bar{z}^n_{i,h}) + \frac{1}{\Delta t} ||\bar{z}^n_{i,h}||^2_{0,\Omega_i} \right). \quad (3.7.73)$$

From the estimate (3.7.73), we obtain that (3.7.52), i.e., $4Q = R_1 C_1^{2N} h^{-1/2} H^{1/2} \left(2 + \frac{h^2}{\Delta t}\right)$.

Theorem 3.7.4 *Assume that the parameter $\beta = \beta_{ij} = \beta_{ji}$ in the iterative procedure (3.7.5)-(3.7.6) satisfies $\beta = O(h^{-1/2}H^{-1/2})$. Then, for $\Delta t = O(h^2)$ the spectral radius $\rho(\bar{T}^n_0)$ satisfies*

$$\rho(\bar{T}^n_0) \leq 1 - C h^{1/2} H^{-1/2} \equiv \gamma^n_0, \quad (3.7.74)$$

where $C = \frac{4}{R_1 C_1^{2N} C_4}$ with $C_4 > 2$ and the iteration (3.7.5)-(3.7.6) converges with an error at the k^{th} iteration bounded asymptotically by $O(\gamma_0^{k_n})$.

Remark 3.7.1 *From Theorem 2.7.2, Theorem 3.7.1 and Theorem3.7.4, we conclude that*

$$||u^n - U^{n,k}||_{0,\Omega} \leq ||u^n - U^n||_{0,\Omega} + ||U^n - U^{n,k}||_{0,\Omega} \leq O(\Delta t + h^2) + O(\gamma_0^{k_n}). \quad (3.7.75)$$

Since the overall error estimate at time level t_n *is of order* $O(\Delta t + h^2)$. *We need to stop the iterative procedure in* k *when we achieve* $\gamma_0^{k_n} \leq O(\Delta t + h^2)$, *that is with* $\Delta t \approx h^2$, k_n *satisfies*

$$k_n \approx \frac{log(\Delta t)}{log(\gamma_0)}. \quad (3.7.76)$$

Remark 3.7.2 *In this iterative procedure, both theoretical and computational result shows the parabolic problem has faster convergence than the elliptic problem.*

3.8 Numerical experiments

In this section, we have discussed the implementation procedure of the present results to a model problem. The numerical implementation scheme has been performed in a sequential machine using MATLAB.

Consider the parabolic initial boundary value problem (3.7.1) with $f(x, y, t) = e^t[x(1 - x) + y(1 - y) + 2x(1 - x) + 2y(1 - y)]$ and $u(x, y, 0) = u_0(x, y)$. The exact solution of the problem (3.7.1) problem is given by $u = e^t x(1 - x)y(1 - y)$.

Here we take the square domain $\Omega = (0, 1) \times (0, 1)$. We decompose the square into $[0, 3/4] \times [0, 1]$ and $[3/4, 1] \times [0, 1]$, with interface $\Gamma = \{3/4\} \times (0, 1)$. We triangulate the domain uniformly and mesh size is h. Here, we consider the winding number $N = 1$. We choose the initial guess $\{u_{i,h}^{n,0}, \lambda_{ij,h}^{n,0}\} = \{u_{i,h}^{n-1}, \lambda_{ij,h}^{n-1}\}$, where $n - 1$ is the previous time step. The stop criterion is $\|u_h^{n,k} - u_h^n\|_\infty \leq 10^{-4}$, where iteration number is k. We choose the relaxation parameter $\beta = O(h^{-1/2}H^{-1/2})$.

In Figure 3.3, the graph of the L^2 error $\|u - u_h\|$ is plotted as a function of the discretization step $'h'$ in the $log - log$ scale. The slope of the graph provides the computed order of convergence as approximately 2.0.

In Table 3.2, the iteration number, order of convergence and L^2 error $e_h = \|u - u_h\|$ for $h = 1/8$, $h = 1/12$, $h = 1/16$, $h = 1/20$, $h = 1/24$ and $h = 1/28$, and $\Delta t = h^2$ at time $t = 1$ are given. The numerical result confirms our theoretical result.

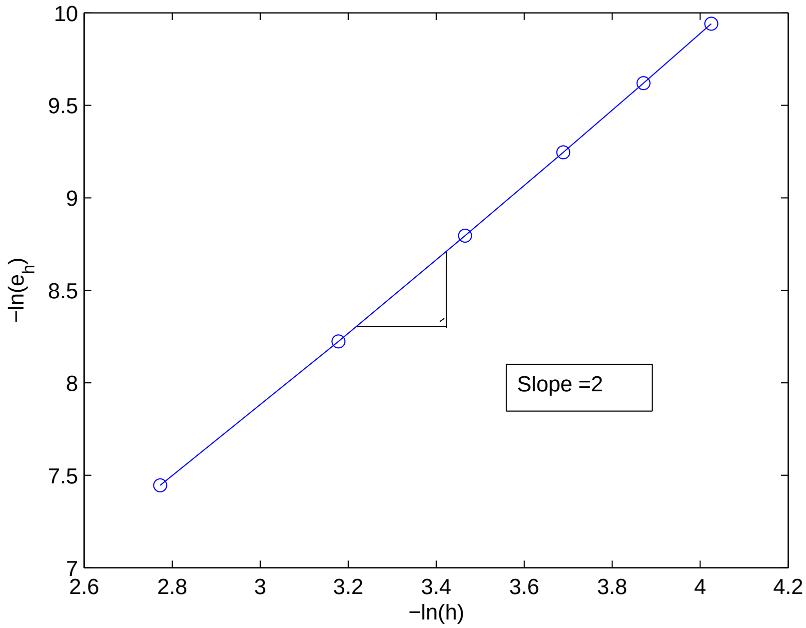

Figure 3.3: The order of convergence

h	H	D.O.F. in Ω_1	D.O.F. in Ω_2	k = No. of Iter.	$e_h = \|u - u_h\|_{0,\Omega}$	Rate
1/8	1	138	46	5	5.83814998×10^{-4}	-
1/12	1	315	105	5	2.68160215×10^{-4}	1.9188
1/16	1	564	188	5	1.51485628×10^{-4}	1.9851
1/20	1	885	295	5	9.65067546×10^{-5}	2.0206
1/24	1	1278	426	5	6.63850676×10^{-5}	2.0521
1/28	1	1743	581	5	4.81675258×10^{-5}	2.0810

Table 3.2: L^2 error and the rate of convergence for the 2-domain case

Chapter 4

Parallel Iterative Procedures Using Mixed Finite Element Methods

4.1 Introduction

In this chapter, we discuss an iterative method based on mixed finite element methods using Robin-type boundary condition as transmission data on the artificial interface (inter-subdomain boundary) for nonoverlapping DDM. We now consider the following second order elliptic problem, which models single phase Darcy flow in a porous medium: Find pressure p and velocity $\mathbf{u}$ satisfying

$$\mathbf{u} = -K\,\nabla p \qquad \text{in} \qquad \Omega, \tag{4.1.1}$$

$$\nabla \cdot \mathbf{u} + b\,p = f \qquad \text{in} \qquad \Omega, \tag{4.1.2}$$

$$p = g \qquad \text{on} \qquad \partial\Omega, \tag{4.1.3}$$

where $\Omega \subset \mathbb{R}^d,\ d = 2$ or 3, is the bounded domain, K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$-components representing the permeability divided by the viscosity and $b(x) \geq 0$, $b(x) \in L^\infty(\Omega)$. The Dirichlet boundary conditions are considered for simplicity. In the proposed method, the problem (4.1.1)-(4.1.3) is decomposed into a series of small, local (or subdomain) problems. On each artificial interface, Robin-type boundary are considered as transmission conditions and then the subproblems are solved using mixed finite element methods.

Other domain decomposition methods with nonoverlapping partitions for mixed finite element methods are discussed by Glowinski and Wheeler [74, 37], Cowsar, Mandel, Wheeler [38], and Douglas et al [49, 51]. The balancing domain decomposition method is proposed in [38, 94]. In [49], Douglas et al. have also proposed and analyzed a parallel iterative domain decomposition method with Robin-type boundary conditions as transmission conditions on the interface. While the proposed iterative method is different from the one introduced by Douglas et al. [49], it is closely related to one proposed by Deng [43, 44].

The organization of this chapter is as follows. In Section 4.2, we have discussed DD procedures based on mixed finite element methods. In this section, we have introduced iterative method for multidomain problem. In Section 4.3, we have discussed the convergence analysis for the iterative mixed finite element multidomain formulation. In Section 4.4, we have estimated rate of convergence using spectral radius of the matrix associated with the fixed point iterations.

4.2 Domain decomposition and finite element framework

In this section, we discuss the variational and mixed finite element formulations for the multidomain problems.

Before stating the weak formulation of (4.1.1)-(4.1.3), we recall the usual velocity space [26]. Let

$$H(\mathrm{div};\Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^d \ : \ \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}, \tag{4.2.1}$$

be equipped with the norm

$$||\mathbf{v}||_{H(\mathrm{div};\Omega)} = \left(||\mathbf{v}||^2 + ||\nabla \cdot \mathbf{v}||^2 \right)^{1/2}. \tag{4.2.2}$$

The weak formulation corresponding to (4.1.1)-(4.1.3) is to find $\{\mathbf{u}^\star, p^\star\} \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u}^\star, \mathbf{v})_\Omega - (p^\star, \nabla \cdot \mathbf{v})_\Omega = -\langle g, \mathbf{v} \cdot \nu \rangle_{\partial\Omega}, \qquad \mathbf{v} \in H(\mathrm{div};\Omega), \tag{4.2.3}$$

$$(\nabla \cdot \mathbf{u}^\star, q)_\Omega + (b\, p^\star, q)_\Omega = (f, q)_\Omega, \qquad q \in L^2(\Omega), \tag{4.2.4}$$

where ν is the outward unit normal vector to $\partial\Omega$. Under the assumption of coercivity and LBB condition (see, the References [26, 114]) the problem (4.2.3)-(4.2.4) has a unique solution. We now assume that the problem (4.1.1)-(4.1.3) is H^2-regular, i.e., there exists a positive constant C depending only on K and Ω such that

$$||p||_2 \leq C\left(\,||f|| + ||g||_{3/2,\partial\Omega}\right). \tag{4.2.5}$$

We refer the reader to [71, 75, 93] for sufficient conditions for H^2-regularity.

In order to obtain a discretization of (4.2.3)-(4.2.4), we assume that there are two standard mixed finite element spaces $\bar{\mathbf{V}}_h \subset H(\text{div};\Omega)$ and $\bar{W}_h \subset L^2(\Omega)$ (see, the References [26, 114]). Now the approximation of $(\mathbf{u}^\star, p^\star)$ is to find $(\mathbf{u}_h^\star, p_h^\star) \in \bar{\mathbf{V}}_h \times \bar{W}_h$ satisfying

$$(K^{-1}\mathbf{u}_h^\star, \mathbf{v})_\Omega - (p_h^\star, \nabla\cdot\mathbf{v})_\Omega \;=\; \langle g\,, \mathbf{v}\cdot\nu\rangle_{\partial\Omega}, \qquad \mathbf{v} \in \bar{\mathbf{V}}_h, \tag{4.2.6}$$

$$(\nabla\cdot\mathbf{u}_h^\star, q)_\Omega + (b\,p_h^\star, q)_\Omega \;=\; (f, q), \qquad q \in \bar{W}_h. \tag{4.2.7}$$

Under the assumption of coercivity and discrete LBB condition (see, the References [26, 114]) the discrete problem (4.2.6)-(4.2.7) has a unique solution in $(\mathbf{u}_h^\star, p_h^\star) \in \bar{\mathbf{V}}_h \times \bar{W}_h$.

For the multidomain formulation, let the domain Ω be partitioned into a finite number of non-overlapping sub-domains $\Omega_i (i = 1, 2, \cdots, M)$ with $\bar{\Omega} = \bigcup_{i=1}^{M} \Omega_i$, and let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j = \Gamma_{ji}$ with $|\Gamma_{ij}|$ as the measure of Γ_{ij}. Further, let $\Gamma = \bigcup_{i=1,\ j\in N(i)}^{M} \Gamma_{ij}$, and $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ denote the interior interfaces, where $N(i) = \{j \neq i \,|\, |\Gamma_{ij}| > 0\}$.

We define a sequence of sets D_i whose elements are subdomain by induction: $D_1 = \{\Omega_i \mid$ at least one face of Ω_i belongs to $\partial\Omega\}$, $D_{r+1} = \{\Omega_i \mid \Omega_i \notin D_r, \Omega_i$ share one face with some $\Omega_j \in D_r$ at least$\}$ (see the definition 3.2.1 in chapter 3).

Now we are in a position to write the following multidomain formulation corresponding to the problem (4.1.1)-(4.1.3). Find $(\mathbf{u}_i, p_i)$, $i = 1, 2, \cdots, M$ satisfying the following subproblems:

$$\mathbf{u}_i + K\,\nabla p_i = 0 \qquad \text{in} \qquad \Omega_i, \tag{4.2.8}$$

$$\nabla\cdot\mathbf{u}_i + b\,p_i = f \qquad \text{in} \qquad \Omega_i, \tag{4.2.9}$$

$$p_i = g \qquad \text{on} \qquad \partial\Omega_i \cap \partial\Omega. \tag{4.2.10}$$

The consistency conditions which need to be imposed on the artificial interface Γ of the problem are

$$p_i = p_j \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i), \tag{4.2.11}$$

$$K\,\nabla p_i \cdot \nu = K\,\nabla p_j \cdot \nu \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i), \tag{4.2.12}$$

where $\nu = \nu^{ij} = -\nu^{ji}$ on Γ_{ij} and ν^{ij} and ν^{ji} are the unit outward normal vectors to $\partial\Omega_i$ and $\partial\Omega_j$, respectively. The equation (4.2.12) can be written as

$$\mathbf{u}_i \cdot \nu^{ij} = -\mathbf{u}_j \cdot \nu^{ji} \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i). \tag{4.2.13}$$

We need the following spaces for our future use. Let

$$\mathbf{V}_i = H(\mathrm{div};\Omega_i), \qquad \mathbf{V} = \prod_{i=1}^{M} \mathbf{V}_i, \tag{4.2.14}$$

$$W_i = L^2(\Omega_i), \qquad W = \prod_{i=1}^{M} W_i = L^2(\Omega). \tag{4.2.15}$$

For $\mathbf{v} \in \mathbf{V}$, $q \in W$ and $\eta \in L^2(\Gamma)$, the multidomain weak formulation corresponding to (4.2.3)-(4.2.4) becomes :

$$\sum_{i=1}^{M} \left\{(K^{-1}\mathbf{u}_i, \mathbf{v})_{\Omega_i} - (p_i, \nabla\cdot\mathbf{v})_{\Omega_i}\right\} = -\sum_{i=1}^{M} \left\{ \sum_{j\in N(i)} \langle p_i\,, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} - \langle g\,, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma} \right\}, \tag{4.2.16}$$

$$\sum_{i=1}^{M} \left\{(\nabla\cdot\mathbf{u}_i, q)_{\Omega_i} + (b\,p_i, q)_{\Omega_i}\right\} = \sum_{i=1}^{M} (f, q)_{\Omega_i}, \tag{4.2.17}$$

$$\sum_{i=1}^{M} \sum_{j\in N(i)} \langle \mathbf{u}_i\cdot\nu^{ij}, \eta\rangle_{\Gamma_{ij}} = 0, \tag{4.2.18}$$

where ν^{ij} is the outward unit normal vector on $\partial\Omega_i$ (see, [26, pp. 91-92]), $\mathbf{u}_i = \mathbf{u}^\star_{|\,\Omega_i}$ and $p_i = p^\star_{|\,\Omega_i}$. There may be problem in assigning a meaning of the traces in (4.2.16)-(4.2.18), but this formulation will be useful for discrete formulation.

To describe finite element approximations for (4.2.16)-(4.2.18), we begin with the triangulation of $\Omega_i,\ i = 1, 2, \cdots, M$. Let $\mathcal{T}_{h,i}$ be a conforming and regular triangulation of $\bar{\Omega}_i$ into triangles (resp. tetrahedrons) satisfying $\forall\, i$

$$T \subset \bar{\Omega}_i \quad \forall T \in \mathcal{T}_{h,i}, \quad \bar{\Omega}_i = \bigcup_{T\in\mathcal{T}_{h,i}} T. \tag{4.2.19}$$

We also assume that the triangles (resp. tetrahedrons) T should not cross the interface Γ, and thus, each element is either contained in $\bar{\Omega}_i$ or in $\bar{\Omega}_j$, where $1 \le i, j \le M$ and they share the same edges of Γ_{ij}. This implies that the global triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ induces the triangulations $\mathcal{T}_{h,i}$ of $\bar{\Omega}_i$ and $\mathcal{T}_{h,j}$ of $\bar{\Omega}_j$, $1 \le i, j \le M$. i.e., $\mathcal{T}_h = \bigcup_{i=1}^{M} \mathcal{T}_{h,i}$. Let

$$\mathbf{V}_{i,h} \times W_{i,h} \subset \mathbf{V}_i \times W_i, \tag{4.2.20}$$

be any of the usual mixed finite element spaces defined on $\mathcal{T}_{h,i}$ (see for the RT spaces [104, 112], the BDM spaces [25], the BDFM spaces [24], the BDDF spaces [23], or the CD spaces [31]). The velocity and pressure mixed finite element spaces on Ω are defined as follows:

$$\mathbf{V}_h = \prod_{i=1}^{M} \mathbf{V}_{i,h}, \qquad W_h = \prod_{i=1}^{M} W_{i,h}. \tag{4.2.21}$$

For example, let T be a d- simplicial (triangular or tetrahedral) element. Define the Raviart-Thomas spaces [104, 112]

$$RT_r(T) = (P_r(T))^d + \mathbf{x}\, P_r(T), \tag{4.2.22}$$

where $\mathbf{x} = (x_1, x_2)$ for $d = 2$, $\mathbf{x} = (x_1, x_2, x_3)$ for $d = 3$ and $P_r(T)$ is the polynomial of degree $\le r$ over T.

Lemma 4.2.1 *[26, pp. 116] For any d-simplicial element T we have for $\mathbf{v} \in RT_r(T)$*

$$\operatorname{div} \mathbf{v} \in P_r(T), \qquad \mathbf{v} \cdot \nu_{|\partial T} \in R_r(\partial T), \tag{4.2.23}$$

where ν is the outward unit normal vector on ∂T and

$$R_r(\partial T) = \left\{ \phi \,|\, \phi \in L^2(\partial T),\, \phi_{|e_i} \in P_r(e_i) \forall e_i, \text{ and } e_i \text{ are the edges of triangles} \right\}.$$

In the remaining part of the paper, we have used the Raviart-Thomas spaces of lowest order RT_0 [104, 112]. Define the finite dimensional spaces

$$\mathbf{V}_h = \left\{ \mathbf{v} = (v_1, \cdots, v_d) \in \mathbf{V} \,:\, \mathbf{v}_{|T} = v_l = \alpha_l + \beta x_l \,;\, \alpha_l, \beta \in \mathbb{R},\ l = 1, \cdots, d \right\}, \tag{4.2.24}$$

$$W_h = \left\{ w \in L^2(\Omega) \,:\, w_{|T} = \text{constant} \right\}. \tag{4.2.25}$$

Note that for any element $T \in \mathcal{T}_h$, the degrees of freedom for a vector $\mathbf{v} \in \mathbf{V}_h$ can be specified by the values of its normal components $\mathbf{v} \cdot \nu_T$ at the midpoints of all edges (faces) of T, where ν_T is the outward unit normal vector on ∂T. The degree of freedom for a function $w \in W_h$ is its value at the center of T.

Remark 4.2.1 *The normal components of vectors in $\mathbf{V}_h$ are continuous between the inter element faces within each subdomain Ω_i and there is no such restriction across Γ, that is, the normal component of the flux variable may not be continuous across the inter-subdomain boundaries Γ_{ij} and hence, $\mathbf{V}_h$ may not be a subspace of $H(\mathrm{div};\Omega)$.*

Let $\mathcal{T}_{ij,h}$ be a quasi-uniform finite element partition of Γ_{ij}. From Proposition 4.2.1, we find that r is the degree of the polynomials in $\mathbf{V}_h \cdot \nu^{ij}$. In order to construct the Lagrange multiplier space on Γ_{ij}, let $\Lambda_{ij,h} \subset L^2(\Gamma_{ij})$ consist of either the continuous or discontinuous piecewise polynomials of degree r on $\mathcal{T}_{ij,h}$, where r is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu^{ij}$. For example, in the case of RT_0, $\Lambda_{ij,h}$ is the space of all piecewise constants (linear, if $d = 3$ and the grids are hexahedral) polynomials on $\mathcal{T}_{ij,h}$. Let

$$\Lambda_h = \prod_{i=1}^{M} \prod_{j \in N(i)} \Lambda_{ij,h}. \tag{4.2.26}$$

be the Lagrange multiplier space on Γ. For convenience, we interpret any function $\eta \in \Lambda_h$ to be extended by zero on $\partial\Omega$. The mixed finite element formulation corresponding to (4.2.16)-(4.2.18) is to seek $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$ such that, for $\mathbf{v} \in \mathbf{V}_h$, $q \in W_h$ and $\eta \in \Lambda_h$

$$\sum_{i=1}^{M} \left\{ (K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} - (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} \right\} = -\sum_{i=1}^{M} \left\{ \sum_{j \in N(i)} \langle \mathbf{v} \cdot \nu^{ij}, \lambda_h \rangle_{\Gamma_{ij}} - \langle g, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma} \right\}, \tag{4.2.27}$$

$$\sum_{i=1}^{M} \{ (\nabla \cdot \mathbf{u}_h, q)_{\Omega_i} + (b\, p_h, q)_{\Omega_i} \} = \sum_{i=1}^{M} (f, q)_{\Omega_i}, \tag{4.2.28}$$

$$\sum_{i=1}^{M} \sum_{j \in N(i)} \langle \mathbf{u}_h \cdot \nu^{ij}, \eta \rangle_{\Gamma_{ij}} = 0. \tag{4.2.29}$$

Here on each subdomain Ω_i, we have a standard mixed finite element method, and (4.2.28) enforces local conservation over each degree of freedom. Moreover, since $\mathbf{u}_h \cdot \nu$ is continuous at any element face (or edge) $\tau \not\subset \Gamma \cup \partial\Omega$, the local mass conservation property across interior element faces is satisfied. By considering the Dirichlet boundary condition, it is clear from (4.2.16) and (4.2.27) that the Lagrange multiplier $\lambda_h \in \Lambda_h$ actually replaces the pressure p on the boundary Γ_{ij}. The equation (4.2.29) enforces weak continuity of the flux across the interfaces (weakly with respect to the Lagrange multiplier space Λ_h). The matrix associated with (4.2.27)-(4.2.29) takes the form

$$\begin{bmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{B}^T & \hat{E} & 0 \\ \hat{C}^T & 0 & 0 \end{bmatrix}, \tag{4.2.30}$$

where $\hat{A}$ is a block diagonal matrix and $\hat{B}$ also has a block structure. Actually, by introducing the Lagrange multiplier, we easily eliminate the flux and obtain a reduced problem for the pressure unknowns only. Thus, the variable $\mathbf{u}_h$ can be eliminated by computing the inverse of $\hat{A}$ which is trivial. The reduced matrix takes the form

$$\hat{D} = \begin{bmatrix} \hat{B}^T\hat{A}^{-1}\hat{B} + \hat{E} & \hat{B}^T\hat{A}^{-1}\hat{C} \\ \hat{C}^T\hat{A}^{-1}\hat{B} & \hat{C}^T\hat{A}^{-1}\hat{C} \end{bmatrix}. \tag{4.2.31}$$

Notice that the simplification of the matrix (4.2.30) cannot in general be done in practice. Moreover, the matrix $\hat{D}$ in (4.2.31) is also ill-conditioned. Therefore, efficient iterative methods need be introduced to handle such a difficult situation. In the next section, we are going to introduce mixed iterative domain decomposition method.

4.2.1 Iterative method for multidomain problem

In this subsection, we discuss the iterative method based on the multidomain subproblems, and also derive the weak formulation for both continuous and discrete problems.

It is easy (cf. [45, 46]) to replace (4.2.11) and (4.2.13) by the following Robin-type boundary condition on the artificial interfaces Γ_{ij} as :

$$-\beta_{ij}\,\mathbf{u}_i \cdot \nu^{ij} + p_i = \beta_{ji}\,\mathbf{u}_j \cdot \nu^{ji} + p_j, \quad x \in \Gamma_{ij} \subset \partial\Omega_i, \tag{4.2.32}$$

$$-\beta_{ji}\,\mathbf{u}_j \cdot \nu^{ji} + p_j = \beta_{ij}\,\mathbf{u}_i \cdot \nu^{ij} + p_i, \quad x \in \Gamma_{ji} \subset \partial\Omega_j, \tag{4.2.33}$$

where $\beta_{ij} = \beta_{ji} > 0$ are parameters. Now, we define an iterative procedure based on the nonoverlapping multidomain problems as follows: for all $i = 1, 2, \cdots, M$

(i) given $l^0_{ij},\ 1 \le i \ne j \le M$, arbitrarily.

(ii) recursively compute $\mathbf{u}^k_i,\ p^k_i,\ i = 1, 2, \cdots, M$, by solving in parallel

$$\alpha\, \mathbf{u}^k_i + \nabla p^k_i = 0 \qquad \text{in} \qquad \Omega_i, \tag{4.2.34}$$

$$\nabla \cdot \mathbf{u}^k_i + b\, p^k_i = f \qquad \text{in} \qquad \Omega_i, \tag{4.2.35}$$

$$-\beta_{ij}\, \mathbf{u}^k_i \cdot \nu^{ij} + p^k_i = l^k_{ij} \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i), \tag{4.2.36}$$

$$p^k_i = g \qquad \text{on} \qquad \partial\Omega_i \cap \partial\Omega, \tag{4.2.37}$$

where $\alpha = K^{-1}$.

(iii) for $i = 1, 2, \cdots, M$ update the Robin-type transmission condition

$$l^{k+1}_{ij} = 2\, \beta_{ji}\, \mathbf{u}^k_j \cdot \nu^{ji} + l^k_{ji} \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i). \tag{4.2.38}$$

The weak formulation corresponding to the problem (4.2.34)-(4.2.38) may be stated as follows: For all i and j, given $l^0_{ij} \in \Lambda_{ij},\ l^0_{ji} \in \Lambda_{ji}$ arbitrarily, find $\left\{\mathbf{u}^k_i,\ p^k_i,\ l^{k+1}_{ij}\right\} \in \mathbf{V}_i \times W_i \times \Lambda_{ij}$ such that

$$(\alpha\, \mathbf{u}^k_i, \mathbf{v})_{\Omega_i} - (p^k_i, \nabla \cdot \mathbf{v})_{\Omega_i} + \sum_{j \in N(i)} \beta_{ij} \langle \mathbf{u}^k_i \cdot \nu^{ij}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}}$$
$$= - \sum_{j \in N(i)} \langle l^k_{ij}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} - \langle g\,, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_i, \tag{4.2.39}$$

$$(\nabla \cdot \mathbf{u}^k_i, q)_{\Omega_i} + (b\, p^k_i, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad q \in W_i, \tag{4.2.40}$$

and

$$\langle l^{k+1}_{ij}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} = 2\, \beta_{ji} \langle \mathbf{u}^k_j \cdot \nu^{ji}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} + \langle l^k_{ji}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}}, \quad \mathbf{v} \cdot \nu^{ij} \in L^2(\Gamma_{ij}). \tag{4.2.41}$$

There may be some difficulty in assigning a meaning to (4.2.41) regarding the product $\langle \mathbf{u}^k_j \cdot \nu^{ji}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}}$ if $\mathbf{u}^k_j \in \mathbf{V}_j$ and $\mathbf{v} \in \mathbf{V}_i$. Similar difficulties may arise while attaching a meaning to some of the term in (4.2.39). However, the problem (4.2.39)-(4.2.41) may be viewed a motivation for the following iterative mixed finite element multidomain formulation.

For all i and j, given $l^0_{ij,h} \in \Lambda_{ij,h},\ l^0_{ji,h} \in \Lambda_{ji,h}$ arbitrarily, find $\left\{\mathbf{u}^k_{i,h},\ p^k_{i,h},\ l^{k+1}_{ij,h}\right\} \in \mathbf{V}_{i,h} \times$

$W_{i,h} \times \Lambda_{ij,h}$ such that

$$(\alpha\, \mathbf{u}_{i,h}^k, \mathbf{v})_{\Omega_i} - (p_{i,h}^k, \nabla \cdot \mathbf{v})_{\Omega_i} + \sum_{j \in N(i)} \beta_{ij} \langle \mathbf{u}_{i,h}^k \cdot \nu^{ij} \,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}}$$
$$= - \sum_{j \in N(i)} \langle l_{ij,h}^k \,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} - \langle g \,, \mathbf{v} \cdot \nu_i \rangle_{\partial \Omega_i \setminus \Gamma}, \;\; \mathbf{v} \in \mathbf{V}_{i,h}, \qquad (4.2.42)$$
$$(\nabla \cdot \mathbf{u}_{i,h}^k, q)_{\Omega_i} + (b\, p_{i,h}^k, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad q \in W_{i,h}, \qquad (4.2.43)$$

and

$$l_{ij,h}^{k+1} = 2\, \beta_{ji}\, \mathbf{u}_{j,h}^k \cdot \nu^{ji} + l_{ji,h}^k \qquad \text{on} \qquad \Gamma_{ij}, \;\forall j \in N(i). \qquad (4.2.44)$$

Note that the two spaces $\Lambda_{ij,h}$ and $\Lambda_{ji,h}$ are different on the edge or side Γ_{ij}.

4.3 Convergence analysis

In this section, we discuss the convergence of the iterative method defined by (4.2.42)-(4.2.44).

Below, we first discuss the equivalence between the mixed finite element multidomain formulation and the single domain formulation (4.2.6)-(4.2.7).

Theorem 4.3.1 *Let $(\mathbf{u}_h^\star, p_h^\star) \in \bar{\mathbf{V}}_h \times \bar{W}_h$ be the solution of (4.2.6)-(4.2.7), and $\mathbf{u}_{i,h} = \mathbf{u}^\star_{h|_{\Omega_i}}$ and $w_{i,h} = w^\star_{h|_{\Omega_i}}$. Then for all $1 \le i \le M$ and $j \in N(i)$, there exists $l_{ij,h} \in \Lambda_{ij,h}$, such that $(\mathbf{u}_{i,h}, p_{i,h}) \in \mathbf{V}_{i,h} \times W_{i,h}$ satisfies*

$$(\alpha\, \mathbf{u}_{i,h}, \mathbf{v})_{\Omega_i} - (p_{i,h}, \nabla \cdot \mathbf{v})_{\Omega_i} + \sum_{j \in N(i)} \beta_{ij} \langle \mathbf{u}_{i,h} \cdot \nu^{ij} \,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}}$$
$$= - \sum_{j \in N(i)} \langle l_{ij,h} \,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} - \langle g \,, \mathbf{v} \cdot \nu_i \rangle_{\partial \Omega_i \setminus \Gamma}, \;\; \mathbf{v} \in \mathbf{V}_{i,h}, \qquad (4.3.1)$$
$$(\nabla \cdot \mathbf{u}_{i,h}, q)_{\Omega_i} + (b\, p_{i,h}, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad q \in W_{i,h}, \qquad (4.3.2)$$

and

$$l_{ij,h} = 2\, \beta_{ji}\, \mathbf{u}_{j,h} \cdot \nu^{ji} + l_{ji,h} \qquad \text{on} \qquad \Gamma_{ij}, \;\forall j \in N(i), \qquad (4.3.3)$$

where $\alpha = K^{-1}$ and $\beta = \beta_{ij} = \beta_{ji} > 0$.

Proof. For simplicity, we prove the above theorem for the two fixed subdomains, i.e.,

$M = 2$. For example, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Gamma = \Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. Let

$$\bar{\mathbf{V}}_h = \mathbf{V}^0_{1,h} \bigoplus \mathbf{V}^0_{2,h} \bigoplus \mathbf{V}_{\Gamma_{12},h}, \tag{4.3.4}$$

where

$$\mathbf{V}^0_{i,h} = \left\{ \mathbf{v}_{i,h} \,|\, \mathbf{v}_{i,h} = \mathbf{v}_{h|_{\Omega_i}} \in \bar{\mathbf{V}}_h \text{ and } \mathbf{v}_h \cdot \nu_i = 0 \text{ on } \Gamma_{ij} \right\},\ i = 1,\ 2, \tag{4.3.5}$$

and we associate to Γ_{12} a complementary subspace $\mathbf{V}_{\Gamma_{12},h}$ of $\mathbf{V}^0_{1,h} \bigoplus \mathbf{V}^0_{2,h}$ in $\bar{\mathbf{V}}_h$. Now equation (4.2.6)-(4.2.7) can be written in an equivalent split form: Find $(\mathbf{u}_{i,h}, p_{i,h}) \in \mathbf{V}_{i,h} \times W_{i,h}$ such that

$$(\alpha\, \mathbf{u}_{i,h}, \mathbf{v})_{\Omega_i} - (p_{i,h}, \nabla \cdot \mathbf{v})_{\Omega_i} = -\langle g\,, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \cap \partial\Omega}, \qquad \forall \mathbf{v} \in \mathbf{V}^0_{i,h},\ i = 1,\ 2, \tag{4.3.6}$$

$$(\nabla \cdot \mathbf{u}_{i,h}, q)_{\Omega_i} + (b\, p_{i,h}, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad \forall q \in W_{i,h},\ i = 1,\ 2, \tag{4.3.7}$$

and

$$(\alpha\, \mathbf{u}_{2,h}, \mathbf{v})_{\Omega_2} - (p_{2,h}, \nabla \cdot \mathbf{v})_{\Omega_2} = -\left[(\alpha\, \mathbf{u}_{1,h}, \mathbf{v})_{\Omega_1} - (p_{1,h}, \nabla \cdot \mathbf{v})_{\Omega_1} \right], \quad \forall \mathbf{v} \in \mathbf{V}_{\Gamma_{12},h}. \tag{4.3.8}$$

Now we consider first Ω_1. Then for $\mathbf{v} \in \mathbf{V}_{\Gamma_{12},h}$, we define $L_{12,h}$ as

$$L_{12,h} = (\alpha\, \mathbf{u}_{2,h}, \mathbf{v})_{\Omega_2} - (p_{2,h}, \nabla \cdot \mathbf{v})_{\Omega_2}. \tag{4.3.9}$$

Therefore, we can construct $l_{12,h} \in \Lambda_{12,h}$ such that

$$\langle l_{12,h}\,,\, \mathbf{v} \cdot \nu^{12} \rangle_{\Gamma_{12}} = -\beta \langle \mathbf{u}_{1,h} \cdot \nu^{12}\,,\, \mathbf{v} \cdot \nu^{12} \rangle_{\Gamma_{12}} + L_{12,h} \quad \forall \mathbf{v} \in \mathbf{V}_{\Gamma_{12},h}. \tag{4.3.10}$$

The space $\Lambda_{12,h}$ consists of polynomials of degree $\leq r$ and also the normal component $\mathbf{v} \cdot \nu^{12}$ on Γ_{12} is a polynomial of fixed degree $\leq r$. So, existence of a unique $l_{12,h}$ follows from the equation (4.3.10). Similarly, we now consider Ω_2. Then we define $L_{21,h}$ as

$$L_{21,h} = (\alpha\, \mathbf{u}_{1,h}, \mathbf{v})_{\Omega_1} - (p_{1,h}, \nabla \cdot \mathbf{v})_{\Omega_1}. \tag{4.3.11}$$

Therefore, we can construct $l_{21,h} \in \Lambda_{21,h}$ such that

$$\langle l_{21,h}\,,\, \mathbf{v} \cdot \nu^{21} \rangle_{\Gamma_{12}} = -\beta \langle \mathbf{u}_{2,h} \cdot \nu^{21}\,,\, \mathbf{v} \cdot \nu^{21} \rangle_{\Gamma_{12}} + L_{21,h} \quad \forall \mathbf{v} \in \mathbf{V}_{\Gamma_{12},h}. \tag{4.3.12}$$

Existence of a unique $l_{21,h}$ follows from the equation (4.3.12). Thus, it follows from (4.3.6), (4.3.8), (4.3.10) and (4.3.12) that

$$(\alpha\,\mathbf{u}_{1,h},\mathbf{v})_{\Omega_1} - (p_{1,h},\nabla\cdot\mathbf{v})_{\Omega_1} + \beta\langle \mathbf{u}_{1,h}\cdot\nu^{12}\,,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}}$$
$$= -\langle l_{12,h}\,,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}} - \langle g\,,\mathbf{v}\cdot\nu_1\rangle_{\partial\Omega_1\cap\partial\Omega},\ \ \mathbf{v}\in\mathbf{V}_{1,h}, \quad (4.3.13)$$
$$(\alpha\,\mathbf{u}_{2,h},\mathbf{v})_{\Omega_2} - (p_{2,h},\nabla\cdot\mathbf{v})_{\Omega_2} + \beta\langle \mathbf{u}_{2,h}\cdot\nu^{21}\,,\mathbf{v}\cdot\nu^{21}\rangle_{\Gamma_{12}}$$
$$= -\langle l_{21,h}\,,\mathbf{v}\cdot\nu^{21}\rangle_{\Gamma_{12}} - \langle g\,,\mathbf{v}\cdot\nu_2\rangle_{\partial\Omega_2\cap\partial\Omega},\ \ \mathbf{v}\in\mathbf{V}_{2,h}. \quad (4.3.14)$$

Clearly, (4.3.13)-(4.3.14) and (4.3.7) implies (4.3.1)-(4.3.2) with $\beta=\beta_{ij}=\beta_{ji}$. Adding (4.3.12) and (4.3.10), we obtain

$$\langle l_{12,h}\,,\,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}} + \langle l_{21,h}\,,\,\mathbf{v}\cdot\nu^{21}\rangle_{\Gamma_{12}} = -\beta\langle\mathbf{u}_{1,h}\cdot\nu^{12}\,,\,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}}$$
$$-\beta\langle\mathbf{u}_{2,h}\cdot\nu^{21}\,,\,\mathbf{v}\cdot\nu^{21}\rangle_{\Gamma_{12}} + (L_{12,h}+L_{21,h})\quad \forall\mathbf{v}\in\mathbf{V}_{\Gamma_{12},h}. \quad (4.3.15)$$

Since $\mathbf{u}_{1,h}=u^{\star}_{h|_{\Omega_1}}$, $\mathbf{u}_{2,h}=u^{\star}_{h|_{\Omega_2}}$ and $\mathbf{u}^{\star}_h$ is the solution of (4.2.6)-(4.2.7), therefore, we obtain

$$\mathbf{u}_{1,h}\cdot\nu^{12} = -\mathbf{u}_{2,h}\cdot\nu^{21} \text{ on } \Gamma_{12}, \quad (4.3.16)$$

where ν^{12} and ν^{21} are outward normals to Ω_1 and Ω_2, respectively. From (4.3.9) and (4.3.11), we find that

$$L_{12,h}+L_{21,h}=0. \quad (4.3.17)$$

Substituting (4.3.17) in (4.3.15), we arrive at

$$\langle l_{12,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}\,,\,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}} + \langle l_{21,h}+\beta\ \mathbf{u}_{2,h}\cdot\nu^{21},\,\mathbf{v}\cdot\nu^{21}.\rangle_{\Gamma_{12}} = 0 \quad (4.3.18)$$

We rewrite the equation (4.3.18) to obtain

$$\langle l_{12,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}\,,\,\mathbf{v}\cdot\nu^{12}\rangle_{\Gamma_{12}} - \langle l_{21,h}+\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}\,,\,-\mathbf{v}\cdot\nu^{21}\rangle_{\Gamma_{12}} = 0. \quad (4.3.19)$$

Now choose $\mathbf{v}\cdot\nu^{12} = l_{12,h}-l_{21,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}-\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}$ and $\mathbf{v}\cdot\nu^{21} = -l_{12,h}+l_{21,h}-\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}+\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}$, and substituting in (4.3.19), we arrive at

$$\langle l_{12,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}\,,\,l_{12,h}-l_{21,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}-\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}\rangle_{\Gamma_{12}}$$
$$-\langle l_{21,h}+\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}\,,\,l_{12,h}-l_{21,h}+\beta\,\mathbf{u}_{1,h}\cdot\nu^{12}-\beta\,\mathbf{u}_{2,h}\cdot\nu^{21}\rangle_{\Gamma_{12}} = 0. \quad (4.3.20)$$

Using (4.3.16) in (4.3.20), we find that

$$l_{12,h} = 2\,\beta\,\mathbf{u}_{2,h} + l_{21,h}. \tag{4.3.21}$$

Similarly, we obtain

$$l_{21,h} = 2\,\beta\,\mathbf{u}_{1,h} + l_{12,h}. \tag{4.3.22}$$

Clearly, (4.3.21)-(4.3.22) implies (4.3.3) with $\beta = \beta_{ij} = \beta_{ji}$. Here, we have proved for two subdomain cases with $\beta = \beta_{ij} = \beta_{ji}$. Similarly we can proceed for more than two subdomains with $\beta = \beta_{ij} = \beta_{ji}$. This completes the rest of the proof. ∎

Now we are in a position to discuss the convergence of the iterative method defined by (4.2.42)-(4.2.44). Define

$$\mathbf{e}^k_{i,h} = \mathbf{u}^k_{i,h} - \mathbf{u}_{i,h},\ r^k_{i,h} = p^k_{i,h} - p_{i,h},\ \mu^k_{ij,h} = l^k_{ij,h} - l_{ij,h} \text{ and } \mu^k_{ji,h} = l^k_{ji,h} - l_{ji,h}. \tag{4.3.23}$$

Then, subtracting (4.3.1)-(4.3.3) from (4.2.42)-(4.2.44), we obtain the following equations:

$$\begin{aligned}(\alpha\,\mathbf{e}^k_{i,h},\mathbf{v})_{\Omega_i} - (r^k_{i,h},\nabla\cdot\mathbf{v})_{\Omega_i} &+ \sum_{j\in N(i)} \beta_{ij}\langle \mathbf{e}^k_{i,h}\cdot\nu^{ij}\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} \\ &= -\sum_{j\in N(i)} \langle \mu^k_{ij,h}\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}}, \qquad \mathbf{v}\in\mathbf{V}_{i,h}, \qquad (4.3.24)\\ (\nabla\cdot\mathbf{e}^k_{i,h},q)_{\Omega_i} + (b\,r^k_{i,h},q)_{\Omega_i} &= 0, \qquad q\in W_{i,h}, \qquad (4.3.25)\end{aligned}$$

and

$$\mu^{k+1}_{ij,h} = 2\,\beta_{ji}\,\mathbf{e}^k_{j,h}\cdot\nu^{ji} + \mu^k_{ji,h} \qquad \text{on} \qquad \Gamma_{ij},\ \forall j\in N(i). \tag{4.3.26}$$

Setting $\mathbf{v} = \mathbf{e}^k_{i,h}$ in (4.3.24) and $q = r^k_{i,h}$ in (4.3.25), and adding the resulting equations, we arrive at the following equality:

$$(\alpha\,\mathbf{e}^k_{i,h},\mathbf{e}^k_{i,h})_{\Omega_i} + (b\,r^k_{i,h},r^k_{i,h})_{\Omega_i} + \sum_{j\in N(i)} \beta_{ij}\langle \mathbf{e}^k_{i,h}\cdot\nu^{ij}\,,\mathbf{e}^k_{i,h}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} = -\sum_{j\in N(i)} \langle \mu^k_{ij,h}\,,\mathbf{e}^k_{i,h}\cdot\nu^{ij}\rangle_{\Gamma_{ij}}.$$

Lemma 4.3.1 *Let $\{\mathbf{e}^k_{i,h},\ r^k_{i,h},\mu^k_{ij,h}\}$ for all i and $j\in N(i)$ satisfy (4.3.24)-(4.3.26). Then, the following identity holds true :*

$$||\mu^k_h||^2_{0,\Gamma} = ||\mu^{k-1}_h||^2_{0,\Gamma} - 4\,\beta \sum_{i=1}^{M} \left\{(\alpha\,\mathbf{e}^{k-1}_{i,h},\mathbf{e}^{k-1}_{i,h})_{\Omega_i} + (b\,r^{k-1}_{i,h},r^{k-1}_{i,h})_{\Omega_i}\right\}, \tag{4.3.27}$$

where $\beta = \beta_{ij} = \beta_{ji}$ *and*

$$||\mu_h^k||_{0,\Gamma}^2 = \sum_{i=1}^{M} \sum_{j\in N(i)} ||\mu_{ij,h}^k||_{0,\Gamma_{ij}}^2. \tag{4.3.28}$$

Proof. From (4.3.26), we arrive at

$$\begin{aligned}\sum_{j\in N(i)} ||\mu_{ij,h}^k||_{0,\Gamma_{ij}}^2 &= \sum_{j\in N(i)} \int_{\Gamma_{ij}} |\mu_{ij,h}^k|^2 ds = \sum_{j\in N(i)} \int_{\Gamma_{ij}} |2\,\beta\,\mathbf{e}_{j,h}^{k-1}\cdot\nu^{ji} + \mu_{ji,h}^{k-1}|^2 ds \\ &= \sum_{j\in N(i)} \int_{\Gamma_{ij}} |\mu_{ij,h}^{k-1}|^2 ds + 4\,\beta \sum_{j\in N(i)} \int_{\Gamma_{ij}} \left(\mu_{ij,h}^{k-1} + \beta\,\mathbf{e}_{i,h}^{k-1}\cdot\nu^{ji}\right)\mathbf{e}_{i,h}^{k-1}\cdot\nu^{ji} ds \\ &= \sum_{j\in N(i)} \int_{\Gamma_{ij}} |\mu_{ij,h}^{k-1}|^2 ds - 4\,\beta\,\left\{(\alpha\,\mathbf{e}_{i,h}^{k-1}, \mathbf{e}_{i,h}^{k-1})_{\Omega_i} + (b\,r_{i,h}^{k-1}, r_{i,h}^{k-1})_{\Omega_i}\right\}. \end{aligned} \tag{4.3.29}$$

Sum up over $i = 1, \cdots, M$ to complete the rest of the proof. ∎

Below, we discuss some lemmas for our future use.

Lemma 4.3.2 (**Local inverse inequality**) *[2, Lemma 4.1, pp. 1304] For any function* $\mathbf{v} \in \mathbf{V}_{i,h}$*, there exists a positive constant* C *independent of* h *and* Ω_i *such that*

$$||\mathbf{v}\cdot\nu^{ij}||_{0,\partial\Omega_i} \le Ch^{-1/2}||\mathbf{v}||_{0,\Omega_i}. \tag{4.3.30}$$

Lemma 4.3.3 *[49, pp. 102] For any function* $\mathbf{v} \in \mathbf{V}_{i,h}$*, there exists a positive constant* C_1 *independent of* h *such that*

$$||\mathbf{v}||_{0,\Omega_i} \le C_1 \left(||\nabla\cdot\mathbf{v}||_{0,\Omega_i} + \sum_{j\in N(i)} ||\mathbf{v}\cdot\nu^{ij}||_{0,\Gamma_{ij}} \right). \tag{4.3.31}$$

Lemma 4.3.4 *[49, pp. 102] Let* $\mathcal{T}_{h,i}$ *be a regular triangulation of* Ω_i *and let* Γ_{ij} *and* Γ_{im} *be the two faces of* Ω_i*, then for any function* $\mathbf{v} \in \mathbf{V}_{i,h}$*, there exists a positive constant* C_2 *such that*

$$||\mathbf{v}\cdot\nu^{ij}||_{0,\Gamma_{ij}} \le C_2 \left(||\nabla\cdot\mathbf{v}||_{0,\Omega_i} + ||\mathbf{v}\cdot\nu^{im}||_{0,\Gamma_{im}}\right). \tag{4.3.32}$$

Theorem 4.3.2 *Let* $\{\mathbf{u}_{i,h}, p_{i,h}, l_{ij,h}\}$*,* $i = 1, 2, \cdots, M$*,* $j \in N(i)$*, be the solutions of the problem (4.3.1)-(4.3.3) and let* $\left\{\mathbf{u}_{i,h}^k, p_{i,h}^k, l_{ij,h}^k\right\}$*,* $i = 1, 2, \cdots, M$*,* $j \in N(i)$*, be the solutions* *of the discrete iterative problem (4.2.42)-(4.2.44) at iterative step k. Then, for any initial guess $\{l^0_{ij,h}, l^0_{ji,h}\} \in \{\Lambda_{ij,h}, \Lambda_{ji,h}\}$, $\forall i$, $\forall j \in N(i)$, the iterative method converges in the sense that*

$$||\mathbf{u}_h^k - \mathbf{u}_h||_{0,\Omega} = \left(\sum_{i=1}^{M} ||\mathbf{u}_{i,h}^k - \mathbf{u}_h||^2_{0,\Omega_i} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty, \tag{4.3.33}$$

$$||p_h^k - p_h||_{0,\Omega} = \left(\sum_{i=1}^{M} ||p_{i,h}^k - p_h||^2_{0,\Omega_i} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty, \tag{4.3.34}$$

and

$$||l_h^k - l_h||_{0,\Gamma} = \left(\sum_{i=1}^{M} \sum_{j \in N(i)} ||l_{ij,h}^k - l_{ij,h}||^2_{0,\Gamma_{ij}} \right)^{1/2} \to 0, \quad \text{as} \quad k \to \infty. \tag{4.3.35}$$

Proof. Since $\mathbf{e}_{i,h}^k = \mathbf{u}_{i,h}^k - \mathbf{u}_{i,h}$, $r_{i,h}^k = p_{i,h}^k - p_{i,h}$ and $\mu_{ij,h}^k = l_{ij,h}^k - l_{ij,h}$, it is enough to show that for each i,

$$||\mathbf{e}_{i,h}^k||^2_{0,\Omega_i} \to 0, \quad \text{as} \quad k \to \infty, \tag{4.3.36}$$

$$||r_{i,h}^k||^2_{0,\Omega_i} \to 0, \quad \text{as} \quad k \to \infty, \tag{4.3.37}$$

$$||\mu_{ij,h}^k||^2_{0,\Gamma_{ij}} \to 0, \quad \text{as} \quad k \to \infty, \ \forall j \in N(i). \tag{4.3.38}$$

From (4.3.28), we note that

$$||\mu_h^k||^2_{0,\Gamma} + 4\,\beta \sum_{i=1}^{M} \left\{ (\alpha\, \mathbf{e}_{i,h}^{k-1}, \mathbf{e}_{i,h}^{k-1})_{\Omega_i} + (b\, r_{i,h}^{k-1}, r_{i,h}^{k-1})_{\Omega_i} \right\} = ||\mu_h^{k-1}||^2_{0,\Gamma}. \tag{4.3.39}$$

Since the second term on the right hand side of (4.3.39) is non-negative, $0 \leq ||\mu_h^k||^2_{0,\Gamma} \leq ||\mu_h^{k-1}||^2_{0,\Gamma}$ and hence, $\{||\mu_h^k||_{0,\Gamma}\}$ is a decreasing sequence of non-negative terms which is bounded above by $||\mu_h^0||_{0,\Gamma}$. Therefore, $\lim\limits_{k\to\infty} ||\mu_h^k||_{0,\Gamma}$ converges. Moreover,

$$4\,\beta \sum_{i=1}^{M} \left\{ (\alpha\, \mathbf{e}_{i,h}^{k-1}, \mathbf{e}_{i,h}^{k-1})_{\Omega_i} + (b\, r_{i,h}^{k-1}, r_{i,h}^{k-1})_{\Omega_i} \right\} = ||\mu_h^{k-1}||^2_{0,\Gamma} - ||\mu_h^k||^2_{0,\Gamma}. \tag{4.3.40}$$

On summing up $k = 1$ to N_s, we obtain

$$\begin{aligned} 4\,\beta \sum_{k=1}^{N_s} \sum_{i=1}^{M} \left\{ (\alpha\, \mathbf{e}_{i,h}^{k-1}, \mathbf{e}_{i,h}^{k-1})_{\Omega_i} + (b\, r_{i,h}^{k-1}, r_{i,h}^{k-1})_{\Omega_i} \right\} &= \sum_{k=1}^{N_s} \left(||\mu_h^{k-1}||^2_{0,\Gamma} - ||\mu_h^k||^2_{0,\Gamma} \right) \\ &= ||\mu_h^0||^2_{0,\Gamma} - ||\mu_h^{N_s}||^2_{0,\Gamma} \leq 2\,||\mu_h^0||^2_{0,\Gamma} \end{aligned} \tag{4.3.41}$$

and, hence,

$$\sum_{k=1}^{\infty}\sum_{i=1}^{M}\left\{(\alpha\,\mathbf{e}_{i,h}^{k},\mathbf{e}_{i,h}^{k})_{\Omega_j}+(b\,r_{i,h}^{k},r_{i,h}^{k})_{\Omega_j}\right\}<\infty. \tag{4.3.42}$$

Thus,

$$(\alpha\,\mathbf{e}_{i,h}^{k},\mathbf{e}_{i,h}^{k})_{\Omega_j}\to 0 \quad\text{as}\quad k\to\infty, \qquad i=1,2,\cdots,M. \tag{4.3.43}$$

Therefore,

$$\mathbf{e}_{i,h}^{k}\to 0 \quad\text{in}\quad L^2(\Omega_i) \quad\text{as}\quad k\to\infty, \qquad i=1,2,\cdots,M. \tag{4.3.44}$$

Using Lemma 4.3.2 and (4.3.44), we find that for fixed h

$$||\mathbf{e}_{i,h}^{k}\cdot\nu^{ij}||_{0,\partial\Omega_i}\to 0 \quad\text{as}\quad k\to\infty, \qquad i=1,2,\cdots,M. \tag{4.3.45}$$

In particular,

$$\mathbf{e}_{i,h}^{k}\cdot\nu^{ij}\to 0 \quad\text{in}\quad L^2(\Gamma_{ij}) \quad\text{as}\quad k\to\infty, \qquad \forall\, i,\ \forall\, j\in N(i). \tag{4.3.46}$$

If the function $b(x)\geq b_0>0$ on Ω, then it follows from (4.3.42) that

$$r_{i,h}^{k}\to 0 \quad\text{in}\quad L^2(\Omega_i) \quad\text{as}\quad k\to\infty, \qquad i=1,2,\cdots,M. \tag{4.3.47}$$

But we have to prove this in general case, i.e., $b(x)\geq 0$. First we consider the subdomains $\Omega_i\in D_1$, that is, one face of the subdomains Ω_i, which belongs to the boundary $\partial\Omega$. Choose $\mathbf{v}\in\mathbf{V}_{i,h}$, for all i, $\Omega_i\in D_1$, such that

$$\nabla\cdot\mathbf{v}=r_{i,h}^{k} \qquad \text{on} \qquad \Omega_i \qquad \text{and} \qquad \mathbf{v}\cdot\nu^{ij}=0 \qquad \text{on} \qquad \Gamma_i. \tag{4.3.48}$$

Substituting (4.3.48) in (4.3.24) and using Lemma 4.3.3, we obtain

$$||r_{i,h}^{k}||_{0,\Omega_i}^{2}=(\alpha\,\mathbf{e}_{i,h}^{k},\mathbf{v})_{\Omega_i}\leq C||\mathbf{e}_{i,h}^{k}||_{0,\Omega_i}||\mathbf{v}||_{0,\Omega_i}\leq C||\mathbf{e}_{i,h}^{k}||_{0,\Omega_i}||r_{i,h}^{k}||_{0,\Omega_i}. \tag{4.3.49}$$

Therefore,

$$||r_{i,h}^{k}||_{0,\Omega_i}\to 0, \quad\text{as}\quad k\to\infty, \quad\text{for all } i, \quad\text{where}\quad \Omega_i\in D_1. \tag{4.3.50}$$

Other way around, we choose $\mathbf{v} \in \mathbf{V}_{i,h}$, for all i, $\Omega_i \in D_1$, such that

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{on} \qquad \Omega_i \;\; \text{and} \;\; \mathbf{v} \cdot \nu^{ij} = \begin{cases} -\mu^k_{ij,h} & \text{on } \; \Gamma_{ij}, \\ 0 & \text{on } \; \Gamma_{im}, \; m \neq j. \end{cases} \tag{4.3.51}$$

Substituting (4.3.51) in (4.3.24) and using Lemma 4.3.3, we arrive at

$$\begin{aligned} ||\mu^k_{ij,h}||^2_{0,\Gamma_{ij}} &= (\alpha\, \mathbf{e}^k_{i,h}, \mathbf{v})_{\Omega_i} + \beta\, \langle \mathbf{e}^k_{i,h} \cdot \nu^{ij} \,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} \\ &\leq C||\mathbf{e}^k_{i,h}||_{0,\Omega_i} ||\mathbf{v}||_{0,\Omega_i} + C\beta\, ||\mathbf{e}^k_{i,h} \cdot \nu^{ij}||_{0,\Gamma_{ij}} ||\mathbf{v} \cdot \nu^{ij}||_{0,\Gamma_{ij}} \\ &\leq C \left(||\mathbf{e}^k_{i,h}||_{0,\Omega_i} + \beta\, ||\mathbf{e}^k_{i,h} \cdot \nu_i||_{0,\Gamma_{ij}} \right) ||\mathbf{v} \cdot \nu_i||_{0,\Gamma_{ij}} \\ &= C \left(||\mathbf{e}^k_{i,h}||_{0,\Omega_i} + \beta\, ||\mathbf{e}^k_{i,h} \cdot \nu_i||_{0,\Gamma_{ij}} \right) ||\mu^k_{ij,h}||_{0,\Gamma_{ij}}. \end{aligned} \tag{4.3.52}$$

Using (4.3.44) and (4.3.46) in (4.3.52), we find that for all i, $\Omega_i \in D_1$

$$||\mu^k_{ij,h}||_{0,\Gamma_{ij}} \to 0 \;\; k \to \infty \qquad j \in N(i). \tag{4.3.53}$$

Thus, we have proved convergence of $\mathbf{u}^k_{i,h}$, $p^k_{i,h}$, $l^k_{ij,h}$ on boundary subdomains ($\Omega_i \in D_1$). Now, we consider a subdomain, which shares at least one interface with boundary subdomains, and having a common face Γ_{im} with one of the boundary elements, i.e., for all i, $\Omega_i \in D_2$. From (4.3.26) with $\beta = \beta_{ij} = \beta_{ji}$, it follows that

$$\mu^k_{ij,h} = 2\,\beta\, \mathbf{e}^{k-1}_{j,h} \cdot \nu^{ji} + \mu^{k-1}_{ji,h} \qquad \forall x \in \Gamma_{ij}, \; j \in N(i). \tag{4.3.54}$$

Using (4.3.46) and (4.3.53) in (4.3.54), we obtain for all i, $\Omega_i \in D_2$

$$||\mu^k_{ij,h}||_{0,\Gamma_{ij}} \to 0, \;\; \text{as} \;\; k \to \infty, \;\; \text{where} \;\; \Omega_j \in D_1. \tag{4.3.55}$$

Now, we choose $\mathbf{v} \in \mathbf{V}_{i,h}$, for all i, $\Omega_i \in D_2$, such that

$$\nabla \cdot \mathbf{v} = r^k_{i,h} \qquad \text{on} \qquad \Omega_i \;\; \text{and} \;\; \mathbf{v} \cdot \nu^{ij} = \begin{cases} 0 & \text{on} \quad \Gamma_{ij}, \; m \neq j \in N(i), \\ \mathbf{v} \cdot \nu^{im} & \text{on} \quad \Gamma_{im}, \; m \neq j. \end{cases} \tag{4.3.56}$$

Substituting (4.3.56) in (4.3.24) and using Lemma 4.3.3, we obtain

$$\begin{aligned} ||r^k_{i,h}||^2_{0,\Omega_i} &= (\alpha\, \mathbf{e}^k_{i,h}, \mathbf{v})_{\Omega_i} + \beta\, \langle \mathbf{e}^k_{i,h} \cdot \nu^{im} \,, \mathbf{v} \cdot \nu^{im} \rangle_{\Gamma_{im}} + \langle \mu^k_{ij,h} \,, \mathbf{v} \cdot \nu^{im} \rangle_{\Gamma_{im}} \\ &\leq C||\mathbf{e}^k_{i,h}||_{0,\Omega_i} ||\mathbf{v}||_{0,\Omega_i} + C \left(\beta\, ||\mathbf{e}^k_{i,h} \cdot \nu^{im}||_{0,\Gamma_{im}} + ||\mu^k_{im,h}||_{0,\Gamma_{im}} \right) ||\mathbf{v} \cdot \nu^{im}||_{0,\Gamma_{im}} \\ &\leq C||\mathbf{e}^k_{i,h}||_{0,\Omega_i} ||r^k_{i,h}||_{0,\Omega_i} \\ &+ C \left(||\mathbf{e}^k_{i,h}||_{0,\Omega_i} + \beta\, ||\mathbf{e}^k_{i,h} \cdot \nu^{im}||_{0,\Gamma_{im}} + ||\mu^k_{im,h}||_{0,\Gamma_{im}} \right) ||\mathbf{v} \cdot \nu^{im}||_{0,\Gamma_{im}}. \end{aligned} \tag{4.3.57}$$

First we have to use Lemma 4.3.4 in (4.3.57) and then using (4.3.56), (4.3.44), (4.3.46) and (4.3.55), we arrive at

$$||r_{i,h}^k||_{0,\Omega_i} \to 0 \quad \text{as} \quad k \to \infty \quad \text{for all } i, \quad \text{where} \quad \Omega_i \in D_2. \tag{4.3.58}$$

Similarly, we can continue the argument until the domain is exhausted and this completes the rest of the proof. ■

We now recall the spaces defined earlier in (4.2.21) and (4.2.26),

$$\mathbf{V}_h = \prod_{i=1}^{M} \mathbf{V}_{i,h}, \qquad W_h = \prod_{i=1}^{M} W_{i,h}, \qquad \Lambda_h = \prod_{i=1}^{M} \prod_{j\in N(i)} \Lambda_{ij,h}.$$

Also, let $T_{f,g} : \mathbf{V}_h \times W_h \times \Lambda_h \to \mathbf{V}_h \times W_h \times \Lambda_h$ be an affine mapping such that for any $(\mathbf{z}_h, w_h, \eta_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$, $(\mathbf{m}_h, d_h, \theta_h) \equiv T_{f,g}(\mathbf{z}_h, w_h, \eta_h)$ is the solution, for all i, of

$$\begin{aligned}(\alpha\, \mathbf{m}_{i,h}, \mathbf{v})_{\Omega_i} - (d_{i,h}, \nabla\cdot\mathbf{v})_{\Omega_i} + \sum_{j\in N(i)} \beta_{ij} \langle \mathbf{m}_{i,h}\cdot\nu^{ij}, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} \\ = -\sum_{j\in N(i)} \langle \theta_{ij,h}, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma}, \ \mathbf{v}\in\mathbf{V}_{i,h}, \quad (4.3.59) \\ (\nabla\cdot\mathbf{m}_{i,h}, q)_{\Omega_i} + (b\, d_{i,h}, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad q \in W_{i,h}, \quad (4.3.60)\end{aligned}$$

and

$$\theta_{ij,h} = 2\,\beta_{ji}\, \mathbf{z}_{j,h}\cdot\nu^{ji} + \eta_{ji,h} \qquad \text{on} \qquad \Gamma_{ij}, \ \forall j \in N(i), \tag{4.3.61}$$

where $\alpha = K^{-1}$, $\mathbf{m}_{i,h} = \mathbf{m}_{h|_{\Omega_i}}$, $\mathbf{z}_{i,h} = \mathbf{z}_{h|_{\Omega_i}}$, $d_{i,h} = d_{h|_{\Omega_i}}$, $w_{i,h} = w_{h|_{\Omega_i}}$, $\theta_{ij,h} = \theta_{h|_{\Gamma_{ij}}}$, $\theta_{ji,h} = \theta_{h|_{\Gamma_{ij}}}$, $\eta_{ij,h} = \eta_{h|_{\Gamma_{ij}}}$ and $\eta_{ji,h} = \eta_{h|_{\Gamma_{ij}}}$.

Lemma 4.3.5 *The triple* $(\mathbf{u}_h, p_h, l_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$ *is the solution of (4.3.1)-(4.3.3) if and only if it is a fixed point of* $T_{f,g}$. *Moreover, if* $(\mathbf{u}_h, p_h, l_h)$ *is a fixed point of* $T_{f,g}$, *then* $\mathbf{u}_{i,h}\cdot\nu^{ij} = -\mathbf{u}_{j,h}\cdot\nu^{ji}$ *for all* Γ_{ij}.

Proof. Observe that if $(\mathbf{u}_h, p_h, l_h)$ is a fixed point of $T_{f,g}$, then $T_{f,g}(\mathbf{u}_h, p_h, l_h) = (\mathbf{u}_h, p_h, l_h)$ and hence, $(\mathbf{u}_h, p_h, l_h)$ is a solution of (4.3.1)-(4.3.3). Conversely, if $(\mathbf{u}_h, p_h, l_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$ is a solution of (4.3.1)-(4.3.3), then, it is straight forward to check that it is a fixed point of $T_{f,g}$. For the second part, let $(\mathbf{u}_h, p_h, l_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$ be a fixed point of $T_{f,g}$, i.e.,

$T_{f,g}(\mathbf{u}_h, p_h, l_h) = (\mathbf{u}_h, p_h, l_h)$. Then replacing θ_h by l_h and η_h by l_h from (4.3.61), we note that

$$l_{ij,h} = 2\,\beta_{ji}\,\mathbf{u}_{j,h}\cdot\nu^{ji} + l_{ji,h}, \tag{4.3.62}$$
$$l_{ji,h} = 2\,\beta_{ij}\,\mathbf{u}_{i,h}\cdot\nu^{ij} + l_{ij,h}. \tag{4.3.63}$$

Summing (4.3.62) and (4.3.63), we arrive at

$$\beta_{ij}\,\mathbf{u}_{i,h}\cdot\nu^{ij} + \beta_{ji}\,\mathbf{u}_{j,h}\cdot\nu^{ji} = 0. \tag{4.3.64}$$

Here $\beta = \beta_{ij} = \beta_{ji}$ and this completes the rest of the proof. ■

Since the operator $T_{f,g}(\mathbf{z}_h, w_h, \eta_h)$ is linear, we can split the operator $T_{f,g}(\mathbf{z}_h, w_h, \eta_h)$ into a sum of two operators $T_{0,0}(\mathbf{z}_h, w_h, \eta_h)$ and $T_{f,g}(0,0,0)$, i.e.,

$$T_{f,g}(\mathbf{z}_h, w_h, \eta_h) = T_{0,0}(\mathbf{z}_h, w_h, \eta_h) + T_{f,g}(0,0,0), \tag{4.3.65}$$

where $T_{0,0}(\mathbf{z}_h, w_h, \eta_h)$ and $T_{f,g}(0,0,0)$ are defined as follows: Given $(\mathbf{z}_h, w_h, \eta_h)$, the operator $(\mathbf{m}_h^\star, d_h^\star, \theta_h^\star) = T_{0,0}(\mathbf{z}_h, w_h, \eta_h)$ is defined for all i through

$$(\alpha\,\mathbf{m}_{i,h}^\star, \mathbf{v})_{\Omega_i} - (d_{i,h}^\star, \nabla\cdot\mathbf{v})_{\Omega_i} + \sum_{j\in N(i)} \beta_{ij}\langle \mathbf{m}_{i,h}^\star\cdot\nu^{ij}\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} = -\sum_{j\in N(i)} \langle \theta_{ij,h}^\star\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}},\ \ \mathbf{v}\in\mathbf{V}_{i,h}, \tag{4.3.66}$$
$$(\nabla\cdot\mathbf{m}_{i,h}^\star, q)_{\Omega_i} + (b\,d_{i,h}^\star, q)_{\Omega_i} = 0, \qquad q\in W_{i,h}, \tag{4.3.67}$$

and

$$\theta_{ij,h}^\star = 2\,\beta_{ji}\,\mathbf{z}_{j,h}\cdot\nu^{ji} + \eta_{ji,h} \qquad \text{on} \qquad \Gamma_{ij},\ \forall j\in N(i), \tag{4.3.68}$$

and $(\mathbf{m}_h^\star, d_h^\star, \theta_h^\star) = T_{f,g}(0,0,0)$ satisfies for all i,

$$(\alpha\,\mathbf{m}_{i,h}^o, \mathbf{v})_{\Omega_i} - (d_{i,h}^o, \nabla\cdot\mathbf{v})_{\Omega_i} + \sum_{j\in N(i)} \beta_{ij}\langle \mathbf{m}_{i,h}^o\cdot\nu^{ij}\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} = \sum_{j\in N(i)} \langle \theta_{ij,h}^o\,,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} - \langle g\,,\mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma},\ \ \mathbf{v}\in\mathbf{V}_{i,h}, \tag{4.3.69}$$
$$(\nabla\cdot\mathbf{m}_{i,h}^o, q)_{\Omega_i} + (b\,d_{i,h}^o, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad q\in W_{i,h}, \tag{4.3.70}$$

and

$$\theta^o_{ij,h} = 0 \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i), \tag{4.3.71}$$

Then $(\mathbf{m}_h, d_h, \theta_h) = (\mathbf{m}^\star_h, d^\star_h, \theta^\star_h) + (\mathbf{m}^o_h, d^o_h, \theta^o_h)$.
Then the fixed point $(\mathbf{u}_h, p_h, l_h)$ of $T_{f,g}$, that is, $T_{f,g}(\mathbf{u}_h, p_h, l_h) = (\mathbf{u}_h, p_h, l_h)$ is characterized as a solution of

$$(I - T_{0,0})(\mathbf{u}_h, p_h, l_h) = T_{f,g}(0, 0, 0). \tag{4.3.72}$$

Observe that the problem (4.3.24) - (4.3.26) can be written in abstract form as

$$(\mathbf{e}^k_h, r^k_h, \mu^k_h) = T_{0,0}(\mathbf{e}^{k-1}_h, r^{k-1}_h, \mu^{k-1}_h). \tag{4.3.73}$$

Now our next aim to find the spectral radius of $T_{0,0}$.

Remark 4.3.1 *Here $\mathbf{V}_h \times W_h \times \Lambda_h$ is a real linear space and $T_{0,0}$ is a real linear operator. In general, the spectral radius formula does not hold for the real case. So the complexification of the real linear spaces and the real linear operators are necessary.*

Now, we recall the linear operator $T_{0,0}$ defined in (4.3.73) and the linear space $\mathbf{V}_h \times W_h \times \Lambda_h$ defined in (4.2.21) and (4.2.26). Our main idea to find $||T^k_{0,0}||$, i.e., $||T^k_{0,0}||$ is dominated by $\rho(\bar{T}_{0,0})$, where $\bar{T}_{0,0} = 1 \otimes T_{0,0}$ is the complexification of $T_{0,0}$ (see, subsection 1.2.2) and $\rho(\bar{T}_{0,0})$ is the spectral radius of $\bar{T}_{0,0}$. The next lemma shows the relation between $||T^k_{0,0}||$ and $\rho(\bar{T}_{0,0})$.

Lemma 4.3.6 *If $\mathbf{V}_h \times W_h \times \Lambda_h$ is equipped with an inner-product and*

$$\rho(\bar{T}_{0,0}) \leq 1 - R, \qquad R \in (0, 1), \tag{4.3.74}$$

then for each positive integer k, there exists a constant C independent of k such that

$$||T^k_{0,0}|| \leq C(1 - R/2)^k. \tag{4.3.75}$$

Proof. From Lemmas 1.2.13 and 1.2.14, we find that

$$||\bar{T}^k_{0,0}|| = ||T^k_{0,0}||. \tag{4.3.76}$$

Since $\bar{T}_{0,0}$ is a complex linear operator on the complex linear space $\mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$, by the spectral radius formula

$$\rho(\bar{T}_{0,0}) = \lim_{k\to\infty} ||\bar{T}_{0,0}^k||^{1/k}, \tag{4.3.77}$$

that is for $\epsilon > 0$, there exists a natural number N_m such that when $k > N_m$, we arrive at

$$||\bar{T}_{0,0}^k||^{1/k} \leq \rho(\bar{T}_{0,0}) + \epsilon,$$

and hence,

$$||\bar{T}_{0,0}^k|| \leq (\rho(\bar{T}_{0,0}) + \epsilon)^k.$$

Choose a constant $C > 1$ such that

$$||\bar{T}_{0,0}^k|| \leq C(\rho(\bar{T}_{0,0}) + \epsilon)^k$$

for $k = 1, 2, \cdots, N$. Then $\forall k$

$$||T_{0,0}^k|| = ||\bar{T}_{0,0}^k|| \leq C(\rho(\bar{T}_{0,0}) + \epsilon)^k. \tag{4.3.78}$$

With $\epsilon = R/2$ in (4.3.78), we complete the rest of the proof. ∎

We have complexify only the operator $T_{0,0}$ and the space $\mathbf{V}_h \times W_h \times \Lambda_h$. In our subsequent analysis, we need also the complexification of other real linear spaces such as $\mathbf{V}_{i,h}, W_{i,h}$, $\Lambda_{ij,h}$ and $\Lambda_{ji,h}$.

4.4 Spectral radius

In Section 4.3, we have discussed the convergence of the proposed iterative scheme in Theorem 4.3.2. Now in this section, we plan to derive the rate of convergence of the iterative procedure.

4.4.1 Spectral radius without quasi-uniformity assumptions

Let $(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) \in \mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$, i.e.,

$$(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) = (\tilde{\mathbf{m}}_h, \tilde{d}_h, \tilde{\theta}_h) + \sqrt{(-1)}(\hat{\mathbf{m}}_h, \hat{d}_h, \hat{\theta}_h), \tag{4.4.1}$$

where $(\tilde{\mathbf{m}}_h, \tilde{d}_h, \tilde{\theta}_h), (\hat{\mathbf{m}}_h, \hat{d}_h, \hat{\theta}_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$. Using Lemma 1.2.12, we obtain the following identities.

Lemma 4.4.1 *Let* $(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) \in \mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$, *and* $(\tilde{\mathbf{m}}_h, \tilde{d}_h, \tilde{\theta}_h), (\hat{\mathbf{m}}_h, \hat{d}_h, \hat{\theta}_h) \in \mathbf{V}_h \times W_h \times \Lambda_h$ *satisfy (4.4.1). Then*

$$||\bar{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} = ||\tilde{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} + ||\hat{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} \tag{4.4.2}$$

$$||\nabla \cdot \bar{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} = ||\nabla \cdot \tilde{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} + ||\nabla \cdot \hat{\mathbf{m}}_{i,h}||^2_{0,\Omega_i} \tag{4.4.3}$$

$$||\bar{d}_{i,h}||^2_{0,\Omega_i} = ||\tilde{d}_{i,h}||^2_{0,\Omega_i} + ||\hat{d}_{i,h}||^2_{0,\Omega_i} \tag{4.4.4}$$

$$||\bar{\theta}_{ij,h}||^2_{0,ij} = ||\tilde{\theta}_{ij,h}||^2_{0,ij} + ||\hat{\theta}_{ij,h}||^2_{0,ij}, \tag{4.4.5}$$

and

$$||\bar{\mathbf{m}}_{i,h} \cdot \nu^{ij}||^2_{0,ij} = ||\tilde{\mathbf{m}}_{i,h} \cdot \nu^{ij}||^2_{0,ij} + ||\hat{\mathbf{m}}_{i,h} \cdot \nu^{ij}||^2_{0,ij}. \tag{4.4.6}$$

Lemma 4.4.2 *Let* $(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) \in \mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$, *with* $(\bar{\mathbf{m}}_{i,h}, \bar{d}_{i,h}, \bar{\theta}_{ij,h}) = (\tilde{\mathbf{m}}_{i,h}, \tilde{d}_{i,h}, \tilde{\theta}_{ij,h}) + \sqrt{(-1)}(\hat{\mathbf{m}}_{i,h}, \hat{d}_{i,h}, \hat{\theta}_{ij,h})$, *where* $(\tilde{\mathbf{m}}_{i,h}, \tilde{d}_{i,h}, \tilde{\theta}_{ij,h}), (\hat{\mathbf{m}}_{i,h}, \hat{d}_{i,h}, \hat{\theta}_{ij,h}) \in \mathbf{V}_{i,h} \times W_{i,h} \times \Lambda_{ij,h}$ *are the solutions of (4.3.66)-(4.3.68). Then the following identity holds true :*

$$||\bar{\theta}_h||^2_{0,\Gamma} = ||\bar{\theta}_h||^2_{0,\Gamma} - 4\,\beta \sum_{i=1}^{M} \left\{ (\alpha\, \bar{\mathbf{m}}_{i,h}, \bar{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \bar{d}_{i,h}, \bar{d}_{i,h})_{\Omega_i} \right\}, \tag{4.4.7}$$

where $\beta = \beta_{ij} = \beta_{ji}$ *and*

$$||\bar{\theta}_h||^2_{0,\Gamma} = \sum_{i=1}^{M} \sum_{j \in N(i)} ||\bar{\theta}_{ij,h}||^2_{0,\Gamma_{ij}}. \tag{4.4.8}$$

Proof. By Lemma 4.4.1, we find that

$$\begin{aligned} \sum_{j \in N(i)} ||\bar{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} &= \sum_{j \in N(i)} ||\tilde{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} + \sum_{j \in N(i)} ||\hat{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} \\ &= I_1 + I_2. \end{aligned} \tag{4.4.9}$$

Since $(\tilde{\mathbf{m}}_{i,h}, \tilde{d}_{i,h}, \tilde{\theta}_{ij,h}) and (\hat{\mathbf{m}}_{i,h}, \hat{d}_{i,h}, \hat{\theta}_{ij,h}) \in \mathbf{V}_{i,h} \times W_{i,h} \times \Lambda_{ij,h}$, then by Lemma 4.3.1, we obtain

$$I_1 = \sum_{j \in N(i)} ||\tilde{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} ds - 4\,\beta \left\{ (\alpha\, \tilde{\mathbf{m}}_{i,h}, \tilde{\mathbf{m}}_{i,h})_{\Omega_j} + (b\, \tilde{d}_{i,h}, \tilde{d}_{i,h})_{\Omega_j} \right\}, \tag{4.4.10}$$

and

$$I_2 = \sum_{j\in N(i)} ||\hat{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} ds - 4\,\beta \left\{ (\alpha\, \hat{\mathbf{m}}_{i,h}, \hat{\mathbf{m}}_{i,h})_{\Omega_j} + (b\, \hat{d}_{i,h}, \hat{d}_{i,h})_{\Omega_j} \right\}. \tag{4.4.11}$$

From (4.4.9)-(4.4.11) and Lemma 4.4.1, we arrive at (4.4.7) and this completes the rest of the proof. ■

Lemma 4.4.3 *Let $(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) \in \mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$ be an eigenvector of $\bar{T}_{0,0}$ such that $\bar{T}_{0,0}(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) = \gamma\, (\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h)$. Then the following identity holds true :*

$$\gamma\, \bar{\theta}_{ij,h} = 2\,\beta\, \bar{\mathbf{m}}_j \cdot \nu^{ji} + \bar{\theta}_{ji,h} \qquad \forall x \in \Gamma_{ij},\ j \in N(i). \tag{4.4.12}$$

Theorem 4.4.1 *Let $\rho(\bar{T}_{0,0})$ be the spectral radius of $\bar{T}_{0,0}$. Then*

$$\rho(\bar{T}_{0,0}) < 1. \tag{4.4.13}$$

Proof. Let γ be an eigenvalue of $\bar{T}_{0,0}$ and let $(\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h) \neq (0,0)$ be its corresponding eigenvector, i.e.,

$$\bar{T}_{0,0}(\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h) = \gamma\, (\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h). \tag{4.4.14}$$

It follows from (4.4.12) and Lemma 4.4.2 that

$$\gamma^2\, ||\bar{\theta}_h||^2_{0,\Gamma} = ||\bar{\theta}_h||^2_{0,\Gamma} - 4\,\beta \sum_{i=1}^{M} \left\{ (\alpha\, \bar{\mathbf{m}}_{i,h}, \bar{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \bar{d}_{i,h}, \bar{d}_{i,h})_{\Omega_i} \right\}, \tag{4.4.15}$$

Therefore,

$$|\gamma|^2 = 1 - \frac{4\,\beta}{||\bar{\theta}_h||^2_{0,\Gamma}} \sum_{i=1}^{M} \left\{ (\alpha\, \bar{\mathbf{m}}_{i,h}, \bar{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \bar{d}_{i,h}, \bar{d}_{i,h})_{\Omega_i} \right\}. \tag{4.4.16}$$

From (4.4.16), we concluded that $|\gamma| \leq 1$. Note that $|\gamma| = 1$ if and only if

$$(\alpha\, \tilde{\mathbf{m}}_{i,h}, \tilde{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \tilde{d}_{i,h}, \tilde{d}_{i,h})_{\Omega_i} = 0 \qquad \forall\, i = 1, 2, \cdots, M, \tag{4.4.17}$$

and

$$(\alpha\, \hat{\mathbf{m}}_{i,h}, \hat{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \hat{d}_{i,h}, \hat{d}_{i,h})_{\Omega_i} = 0 \qquad \forall\, i = 1, 2, \cdots, M. \tag{4.4.18}$$

Then applying the argument used in the proof of Theorem 4.3.2, it is easy to show that $(\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h)$ is trivial, i.e., $(\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h) = (0,0,0)$ and this leads to a contradiction as $(\bar{\mathbf{m}}_h, \bar{w}_h, \bar{\theta}_h)$ is an eigenvector of $T_{0,0}$. Hence, $|\gamma| < 1$ and this completes the rest of the proof. ■

4.4.2 Rate of convergence with quasi-uniformity assumption on the mesh

In this subsection, we estimate the spectral radius and derive the rate of convergence of the iterative method under the quasi-uniformity assumption on the mesh in each Ω_i.
From (4.4.20), we obtain

$$|\gamma|^2 \leq 1 - \frac{1}{Q_0}, \tag{4.4.19}$$

where $1 < Q_0 < \infty$ is such that

$$||\bar{\theta}_h||^2_{0,\Gamma} \leq 4\, Q_0\, \beta \sum_{i=1}^{M} \left\{ (\alpha\, \bar{\mathbf{m}}_{i,h}, \bar{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \bar{d}_{i,h}, \bar{d}_{i,h})_{\Omega_i} \right\}. \tag{4.4.20}$$

Note that estimation of Q_0 with yields the convergence rate for the iterative procedure (4.2.42)-(4.2.44).

Lemma 4.4.4 *Let $(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) \in \mathbb{C} \otimes (\mathbf{V}_h \times W_h \times \Lambda_h)$ be an eigenvector of $\bar{T}_{0,0}$ and let γ be its corresponding to an eigenvalue, i.e., $\bar{T}_0(\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h) = \gamma\, (\bar{\mathbf{m}}_h, \bar{d}_h, \bar{\theta}_h)$. Then*

$$\sum_{i=1}^{M} \sum_{j \in N(i)} ||\bar{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} \leq C\, \beta^{-1} \left(C_3 + \beta^2\, h^{-1} + H_\star^{-1}\, b^{-1} \right) \beta \sum_{i=1}^{M} \{ (\alpha\, \bar{\mathbf{m}}_{i,h}, \bar{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \bar{d}_{i,h}, \bar{d}_{i,h})_{\Omega_i} \}, \tag{4.4.21}$$

where C is independent of Γ_{ij} and β and $H_\star$ is the minimum diameter of the subdomains.
Proof. It is enough to show that

$$\sum_{j \in N(i)} ||\tilde{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} \leq C\, \beta^{-1} \left(C_3 + \beta^2\, h^{-1} + H_\star\, b^{-1} \right) \beta \left\{ (\alpha\, \tilde{\mathbf{m}}_{i,h}, \tilde{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \tilde{d}_{i,h}, \tilde{d}_{i,h})_{\Omega_i} \right\}, \tag{4.4.22}$$

and

$$\sum_{j \in N(i)} ||\hat{\theta}_{ij,h}||^2_{0,\Gamma_{ij}} \leq C\, \beta^{-1} \left(C_3 + \beta^2\, h^{-1} + H_\star\, b^{-1} \right) \beta \left\{ (\alpha\, \hat{\mathbf{m}}_{i,h}, \hat{\mathbf{m}}_{i,h})_{\Omega_i} + (b\, \hat{d}_{i,h}, \hat{d}_{i,h})_{\Omega_i} \right\}. \tag{4.4.23}$$

From (4.3.66), we observe that

$$(\alpha\, \mathbf{m}_{i,h}, \mathbf{v})_{\Omega_i} - (d_{i,h}, \nabla\cdot\mathbf{v})_{\Omega_i} + \sum_{j\in N(i)} \beta\langle \mathbf{m}_{i,h}\cdot\nu^{ij}\,, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} = -\sum_{j\in N(i)} \langle \theta_{ij,h}, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}}. \tag{4.4.24}$$

Now, we choose $\mathbf{v}\in\mathbf{V}_{i,h}$, for all i, such that

$$\mathbf{v}\cdot\nu^{ij} = -\theta_{ij,h}, \qquad \nabla\cdot\mathbf{v} = \tilde{S}_i = -\frac{1}{|\Omega_i|}\sum_{j\in N(i)}\int_{\Gamma_{ij}} \theta_{ij,h}\, ds, \tag{4.4.25}$$

then

$$||\mathbf{v}||^2_{0,\Omega_i} \le C_3 \sum_{j\in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}}. \tag{4.4.26}$$

Substituting (4.4.25) in (4.4.24), we obtain

$$\begin{aligned} \sum_{j\in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}} &= (\alpha\, \mathbf{m}_{i,h}, \mathbf{v})_{\Omega_i} + |\tilde{S}_i|(d_{i,h}, 1)_{\Omega_i} + \sum_{j\in N(i)} \beta\,\langle \mathbf{m}_{i,h}\cdot\nu^{ij}\,, \mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}} \\ &\le C\Bigg(||\mathbf{m}_{i,h}||_{0,\Omega_i}||\mathbf{v}||_{0,\Omega_i} + |\tilde{S}_i|\,\sqrt{|\Omega_i|}\;||d_{i,h}||_{0,\Omega_i} \\ &\qquad + \sum_{j\in N(i)} \beta\,||\mathbf{m}_{i,h}\cdot\nu^{ij}||_{0,\Gamma_{ij}}||\mathbf{v}\cdot\nu^{ij}||_{0,\Gamma_{ij}}\Bigg). \end{aligned} \tag{4.4.27}$$

Using Cauchy-Schwarz inequality, we find that

$$|\tilde{S}_i| \le \frac{|\partial\Omega_i|^{1/2}}{|\Omega_i|}\left(\sum_{j\in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}}\right)^{1/2}. \tag{4.4.28}$$

Together with (4.4.28), (4.4.26) in (4.4.27) and then applying Lemma 4.3.2, we arrive at

$$\begin{aligned} \sum_{j\in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}} \le C\,\Big(C_3||\mathbf{m}_{i,h}||_{0,\Omega_i} + \sqrt{|\partial\Omega_i|/|\Omega_i|}\;||d_{i,h}||_{0,\Omega_i} + C\,\beta\, h^{-1/2}\,||\mathbf{m}_{i,h}||_{0,\Omega_i}\Big) \\ \times\left(\sum_{j\in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}}\right)^{1/2}. \end{aligned} \tag{4.4.29}$$

Now eliminating first $\left(\sum_{j \in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}} \right)^{1/2}$ from right hand side of (4.4.29) and squaring both sides, it follows that

$$\sum_{j \in N(i)} ||\theta_{ij,h}||^2_{0,\Gamma_{ij}} \leq C\,\beta^{-1}\left(C_3 + \beta^2\,h^{-1} + |\partial\Omega_i|/|\Omega_i|\,b^{-1}\right)\,\beta\,\{(\alpha\,\mathbf{m}_{i,h}, \mathbf{m}_{i,h})_{\Omega_i} + (b\,d_{i,h}, d_{i,h})_{\Omega_i}\}\,. \quad (4.4.30)$$

The bound of $|\partial\Omega_i|/|\Omega_i|$ is less than $C\,H_\star^{-1}$, where $H_\star$ is minimum diameter of the subdomains. Since $(\tilde{\mathbf{m}}_{i,h}, \tilde{d}_{i,h}, \tilde{\theta}_{ij,h}), (\tilde{\mathbf{m}}_{i,h}, \tilde{d}_{i,h}, \tilde{\theta}_{ij,h}) \in \mathbf{V}_{i,h} \times W_{i,h} \times \Lambda_{ij,h}$ and satisfies the equation (4.4.24). We, therefore obtain (4.4.22) and (4.4.23) from (4.4.30). This completes the rest of the proof. ■

From the estimate (4.4.21), we obtain

$$4\,Q_0 = C\,\beta^{-1}\left(C_3 + \beta^2\,h^{-1} + H_\star^{-1}\,b^{-1}\right). \quad (4.4.31)$$

Theorem 4.4.2 *Let the parameter $\beta = \beta_{ij} = \beta_{ji}$, $b(x) \geq b_0 > 0$, in the iterative procedure (4.2.42)-(4.2.44) satisfy $\beta = O(\sqrt{h})$. Then, the spectral radius $\rho(\bar{T}_{0,0})$ of the operator is bounded as follows:*

$$\rho(\bar{T}_{0,0}) \leq 1 - C\sqrt{h}H_\star \equiv \gamma_0, \quad (4.4.32)$$

where $H_\star$ is minimum diameter of the subdomains and $C = \dfrac{4}{C\,(C^\star + b^{-1})}$ with $C^\star$ depends on fixed constant $H_\star$, and the iteration (4.2.42)-(4.2.44) converges with an error at the k^{th} iteration bounded asymptotically by $O(\gamma_0^k)$.

Chapter 5

Conclusions

In this concluding chapter, we highlight the main results obtained in the present dissertation. Further, we discuss the possible extensions and the scope for further investigations in this direction.

5.1 Summary and some observations

In this thesis, we have studied nonoverlapping DD methods for second order elliptic and parabolic problems for both iterative and non-iterative cases. We also have analyzed the iterative DD methods using the mixed finite elements for elliptic problems with a scope to apply mixed finite element methods for parabolic problems.

In Chapter 2, we have discussed a DD method with Lagrange multipliers for elliptic problems (1.3.1), when $b(x) = 0$ and parabolic initial and boundary value problems (2.5.1). In this context, we note that the bilinear form $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$ of the Lagrange multipliers for (2.2.16)-(2.2.17) satisfies naturally the following continuous inf-sup condition

$$\inf_{0\neq\mu\in Y} \sup_{0\neq v\in X} \frac{b(v,\mu)}{||v||_X||\mu||_Y} \geq K_0, \tag{5.1.1}$$

where $K_0 > 0$, see [8, Lemma 3.1(c), pp. 614], with the spaces X and Y defined as in Chapter 2.

In the discrete case with $V_h^\star \subset X$ and $\Lambda_h^\star \subset L^2(\Gamma)$ as a finite-dimensional subspaces of X and Y, respectively, we derive the following discrete form of the inf-sup condition (5.1.1)

$$\inf_{0\neq\mu_h\in\Lambda_h^\star} \sup_{0\neq v_h\in V_h^\star} \frac{b(v_h,\mu_h)}{||v_h||_X||\mu||_Y} \geq K_1, \tag{5.1.2}$$

where $K_1 > 0$. While this discrete inf-sup condition (5.1.2), which plays a crucial role in deriving the error estimates, is taken as a hypothesis in Bamberger et al. [8, pp. 618], in the context of mortar finite element method, Belgacem [11] and Wohlmuth [126] have proved discrete inf-sup condition (5.1.2) with appropriate compatibility condition on $V_h^\star$ and $\Lambda_h^\star$. Based on nonconforming Crouzeix-Raviart space (cf. [39]), an attempt has been made in Chapter 2 to discuss DD method with Lagrange multipliers for the discretization of the problem (2.2.16)-(2.2.17). It was shown in [8, 11, 126] that the choice of the discrete Lagrange multiplier spaces $\Lambda_h^\star \subset L^2(\Gamma) \subset Y$, but in our analysis, we have chosen discrete Lagrange multiplier spaces Y_h, which are piecewise constants on the elements of the triangulations over interfaces Γ and Y_h is not a subspace of Y. The emphasis throughout this study is on the existence and uniqueness of the approximate solutions (2.2.36)-(2.2.37) and the order of convergence in the broken H^1 norm (2.2.26) and L^2-norm using Strang's second lemma [34, 121, 122]. For finding the consistency error, a projection operator $Q_h : L^2(\Gamma_{ij}) \to Y_{ij,h}$, which is defined in (2.3.7) as

$$\int_{\Gamma_{ij}} (Q_h\mu)\, \pi_{ij} v_h \, ds = \int_{\Gamma_{ij}} \mu\, (\pi_{ij} v_h)\, ds \qquad \forall\, v_h \in X_{i,h}. \tag{5.1.3}$$

is introduced and optimal order of estimates in the broken H^1-norm (2.2.26) and L^2-norm are derived. The error estimates have been illustrated with numerical experiments for each of these methods. Further, we have discussed a DD method with Lagrange multipliers for parabolic problems (2.5.1). Both semidiscrete and fully discrete schemes are discussed. Based on backward Euler method, a completely discrete scheme is analyzed. For optimal error estimates in semidiscrete case, we first split the error $u - u_h = u - R_h u + R_h u - u_h$ and $\lambda - \lambda_h = \lambda - G_h\lambda + G_h\lambda - \lambda_h$, using intermediate projection $R_h u$ and $G_h\lambda$, where $R_h u \in X_h$ and $G_h\lambda \in Y_h$ are defined in (2.6.19)-(2.6.20) as : for given u and λ,

$$a^h(u - R_h u, v_h) - \sum_{i=1}^{M} \sum_{i<j\in N(i)} \left[\int_{\Gamma_{ij}} \lambda_{ij}\, [v_h]\, ds - G_h\lambda_{ij}\, [\pi\, v_h]\, ds \right]$$
$$= \sum_{i=1}^{M} \sum_{T\in\mathcal{T}_{h,i}} \int_{\partial T_{int}} \frac{\partial u_i}{\partial \nu^T} v_{i,h}\, ds \qquad \forall v_h \in X_h, \tag{5.1.4}$$

$$\sum_{i=1}^{M} \sum_{i<j\in N(i)} \int_{\Gamma_{ij}} [u - \pi\, R_h u]\, \mu_h\, ds = 0 \qquad \forall \mu_h \in Y_h. \tag{5.1.5}$$

After deriving the estimates of $u-R_h u$ and $\lambda-G_h\lambda$, the estimates of $R_h u-u_h$ and $G_h\lambda-\lambda_h$ can be derived in terms of $u-R_h u$ and $\lambda-G_h\lambda$ and then use of triangle inequality completes the rest of the estimates. Similar procedures are also adopted for the complete discrete scheme. This chapter is concluded with some numerical experiments.

We observe that the nonconforming multidomain approximation related to the elliptic problem leads to a discrete system (2.2.36)-(2.2.37) with a saddle point structure of the form

$$\begin{cases} A\xi + B\eta &= b, \\ B^T\xi &= c. \end{cases} \tag{5.1.6}$$

Here, $A \in \mathbb{R}^{m\times m}$ a block diagonal matrix, which is symmetric and positive definite, and $B \in \mathbb{R}^{m\times n}$ also has a block structure with $n \leq m$. Now, the coefficient matrix $\mathcal{A}$ associated with the system (5.1.6) is given by

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}, \tag{5.1.7}$$

and it is symmetric, nonsingular, and indefinite. However, the matrix A is invertible, and the system (5.1.6) can be reduced to a positive definite system in variable η as

$$B^T A^{-1} B\eta = B^T A^{-1} b - c \tag{5.1.8}$$

which first yields η on the interface. Using η in (5.1.6), it is easy to obtain ξ. However, the matrix $B^T A^{-1} B$ is dense and has a high condition number. Note that, the construction of effective iterative methods for the discrete system (5.1.6) is not as well studied compared to the systems arising from conforming finite element methods. Therefore, it is desirable to introduce iterative methods to compute a good preconditioner and this is a part of our future plan.

In Chapter 3, we have discussed a nonoverlapping iterative DD method for the elliptic problems (1.3.1) and parabolic initial and boundary value problems (2.5.1). The iterative method has been defined with the help of Robin-type boundary conditions on the artificial interfaces Γ_{ij} as

$$\nabla u_i \cdot \nu^{ij} + \beta_{ij}\, u_i = -\nabla u_j \cdot \nu^{ji} + \beta_{ji}\, u_j \qquad \text{on} \qquad \Gamma_{ij},\ 1 \leq j \neq i \leq M, \tag{5.1.9}$$

$$\nabla u_j \cdot \nu^{ji} + \beta_{ji}\, u_j = -\nabla u_i \cdot \nu^{ij} + \beta_{ij}\, u_i \qquad \text{on} \qquad \Gamma_{ij},\ 1 \leq j \neq i \leq M, \tag{5.1.10}$$

where $\beta_{ij} = \beta_{ji} > 0$ are parameters and M is the number of subdomains. The Robin-type boundary conditions as interface conditions was earlier proposed by Lions in [92] as a tool for the domain decomposition iterative methods in the context of conforming discretization. As in Chapter 2, we introduce in Chapter 3 the following Lagrange multipliers on the interfaces

$$\lambda_{ij} = \nabla u_i \cdot \nu^{ij}, \quad \lambda_{ji} = \nabla u_j \cdot \nu^{ji} \text{ on } \Gamma_{ij}, \tag{5.1.11}$$

where ν^{ij} is the normal vector oriented from Ω_i to Ω_j. For deriving the discrete case, we have adopted the nonconforming method. A convergence analysis is carried out and the convergence of the iterative algorithm is proved for the elliptic problems (1.3.1) when $b(x) = 0$. In discrete case, the convergence of the iterative scheme is obtained by proving that the spectral radius of the matrix associated with the fixed point iterations is less than 1. Earlier Douglas et al. [52] have established the convergence rate as $1-Ch$ for nonconforming finite element methods by again using the spectral radius estimation of the iterative solution for the elliptic problems (1.3.1) on quasi-uniform partitions when $b(x) \geq b_0 > 0$. Note that, Douglas et al. have considered each triangle as a subdomain. Particular attention is needed when $b(x) = 0$ and this is due to lack of coercivity of the associated bilinear form in the inner subdomains. In case, $b(x) = 0$, we have derived the convergence rate which is shown to be of 1 - $O(h^{1/2}H^{-1/2})$, when the winding number N (see, the definition 3.2.1 given in section 3) is not large and H is the maximum diameter of the subdomains. Note that, we have also assumed quasi-uniform hypothesis for the mesh on every subdomain and not on the global mesh defined on the entire domain. This results suggest that the best choice for the parameter $\beta = \beta_{ij} = \beta_{ji}$ in the iterative procedure satisfies $\beta = O(h^{-1/2}H^{-1/2})$ and this is the best rate of convergence that can be expected using this iterative procedure. Moreover, we have extended this iterative method to parabolic initial-boundary value problems and demonstrated the convergence of the iteration at each time step. Numerical experiments confirm the theoretical results established in Chapter 3.

The matrix associated with (4.2.27)-(4.2.29) corresponding to mixed finite element formulations based on Lagrange multiplier takes the form

$$\begin{bmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{B}^T & \hat{E} & 0 \\ \hat{C}^T & 0 & 0 \end{bmatrix}, \tag{5.1.12}$$

where $\hat{A}$ is a block diagonal matrix and $\hat{B}$ also has a block structure. Actually, by introducing the Lagrange multiplier, we easily eliminate the flux and obtain a reduced problem for the pressure unknowns only. Thus, the variable $\mathbf{u}_h$ can be eliminated by computing the inverse of $\hat{A}$ which is trivial. The reduced matrix takes the form

$$\hat{D} = \begin{bmatrix} \hat{B}^T\hat{A}^{-1}\hat{B} + \hat{E} & \hat{B}^T\hat{A}^{-1}\hat{C} \\ \hat{C}^T\hat{A}^{-1}\hat{B} & \hat{C}^T\hat{A}^{-1}\hat{C} \end{bmatrix}. \tag{5.1.13}$$

and it is a common practice to complete the process by solving (5.1.13) using a direct method. It is observed that the matrix $\hat{D}$ is ill-conditioned, therefore, efficient iterative methods are required computing a good preconditioner and this may be a part of our future investigation.

In Chapter 4, we discuss an iterative scheme based on mixed finite element methods using Robin-type boundary condition as transmission data on the artificial interfaces (inter subdomain boundaries) for nonoverlapping DD method applied to (1.3.1) with nonhomogeneous boundary condition. In this context, it is easy (cf. [45, 46]) to replace (4.2.11) and (4.2.13) by the following Robin-type boundary condition on the artificial interfaces Γ_{ij}:

$$-\beta_{ij}\,\mathbf{u}_i\cdot\nu^{ij} + p_i = \beta_{ji}\,\mathbf{u}_j\cdot\nu^{ji} + p_j, \quad x \in \Gamma_{ij} \subset \partial\Omega_i, \tag{5.1.14}$$

$$-\beta_{ji}\,\mathbf{u}_j\cdot\nu^{ji} + p_j = \beta_{ij}\,\mathbf{u}_i\cdot\nu^{ij} + p_i, \quad x \in \Gamma_{ji} \subset \partial\Omega_j, \tag{5.1.15}$$

where $\beta_{ij} = \beta_{ji} > 0$ are parameters. Then, we have proposed an iterative procedure based on the nonoverlapping multidomain problems in (4.2.34)-(4.2.38). There may be some difficulty in assigning a meaning to (4.2.41) regarding the product $\langle \mathbf{u}_j^k\cdot\nu^{ji}\,,\,\mathbf{v}\cdot\nu^{ij}\rangle_{\Gamma_{ij}}$ if $\mathbf{u}_j^k \in \mathbf{V}_j$ and $\mathbf{v} \in \mathbf{V}_i$, but the problem (4.2.39)-(4.2.41) may be viewed as a motivation for the iterative mixed finite element multidomain formulation (4.2.42)-(4.2.44). In this chapter, we have shown the convergence of the iterative scheme for the discrete problem (4.2.42)-(4.2.44). In the convergence analysis, we have used the spectral radius of the matrix associated with the fixed point iterations which is shown to be less than 1. Further, it is shown that the spectral radius has a bound of the form $1 - C\sqrt{h}H_\star$ for quasi-uniform partitions when $b(x) \geq b_0 > 0$, where h is the mesh size for triangulations and $H_\star$ is the minimum diameter of the subdomains with appropriate parameter $\beta = \beta_{ij} = \beta_{ji} = O(\sqrt{h})$. In this context, Douglas et al. [49] have discussed parallel iterative procedure to approximate the solution of (1.3.1) by using mixed finite element methods and obtained the rate of convergence through a spectral radius estimation of the iterative solution. Note that each triangle is considered as a subdomain. Further, it is shown that the spectral radius has a bound of the form $1 - Ch$ for quasi-regular partitions when $b(x) \geq b_0 > 0$, where h is the mesh size for triangulations. Compared to the iterative method proposed by Douglas et al. [49], the proposed iterative method is also different. In our case, we choose initial guess $l^0_{ij,h} \in \Lambda_{ij,h}$, $l^0_{ji,h} \in \Lambda_{ji,h}$ arbitrarily ($l^0_{ij,h} = l^0_{ji,h}$ seems natural), but in [49], one needs to choose initial guesses $\mathbf{u}^0_{i,h} \in \mathbf{V}_{i,h}$, $p^0_{i,h} \in W_{i,h}$, $\lambda^0_{ij,h} \in \Lambda_{ij,h}$ and $\lambda^0_{ji,h} \in \Lambda_{ji,h}$.

5.2 Possible extensions and future problems

In this section, we discuss possible extension and future problems.

5.2.1 Parallelization

One of the main objective of the DD methods is to parallelize the algorithm naturally. In the entire thesis, we have not touched upon the parallel implementation aspect. Below we present briefly our on going effort in parallelizing the algorithms.

As our first model problem, we have considered a parallel implementation of the second order parabolic initial boundary value problem (2.5.1) using a conforming finite element method with Lagrange multipliers. Parallel numerical computations have been carried out on a Beowulf cluster called "Galaxy" under message passing library. The cluster comprises of 34 compute nodes with the following configuration:

- CPU: Intel(R) Dual Processor Xeon(R) CPU 3.2GHz
- RAM: 2GB per node

- HDD: 40GB IDE

Consider the parabolic problem (2.5.1) with $f(x,y,t) = e^t[x(1-x)+y(1-y)+2x(1-x)+2y(1-y)]$. The exact solution of the problem (2.5.1) problem is given by $u(x,y,t) = e^t x(1-x)y(1-y)$. Here we take $\Omega = (0,1)\times(0,1)$. For a given number of parallel processors, say 'M', we subdivide the original problem into multi-domain problems on 'M' subdomains. Each subdomain is assigned to only one processor and the multidomain problem for that subdomain is fully solved by its assigned processor. Also, whenever one subdomain shares an interface with another subdomain, the interface information is available with both the processors. This kind of subdivision minimizes the inter processor communication which speeds up the computing time. Under SIMD (single instruction multiple data) approach, each processor carries out triangulation for its subdomain, defines matrices for the elements assigned to it and assembles them. For every time step, each processor uses LU decomposition to solve the system of equations. Processors communicate the interface data to its neighboring processor, which contains the same interface, to satisfy the interface condition. Solution obtained at one time step is used as an initial solution for the next time step.

We carried out our computations on 2 and 4 processors by subdividing the problem into 2 and 4 subdomains, respectively. The following table summarizes the total computing time on 2 and 4 processors with increasing number of elements in each processor. Here for 8×8 problem size, computing time using 2 processors is more than 4 processors. This is because of the fact that for small problem size, computing time is very less in comparison to the inter-processor communication time. For a problem size of order 24×24 and more, we obtain an improvement factor of almost 8. Here for a particular problem size, improvement factor is calculated as follows:

$$\text{Improvement factor} = \frac{\text{Total computing time on 4 processors}}{\text{Total computing time on 2 processors}}.$$

Plots in the Figures 5.1 and 5.2 show the time spent in various subroutines of the code for 2 and 4 processors, respectively, with data size 64×64. Here, we notice that the inter processor communication time is very less as compared with the total time. Also matrix calculation and solver is the main time consuming part in the code. Using some sparse

Problem size	For 2 processors		For 4 processors		Improvement Factor
	DOF in each processor	Total computing time	DOF in each processor	Total computing time	
8×8	45	0.06s	25	0.49s	0.12
16×16	153	0.23s	81	0.08s	2.88
24×24	325	3.91s	169	0.47s	8.31
32×32	561	30.06s	289	3.61s	8.35
48×48	1225	9m 35.50s	625	1m 9.08s	8.38
64×64	2145	1h 29m 5.28s	1089	10m 32.44s	8.45

Table 5.1: Parallel computing time

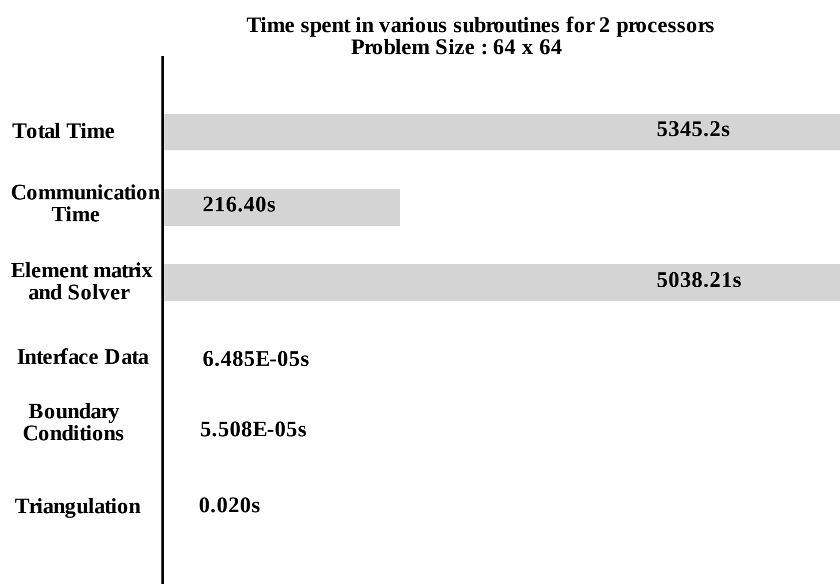

Figure 5.1: Time spent in various subroutines of the program: 2 processors case

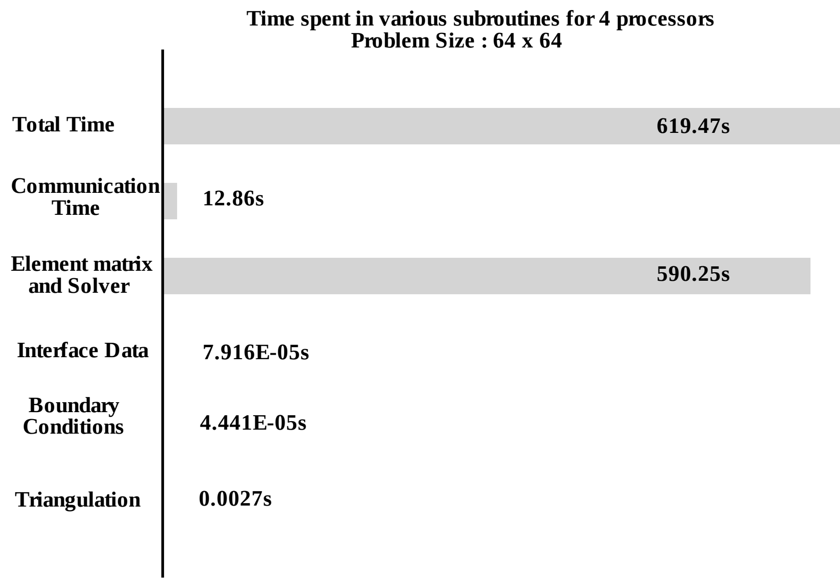

Figure 5.2: Time spent in various subroutines of the program: 4 processors case

storage scheme for the matrix and sparse system solver the performance can be improved in terms of computing time.

Since the initial results of parallel implementation with conforming finite elements are quite encouraging, now we propose parallel algorithms for the problems presented in this thesis. An efficient parallel implementation of these algorithms will be a subject of our immediate future research.

Parallel Algorithm - I **(For elliptic problems in Chapter** 3).

Step 1. Given $\{u^0_{i,h}, \lambda^0_{ij,h}, \lambda^0_{ji,h}\} \in \{X_{i,h}, Y_{ij,h}, Y_{ji,h}\}$, arbitrarily, for all $i = 1, \cdots, M$ and $j \in N(i)$.

for $k = 1, 2, \cdots,$

Step 2. Find $u^k_{i,h} \in X_{i,h}$, $i = 1, \cdots, M$ such that

$$\begin{aligned} a^h_{\Omega_i}(u^k_{i,h}, v_h) \quad &+ \sum_{j\in N(i)} \beta_{ij} \int_{\Gamma_{ij}} \pi_{ij} u^k_{i,h}\, \pi_{ij} v_h \, ds = (f, v_h)_{\Omega_i} \\ &+ \sum_{j\in N(i)} \beta_{ji} \int_{\Gamma_{ij}} \pi_{ji} u^{k-1}_{j,h}\, \pi_{ij} v_h \, ds - \sum_{j\in N(i)} \int_{\Gamma_{ij}} \lambda^{k-1}_{ji,h} \pi_{ij} v_h \, ds \quad \forall v_h \in X_{i,h}. \end{aligned}$$

Step 3. Calculate $\lambda_{ji,h}^k \in Y_{ji,h}$, $i = 1, \cdots, M$

$$\lambda_{ij,h}^k = -(\ \beta_{ij}\pi_{ij}u_{i,h}^k(p) - \beta_{ji}\pi_{ji}u_{j,h}^{k-1}(p)\ ) - \lambda_{ji,h}^{k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i).$$

end for

Remark 5.2.1 *Steps 2 and 3 can be performed in parallel. Step 1 is just providing an initial guess at the interfaces.*

Parallel Algorithm-II (For mixed finite element methods in Chapter 4).

Step 1. Given $l_{ij,h}^0 \in \Lambda_{ij,h}$, $l_{ji,h}^0 \in \Lambda_{ji,h}$ arbitrarily, for all $i = 1, \cdots, M$ and $j \in N(i)$.

for $k = 0, 1, 2, \cdots,$

Step 2. Find $\left\{\mathbf{u}_{i,h}^k,\ p_{i,h}^k\right\} \in \mathbf{V}_{i,h} \times W_{i,h}$, $i = 1, \cdots, M$ such that

$$\begin{aligned}
&(\alpha\, \mathbf{u}_{i,h}^k, \mathbf{v})_{\Omega_i} - (p_{i,h}^k, \nabla \cdot \mathbf{v})_{\Omega_i} + \sum_{j \in N(i)} \beta_{ij} \langle \mathbf{u}_{i,h}^k \cdot \nu^{ij}\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} \\
&\qquad\qquad = - \sum_{j \in N(i)} \langle l_{ij,h}^k\,, \mathbf{v} \cdot \nu^{ij} \rangle_{\Gamma_{ij}} - \langle g\,, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma},\ \ \mathbf{v} \in \mathbf{V}_{i,h}, \\
&(\nabla \cdot \mathbf{u}_{i,h}^k, q)_{\Omega_i} + (b\, p_{i,h}^k, q)_{\Omega_i} = (f, q)_{\Omega_i}, \qquad\qquad q \in W_{i,h}.
\end{aligned}$$

Step 3. Calculate $l_{ij,h}^{k+1} \in \Lambda_{ij,h}$, $i = 1, \cdots, M$

$$l_{ij,h}^{k+1} = 2\, \beta_{ji}\, \mathbf{u}_{j,h}^k \cdot \nu^{ji} + l_{ji,h}^k \qquad \text{on} \qquad \Gamma_{ij},\ \forall j \in N(i).$$

end for

Remark 5.2.2 *Steps 2 and 3 can be performed in parallel. Step 1 is just providing an initial guess at the interfaces.*

Similarly, parallel algorithm can be proposed for the parabolic problem considered in Chapter 3 and elliptic/parabolic problem considered in Chapter 2.

5.2.2 Choice of relaxation parameter

For the improvement in the rate of convergence in Chapter 3, it may be worthwhile to propose an under relaxed version of the transmission condition by replacing (3.2.9) with

$$\lambda_{ij}^k = -\beta\left(\ (u_i^k - u_i^{k-1}) + \delta_k\,(u_i^{k-1} - u_j^{k-1})\ \right) + (1-\delta_k)\lambda_{ij}^{k-1} - \delta_k\,\lambda_{ji}^{k-1} \qquad \forall x \in \Gamma_{ij},\ j \in N(i), \tag{5.2.1}$$

where $\beta = \beta_{ij} = \beta_{ji}$ and for some value of the relaxation parameter $\delta_k \in [0, 1)$. The relaxation parameter approach was introduced by Despres [47] for the Lions iterative method in the context of Helmholtz problems. But the optimal choice of the relaxation parameter was not discussed. In his analysis, the random selection of $\delta \in [0.7, 1)$ for each iteration is reported to yield unexpectedly good results. Subsequently, Guo and Hou [79] have discussed relaxation parameter method and applied it to the iterative method proposed by Deng [43]. They also did not discuss the optimal choice of the relaxation parameter. In general, their observation is that one can choose $\delta \in [0.5, 1)$. In the absence of any further guidance as to a good choice of a constant δ, they have suggested using the golden ration constant $(\sqrt{5} - 1)/2 \approx 0.618$. In our approach (5.2.1), we propose to find the optimal choice of the relaxation parameter δ as a future problem.

5.2.3 Rate of convergence

In Chapter 4, we have shown the convergence of the iterative scheme for the discrete problem (4.2.42)-(4.2.44) when $b(x) = 0$. Further, it is shown that the spectral radius has a bound of the form $1 - C\sqrt{h}H_\star$ for quasi-uniform partitions when $b(x) \geq b_0 > 0$, where h is the mesh size for the triangulations and $H_\star$ is the minimum diameter of the subdomains with appropriate parameter $\beta = \beta_{ij} = \beta_{ji} = O(\sqrt{h})$. To the best of our knowledge, there is no result for DD with mixed finite element method when $b(x) = 0$. Therefore, it is pertinent to discuss the rate of convergence when $b(x) = 0$ and we plan to investigate this in future.

In Chapter 4, we obtain the rate of convergence is of $1 - C\sqrt{h}H_\star$. But in the elliptic case (see Chapter 3), we have derived the rate of convergence is of $1 - Ch^{1/2}H^{-1/2}$. Therefore, it is worth while to explore this in future.

5.2.4 DD for biharmonic problems

Except for [70] and references cited there, there is hardly any literature in the direction of DD method for biharmonic problems. Gervasio [70] has analyzed the DD method for plate bending problems based on spectral element methods and discussed Dirichlet-Neumann iterative scheme as a preconditioner. We consider the biharmonic equation as a model problem. Given f, we are interested to find u such that

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.2.2}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, Δ^2 is the biharmonic operator defined as

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \tag{5.2.3}$$

and $(\partial u/\partial \nu)$ is the exterior normal derivative of u along $\partial\Omega$. This problem arises in fluid mechanics and in solid mechanics (bending of elastic plates).

In a mixed method, the problem is decomposed into problems involving lower order differential equations by introducing new independent variables which are then approximated along with the solution of the original problem. One reason for this is that if one uses a finite element method based on the standard variational principle, i.e., find $u \in H_0^2(\Omega)$ such that for all $v \in H_0^2(\Omega)$, $\int_\Omega \Delta u \Delta v = \int_\Omega f v dx$, then the approximate solution must lie in a subspace of $H_0^2(\Omega)$. Since the construction of such subspaces can be difficult in general, we set $w = -\Delta u$ to obtain the following equivalent system of PDEs in variables u, w :

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega \\ -\Delta u = w & \text{in} \quad \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.2.4}$$

Here, both u and w are taken as primary variables. It is worthwhile to extend the Dirichlet-Neumann and Neumann-Neumann preconditioners to fourth order boundary value problems and discretize with the help of the mixed finite element methods and this will be a part of our future project.

Decomposing Ω into two disjoint subdomains Ω_1 and Ω_2 with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ where Γ is the interface and $\Gamma_i = \partial\Omega_i \cap \partial\Omega$ with Γ_i the external boundaries for each $i = 1,\ 2$, now we split the original problem in the framework of the multi-domain as for each $i = 1,\ 2$, find (w_i, u_i) such that

$$\begin{cases} -\Delta w_i = f & \text{in} \quad \Omega_i \\ -\Delta u_i = w_i & \text{in} \quad \Omega_i \\ u_i = \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on} \quad \Gamma_i \end{cases} \tag{5.2.5}$$

and

$$\begin{cases} u_2 = u_1, \qquad w_2 = w_1 & \text{on} \quad \Gamma \\ \dfrac{\partial u_2}{\partial \nu} = \dfrac{\partial u_1}{\partial \nu}, \qquad \dfrac{\partial w_2}{\partial \nu} = \dfrac{\partial w_1}{\partial \nu} & \text{on} \quad \Gamma. \end{cases} \tag{5.2.6}$$

Here u_i and $w_i,\ i = 1, 2$ both are the restrictions to $\Omega_i,\ i = 1, 2$ of the solution u and w of original the problem (5.2.4) (that means $u_i = u_{|\Omega_i}$ and $w_i = w_{|\Omega_i}, i = 1, 2$) and ν^i is the unit outward normal to $\partial\Omega_i \cap \Gamma$ (oriented outward). The equation (5.2.6) yields the transmission conditions for u_1 and u_2, and w_1 and w_2 on Γ of the mixed problem (5.2.4), where $\nu = \nu^1 = -\nu^2$.

In order to solve the problem (5.2.5)-(5.2.6), we introduce two iterative procedures which entails the solution of a sequence of boundary value problems on each subdomain, along with relaxation conditions at the interface Γ.

Gervasio [70] has introduced the Dirichlet-Neumann type iterative scheme in the context of plate bending problems. We propose Neumann-Neumann iterative scheme for biharmonic problem.

Neumann-Neumann Iterative Scheme. Let $\bar{\lambda}_1^0 \in \Lambda$ and $\bar{\lambda}_2^0 \in \Lambda^0$ be given. For $kn \geq 1$, we construct the sequence of functions : find (w_1^k, u_1^k) such that

$$\begin{cases} -\Delta w_1^k = f & \text{in} \quad \Omega_1, \\ -\Delta u_1^k = w_1^k & \text{in} \quad \Omega_1, \\ u_1^k = \dfrac{\partial u_1^k}{\partial n} = 0 & \text{on} \quad \Gamma_1, \\ u_1^k = \bar{\lambda}_1^{k-1}, \quad w_1^k = \bar{\lambda}_2^{k-1} & \text{on} \quad \Gamma, \end{cases} \tag{5.2.7}$$

and find (w_2^k, u_2^k) such that

$$\begin{cases} -\Delta w_2^k = f & \text{in} \quad \Omega_2, \\ -\Delta u_2^k = w_2^k & \text{in} \quad \Omega_2, \\ u_2^k = \dfrac{\partial u_2^k}{\partial n} = 0 & \text{on} \quad \Gamma_2, \\ \dfrac{\partial u_2^k}{\partial n} = \dfrac{\partial u_1^k}{\partial n}, \quad \dfrac{\partial w_2^k}{\partial n} = \dfrac{\partial w_1^k}{\partial n} & \text{on} \quad \Gamma, \end{cases} \tag{5.2.8}$$

where, for $n \geq 1$, let be given by $\bar{\lambda}_1^0 \in \Lambda$ and $\bar{\lambda}_2^0 \in \Lambda^0$

$$\bar{\lambda}_1^k = \bar{\theta}_1 \; u_{2|\Gamma}^k + (1-\bar{\theta}_1)\bar{\lambda}_1^{k-1} \quad \text{and} \quad \bar{\lambda}_2^k = \bar{\theta}_2 \; w_{2|\Gamma}^k + (1-\bar{\theta}_2)\bar{\lambda}_2^{k-1}. \tag{5.2.9}$$

In (5.2.9), $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2)$ are the (positive) relaxation parameter that will be determined in order to ensure (and possibly, to accelerate) the convergence of the iterative scheme. Variational formulation for the problem (5.2.7)-(5.2.8) given below. Given $\bar{\lambda}_1^0 \in \Lambda$ and $\bar{\lambda}_2^0 \in \Lambda^0$, find $(w_1^k, u_1^k) \in H^1(\Omega_1) \times H^1_{\Gamma_1}(\Omega_1)$ such that

$$\begin{cases} (w_1^k, v_1)_{\Omega_1} - a_1(v_1, u_1^k) = 0 & \forall v_1 \in H^1_\Gamma(\Omega_1), \\ a_1(w_1^k, z_1) = (f, z_1)_{\Omega_1} & \forall z_1 \in H^1_0(\Omega_1), \\ \gamma_0 u_1^k = \bar{\lambda}_1^{k-1}, \quad \gamma_0 w_1^k = \bar{\lambda}_2^{k-1} & \text{on} \quad \Gamma, \end{cases} \tag{5.2.10}$$

and find $(w_2^k, u_2^k) \in H^1(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2)$ such that

$$\begin{cases} (w_2^k, v_2)_{\Omega_2} - a_2(v_2, u_2^k) = 0 \qquad \forall v_2 \in H^2_\Gamma(\Omega_2), \\ a_2(w_2^k, z_2) = (f, z_2)_{\Omega_2} \qquad \forall z_2 \in H^1_0(\Omega_2), \\ (w_2^k, \mathcal{R}_2\mu)_{\Omega_2} - a_2(\mathcal{R}_2\mu, u_2^k) = -(w_1^k, \mathcal{R}_1\mu)_{\Omega_1} + a_1(\mathcal{R}_1\mu, u_1^k) \quad \forall\, \mu \in \Lambda, \\ a_2(w_2^k, \mathcal{R}_2^0\eta) = (f, \mathcal{R}_2^0\eta)_{\Omega_2} + (f, \mathcal{R}_1^0\eta)_{\Omega_1} - a_1(w_1^k, \mathcal{R}_1^0\eta) \quad \forall\, \eta \in \Lambda^0, \end{cases} \tag{5.2.11}$$

where, for $k \geq 1$, let $\bar{\lambda}_1^k \in \Lambda$ and $\bar{\lambda}_2^k \in \Lambda^0$ be given by

$$\bar{\lambda}_1^k = \bar{\theta}_1 \; \gamma_0 u_2^k + (1-\bar{\theta}_1)\bar{\lambda}_1^{k-1} \quad \text{and} \quad \bar{\lambda}_2^k = \bar{\theta}_2 \; \gamma_0 w_2^k + (1-\bar{\theta}_2)\bar{\lambda}_2^{k-1}, \tag{5.2.12}$$

and $\mathcal{R}_i$ $(i = 1, 2)$ denotes any possible extension operator from Λ to $H^1(\Omega_i)$ that satisfies $(\mathcal{R}_i\mu)_{|\Gamma} = \mu$ and $\mathcal{R}_i^0$ $(i = 1, 2)$ denotes any possible extension operator from Λ^0 to $H^1_{\Gamma_i}(\Omega_i)$ that satisfies $(\mathcal{R}_i^0\eta)_{|\Gamma} = \eta$.

We are working on the convergence analysis, finite element formulation and implementation of the iterative scheme (5.2.7)-(5.2.8).

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Acknowledgments

It is my great pleasure to take this opportunity to acknowledge the people who have been very helpful to me during the entire Ph.D work.

I would like to express my deep appreciation to my supervisor, Prof. Neela Nataraj, for her encouragement in preparation of this thesis, her support, guidance, invaluable suggestions and time spent in discussions throughout my research. Her detailed and constructive lectures on 'MATLAB implementations for finite element methods', helped me a lot in understanding the basics of numerical implementations. This thesis would not have been possible without her guidance.

My deepest gratitude is to my co-supervisor Prof. Amiya Kumar Pani for his continuous support and ideas in the entire Ph.D program. He showed me different ways to approach a problem and the need to be persistent to accomplish my goal. More importantly, he taught me how to work hard. Without his encouragement and constant guidance, I could not have finished this thesis. I have been very fortunate with my co-supervisors. His unlimited energy and availability will be warmly remembered. I am also very thankful to his wife Mrs. Tapaswani Pani for her support and providing family environment.

Besides my supervisors, I would also like to thank my RPC (Research Progress Committee) members Prof. Rekha P. Kulkarni and Prof. K. Suresh Kumar for their valuable suggestions and comments.

I am very thankful to the Department of Mathematics IIT Bombay, for providing me the working place and computing facilities during my research work. In addition, I am also thankful to all the faculty, research scholar and staffs of the Department of Mathematics. I would also like to thank Department of Aerospace Engineering IIT Bombay, for use and access to Beowulf cluster "GALAXY" for parallel computing.

I am very grateful to Prof. Swadhen Pattanayak, Director, Institute of Mathematics and Applications, Bhubaneswar for his constant encouragement, who is the main source of inspiration to pursue research work.

I would like to thank Shalini Bhaskar for helping me to provide the parallel computational results in this thesis. In addition, I would like to thank Ajit Patel, Dr. Raja Sekhar, Sarvesh Kumar and Dr. Tirupathi Gudi for their ever-present support. They made the IIT, Bombay a wonderful workplace and home for the past six years by their presence whenever I was in need. I also like to thank Saumya Bajpai for her help.

I have been fortunate to come across many good friends, which will be a memorable time in my life. Special thanks goes to Rajendra, Jayant, Suru, Subash, Banshi, Pravat, Sudarsan, Narmada, Sonalika, Anita, Bikash and Benudhar for their support and help during my study.

I would like to express my heartfelt thanks to my parents Shri Haradhan Pradhan and Smt. Gadamani Pradhan for their unconditional love, support and encouragement throughout my study. My special thanks to my father-in-law Shri Brahmananda Sahoo and my mother-in-law Smt. Satyabhama Dei for their affection and moral support during my research work.

Most of all, I wish to thank my wife Kanan Bala who has provided me an extra hand with her patience, for her personal support throughout the research, for which my mere expression of thanks does not suffice.

Thanks to Anup, Pinky, Mili, Lily, Debu, Rajani, Paramananda and Ajay bhai for their constant love and support whenever I was missing my family. Their affection made my stay comfortable at IIT Bombay, which will be a memorable time in my life.

Many people helped me during my PhD work directly or indirectly, for which I would like to thank them wholeheartedly.

IIT Bombay **Debasish Pradhan**