

SYMPLECTIC OPERAD GEOMETRY AND GRAPH HOMOLOGY

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ABSTRACT. A theorem of Kontsevich relates the homology of certain infinite dimensional Lie algebras to graph homology. We formulate this theorem using the language of reversible operads and mated species. All ideas are explained using a pictorial calculus of cuttings and matings. The Lie algebras are constructed as Hamiltonian functions on a symplectic operad manifold. And graph complexes are defined for any mated species. The general formulation gives us many examples including a graph homology for groups. We also speculate on the role of deformation theory for operads in this setting.

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1. INTRODUCTION

This paper is my humble tribute to the genius of Maxim Kontsevich. Needless to say, the credit for any new ideas that occur here goes to him, and not me. For how I got involved in this wonderful project, see the historical note (1.3).

In the papers [22, 23], Kontsevich defined three Lie algebras and related their homology with classical invariants, including the homology of the group of outer

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automorphisms of a free group and mapping class groups. He showed this by reducing the homology computation in each case to three graph complexes. His main theorem can be informally stated as

Theorem. $H_*(\text{Lie algebra}) = H_*(\text{Graph complex}) = H_*(\text{Group})$.

In this paper, we consider only the first part of this theorem. The definitions of the Lie algebras are motivated by classical symplectic geometry. Kontsevich considered three worlds - commutative, associative and Lie. He developed formal analogues of classical symplectic geometry (which is the commutative case) for the associative and Lie worlds. Each of the three Lie algebras was then defined as Hamiltonian functions on a “symplectic non-commutative manifold” with the bracket being the analogue of the usual Poisson bracket. The symplectic Lie algebra $\mathfrak{sp}(2n)$ was a Lie subalgebra of all three Lie algebras.

In the commutative case, Kontsevich defined a chain complex using graphs. The homology of this chain complex is known as graph homology. He also gave an analogous definition in the associative case using ribbon graphs and in the Lie case using more complicated graphs, which one may call Lie graphs.

The connection between an algebraic or geometric object like a Lie algebra on one side and a combinatorial object like a graph on the other side is the content of Kontsevich’s theorem. The main tool in proving this connection is to use classical invariant theory of $\mathfrak{sp}(2n)$.

1.1. The goal of this paper. From the unified way in which Kontsevich treated the three cases, one expected them to be a part of a more general theory. In fact, in [22], Kontsevich made the following statement.

Our formalism could be extended to the case of “Koszul dual pairs of quadratic operads” (see [17]) including Poisson algebras and, probably, operator algebras etc.

This paper arose in an attempt to understand this statement. In mathematics, just as we have the notion of a theorem, proposition, corollary, etc, so should we have the notion of a “Kontsevich sentence”, the above being a classic example.

Earlier we used the term “world” to stand for one of the three words - commutative, associative and Lie. A possible mathematical substitute for this term is the word “operad”. It is not possible to develop an analogue of symplectic geometry for every operad. However, one can do so if the operad is “reversible”. The commutative, associative and Lie operads are three examples. The notion of a reversible operad was developed independently by Getzler and Kapranov under the name of cyclic operad [14]. For some information on the history, see the discussion in (1.3).

The goal of this paper is to formulate the first part of Kontsevich’s theorem in what appears to be its most natural setting - the world of a reversible operad. To such an operad P , we associate a Lie algebra QA_∞ and a graph complex $(Q\mathcal{G}, \partial_E)$ and then show that they have the same homology. A precise statement is given in Theorem 1 in Section 11.

1.2. Organisation of the paper. We now explain the structure of the paper and give a guide to how readers with different interests and backgrounds can read it.

The two words that dominate the paper are species and operads. These notions are explained in Section 2. In the same section we define reversible operads. We give plenty of simple examples with stress laid on drawing pictures. More examples

are discussed in Sections 7 and 9. We hope that the amount of detail given along with the examples will be sufficient for the reader unfamiliar with these concepts.

In Section 3, we construct a

$$\text{Mating Functor} : \{\text{Reversible operads}\} \longrightarrow \{\text{Species}\}.$$

Let P be a reversible operad and Q its image under this functor. Remember that our first goal is to associate a Lie algebra and a graph complex to P . The fact is that the mating functor is the main step in constructing these two objects, and the species Q plays a key role. So we call these two objects QA_∞ and $(Q\mathcal{G}, \partial_E)$ respectively, stressing the close connection to the species Q . The letters A and \mathcal{G} stand for algebra and graph respectively. The subscript ∞ refers to the fact that QA_∞ will be defined as a direct limit of a family of Lie algebras QA_n .

Sections 2 and 3, on which we have elaborated so far, form the basic structure on which the rest of the paper is built. It is a good idea to avoid technical details on a first reading and just keep a few concrete examples in mind. The rest of the paper can be split into three parts. Part A is about the left hand side of Theorem 1 (Lie algebras), Part B is about the right hand side (graph homology) and Part C is about stating and proving the theorem.

Part A : Sections 4-7. This deals with symplectic operad geometry. We start with an overview in Section 4. The goal of Section 5 is to define the Lie algebra QA_∞ . This is done via a simple pictorial calculus of cuttings and matings. The Lie algebra homology $H_*(QA_\infty)$ has the structure of a Hopf algebra, which we then explain. This completes the definition of the left hand side of Theorem 1.

In Section 6, we explain Kontsevich's symplectic mini-theory. This is logically not essential to the rest of the paper. However, it is conceptually the most important section in Part A. It shows how the Lie structure on QA_∞ emerges naturally from a symplectic form. This allows us to think of QA_∞ as Hamiltonian functions on a "symplectic operad manifold".

Part B : Sections 8-10. This deals with graph homology. The reader, who is mainly interested in graph homology, can directly start with this part even skipping Sections 2 and 3. We first define a general graph complex $(\mathcal{G}, \partial_E)$ and then look at other relevant (and smaller) graph complexes, $(Q\mathcal{G}, \partial_E)$ being one of them. This completes the definition of the right hand side of Theorem 1.

Section 9 is optional. It contains an interesting example of the theory which is based on groups. In Section 10, we continue with graph homology and develop some tools that will be used in the proof of the main theorem. This involves defining a cochain complex (\mathcal{G}, δ_E) , i.e. graph cohomology. The homology and cohomology are related by an interesting and highly non-trivial pairing on graphs (10.2).

Part C : Sections 11-14. This deals with the connection between symplectic geometry and graph homology. Section 11 contains a precise formulation of the main theorem. To digest the statement completely, the material on Lie algebras and graph homology in Sections 5 and 8 respectively is a prerequisite. The next three sections deal with the proof of the theorem.

The main ideas of the proof are already present in the commutative case. Since the proof is quite involved, we suggest that the reader specialise to this case on a first reading. The Lie algebra QA_∞ in this case is easy to define directly. So Sections 2-7 are not necessary. Thus the reader, who mainly wants to see the ideas

in the proof, may start with Part C directly, skipping Part A and referring back to Part B as necessary.

The proof of Theorem 1 is given in three parts (Sections 12, 13 and 14). In each of these sections, we state and prove a theorem of the form

$$H_*(\text{Lie algebra}) = H_*(\text{Graph complex}),$$

gradually getting closer to our goal. As mentioned earlier, the main tool in the proof is classical invariant theory of $\mathfrak{sp}(2n)$. This part is done in Section 12. The second part of the theorem given in Section 13 deals with the stability issue ($n \rightarrow \infty$) and makes strong use of the ideas of Section 10. In Section 14, we understand the space of primitive elements of the Hopf algebra $H_*(QA_\infty)$. This is largely a matter of unwinding definitions and there are no new ideas here. Since Hopf algebras play a minor role in this paper, the reader unfamiliar with them may simply omit this section without losing any of the main ideas.

Appendix. We have included two appendices, which explain how deformation theory relates to the ideas of this paper. They are not logically essential to the understanding of the main theorem. Appendix A speculates on the deformation quantisation problem for operads in this setting. In Appendix B, we show that a certain Lie bracket on graph homology defined in [7] is zero.

1.3. A historical note. This work grew out of a seminar organised by Karen Vogtmann, devoted to understanding Kontsevich's work (Fall 2000). Other participants included D. Ciubotaru, F. Gerlits, D. Brown, J. Conant, M. Horak, F. Schwartz, M. Cohen and J. West. My understanding of Kontsevich's ideas, particularly the proof of the commutative case, is due to them. The material in (8.2-8.3) and (12.1-12.3) is based on the seminar notes. Conant and Vogtmann are writing an exposition that has some overlap with this.

The notion of a reversible operad and its role in symplectic operad geometry was done in July 2001 (Sections 2-7). The rest of the paper, namely, defining graph homology for a mated species and Theorem 1, was then relatively easy. The result in Appendix B was proved later in November 2001. These ideas, in some form or the other, had already appeared in the works of Getzler and Kapranov [14, 15], Ginzburg [16] and Markl [29]. Being ignorant of the operad literature at that time, this was a gradual discovery for me. In this paper, I give my own viewpoint of the subject, which was formed by reading Kontsevich.

In [14], Getzler and Kapranov introduced the notion of cyclic operads in order to extend the formalism of cyclic homology for associative algebras [27] to operad algebras. This notion coincides with what we call a reversible operad. For such an operad P , they introduced a functor

$$\lambda(P, -) : \{P\text{-algebras}\} \longrightarrow \{\text{Vector spaces}\}.$$

In our notation (see Sections 4 and 5), $\lambda(P, PA) = QA$, and equation (3) in (5.1) can be taken as the definition of λ . The mating functor is similar in spirit to λ , except that we deal with operads and species rather than algebras and vector spaces.

In [16], Ginzburg explained Kontsevich's symplectic mini-theory for a cyclic Koszul operad. He denoted the functor λ by the letter R . Since a Koszul operad is quadratic, the relations in equation (3) take the simpler form

$$a \otimes b = b \otimes a \quad \text{and} \quad \mu(a, b) \otimes c = a \otimes \mu(b, c) \quad \text{for} \quad \mu \in P[2].$$

These are precisely the relations that Kontsevich wrote in [22]. Elements of associative and Lie calculus had appeared earlier in the works of Karoubi [20] and Drinfeld [8].

However, as we show, to do symplectic geometry, it is sufficient to assume that P is reversible, not necessarily Koszul or quadratic. In fact, all the examples in Section 7 and some earlier ones too are non-quadratic. We discovered reversibility by requiring that the notion of a partial derivative make sense (Proposition 1). So in this sense, it is also a necessary condition to do symplectic geometry.

In [29], Markl considered graph complexes associated to cyclic operads. And for this, he referred back to the Feynman transform construction of Getzler and Kapranov [15].

We conclude the introduction with two remarks. Koszul operads will play an important role in the second part of Kontsevich's theorem, which we do not consider here. If we switch from Lie to Liebniz algebras [26] then we will end up with another avatar of graph homology. Now vertices of graphs will have honest labels, unlike in usual graph homology, where vertices are labelled only upto orientation.

2. SPECIES, OPERADS AND REVERSIBLE OPERADS

In this section, we give a brief introduction to species and operads and then define reversible operads. An excellent exposition on species can be found in the book by Bergeron, Labelle and Leroux [3]. The notes by Voronov [38] contain a good review on operads. A more comprehensive reference is the seminal work of Ginzburg and Kapranov [17]. For some recent developments on operads, see [13, 14, 15, 28, 25]. For earlier literature, see [19, 30, 35, 24], where the concepts of species and operads first originated.

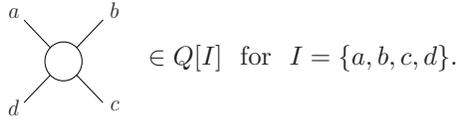
2.1. Species. A species Q is a functor from the category of (Finite sets, bijections) to the category of (Sets, maps). We denote the image of a set I by $Q[I]$ and say that $Q[I]$ is the set of Q -structures on the set I . The set $Q[\emptyset]$ will always be empty.

Species Q	Picture of an element of $Q[I]$ for $I = \{a, b, c, d\}$
c	
a	
aa	
t	
tt	

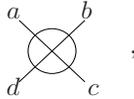
We now give some examples of species. The pictures that go along with the examples are shown in the table.

- $c[I] = \{I\}$, that is, there is exactly one c -structure on the set I . There are a variety of ways to show this via pictures.
- $a[I]$ = the set of linear orders on the set I . We have shown two pictures for it.
- $aa[I]$ = the set of cyclic orders on the set I if $|I| \geq 2$ and empty otherwise.
- $t[I]$ = the set of rooted trees with leaves labelled by elements of the set I .
- $tt[I]$ = the set of trees with leaves labelled by elements of the set I if $|I| \geq 2$ and empty otherwise.

An element of $Q[I]$ can be schematically drawn as



Note that the picture for the species “ aa ” fits in with this representation perfectly. And the remaining examples can also be made to fit in without difficulty. For example, the species “ c ” can be drawn as



and so on for the others too.

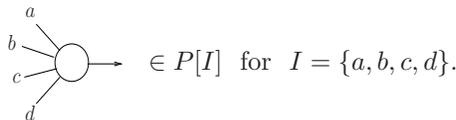
A species can be equivalently defined as a sequence $Q[0], Q[1], Q[2], \dots$, where $Q[n]$ is a Σ_n set, where Σ_n is the symmetric group on n letters. Here we have abbreviated our notation so that $Q[n]$ stands for $Q[\{1, 2, \dots, n\}]$. In the pictures, the group Σ_n acts by permuting the n letters.

It is also useful to use a more general definition of species where the target category is replaced by the category of vector spaces. In this case, a species is a sequence as above with $Q[n]$ being a linear representation of Σ_n . It is clear that using vector spaces as the target category is more general because one can go from sets to vector spaces by linearising the representation.

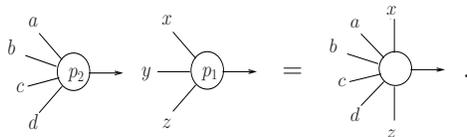
Remark. Throughout this paper, we use only sets and vector spaces as the target categories. However, this restriction is mainly for simplicity. For example, one may consider a species in the category of topological spaces. It would be a sequence $X[0], X[1], X[2], \dots$ of topological spaces with an action of Σ_n on $X[n]$.

2.2. Operads. Note that for each of the examples c, a, t , we suggested a picture with an arrow “ \rightarrow ” drawn in it. This is possible because these species have the additional structure of an operad.

We begin with an informal discussion on operads. An operad P is a species in which there is a substitution rule. It can be shown schematically as

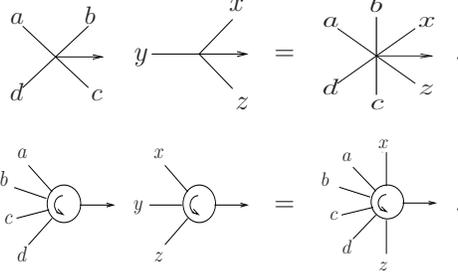


One thinks of a, b, c, d as four inputs and the arrow as an output. The substitution rule allows us to feed the output of one object p_2 into the input of another object p_1 . We write this as $p_2 \rightarrow p_1$ or $p_1 \leftarrow p_2$. This can be shown as

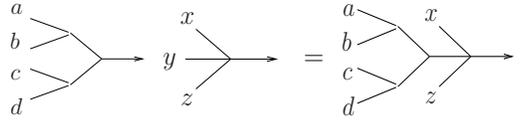


If we wish to be more specific, we will write $p_2 \xrightarrow{y} p_1$. This means that we feed p_2 to that input of p_1 whose label is y .

Now we show how the substitution rule works in each of the examples c, a, t .



For the species t , we graft the root of the first tree into the specified leaf of the second tree.



Note that there is another way possible. One could contract the edge at which the grafting took place. This is what happened for the species “ c ”. There are numerous other variations one can do on this example; for instance, one could use rooted binary trees.

To give a formal definition, an operad P is a species with a substitution rule which is

- associative, and
- compatible with the morphisms in the source category.

The first condition says that if we perform two substitutions, one after the other, then the order in which we do them does not matter. The second condition says that substitution commutes with relabelling of the inputs. For more detail on these conditions, see the references cited earlier.

For convenience of bookkeeping, it is useful to label the output as well. This idea can be formalised as follows. For any operad P , define a bi-functor

$$P : (\text{Sets, bijections}) \times (\text{Singleton sets, bijections}) \rightarrow (\text{Vector spaces, maps})$$

by setting $P[I, U] = P[I]$, where U is a singleton set. One thinks of an element of $P[I, U]$ as an element of $P[I]$ whose output is labelled by the element of U . For the substitution rule in this situation, we demand that the output label of p_2 match the input label of p_1 . So the notation $p_2 \xrightarrow{y} p_1$ now means that the label of the output of p_2 is y and it is fed to an input of p_1 whose label is again y .

Just as for species, we can define an operad as a sequence $P[n]$ of Σ_n modules equipped with a substitution rule that is associative and Σ_n invariant. A unit element of P is an element of $P[1]$ which when substituted into an input of any $p \in P[n]$ gives back p . We assume that our operads have units and call the unit element u . In the second notation, we would write $P[n, 1]$ instead of $P[n]$ and the unit element would lie in $P[1, 1]$.

2.3. Examples. We now organise together a basic set of examples. These include the ones shown in the table (2.1). The motivation for organising them in this manner will become clear when we discuss the mating functor (3.1). We make two quick remarks. All examples here are based on sets. Furthermore, the ones that

are operads have unit elements and satisfy $P[1] = \mathbb{Q}$, i.e. $P[1]$ is just the span of the unit element u of P .

2.3.1. *The unit operad u and species uu .* Define $u[I] = \{I\}$ if I is a singleton and empty otherwise. And similarly, put $uu[I] = \{I\}$ if $|I| = 2$ and empty otherwise. These two examples play a fundamental role in this paper because of their connection to the symplectic Lie algebra $\mathfrak{sp}(2n)$, see (5.4).



2.3.2. *The commutative operad c and species cc .* Define $c[I] = \{I\}$, that is, there is exactly one c -structure on the set I . Also define $cc[I] = \{I\}$ if $|I| \geq 2$ and empty otherwise. Though this is also an operad (without a unit), we call it the commutative species. As far as pictures go, we will use the one with arrow “ \rightarrow ” for “ c ” and the one without it for “ cc ”.

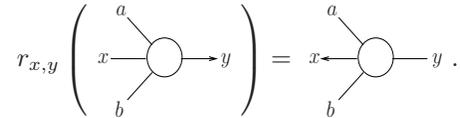
2.3.3. *The associative operad a and species aa .* Let $a[I] =$ the set of linear orders on the set I . Also let $aa[I] =$ the set of cyclic orders on the set I if $|I| \geq 2$ and empty otherwise.

2.3.4. *The tree operad t and species tt .* Let $t[I] =$ the set of rooted trees with leaves labelled by elements of the set I . For the tree species tt , we drop the word “rooted”. In other words, $tt[I] =$ the set of trees with leaves labelled by elements of the set I if $|I| \geq 2$ and empty otherwise. Since there is no root, one cannot define a substitution rule.

2.3.5. *The chord operad k and species kk .* The chord species $kk[I]$ is the set of chord diagrams on the set I . That is, an element of $kk[I]$ specifies a way to pair off the elements of I . Clearly, this is non-empty only when the cardinality of I is even. Define the chord operad $k[I]$ in the same way, except that one of the chords is left hanging at one end. Hence this is non-empty only when the cardinality of I is odd. The substitution rule should be clear.

The species aa, tt, kk have no natural substitution rule; so they are not operads. We will give some more examples later (Sections 7 and 9).

2.4. **Reversible operads.** An operad P is reversible if there is a rule that allows us to switch the output with any given input. One represents this pictorially as



To give a formal definition, for each set I and $x \in I$, there exist reversal maps

$$r_{x,y} : P[I, \{y\}] \rightarrow P[I \setminus \{x\} \cup \{y\}, \{x\}],$$

which are subject to two conditions.

1. We require $r_{y,x} \circ r_{x,y} =$ identity and $r_{x,y} \circ r_{y,z} = r_{x,z}$. It follows from these two relations that the composite of a sequence of reversals is either the identity map or a single reversal.

2. Reversal must be compatible with substitution. That is,

$$r_{x,z}(p_2 \xrightarrow{y} p_1) = \begin{cases} p_2 \xrightarrow{y} r_{x,z}(p_1) & \text{if } x \text{ is an input of } p_1, \\ r_{x,y}(p_2) \xleftarrow{y} r_{y,z}(p_1) & \text{if } x \text{ is an input of } p_2. \end{cases}$$

Here y labels one of the inputs of p_1 as well as the output of p_2 and z labels the output of p_1 .

The examples of operads ($P = u, c, a, t, k$) that we gave in (2.3) are all reversible. For each of these examples, there is a picture of the operad that fits in with the schematic one. The reversal map in all cases then works exactly as shown above in the schematic picture.

We give some intuitive meaning to the conditions on the reversal maps. Condition 1 says that reversal is an external operation. The internal data of the operad remains unchanged. Condition 2 says that operad substitution commutes with operad reversal. This will become clearer from the examples of non-reversible operads given in (2.5).

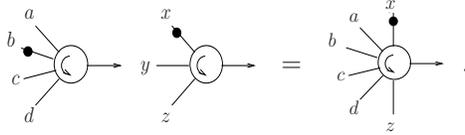
Remark. The reversal rules may not be unique. For example, in (7.2), we give two distinct ways to reverse the graph operad.

2.5. Non-examples. We now give two examples of operads that are not reversible. So they play no role in the rest of the paper. Let $\text{Perm}[I] = I$ and $\text{Dias}[I]$ be the set of linear orders with a distinguished element on the set I .



The pictures show examples for $I = \{a, b, c, d\}$.

We explain the substitution rule for the operad Dias . There are two cases depending on whether the input into which the substitution occurs is circled or not.



The picture illustrates the second case. The first case works as follows. If the input y were circled instead of x then the result would be a circle on the input b . The substitution rule for Perm is similar.

Observe that Perm and Dias are closely related to the commutative (2.3.2) and associative (2.3.3) operads respectively. They are binary quadratic and were introduced by Loday and his collaborators with motivation from algebraic K-theory [25]. For other examples of non-reversible operads, see [14].

3. THE MATING FUNCTOR

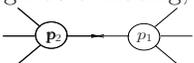
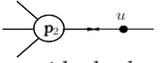
In this section, we introduce the mating functor, which allows us to construct a mated species from a reversible operad. We then explain the concept of a partial derivative for a mated species (Proposition 1). This is the most crucial part of the theory and shows exactly how the conditions on the reversal map given in (2.4) arise.

3.1. The mating functor. Define a species Q starting with a reversible operad P as follows. For any finite set K , let

$$Q[K] = \bigoplus_{I \sqcup J = K} P[I, U] \otimes P[J, U],$$

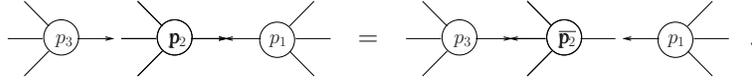
subject to the two relations:

- (R1) $p_2 \otimes p_1 = p_1 \otimes p_2$, and
- (R2) $p_3 \rightarrow p_2 \otimes p_1 = p_3 \otimes r(p_2) \leftarrow p_1$.

The set U is any singleton, all choices being considered equivalent. We interpret the tensor sign as a mating, and say that $p_2 \otimes p_1$ is a mating of p_2 and p_1 . We show it as . For simplicity of notation, we are suppressing the labels. The symmetry of relation (R1) is thus built into the picture. If either of p_2 or p_1 is the unit element u then we say that the mating $p_2 \otimes p_1$ is trivial. We show it as . In a nontrivial mating, we will call the segment joining p_2 and p_1 an *ideal edge*. It has two opposing arrowheads in the centre.

Remark. Ideal edges will play an important role later in the definition of graph cohomology (10.1).

The second relation (R2) is written as under



The notation $\overline{p_2}$ stands for $r(p_2)$.

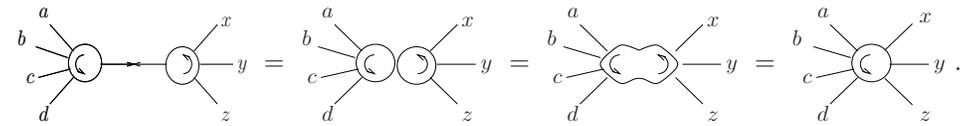
Since we assume that the operad P has a unit, we can use relation (R2) to express any mating $p_2 \otimes p_1$ as a trivial mating as follows. Write $p_1 = p_1 \leftarrow u$, where u is the unit element of P . Then

$$(1) \quad p_2 \otimes p_1 = p_2 \otimes p_1 \leftarrow u = p_2 \rightarrow r(p_1) \otimes u.$$

This allows us to think of a mating $p_2 \otimes p_1$ roughly as a reversal of p_1 followed by a substitution of p_2 into p_1 or vice-versa.

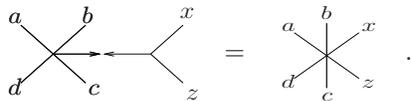
We will refer to a species obtained from a reversible operad by the above procedure as a *mated species*. For obvious reasons, $Q[K]$ is non-empty only if $|K| \geq 2$.

3.2. How elements of a reversible operad mate. In examples, substitutions and reversals usually have a pictorial description. And hence so do matings. We now show the mating in the associative case.



The picture shows that the mating functor maps the associative operad a to the associative species aa .

For the commutative case, mating occurs as follows.



Thus the commutative operad c maps to the commutative species cc .

Similarly, one sees that an operad P maps to the species $Q = PP$ when $P = u, t, k$, see (2.3). The letter repetition is used to indicate that the species is obtained as a mating. A more correct notation would be to write $Q = P\mathfrak{A}$, but we will not bother with that.

3.3. The special role of u and uu . In all examples so far (2.3), we have $P[1] = \mathbb{Q}$. In other words, it is just the span of the unit element u of P . Hence $Q[2] = \mathbb{Q}$ is the span of the element obtained by mating the unit element of P with itself, which is uu . And so $cc[2] = aa[2] = tt[2] = kk[2]$. For example, there is only one cyclic order or unrooted tree or chord diagram on 2 letters. Thus if we ignore the pieces of degree > 1 in P and > 2 in Q then we are left precisely with the unit operad u and species uu .

In general, we always assume that P has a unit. Hence u and uu are always a part of P and Q respectively. For convenience, we use the letter u for both the unit operad and the unit element in an operad. Similarly for uu .

3.4. The partial derivative. Now we introduce the notion of a partial derivative for a mated species.

Proposition 1. *Let P be a reversible operad with unit u and Q be its mated species. For $a \in K$ and $q \in Q[K]$, there is a unique element $p = \frac{\partial q}{\partial a} \in P[K \setminus a]$ such that $q = \frac{\partial q}{\partial a} \otimes u$. In terms of pictures,*

$$\begin{array}{c} b \\ \diagdown \\ \textcircled{q} \\ \diagup \\ d \end{array} \text{---} a = \begin{array}{c} b \\ \diagdown \\ \textcircled{p} \\ \diagup \\ d \end{array} \text{---} \bullet \xleftarrow{u} \text{---} a .$$

Proof. There are two parts to the proposition. The first one is the existence of $\frac{\partial q}{\partial a}$. This is same as saying that q can be written as a trivial mating (at a specified input). We have already derived this in equation (1) as a simple consequence of relation (R2), see (3.1).

The second part is to show the uniqueness of $\frac{\partial q}{\partial a}$. Suppose that $q = p_1 \otimes u = p_2 \otimes u$. This means that one can obtain $p_2 \otimes u$ from $p_1 \otimes u$ by successive applications of relation (R2). Now the reduction lemma below says that this can be done in one step. This implies that $p_1 = p_2$. \square

To complete the proof of the above proposition, we prove a reduction lemma that analyses the relation (R2) in the definition of a mated species (3.1). The proof of the lemma will use the two conditions imposed on the reversal maps in the definition of a reversible operad (2.4).

Local notation. We will follow the convention that subscripts increase from right to left. If $p = p_2 \rightarrow p_1$ then we say that $p_2 \rightarrow p_1$ is a splitting of p . Similarly, $p = p_3 \rightarrow (p_2 \rightarrow p_1) = (p_3 \rightarrow p_2) \rightarrow p_1$ is a splitting of p into three parts. Since operad substitution is associative, we may also write $p = p_3 \rightarrow p_2 \rightarrow p_1$. For simplicity of notation, we are suppressing labels.

Reduction lemma. *Let P be a reversible operad with reversal rule r and unit u . The result obtained by two successive applications of relation (R2) to $p \otimes u$ can, in fact, be obtained by a single application of (R2).*

Proof. We may assume that the first application of relation (R2) to $p \otimes u$ involves splitting p . If we split u then nothing changes and the lemma is proved directly. Hence say $p = p'_2 \rightarrow p_1$ and applying (R2), we obtain

$$p \otimes u = p'_2 \rightarrow p_1 \otimes u \xrightarrow{(R2)} p'_2 \otimes r(p_1) \leftarrow u = p'_2 \otimes p'_1,$$

where $p'_1 = r(p_1) \leftarrow u$.

For the second application, we apply relation (R2) to $p'_2 \otimes p'_1$. This involves two cases.

1. We split p'_2 as say $p'_2 = p_3 \rightarrow p_2$.
2. We split p'_1 .

Let us look at the first case. We have

$$p'_2 \otimes p'_1 = p_3 \rightarrow p_2 \otimes p'_1 \xrightarrow{(R2)} p_3 \otimes r(p_2) \leftarrow p'_1.$$

Substituting $p'_1 = r(p_1) \leftarrow u$ and using the associativity of substitution and the compatibility of reversal with substitution, we get

$$p_3 \otimes r(p_2) \leftarrow p'_1 = p_3 \otimes r(p_2) \leftarrow (r(p_1) \leftarrow u) = p_3 \otimes r(p_2 \rightarrow p_1) \leftarrow u.$$

Hence the net effect of two applications of relation (R2) has been

$$p \otimes u \rightsquigarrow p_3 \otimes r(p_2 \rightarrow p_1) \leftarrow u,$$

where $p = p'_2 \rightarrow p_1 = (p_3 \rightarrow p_2) \rightarrow p_1 = p_3 \rightarrow (p_2 \rightarrow p_1)$. It is clear that the same effect is achieved by

$$p \otimes u = p_3 \rightarrow (p_2 \rightarrow p_1) \otimes u \xrightarrow{(R2)} p_3 \otimes r(p_2 \rightarrow p_1) \leftarrow u,$$

which is just one application of relation (R2) to $p \otimes u$. This completes the analysis for the first case.

For the second case, we just point out a simple subcase and leave the rest out. If we split p'_1 as $p'_1 = r(p_1) \leftarrow u$ and apply relation (R2) then we just reverse the first step and get back $p \otimes u$. And here we used the relation $r_{y,x} \circ r_{x,y} = \text{identity}$. \square

4. OVERVIEW OF SYMPLECTIC OPERAD GEOMETRY

In this section, we explain some of the philosophy of symplectic operad geometry. The reader can just glance through it on a first reading. Complete details are given in Sections 5-7. The reader more interested in graph homology may directly go to Sections 8-10.

4.1. What is an operad manifold? An ordinary manifold X is a commutative object in the following sense. Functions on X form a commutative algebra, the product being pointwise multiplication. To go a little further, differential forms on X form a graded or supercommutative algebra with the usual wedge product. Many geometric notions associated to X can be captured by these commutative algebras. For example, vector fields on X are derivations of the algebra of functions on X . Among all manifolds, \mathbb{R}^n plays a special role. And the space of polynomial functions on \mathbb{R}^n is the free commutative algebra on n generators.

Now for a general operad P , there is a notion of a P -algebra, a P -superalgebra, a free P -algebra, etc. For example, for the associative operad, one has associative algebras; for the Lie operad, one has Lie algebras and so on. We would like to think of a P -algebra as the algebra of functions on a “ P -manifold”, and of the

free P -algebra on n generators as the algebra of functions on the n dimensional “ P -manifold” which is the analogue of \mathbb{R}^n . We emphasise that a “ P -manifold” as a geometric object does not make any sense. However, thinking of a P -algebra in this manner is useful because it allows us to make analogies with the commutative case.

In this paper, we will only deal with free P -algebras.

4.2. What is symplectic operad geometry? Let us start with the commutative case. For an introduction to classical symplectic topology, see the book by McDuff and Salamon [32]. A symplectic manifold (X, ω) is a manifold X with a closed non-degenerate 2 form ω . The existence of such a 2 form implies that X is even dimensional. Further, the algebra of smooth functions $C^\infty(X)$ has the structure of a Lie algebra. The standard example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$. Here ω_0 is the standard symplectic form $\sum dp_i \wedge dq_i$, where $p_1, \dots, p_n, q_1, \dots, q_n$ are coordinates on \mathbb{R}^{2n} . The Lie algebra structure on $C^\infty(\mathbb{R}^{2n})$ is given by the usual Poisson bracket

$$(2) \quad \{F, H\} = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} \quad \text{for } F, H \in C^\infty(\mathbb{R}^{2n}).$$

Instead of smooth functions, it is easier to deal with polynomial functions and henceforth we will always do so.

4.2.1. The general case. We want to do something similar for a general operad P . Let X be a “ P -manifold”. As explained in (4.1), this simply means that we have a P -algebra. And since we are only going to deal with the free case, we further assume that we have a free P -algebra. Call it PA . Since one thinks of PA as polynomial functions on X , one would expect a Lie algebra structure on PA , in analogy with the commutative case. However, this is not true.

It turns out that one needs to consider another algebraic object QA (Section 5). And this object can be constructed only if P is reversible. Then if PA is free on an even number of generators, it is true that QA is a Lie algebra. The two important operations are

$$\frac{\partial}{\partial x_i} : QA \longrightarrow PA \quad \text{and} \quad \{ , \} : QA \otimes QA \longrightarrow QA.$$

The Poisson bracket $\{ , \}$ equips QA with the structure of a Lie algebra. It is given by a formula that is almost identical to equation (2), see equation (4) in (5.3).

Just as PA is related to the operad P , the space QA is related to an object Q . The object Q is not an operad but a simpler object called a species. To give an analogy, operads are like algebras and species are like vector spaces. The species Q is the image of P under the mating functor constructed in Section 3. The space QA can be directly defined in terms of Q .

So far, we have not talked about any symplectic form in the general case. The point is that the Lie structure on QA emerges naturally from a symplectic form. This is explained in Section 6. A “symplectic P -manifold” must be a “ P -manifold” which has a symplectic form. The only example we give is that of a free P -algebra on an even number of generators, for P reversible. When specialised to the commutative case, this says that we are only considering the symplectic manifolds $(\mathbb{R}^{2n}, \omega_0)$.

4.2.2. *Back to the classical case.* We now explain how the general discussion above specialises to the commutative situation. That is, P is the commutative operad c , see (2.3.2). For $P = c$, we have the free (non-unital) P -algebra $PA =$ polynomial functions with no constant terms on the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$. The mated species Q in this case, which is cc , is almost the same. Hence the corresponding algebraic object QA is also similar. We have $QA =$ polynomial functions in $2n$ variables with no constant or linear terms. In this case, the partial derivative has the usual meaning and the Poisson bracket on QA is given by the formula in equation (2). Thus the content of the general discussion is quite simple in this case.

We now give a pictorial description of this calculus. If $F \in QA$ is a monomial, say $F = x_1^2 x_2 x_3$, then we represent it as $F =$ . And if F is a polynomial rather than a monomial then we represent it as a formal sum of pictures. Though PA is almost the same as QA , we will represent its elements slightly differently.

Namely, if $x_1^2 x_2 x_3 \in PA$ then we write it as $F =$ .

We explain how the partial derivative $\frac{\partial}{\partial x_i} : QA \rightarrow PA$ works by an example.

$$\frac{\partial}{\partial x_1} \left(\text{vertex with inputs } x_2, x_1, x_3 \text{ and output } x_1 \right) = \text{vertex with inputs } x_2, x_1, x_3 \text{ and output } x_2 + \text{vertex with inputs } x_2, x_1, x_3 \text{ and output } x_3.$$

In other words, cut all inputs with label x_1 , one at a time. The above pictorial equation just says that $\frac{\partial}{\partial x_1}(x_1^2 x_2 x_3) = 2x_1 x_2 x_3$.

Now we illustrate the Poisson bracket on QA by an example.

$$\left\{ \begin{array}{l} p_1 \\ q_1 \end{array} \right\} \text{---} p_2, \quad q_2 \text{---} \bullet \text{---} p_2 \quad \Bigg\} = \begin{array}{l} p_1 \\ q_1 \end{array} \text{---} \bullet \text{---} p_2 = \begin{array}{l} p_1 \\ q_1 \end{array} \text{---} p_2.$$

In usual terms, this says that $\{p_1 q_1 p_2, p_2 q_2\} = p_1 q_1 p_2$. So to compute $\{F, H\}$, cut a p_i from F , and a q_i from H (or vice-versa), do a mating and then sum over all possibilities. Compare the picture above with the mating suggested for the commutative operad in (3.2).

The pictorial calculus may look silly but the point is that it generalises very nicely to reversible operads (Section 5). The pictorial way of thinking can also be extended to the computation of the homology of the Lie algebra QA . This involves the building of a chain complex of graphs, which leads to graph homology. This connection is made in the first part of the proof of the main theorem (Section 12).

We now record these ideas in a nutshell as follows.

Pictorial	Formal
Cutting of a vertex	Partial derivative
Mating two cut vertices	Poisson bracket
Building a graph	Homology

5. CUTTINGS AND MATINGS

The goal of this section is to define the Lie algebra QA_∞ that occurs in the left hand side of the main theorem. Let V be the vector space with basis x_1, \dots, x_n . Later to do symplectic theory, we will use the basis $p_1, \dots, p_n, q_1, \dots, q_n$. If we want to show the dependence of n explicitly, we will write V_n for V .

Let P be a reversible operad and Q its mated species obtained by applying the mating functor (Section 3). We will first define the algebraic objects QA and PA . Using a calculus of cuttings and matings, we will give QA a Lie algebra structure. The example of the commutative operad ($P = c$) was discussed in (4.2.2). A trivial looking but important example is the unit operad ($P = u$). The Lie algebra in this case is the usual symplectic Lie algebra $\mathfrak{sp}(2n)$. This is explained in (5.4).

As in the case of the underlying vector space V , we will sometimes write QA_n and PA_n , to show the dependence of n explicitly. The Lie algebra QA_∞ will be defined as a direct limit $\lim_{n \rightarrow \infty} QA_n$. After this, we will briefly explain the Hopf algebra structure on $H_*(QA_\infty)$.

5.1. The free algebras. We now define the free algebra QA of a species Q on generators x_1, \dots, x_n . This also includes the case of PA , since every operad is a species.

Operad P	An element of PA	Species Q	An element of QA
c		cc	
a		aa	
t		tt	

To get an element of QA , take any element of $Q[I]$, for some finite non-empty set I and replace each element of I by one of the generators x_1, \dots, x_n . Call such an element a monomial. To get a general element of QA , take linear combinations of monomials.

$$\begin{array}{c} a \\ \diagdown \\ \circ \\ \diagup \\ b \\ \diagdown \\ \circ \\ \diagup \\ d \\ \diagdown \\ \circ \\ \diagup \\ c \end{array} \in Q[I] \quad \rightsquigarrow \quad \begin{array}{c} x_1 \\ \diagdown \\ \circ \\ \diagup \\ x_2 \\ \diagdown \\ \circ \\ \diagup \\ x_4 \\ \diagdown \\ \circ \\ \diagup \\ x_1 \end{array} \in QA^4.$$

We may also write this as $QA = \bigoplus_{j \geq 2} (Q[j] \otimes V^{\otimes j})_{\Sigma_j}$, where Σ_j acts on $Q[j]$ by permuting letters and on $V^{\otimes j}$ by permuting the tensor factors. Strictly speaking, the grading starts in degree 1. But since we are mainly interested in mated species, we started the grading in degree 2. We will sometimes write QA^j for the degree j piece. Though we call QA the free algebra of Q , it only has the structure of a graded vector space.

If the species Q has the structure of an operad P then we emphasise it by drawing instead

$$\begin{array}{c} a \\ \diagdown \\ \circ \\ \diagup \\ b \\ \diagdown \\ \circ \\ \diagup \\ c \\ \diagdown \\ \circ \\ \diagup \\ d \end{array} \in P[I] \quad \rightsquigarrow \quad \begin{array}{c} x_1 \\ \diagdown \\ \circ \\ \diagup \\ x_4 \\ \diagdown \\ \circ \\ \diagup \\ x_1 \\ \diagdown \\ \circ \\ \diagup \\ x_2 \end{array} \in PA^4.$$

Just as for a species, we get $PA = \bigoplus_{j \geq 1} (P[j] \otimes V^{\otimes j})_{\Sigma_j}$. However, the situation for operads is more interesting. The substitution rule for operads allows us to define binary (and higher) products on PA . Hence PA looks more like the algebras that we are used to. For example, $cA =$ free non-unital commutative algebra and $aA =$ free non-unital associative algebra on the generators x_1, \dots, x_n .

As this discussion suggests, it is possible to give a more abstract definition of PA . To every operad P , one can associate a category of P -algebras (4.1); the object PA is then the free object, with respect to V , in this category.

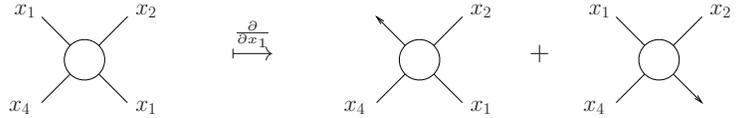
Remark. Recall that, roughly speaking, the mated species Q was defined as a quotient of $P \otimes P$, see (3.1). In the same way, one can define QA as $PA \otimes PA$ subject to the two relations:

$$(3) \quad a \otimes b = b \otimes a \quad \text{and} \quad a \rightarrow b \otimes c = a \otimes r(b) \leftarrow c.$$

Examples. Refer to the table in (5.1). In the commutative case, the pictures for c and cc contain the same amount of information, they represent a monomial in the commutative variables x_1, x_2, \dots, x_n . In the associative case, the pictures for a and aa are different. They represent linear and cyclic orders respectively in non-commutative variables x_1, x_2, \dots, x_n . In the tree case, the objects PA and QA are best described using pictures rather than words.

Following the analogy with the commutative case discussed in (4.2.2), we now discuss the calculus of cuttings and matings in the general case. The reader may keep the above examples in mind.

5.2. Cutting. We define $\frac{\partial}{\partial x_i} : QA \rightarrow PA$ by showing how it works on a schematic example.



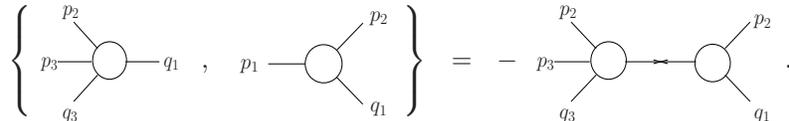
Namely, to define $\frac{\partial}{\partial x_i}$, cut all inputs with label x_i , one at a time. The justification for such a definition is given by Proposition 1 and the above remark.

As an example, in the associative case, the derivative with respect to x_1 of the cyclic word $x_1x_4x_1x_2$ is the sum of two linear words, $x_4x_1x_2$ and $x_2x_1x_4$.

5.3. Mating. In order to perform matings, one requires an even number of characters. So assume that V is even dimensional and has a basis $p_1, \dots, p_n, q_1, \dots, q_n$. Define the Poisson bracket $\{ , \} : QA \otimes QA \rightarrow QA$ using the formula

$$(4) \quad \{F, H\} = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \otimes \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \otimes \frac{\partial H}{\partial p_i}.$$

In (3.2), we explained how the tensor sign can be interpreted as a mating. The only difference now is that the objects being mated, instead of being labelled by elements of a set I , are labelled by elements of a vector space V . So one can continue with the same interpretation.



In the above example, there is only one mating possible. So there is only one term on the right hand side. The minus sign reflects the fact that the “ q ” was cut from the first term and the “ p ” from the second term.

Proposition 2. *The bracket $\{ , \}$ equips QA with the structure of a Lie algebra.*

This can be verified directly. It will also follow from the symplectic operad theory that we will present in Section 6. So we assume this result for the moment. Note that the Lie algebra QA is graded and the bracket $\{ , \}$ is a map of degree -2 .

Remark. If the reversible operad P is based on sets then QA also has a Lie coalgebra structure defined using breakups (rather than matings). This parallels the definition of the coboundary operator on graphs (10.1).

5.4. Connection to the symplectic Lie algebra $\mathfrak{sp}(2n)$. We now look at an important example that is already present in the classical theory. This is the simplest case, i.e., $P = u$ and $Q = uu$, see (3.3). It is easy to see that $uA_n = V$, the underlying vector space and $uuA_n =$ homogeneous polynomials of degree 2 in the variables $p_1, \dots, p_n, q_1, \dots, q_n$. In this case, $\{ , \}$ is the usual Poisson bracket with the tensor sign in equation (4) being ordinary multiplication.

Going to the general case, note that uuA is always a subalgebra of QA that sits inside QA^2 , the piece of degree 2. This is because uu is a subspecies of Q , see (3.3). Moreover, since the bracket $\{ , \}$ has degree -2 , it restricts to a map $QA^j \otimes uuA \rightarrow QA^j$. Hence each graded piece QA^j is a right uuA module. As an example,

$$\left\{ \begin{array}{c} p_2 \\ \diagup \quad \diagdown \\ \text{---} \bigcirc \text{---} \\ \diagdown \quad \diagup \\ q_1 \end{array} , \quad p_1 \text{---} \bullet \text{---} q_2 \right\} = p_1 \text{---} \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \bigcirc \text{---} \\ \diagdown \quad \diagup \\ q_1 \end{array} - p_1 \text{---} \begin{array}{c} p_2 \\ \diagup \quad \diagdown \\ \text{---} \bigcirc \text{---} \\ \diagdown \quad \diagup \\ q_2 \end{array} .$$

Observe that the action only involves changing labels. It does not change the internal structure of the vertex. This is because the matings are all trivial.

The symplectic Lie algebra $\mathfrak{sp}(2n)$ is defined as the space of linear maps on V that kill the symplectic form $\sum dp_i \wedge dq_i$ (the action is as a derivation); the bracket being the usual commutator. Note that $QA^j = (Q[j] \otimes V^{\otimes j})_{\Sigma_j}$ is a left $\mathfrak{sp}(2n)$ module with the action induced by the left action on V and the trivial action on $Q[j]$.

We now explain the relation between the Lie algebras uuA_n and $\mathfrak{sp}(2n)$.

Proposition 3. *There is a Lie algebra anti-isomorphism $uuA_n \rightarrow \mathfrak{sp}(2n)$ given by $H \mapsto \xi_H$, where $\xi_H(p_i) = \frac{\partial H}{\partial q_i}$ and $\xi_H(q_i) = -\frac{\partial H}{\partial p_i}$.*

Again this can be checked directly but will be a consequence of the general theory (Section 6). The definition of the module structures on QA together with the above proposition gives us the following.

Corollary. *The right uuA_n and the left $\mathfrak{sp}(2n)$ module structures on QA^j are compatible via the above anti-isomorphism $uuA_n \rightarrow \mathfrak{sp}(2n)$. In other words, $\{F, H\} = \xi_H \cdot F$ for $F \in QA^j$ and $H \in uuA_n$.*

5.5. The Lie algebra QA_∞ . We have the underlying vector space $V_n \subset V_{n+1}$, with the basis of V_{n+1} extending the basis on V_n to $p_1, \dots, p_{n+1}, q_1, \dots, q_{n+1}$. This gives us a sequence of Lie algebra inclusions

$$QA_1 \subset \dots \subset QA_n \subset QA_{n+1} \subset \dots$$

We denote the direct limit by QA_∞ .

A Hopf algebra is a vector space which has compatible algebra and coalgebra structures and an inverse which is called the antipode. For basic information on Hopf algebras, see [36, 21]. The graded vector space $H_*(QA_\infty)$ has the structure of a Hopf algebra, which we now explain.

To describe the product on $H_*(QA_\infty)$, we start with a morphism of Lie algebras $QA_n \oplus QA_m \hookrightarrow QA_{n+m}$, defined by $(F, H) \mapsto F + H$, where we shift up the indices of H by n . In other words, we think of QA_n and QA_m as disjoint Lie subalgebras of QA_{n+m} . This defines a map $H_*(QA_n) \otimes H_*(QA_m) \rightarrow H_*(QA_{n+m})$. Taking direct limits, we get a product on $H_*(QA_\infty)$. We point out that the diagram

$$\begin{array}{ccc} QA_n \oplus QA_m & \longrightarrow & QA_{n+m} \\ \downarrow & & \downarrow \\ QA_{n+1} \oplus QA_m & \longrightarrow & QA_{n+m+1} \end{array}$$

commutes only upto index shifting. However, this cannot be detected on the homology level.

The coproduct is relatively straightforward to define. We start with the diagonal map $QA_\infty \rightarrow QA_\infty \oplus QA_\infty$, which is a morphism of Lie algebras. This induces a cocommutative coproduct on $H_*(QA_\infty)$.

6. SYMPLECTIC OPERAD THEORY

In this section, we give a self-contained treatment of Kontsevich's symplectic mini-theory. But in order to appreciate it, it is important to be familiar with classical symplectic theory. This involves the principles of Hamiltonian mechanics [32] and basic notions of differential topology such as vector fields, differential forms, Lie derivatives and contraction operators [5, 34].

For some recent ideas and applications in the associative case, see the papers of Ginzburg [16] and Bocklandt and Le Bruyn [4]. For a detailed exposition, see Chapter 7 of Le Bruyn's book "Non-commutative geometry", which is available on his homepage.

The basic objects of interest are summarised in the table. We have already seen PA and QA before. The remaining ones will be defined in this section.

Geometric objects	Algebraic objects	
X	PA	QA
$\mathfrak{X}(X)$	$\text{Der}(PA)$	$\text{Der}(QA)$
ΠTX	$\Omega(PA)$	$\Omega(QA)$
$\mathfrak{X}(\Pi TX)$	$\text{Der}(\Omega(PA))$	$\text{Der}(\Omega(QA))$

Intuition. The geometric objects in the first column do not exist except in the commutative case ($P = c$) in which case $X = \mathbb{R}^n$. One may say that they are defined via their algebraic counterparts. It is useful to have the following dictionary:

X = operad manifold, $\mathfrak{X}(X)$ = vector fields on X , ΠTX = total space of the odd tangent bundle to X , $\mathfrak{X}(\Pi TX)$ = vector fields on ΠTX .

Let X be an operad manifold with the free P -algebra PA as its algebra of functions (4.1). Starting with this purely algebraic data, we want to construct

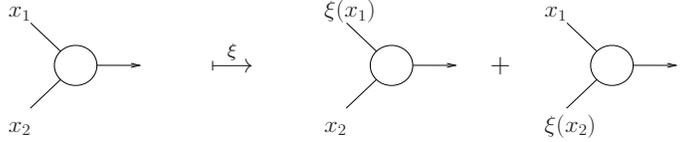
an algebra of “differential forms on X ”, which is defined as some differential P -superalgebra $\Omega(PA)$. One would guess that the symplectic form ω should be a suitable element of $\Omega^2(PA)$. However, this is not true.

It turns out that one needs to consider $\Omega(QA)$, which is the corresponding object for QA . And the symplectic form ω will lie in $\Omega^2(QA)$. So in this sense, it is QA and $\Omega(QA)$, which should be regarded as functions and forms on X .

6.1. Algebraic objects for an operad. We first look more generally at operad manifolds (4.1). To every operad P , one can associate a category of P -algebras. Similarly, one can consider the graded version, which is the category of P -superalgebras. And one can talk of derivations of P -algebras or superderivations of P -superalgebras [17].

In what follows, we will freely use this language. For example, PA is the free object in the category of P -algebras. However, just as we did for PA , we will also give explicit descriptions of all the objects we deal with. This should help the reader who is unfamiliar with the above language. We also recommend that the reader keep a concrete case, say commutative or associative, in mind.

Definition 1. Let $\text{Der}(PA)$ be the space of derivations of PA . It is a Lie algebra with the bracket being the commutator.



A derivation $\xi \in \text{Der}(PA)$ is an operator on PA . It is uniquely determined by its values $\xi(x_1), \dots, \xi(x_n) \in PA$ on the generators x_1, \dots, x_n . An example of how this works is shown above; for each input, we replace the value x_i on it by $\xi(x_i)$. In the first term on the right, $\xi(x_1)$ is written on one input. This is to be understood as a substitution of $\xi(x_1)$ into that input.

Definition 2. The algebra of forms $\Omega(PA)$ is the free differential superalgebra on the generators x_1, \dots, x_n . The term “differential” means that there is a superderivation $d : \Omega(PA) \rightarrow \Omega(PA)$ of odd degree such that $d^2 = 0$.

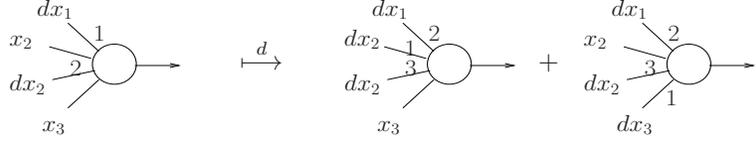
The algebra of forms $\Omega(PA)$ is freely generated as a P -superalgebra by the symbols $x_1, \dots, x_n, dx_1, \dots, dx_n$. Hence a superderivation on $\Omega(PA)$ is uniquely determined by its values on these generators. Note that the differential d is specified by $d(x_i) = dx_i$ and $d(dx_i) = 0$.

Just as for PA , one can give an explicit description of $\Omega(PA)$. To get a monomial in $\Omega(PA)$, take any element of $P[I]$, for some finite non-empty set I and replace each element of I by one of the generators $x_1, \dots, x_n, dx_1, \dots, dx_n$. Since we are now dealing with a super-object, we need to be more careful. We order the inputs with differential symbols in the sense of orientation as follows.

$$(5) \quad \begin{array}{c} dx_1 \\ x_2 \\ dx_2 \\ x_3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} \circ \\ \rightarrow \end{array} = - \begin{array}{c} dx_1 \\ x_2 \\ dx_2 \\ x_3 \end{array} \begin{array}{c} 2 \\ 1 \end{array} \begin{array}{c} \circ \\ \rightarrow \end{array}$$

In other words, an even permutation of the order leaves an element unchanged while an odd permutation gives its negative. We now explain how the differential

d works on this example.



In general, we replace each x_i by a dx_i , one at a time. And the input at which the replacement occurs is given label 1 and the remaining labels are shifted up by 1.

It is clear that $\Omega(PA)$ is in fact \mathbb{Z} graded with the grading given by the number of differential symbols and $\Omega^0(PA) = PA$.

Definition 3. Let $\text{Der}(\Omega(PA))$ be the space of superderivations of $\Omega(PA)$. It is a Lie superalgebra with the bracket being the supercommutator.

To act by a superderivation L on an element of $\Omega(PA)$, one replaces the value v at each input by $L(v)$, one at a time. If $v = x_i$ then we shift the labels by $\deg(L)$. If $v = dx_i$ then we first reorder such that the dx_i being substituted into has label 1 and then shift the remaining labels by $\deg(L)$. The special case when L is the differential d was shown above.

Definition 4. For any derivation $\xi \in \text{Der}(PA)$, we define the Lie derivative $L_\xi \in \text{Der}(\Omega(PA))$ and the contraction operator $i_\xi \in \text{Der}(\Omega(PA))$ by specifying them on the generators as under

$$L_\xi(x_i) = \xi(x_i), \quad L_\xi(dx_i) = d\xi(x_i) \quad \text{and} \quad i_\xi(x_i) = 0, \quad i_\xi(dx_i) = \xi(x_i).$$

They have degrees 0 and -1 respectively.

Lemma 1. For derivations $\xi, \eta \in \text{Der}(PA)$, the superderivations L_ξ, i_ξ, d satisfy the following relations.

- (1) $[i_\xi, d] = L_\xi$ (Cartan's formula).
- (2) $[i_\xi, i_\eta] = 0$.
- (3) $[L_\xi, i_\eta] = i_{[\xi, \eta]}$.
- (4) $[L_\xi, L_\eta] = L_{[\xi, \eta]}$.

Proof. As mentioned earlier, superderivations are preserved by the supercommutator. Hence in all cases, both sides are superderivations, so we need to check (1) – (4) only on the generators $x_1, \dots, x_n, dx_1, \dots, dx_n$. This is a straightforward check that is independent of the operad P . \square

Lemma 2. The algebra of forms $\Omega(PA)$ is exact.

Proof. Consider the Euler vector field $e \in \text{Der}(PA)$, defined by $e(x_i) = x_i$ for all i . Then $L_e(x_i) = x_i$ and $L_e(dx_i) = dx_i$ and hence L_e maps every monomial to a non-zero multiple of itself. Therefore it is invertible on $\Omega(PA)$. So it induces an isomorphism on cohomology. On the other hand, Cartan's formula (item (1) in Lemma 1) shows that it induces the zero map on cohomology. So we conclude that $\Omega(PA)$ is exact. \square

For the operad $P = c$, the lemma just says that the algebra of polynomial forms on \mathbb{R}^n , that vanish at the origin, is exact.

6.2. Algebraic objects for a species. Now assume that the operad P is reversible; so we can talk of its mated species Q . Analogues for Q of the objects in (6.1) can be defined without difficulty. As far as pictures are concerned, we draw the same ones as for operads except that the output arrow is now omitted.

We have seen that QA can be written as a quotient of $PA \otimes PA$ (equation (3)). Similarly, one can define $\Omega(QA)$ as a quotient of $\Omega(PA) \otimes \Omega(PA)$ subject to the super-relations:

$$(6) \quad a \otimes b = (-1)^{|a||b|} b \otimes a \quad \text{and} \quad a \rightarrow b \otimes c = a \otimes r(b) \leftarrow c.$$

Just like $\Omega(PA)$, we see that $\Omega(QA)$ is \mathbb{Z} graded and $\Omega^0(QA) = QA$. We also define $\text{Der}(QA)$, $\text{Der}(\Omega(QA))$ and for $\xi \in \text{Der}(PA)$, the operators $L_\xi, i_\xi \in \text{Der}(\Omega(QA))$. The relations in Lemma 1 continue to hold. Hence it follows that $\Omega(QA)$ is also exact.

Note that any element of $\Omega^1(QA)$ can be uniquely written as $\sum_{i=1}^n f_i \otimes dx_i$, with $f_i \in PA$. And for $H \in QA$, we get $dH = \sum_{i=1}^n \frac{\partial H}{\partial x_i} \otimes dx_i$. These facts are again a consequence of Proposition 1 and will be crucial in what follows.

6.3. The symplectic form ω . Since P has a unit element u , its mated species Q has the special element uu in degree 2, see (3.3). It allows us to define the symplectic form $\omega = \sum dp_i \otimes dq_i \in \Omega^2(QA)$. For a picture, we may draw $\sum dp_i \overset{1}{\bullet} \overset{2}{\bullet} dq_i$. Note that if we switch the order of the factors, we pick a minus sign (equation (6)). This can also be seen from the picture and the analogue of equation (5) for species.

Also $d\omega = 0$ and hence the form is closed. This gives us a symplectic operad manifold X . We think of $\text{Der}(PA)$ as vector fields on X that vanish at a point and $\Omega(QA)$ as differential forms on X with no constant or linear terms, with the 0-forms QA being Hamiltonian functions on X .

Lemma 3. *The map $\text{Der}(PA) \rightarrow \Omega^1(QA)$ which sends ξ to $i_\xi w$ is an isomorphism.*

Proof. We have the chain of equalities

$$i_\xi w = \sum_{i=1}^n i_\xi(dp_i \otimes dq_i) = \sum_{i=1}^n \xi(p_i) \otimes dq_i - \xi(q_i) \otimes dp_i.$$

This shows that $i_\xi w$ is determined by $\xi(p_i)$ and $\xi(q_i)$, which also determine ξ . Since one can represent any element of $\Omega^1(QA)$ uniquely by a sum of the form $\sum_{i=1}^n (\) \otimes dp_i + (\) \otimes dq_i$, the map in the lemma is an isomorphism. \square

Definition 5. Let $\text{Der}(PA, w) = \{\xi \in \text{Der}(PA) \mid L_\xi w = 0\}$. In analogy with the classical case, we call these the symplectic vector fields.

It follows from Cartan's formula (Lemma 1) that

$$\xi \in \text{Der}(PA, w) \iff d(i_\xi w) = 0.$$

Hence under the isomorphism of Lemma 3, one sees that symplectic vector fields correspond to closed 1 forms. Define Hamiltonian vector fields to be the ones that correspond to exact 1 forms. However, since $\Omega(QA)$ is exact, the Hamiltonian and symplectic vector fields coincide in this case. And there is an isomorphism $QA \xrightarrow{d} \Omega^1(QA)_{\text{closed}}$. Putting all facts together, there is an isomorphism $\xi : QA \xrightarrow{\cong} \text{Der}(PA, w)$, where $H \mapsto \xi_H$ is defined by the equation $dH = i_{\xi_H}(w)$. Writing both

sides in the unique form $\sum_{i=1}^n () \otimes dp_i + () \otimes dq_i$, we derive the Hamiltonian equations

$$(7) \quad \xi_H(p_i) = \frac{\partial H}{\partial q_i} \quad \text{and} \quad \xi_H(q_i) = -\frac{\partial H}{\partial p_i}.$$

We now derive the isomorphism between Hamiltonian functions and Hamiltonian vector fields. By Kontsevich kimaya, the proof is same as in the classical case; we give it here for completeness.

Proposition 4. *The map $QA \rightarrow \text{Der}(PA, w)$ defined above is an anti-isomorphism of Lie algebras, with the bracket on QA being the Poisson bracket and on $\text{Der}(PA, w)$ being the usual commutator.*

Proof. First note that $\text{Der}(PA, w)$ is closed under taking commutators (item (4) in Lemma 1). This gives it a Lie algebra structure. So to prove the proposition, we trace through the isomorphism $QA \xrightarrow{\cong} \text{Der}(PA, w)$, compute the induced Lie structure on QA and see that upto a minus sign it coincides with the Poisson bracket (5.3). We do the computation in two steps.

Claim 1. $i_{[\xi_F, \xi_H]} w = d(i_{\xi_F}(dH)).$

$$\begin{aligned} i_{[\xi_F, \xi_H]} w &= [L_{\xi_F}, i_{\xi_H}] w && \text{(item (3) in Lemma 1)} \\ &= L_{\xi_F}(i_{\xi_H} w) && (L_{\xi_F} w = 0 \text{ since } \xi_F \text{ is symplectic}) \\ &= L_{\xi_F}(dH) \\ &= [i_{\xi_F}, d](dH) && \text{(item (1) in Lemma 1)} \\ &= d(i_{\xi_H}(dH)). \end{aligned}$$

Claim 2. $i_{\xi_F}(dH) = -\{F, H\}.$

$$\begin{aligned} i_{\xi_F}(dH) &= i_{\xi_F}(\sum_{i=1}^n \frac{\partial H}{\partial p_i} \otimes dp_i + \frac{\partial H}{\partial q_i} \otimes dq_i) \\ &= \sum_{i=1}^n \frac{\partial H}{\partial p_i} \otimes \xi_F(p_i) + \frac{\partial H}{\partial q_i} \otimes \xi_F(q_i) \\ &= \sum_{i=1}^n \frac{\partial H}{\partial p_i} \otimes \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \otimes \frac{\partial F}{\partial p_i} && \text{(equation (7))} \\ &= -\{F, H\}. && \text{(equation (4))} \end{aligned}$$

The proposition now follows from the two claims. \square

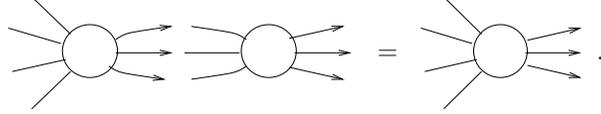
As a corollary, we obtain Proposition 2. Also, if we specialise to $P = u$ and $Q = uu$, we obtain Proposition 3.

7. EXAMPLES MOTIVATED BY PROPS

We have, so far, only defined the notion of a derivative. If one wants to define higher order derivatives then one should consider a more general structure, namely, a PROP. For some motivation for PROPs from the viewpoint of homotopy theory, see [1].

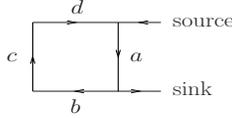
7.1. PROP. We denote a PROP by Pr . It consists roughly of objects with many inputs and many outputs. For a precise definition, see the notes by Voronov [38]. We write $Pr[I, J]$ for the “set of Pr -structures on the set $I \sqcup J$ that have I inputs

and J outputs”. There is also a composition law which allows to feed the outputs of one object to the inputs of another.

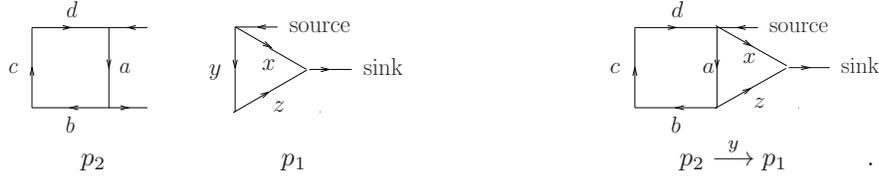


It is useful to think of species and operads as parts of a PROP corresponding to no output (J empty) and a single output (J singleton) respectively. We saw that a partial derivative (Proposition 1) sends a species element to an operad element. Stated differently, it just converts an input to an output. If this is to be the meaning of a derivative then to define higher derivatives one is forced to consider objects with many outputs. Thus one sees that PROPs provide a natural framework to talk of higher derivatives. To make the theory of higher matings work, one would need a “reversible PROP”. We do not make these ideas precise. Instead, we look at some examples that fit this pattern. They may be of independent interest.

7.2. The graph operad g and species gg . Let $g[I] =$ set of graphs with a specified source and sink vertex and directed edges labelled by elements of the set I . The figure shows an element of $g[\{a, b, c, d\}]$.



We indicate the source (resp. sink) vertex by a half-edge with an arrow going in (resp. out). We regard the two half-edges together as constituting the output edge of the operad. Substitution works as under.



In other words, to obtain $p_2 \xrightarrow{y} p_1$, we identify the source (resp. sink) vertex of p_2 with the initial (resp. terminal) vertex of the edge y in p_1 .

7.2.1. Sub-examples. The commutative, associative, chord (2.3) and many other operads can be obtained as special cases by simply restricting the type of graphs allowed. There is seemingly no end to the number of interesting examples one can obtain in this way. We illustrate the commutative and associative cases.



Thus to get the commutative operad, we only allow graphs with two vertices (source and sink) and all edges are directed from the source to the sink. For the associative operad, we allow graphs whose edges form a directed path from the source to the sink.

There is a variation on the associative case (and the commutative), where one allows the arrows on the edges to point in either direction. This is a binary quadratic operad. Axioms for an algebra over this operad can be written explicitly. We do not know whether they have been considered before.

7.2.2. *The reversal maps.* The graph operad is reversible with the reversal map as shown.

$$r_{x,y} \left(\begin{array}{c} \text{source} \\ \leftarrow x \quad \leftarrow y \\ \begin{array}{c} \nearrow a \\ \searrow b \end{array} \\ \rightarrow \text{sink} \end{array} \right) = \begin{array}{c} \text{source} \leftarrow \begin{array}{c} \nearrow a \\ \searrow b \end{array} \rightarrow y \\ x \quad \text{sink} \end{array} .$$

Namely, we open the input edge x and close the output edge y .

There is another way to reverse the graph operad, where in addition to the above, we also switch the directions on x and y . The two methods of reversal are compatible with the associative and commutative cases respectively. Since the two ways are analogous, we will work with the first definition.

7.2.3. *How graph objects mate.* So far, we have only defined the graph operad. But the definition of the graph PROP should be clear; we simply allow more output edges. In other words, graphs can now have many pairs of source and sink vertices. And for the composition law in the PROP, we perform simultaneous substitutions. The graph species gg is the part of the graph PROP with no outputs. So these are graphs with no source and sink vertex. We now show how graph objects mate.

Namely, the sink of one merges (mates) with the source of the other and vice-versa. The result is a graph with no source and sink vertex. Thus the mating functor maps the graph operad g to the graph species gg .

7.2.4. *The algebraic objects.* The free algebras gA and ggA consist of graphs with directed edges labelled by one of the generators x_1, \dots, x_n . Of course, for elements of gA , graphs also have a specified source and sink vertex. The partial derivative $\frac{\partial}{\partial x_i} : ggA \rightarrow gA$ works very nicely.

In words, $\frac{\partial F}{\partial x_i}$ is the sum of graphs obtained by cutting edges of F that are labelled x_i , one at a time. The half-edges that result from a cut get induced directions and provide the source and sink vertex of the graph created. We leave it to the reader to explicitly describe the algebra of forms $\Omega(gA)$, $\Omega(ggA)$ and other notions described in Section 6.

For higher order derivatives, we consider the free algebra of the graph PROP. The order of the derivative determines the number of edges that are cut. And for higher order matings, we take two graphs with the same number of cuts (outputs)

and the result is a graph with no cuts. This generalises the first order mating given by the mating functor.

7.3. A generalisation of the tree example (2.3.4). We now present a second example based on graphs. It complements the earlier example in the sense that (univalent) vertices rather than edges play the role of inputs. Let $hh[I]$ = set of graphs with univalent vertices labelled by elements of the set I . The operad h is defined as the species hh except that we use one of the univalent vertices as the output or root. And for the PROP, we do not place any restriction on the number of outputs or roots. Operad substitution and reversal and matings work exactly as in the tree case.



Note that for the above composition law of the PROP to make sense, we necessarily have to work with graphs and not just trees. We mention that the graph complexes that arise from this species have been considered in [2].

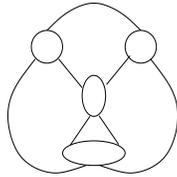
7.4. The surface operad s and species ss . Let $ss[I]$ = set of compact orientable surfaces (not necessarily connected) with boundary circles labelled by elements of the set I . For the operad $s[I]$, we use one of the boundary circles as the output. Operad substitution works by gluing. For the PROP, we allow more outputs as usual. The composition law in the PROP is simultaneous gluing. We leave the details to the reader. Some pictures may be found in A.4.

8. GRAPH HOMOLOGY

In this section, we define a generalisation of Kontsevich's graph homology to mated species. This coincides with the graph complexes considered by Markl [29]. When one specialises to the commutative species (2.3.2), one recovers the usual definition. The case of the associative species (2.3.3) is also explained in [38]. By a graph, we mean a finite 1-dimensional CW complex.

In (8.1-8.3), we define a chain complex $(\mathcal{G}, \partial_E)$ based on graphs. Here \mathcal{G} refers to the chain groups and ∂_E to the boundary operator. The boundary operator ∂_E can be regarded as a limit of a family of boundary operators ∂_n . This family is discussed in (8.4). In (8.6), we list the subcomplexes of $(\mathcal{G}, \partial_E)$ that are relevant to us. The graph complex $(Q\mathcal{G}, \partial_E)$, which appears in the main theorem, is one of them.

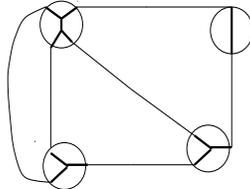
8.1. Graphs. Let Q be any species. Define a Q -graph to be a graph whose vertices are fattened by elements of the species Q . To be more precise, it is a graph Γ such that for every vertex v of Γ , a Q -structure is specified on the set of half-edges incident to v .



Greetings from a Q -graph.

We now give some examples; see the table of pictures and examples of species in (2.1) and (2.3).

- A commutative graph ($Q = cc$) is simply a graph.
- An associative graph ($Q = aa$) is a ribbon graph; that is a cyclic ordering of the half-edges is fixed at each vertex.
- The figure below shows a typical graph for a species based on trees ($Q = tt$).



- For the chord species ($Q = kk$), the vertices instead of being trees are chord diagrams (2.3.5). In our terminology, these are chord graphs.
- For an example with a somewhat different flavour, consider the surface species $Q = ss$, see (7.4). A surface graph is a compact orientable surface with a collection of disjoint loops. In this case, the loops play the role of edges and the pieces obtained by cutting along the loops play the role of vertices. This example was first mentioned to me by J. Conant.

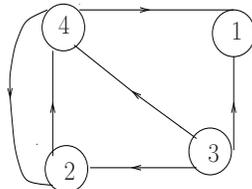
The interested reader can also work out the graphs for the species hh and the graph species gg in Section 7. Note that all the species above have been denoted by a letter repetition. This is to indicate the fact that they are mated species (3.1-3.2).

The reader, who skipped Sections 2 and 3 on species and operads, can read this section and Section 10 by keeping the above concrete examples in mind. For simplicity of notation, from now on, we will just write “graph” instead of “ Q -graph”.

8.2. Oriented graphs. For a graph Γ , we shall denote the set of vertices by $V(\Gamma)$, the set of edges by $E(\Gamma)$ and the set of ends of an edge e by $V(e)$. Finally, $\mathbb{R}X$ will denote the real vector space which has the elements of X as a basis.

Definition 6. An orientation σ of a graph Γ is an orientation of the one-dimensional real vector space $\det \mathbb{R}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \det \mathbb{R}V(e)$. For a vector space W of dimension n , we are using the notation $\det W = \Lambda^n W$. We say that (Γ, σ) is an oriented graph. There is another notion of orientation of a graph equivalent to the above; see [37] for details.

A way to represent an orientation is to order the vertices and orient each edge of the graph. An odd permutation of the labels on the vertices reverses the orientation, and a single change of the orientation of one edge reverses it as well. An even number of these transformations produces an orientation equivalent to the original one. So a graph Γ has two orientations, which we may call σ and $-\sigma$.



A representative of an orientation on a Q -graph.

We say that (Γ, σ) and (Γ', σ') are isomorphic if there is a graph isomorphism $h : \Gamma \rightarrow \Gamma'$ such that h carries the orientation σ to σ' .

Later for simplicity, we will sometimes suppress the orientation from our notation and picture. It is understood that from now on, all graphs are oriented.

8.3. The graph complex. From now on, we restrict to those species Q that arise from the symplectic theory. Namely, we assume that Q is a mated species (3.1-3.2). The examples in (8.1) were all of this type. Hence the vertices necessarily have valence at least 2. We now define a chain complex $(\mathcal{G}, \partial_E)$ for such species.

Definition 7. The k th chain group of \mathcal{G} , which we denote \mathcal{G}_k , is the vector space over \mathbb{Q} generated by all oriented graphs (Γ, σ) with k vertices subject to the relations:

$$(8) \quad (\Gamma, \sigma) = -(\Gamma, -\sigma), \quad \text{and} \quad (\Gamma, \sigma) = (\Gamma', \sigma') \text{ if } (\Gamma, \sigma) \cong (\Gamma', \sigma').$$

A graph can have an orientation reversing automorphism, that is $(\Gamma, \sigma) \cong (\Gamma, -\sigma)$, where σ and $-\sigma$ are the two orientations of Γ . Note that the first relation then makes the class of such a graph equal to zero.

Definition 8. The boundary map $\partial_E : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$ is defined using edge contractions, i.e. matings (3.2). This is where one uses that Q is a mated species. We do not contract loops. More precisely, we have

$$\partial_E(\Gamma, \sigma) = \sum_{e \in E(\Gamma)} (\Gamma/e, \sigma/e),$$

where Γ/e is the graph Γ with the edge e contracted (see figure), and σ/e is obtained the following way: choose a representative of σ where e points from vertex 1 to vertex 2, give the new vertex arising from the contraction of e the label 1, and subtract 1 from the label of each of the other vertices; finally, keep the orientations on the edges (other than e) unchanged.



A mating of two vertices along an edge e .

An equivalent way to describe σ/e is the following: if the labels on the endpoints of e are $i < j$, collapse e , label the resulting vertex i , decrease the labels greater than j by one, and multiply this orientation by $(-1)^j$ if e points from i to j , and by $(-1)^{j+1}$ if it points from j to i .

Note that σ/e is well-defined and that ∂_E respects the relations in equation (8).

Edge contractions in examples. We explain how edge contractions work in the examples in (8.1). A commutative graph is an ordinary graph. And edge contraction has the usual meaning. For associative or ribbon graphs, a (local) picture for an edge contraction is given in (3.2). For tree (resp. chord) graphs, we merge the trees (resp. chords) at the two vertices along the edge being contracted. For a surface graph, an edge is a loop on the surface. To contract an edge, simply delete the loop.

Lemma 4. $\partial_E^2 = 0$.

Proof. If we collapse first the edge $e = (i, j)$ and then the edge $e' = (i', j')$, we get the same graph as by collapsing e' first and then e , but with the opposite orientation. This is easiest to see using the second definition of σ/e above; we may assume that $j < j'$ and $i < j, i' < j'$. We get the same orientation either way, but in the first case with a coefficient $(-1)^j(-1)^{j'-1}$, in the second case with $(-1)^{j'}(-1)^j$. \square

Thus $(\mathcal{G}, \partial_E)$ is a chain complex. We will denote its homology by $H_*(\mathcal{G}, \partial_E)$.

8.4. A family of boundary operators. The reader, who has never seen graph homology before, may omit this section on a first reading. We now define a second boundary operator ∂_H that plays an important part in this theory.

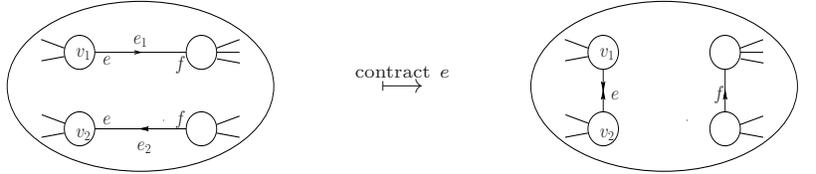
Let $E_q(\Gamma)$ be the set of all *quasi-edges* of Γ . These are unordered pairs of distinct half-edges of Γ . A quasi-edge is specified by two vertices, say v_1 and v_2 , and two edges e_1 and e_2 incident to v_1 and v_2 respectively. We say that e is a *quasi-loop* if $v_1 = v_2$. Observe that $E(\Gamma) \subset E_q(\Gamma)$.

Definition 9. The boundary map $\partial_H : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$ is defined by contracting quasi-edges, which are not edges. We do not contract quasi-loops. We have

$$\partial_H(\Gamma, \sigma) = \sum_{e \in E_q(\Gamma) \setminus E(\Gamma)} (\Gamma/e, \sigma/e),$$

where Γ/e and σ/e are defined as follows.

1. Choose a representative for σ where v_1 and v_2 have labels 1 and 2 respectively and the arrow on edge e_1 (resp. e_2) points out of v_1 (resp. into v_2).
2. Cut the edges e_1 and e_2 and join them so that the quasi-edge e and its partner f turn into edges. In effect, e_1 and e_2 have been replaced by two new edges (which get an induced orientation).



A mating along a quasi-edge e .

3. Collapse the edge joining v_1 and v_2 . Thus v_1 and v_2 become a single vertex, say v . The resulting graph is Γ/e . Note that the edges of Γ/e are already oriented. To get an order on the vertices, label the new vertex v by 1 and subtract 1 from the labels of the other vertices. The resulting orientation is σ/e .

Definition 10. We now define a family of boundary maps by the formula

$$\partial_n = 2n\partial_E + \partial_H.$$

To put in words, $\partial_n(\Gamma)$ is obtained by contracting quasi-edges of Γ . And if the quasi-edge is an edge then we multiply by a factor of $2n$.

The fact that $\partial_n = 2n\partial_E + \partial_H$ is a boundary operator for all n will follow from the proof of Theorem 2 (Section 12). This automatically implies that $\partial_E^2 = \partial_H^2 = 0$ and $\partial_E\partial_H = -\partial_H\partial_E$. In other words, ∂_E and ∂_H together span an abelian Lie

superalgebra of operators on \mathcal{G} . We will call ∂_n the finite boundary operator and write the chain complex as $(\mathcal{G}, \partial_n)$.

Remark. If the boundary operator is not stated explicitly, then we take it to be the stable boundary operator ∂_E .

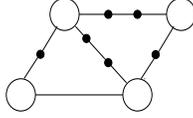
8.5. An important simplification. Note that the boundary maps ∂_E and ∂_H preserve the Euler characteristic χ of any graph Γ . This is because they decrease the number of edges and vertices both by 1. So the graph complex, irrespective of the boundary map used, splits as a direct sum of chain complexes $\mathcal{G}^{(r)}$:

$$\begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{G}_3^{(1)} & \rightarrow & \mathcal{G}_2^{(1)} & \rightarrow & \mathcal{G}_1^{(1)} \rightarrow 0 \\ \cdots & \rightarrow & \mathcal{G}_3^{(2)} & \rightarrow & \mathcal{G}_2^{(2)} & \rightarrow & \mathcal{G}_1^{(2)} \rightarrow 0 \\ \cdots & \rightarrow & \mathcal{G}_3^{(3)} & \rightarrow & \mathcal{G}_2^{(3)} & \rightarrow & \mathcal{G}_1^{(3)} \rightarrow 0 \\ & & \vdots & & \vdots & & \vdots \end{array}$$

The useful point is that the chain groups $\mathcal{G}_k^{(r)}$ are finite dimensional. Instead of χ , we have used the superscript $r = 1 - \chi$ for convenience. If the graph Γ is connected then $r = \dim_{\mathbb{Q}} H_1(\Gamma, \mathbb{Q})$ is just the rank of Γ .

8.6. Other relevant graph complexes. From now on, assume that the species Q is based on sets rather than vector spaces. All the examples in this paper are of this type. We first give a preliminary definition.

Definition 11. We say a vertex in a Q -graph is fake if it corresponds to the element uu in $Q[2]$, the degree 2 piece of Q , see (3.3). Else we say that the vertex is real.



The dots stand for the fake vertices.

Note that the fake vertices are always bivalent and they all look identical. Furthermore, they behave as unit elements in contractions (matings). So in this sense, they are extremely inert.

For all the examples in (8.1), except the surface species ss , $Q[2]$ is a singleton. For example, there is only one cyclic order or chord diagram or unrooted tree on 2 letters. Hence, in these cases, the fake vertices are precisely the bivalent vertices. And real vertices are exactly those with valence greater than 2. For the surface species, $ss[2]$ consists of compact orientable surfaces whose boundary is two disjoint circles. And there are clearly many such. Among them, cylinders are precisely the fake vertices, while the rest are real.

It can even happen that contracting an edge joining two real vertices produces a fake vertex. An example is given in Section 9. In this case, the two real vertices must be necessarily bivalent. In any case, the boundary map can reduce the number of real vertices of a graph by at most 2.

We now look at the various subcomplexes of $(\mathcal{G}, \partial_E)$, that appear in the main theorem and its proof. They are all defined by putting some restriction on the type of graphs that are allowed.

- \mathcal{T} = graphs all of whose vertices are bivalent.
- \mathcal{F} = graphs all of whose vertices are fake.
- \mathcal{C} = connected graphs.
- $Q\mathcal{G}$ = connected graphs all of whose vertices are real and with at least one vertex of degree greater than 2.
- \mathcal{B} = connected graphs all of whose vertices are bivalent.
- \mathcal{P} = connected graphs all of whose vertices are fake.

It is clear that ∂_E preserves the above chain groups except $Q\mathcal{G}$. The trouble with $Q\mathcal{G}$ is that an edge contraction can result in a fake vertex. Hence for $Q\mathcal{G}$, modify the definition of ∂_E such that edge contractions that create fake vertices are ignored. So, strictly speaking, it is not a subcomplex of $(\mathcal{G}, \partial_E)$. In keeping with Kontsevich's original definition, we will call $H_*(Q\mathcal{G}, \partial_E)$ the graph homology of the mated species Q .

Note that \mathcal{F} and \mathcal{P} do not depend on the species Q . The complex \mathcal{P} just consists of polygons and \mathcal{F} consists of disjoint unions of polygons. Further, since fake vertices are always bivalent, the complex \mathcal{F} (resp. \mathcal{P}) is a subcomplex of \mathcal{T} (resp. \mathcal{B}). The containment is strict if $Q[2]$ has more than 1 element, as for the surface or graph species. We discuss another example of this kind in Section 9.

8.7. A reduction step. The chain complex that appears in the statement of the main theorem is $(Q\mathcal{G}, \partial_E)$. However, the one that appears naturally in the proof is $(\mathcal{C}, \partial_E)$. The reduction from the homology of this larger complex to graph homology $H_*(Q\mathcal{G}, \partial_E)$ is the content of Proposition 5. It will be the last step in the proof of the main theorem. Though the statement of the proposition is quite intuitive, its proof is a little technical and may be skipped on a first reading.

Proposition 5. $H_*(\mathcal{C}) = H_*(Q\mathcal{G}) \oplus H_*(\mathcal{B})$.

Proof. The bivalent graph complex \mathcal{B} is a direct summand of the graph complex \mathcal{C} . The complement is the subcomplex consisting of connected graphs containing at least one vertex of degree ≥ 3 . Denote this complex by \mathcal{D} , graded as usual by the number of vertices. The difference between \mathcal{D} and $Q\mathcal{G}$ is that graphs in \mathcal{D} are allowed to contain fake vertices. To finish the proof, we need to show that $H_*(\mathcal{D}) = H_*(Q\mathcal{G})$. This can be done by a spectral sequence argument. For a treatment of spectral sequences, see [31].

We have already defined real and fake vertices. In addition, call an edge fake if it is incident to a fake vertex. Next, define a filtration on \mathcal{D} by $F_p\mathcal{D}_m =$ graphs in \mathcal{D}_m with upto p real vertices. Recall that for the spectral sequence associated to a filtration, $E_{pq}^0 = F_p\mathcal{D}_{p+q}/F_{p-1}\mathcal{D}_{p+q}$. In our case, E_{pq}^0 can be described as the span of oriented graphs with p real and q fake vertices. The vertical maps on the E^0 page, namely, $d_0 : E_{pq}^0 \mapsto E_{p,q-1}^0$ are defined exactly as the boundary map for an oriented graph, except that now we are allowed to collapse only the fake edges. Note that the positive X -axis of the E^0 page consists exactly of graphs with no fake vertices. In other words, they are the chain groups of the graph complex $Q\mathcal{G}$.

Claim. The homology on the E^0 page is trivial except at $q = 0$, that is, the X -axis. Equivalently, $E_{pq}^1 = 0$ for $q > 0$. In addition, we claim that $E_{p0}^1 = Q\mathcal{G}_p$.

Assuming the claim, it is clear that the induced boundary map on the E^1 page $d_1 : E_{p0}^1 \mapsto E_{p-1,0}^1$ coincides with the boundary map defined on $Q\mathcal{G}$. Namely, edge contractions that create fake vertices are ignored. The rest of the E^1 page is zero.

Thus the spectral sequence associated to our filtration of \mathcal{D} collapses at the first term to the graph complex QG . This shows that $H_*(\mathcal{D}) = H_*(QG)$.

Proof of the claim. To every connected graph containing at least one real vertex, one can associate a new graph all of whose vertices are real. The new graph is just the old graph with all the fake vertices removed. This gives us an equivalence relation on the set of all connected graphs containing a real vertex. The equivalence classes are indexed by graphs all of whose vertices are real. Now consider the complex

$$\dots \rightarrow E_{pq}^0 \xrightarrow{d_Q} E_{pq-1}^0 \xrightarrow{d_Q} \dots \xrightarrow{d_Q} E_{p0}^0 = QG_p \xrightarrow{d_Q} 0$$

whose homology we want to compute. This splits as a direct sum of subcomplexes one for each graph Γ in $E_{p0}^0 = QG_p$. Call the subcomplex corresponding to Γ the standard complex of Γ . Note that the graphs that occur in the standard complex of Γ are precisely those that lie in the same equivalence class as Γ . We need to show that the only non-trivial homology occurs in degree 0.

To understand this complex, first look at the standard complex of a single edge. It has dimension 1 in each degree $k \geq 0$, because there is a unique (upto isomorphism) way to put k points in the interior of an edge. The differential in this standard complex is zero for $k = 0$ and all odd k and an isomorphism for positive even k . Hence it has non-trivial homology only in degree 0.

Now the standard complex of Γ is the tensor product of the standard complexes of all its edges modulo the action of the finite group $\text{Aut}(\Gamma)$. Hence by the Kunneth formula and the fact that finite groups have trivial rational homology [6], we are done. \square

Corollary. *Let Q be a mated species, where all bivalent vertices in any graph are fake. Or equivalently, let Q be obtained from a reversible operad satisfying $P[1] = \mathbb{Q}$. Then $H_*(\mathcal{C}) = H_*(QG) \oplus H_*(\mathcal{P})$.*

9. GRAPH HOMOLOGY FOR GROUPS

We now digress to give an example of the theory that is based on groups. It can be defined more generally for algebras with involution; we then recover dihedral homology [27]. We hope that this example will further clarify the concepts discussed in previous sections. An attraction of this example is that it is tractable to computations. Some known results on dihedral homology can be obtained this way.

9.1. Groups as reversible operads. Let K be any finite group. Let P be the operad whose elements are given by $a \xrightarrow{g} \bullet \rightarrow b$ for $g \in K$, with substitution being group multiplication. In other words, $P[1] = \mathbb{Q}K$, the group algebra of K over \mathbb{Q} , and $P[2] = P[3] = \dots = 0$. If K is the trivial group then P is just the unit operad u , see (2.3.1). More generally, one can take $P[1]$ to be any algebra with a unit.

For this example, P is reversible (2.4), if there exists a map $*$: $K \rightarrow K$ which satisfies $1^* = 1$ and $(gh)^* = h^*g^*$ for any $g, h \in K$. Natural candidates for the $*$ map are the identity (if K is abelian) and the inverse map. Thus, for K abelian, there are two distinct ways of reversal. The reversal map is given by

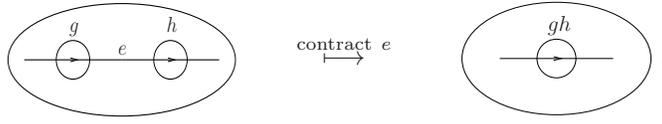
$$r_{a,b} \left(a \xrightarrow{g} \bullet \rightarrow b \right) = a \xleftarrow{g^*} \bullet \leftarrow b .$$

Applying the mating functor to P gives us a mated species Q . This species Q lives entirely in degree 2. We leave it to the reader to check that an element of $Q[\{a, b\}]$ can be specified by the picture

$$(9) \quad \begin{array}{c} g \\ \circlearrowright \\ a \text{ --- } \text{---} b \end{array} = \begin{array}{c} g^* \\ \circlearrowleft \\ a \text{ --- } \text{---} b \end{array} .$$

9.2. Graph homology for groups. For the above species Q , the corresponding graphs are necessarily bivalent (polygons), with each vertex labelled (more or less) by a group element. The vertices labelled by the unit element 1 are fake while the rest are real.

We now explain how an edge contraction (mating) works.



Apply relation (9) if necessary, so that both vertices incident to the edge e point in the same direction. Contract it and make the new vertex also point in the same direction. And label it with the product of the two labels (in the order specified by the direction).

We will call the homology of this bivalent complex as the graph homology of the group K and denote it $H_*(K, *)$. In the notation of Section 8, it coincides with both $H_*(\mathcal{C})$ and $H_*(\mathcal{B})$.

9.3. Graph homology computations. The problem of computing graph homology seems to be difficult in general. The only instance where computations have been made is the commutative case (2.3.2); see the thesis of Ferenc Gerlits [12]. This is the case where we are dealing with usual graphs.

We give two instances where graph homology $H_*(\mathcal{C})$ can be completely computed. They are the first two cases of the example under discussion. It is easy to generalise them but we do not do it here.

9.3.1. The trivial group. If K is the trivial group then $P = u$ and $Q = uu$, see (3.3). The computation for this case, which coincides with $H_*(\mathcal{P})$, see (8.6), is discussed in [22]. The graphs are polygons with only fake vertices. These graphs have two kinds of automorphisms: rotations and reflections. It is easy to see that a simple rotation is orientation reversing if and only if k , the number of vertices, is even, and a reflection reverses the orientation iff $k \equiv 1$ or 2 modulo 4. So there is one non-zero chain in each degree of the form $4i + 3$, and hence all the boundary maps are zero, therefore these are the homology groups also. Hence

$$H_k(\{1\}, \text{identity}) = \begin{cases} \mathbb{Q} & \text{if } k \equiv 3 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary to Theorem 4 (Section 14), this is same as the primitive homology of the Lie algebra $\mathfrak{sp}(2\infty)$.

9.3.2. The group \mathbb{Z}_2 . For this case, the graphs are polygons with two types of vertices corresponding to the two group elements. Computing the homology directly

from the definition is not easy. So we rerun the spectral sequence argument (Proposition 5) with some modification. The stable page is E^2 with non-zero terms only on the positive X and Y axis. The conclusion is

$$H_k(\mathbb{Z}_2, \text{identity}) = \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & k \equiv 3 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

This example, we hope, gives an idea of the complexity of computing graph homology.

10. GRAPH COHOMOLOGY

The ideas in this section are based on a letter of Kontsevich. It leads to the birth of many interesting operations on graphs. I thank Jim Conant for helping me understand its contents and also for providing his notes related to this.

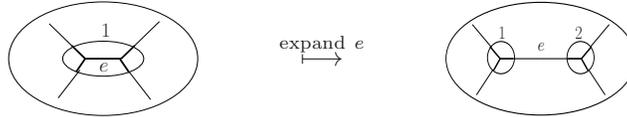
We begin by defining graph cohomology. The homology and cohomology are related by an interesting and highly non-trivial pairing on graphs. This is explained in (10.2-10.3). Later in (10.5), we use it to define a deformation map on graphs. These ideas will be used in Section 13 in the proof of Theorem 3 that deals with stability. Throughout this section, we assume that Q is a mated species based on sets.

10.1. The blowup coboundary operator δ_E . Let Γ be an oriented Q -graph and $I(\Gamma)$ be the set of its ideal edges. These are the edges that are “present” in the internal structure of the vertices of Γ , see (3.1).

Definition 12. The coboundary map $\delta_E : \mathcal{G}_k \rightarrow \mathcal{G}_{k+1}$ is defined using ideal edge expansions, i.e. breakups. It is given by the formula

$$\delta_E(\Gamma, \sigma) = \sum_{e \in I(\Gamma)} (\Gamma \setminus e, \sigma \setminus e),$$

where $\Gamma \setminus e$ is the graph Γ with the ideal edge e expanded (see figure), and $\sigma \setminus e$ is obtained the following way: choose a representative of σ where the vertex of the ideal edge e has label 1, give the two new vertices arising from the breakup the labels 1 and 2, add 1 to the label of each of the other vertices, and direct the newly formed edge e from vertex 1 to 2; finally, keep the orientations on the other edges unchanged.



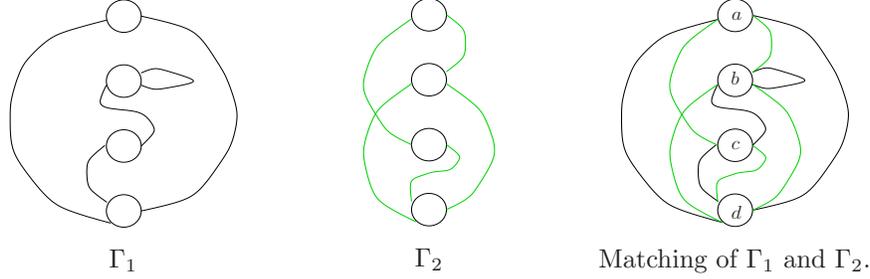
A local picture of a breakup along an ideal edge e .

Similarly, one can define a coboundary operator δ_H as the co-analogue of ∂_H , see (8.4). We will not deal with δ_H in this paper.

10.2. A family of pairings $M(n)$. For every n , we will define a pairing $M(n) : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathbb{Q}$. The definition, though explicit, will be somewhat complicated. A way to understand it is given in (13.1-13.2), where we derive it as a restriction of a simpler pairing defined on a larger space. The reader, who is more interested in Lie algebras or the main theorem rather than just graphs, may read that part first, referring back as necessary.

Let $(\Gamma_1, \sigma_1), (\Gamma_2, \sigma_2) \in \mathcal{G}$ be two oriented graphs. For simplicity, we suppress orientations from the notation.

Definition 13. A matching $m : \Gamma_1 \rightarrow \Gamma_2$ is a bijection between the vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$ that preserves the internal structure of the vertices. We imagine this as an overlaying of the vertex sets of Γ_1 and Γ_2 .



An example of a matching is shown above. For clarity, orientations and the internal structure of the vertices have been omitted. It is clear that for a matching to exist, Γ_1 and Γ_2 must have the same number of vertices and edges.

Definition 14. We now define the number of components $c(m)$ and $\text{sign}(m)$ of a matching $m : \Gamma_1 \rightarrow \Gamma_2$. Delete the vertices from the overlaying of Γ_1 and Γ_2 specified by m . We are left with a disjoint union of even sided polygons whose edges alternate between those of Γ_1 and Γ_2 . The quantity $c(m)$ counts the number of these polygons. In the example above, we get a hexagon and a square; so $c(m) = 2$.



The purpose of $\text{sign}(m)$ is to take orientations into account. Choose representatives for σ_1 and σ_2 . The matching m gives a bijection of the vertex sets. First note the sign of this permutation. Then for each polygon, fix say the clockwise direction for Γ_1 and anticlockwise direction for Γ_2 . And write a minus sign for all edges that are out of order. The product of all signs gives $\text{sign}(m)$.

Definition 15. Define the pairing $M(n) : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathbb{Q}$ by the formula

$$(10) \quad M(n)(\Gamma_1, \Gamma_2) = \sum_{m: \Gamma_1 \rightarrow \Gamma_2} \text{sign}(m)(2n)^{c(m)},$$

where the sum is over all matchings $m : \Gamma_1 \rightarrow \Gamma_2$ and $\text{sign}(m)$ and $c(m)$ are the sign and the number of components of the matching m .

Remark. For Γ_1 and Γ_2 fixed, the maximum value of $c(m)$ is $e =$ number of edges of Γ_1 (and Γ_2). This happens iff every polygon in the matching m has exactly two sides, i.e. iff $m : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism. We will call this a perfect matching. And in this case, the coefficient of $(2n)^e$, upto sign, is $|\text{Aut}(\Gamma_1)| = |\text{Aut}(\Gamma_2)|$.

10.3. The adjoint property. We now relate the finite boundary operator ∂_n in (8.4) to the (stable) coboundary operator δ_E in (10.1). There is a simpler relation between ∂_E and δ_E which can be derived from this one, by looking at the leading coefficient. This is explained in (10.5). Recall that ∂_n is defined by contracting quasi-edges and δ_E is defined by expanding ideal edges.

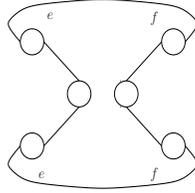
Proposition 6. *The maps ∂_n and δ_E are adjoints with respect to the pairing $M(n)$. In other words, $M(n)(\partial_n\Gamma_1, \Gamma_2) = M(n)(\Gamma_1, \delta_E\Gamma_2)$.*

Proof. To prove the above identity, we express both sides as weighted state sums, and then give a bijection of the state space in the LHS with that in the RHS such that it respects weights. Note that for either side to be nonzero, Γ_1 must have one vertex and edge more than Γ_2 .

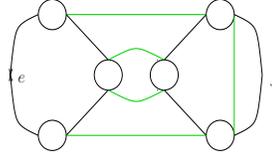
Define a state S_L in the LHS to be a pair (e, m) of a quasi-edge $e \in E_q(\Gamma_1)$ and a matching $m : \Gamma_1/e \rightarrow \Gamma_2$. Then by the definition of the boundary map ∂_n and the pairing $M(n)$, the LHS $= M(n)(\partial_n\Gamma_1, \Gamma_2) = \sum_{S_L} w(S_L)$, where

$$w(S_L) = \begin{cases} \text{sign}(m)(2n)^{c(m)+1} & \text{if } e \in E(\Gamma_1), \\ \text{sign}(m)(2n)^{c(m)} & \text{if } e \in E_q(\Gamma_1) \setminus E(\Gamma_1). \end{cases}$$

An example of a state S_L for $e \in E_q(\Gamma) \setminus E(\Gamma)$ is shown in the figure. The quasi-edge f is the partner of e . The graph Γ_2 is not shown separately, since it is visible from the overlaying. Also, in the graph Γ_1/e , the edge e has been shown as an ideal edge.



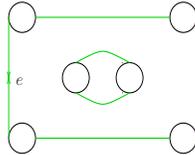
Γ_1 with a quasi-edge e .



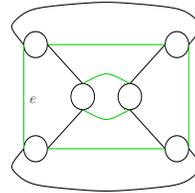
$S_L =$ A matching of Γ_1/e and Γ_2 .

Similarly, a state S_R in the RHS is defined to be a pair (e, m) of an ideal edge $e \in I(\Gamma_2)$ and a matching $m : \Gamma_1 \rightarrow \Gamma_2 \setminus e$. Then by definition the RHS $= M(n)(\Gamma_1, \delta_E(\Gamma_2)) = \sum_{S_R} w(S_R)$, where $w(S_R) = \text{sign}(m)(2n)^{c(m)}$.

We now indicate the bijection between the two state spaces by continuing our example. We draw the state S_R that corresponds to the state S_L shown in the figure above.



Γ_2 with an ideal edge e .



$S_R =$ A matching of Γ_1 and $\Gamma_2 \setminus e$.

The two pictures that represent the states S_L and S_R are identical except for the local behaviour at the edge e .

In general, to see that $w(S_L) = w(S_R)$, note that a state S_R is of two kinds depending on whether the ideal edge $e \in I(\Gamma_2)$ is overlayed on a quasi-edge $e \in E(\Gamma_1)$ or $e \in E_q(\Gamma_1) \setminus E(\Gamma_1)$. In the first case, S_R has one more component than

S_L while in the second they are the same. We illustrated the second case in our pictures. This fits in with the two cases for $w(S_L)$. It can also be checked that the signs work out correctly. \square

10.4. Non-degeneracy of the pairing. Let \mathcal{G}_k^e be the span of all oriented graphs with k vertices and e edges. Observe that the subspaces \mathcal{G}_k^e of \mathcal{G} , as k and e vary, are mutually orthogonal with respect to the pairing $M(n)$. By our earlier notation (8.5), we have $\mathcal{G}_k^e = \mathcal{G}_k^{(r)}$, with $r = 1 - \chi = e - k + 1$. The notation \mathcal{G}_k^e is local to this section and is introduced to improve clarity.

For any k and e , the pairing $M(n)$ defines a map $\mathcal{G}_k^e \rightarrow (\mathcal{G}_k^e)^*$, which we write $M_k^e(n)$. Regard $M_k^e(n)$ as a matrix whose rows and columns are indexed by (isomorphism classes of) oriented graphs Γ with k vertices and e edges and where the (Γ_i, Γ_j) entry is given by $\langle M_k^e(n)(\Gamma_i), \Gamma_j \rangle$. By the definition of the pairing $M(n)$, the entries of this matrix are polynomials in $2n$ of degree $\leq e$ and moreover the degree e entries are precisely the diagonal entries. Furthermore, the coefficient of $(2n)^e$ of $\langle M_k^e(n)(\Gamma), \Gamma \rangle$ is just $|\text{Aut}(\Gamma)|$. This is clear from the remark at the end of (10.2).

With e and k fixed and n large enough, the diagonal entries dominate, so the matrix $M_k^e(n)$ is invertible, and $\mathcal{G}_k^e \xrightarrow{\cong} (\mathcal{G}_k^e)^*$. In other words, the pairing $M(n)$ is non-degenerate for e and k fixed and n large enough.

10.5. The deformation map $D(n)$. The adjoint property of ∂_n and δ_E established in Proposition 6 says that

$$(11) \quad M_{k-1}^{e-1}(n) \circ \partial_n = \delta_E^* \circ M_k^e(n).$$

Let A_k^e denote the diagonal matrix with (Γ, Γ) entry equal to $|\text{Aut}(\Gamma)|$. Then the degree e part of the above equation is just

$$(12) \quad A_{k-1}^{e-1} \circ \partial_E = \delta_E^* \circ A_k^e.$$

Equation (11) (resp. (12)) shows the precise sense in which ∂_n (resp ∂_E) is dual to δ_E . The results for n large enough can be summarised in the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{G}_k^e & \xrightarrow[\cong]{M_k^e(n)} & (\mathcal{G}_k^e)^* & \xleftarrow[\cong]{A_k^e} & \mathcal{G}_k^e \\ 2n\partial_E + \partial_H \downarrow & & \delta_E^* \downarrow & & \partial_E \downarrow \\ \mathcal{G}_{k-1}^{e-1} & \xrightarrow[\cong]{M_{k-1}^{e-1}(n)} & (\mathcal{G}_{k-1}^{e-1})^* & \xleftarrow[\cong]{A_{k-1}^{e-1}} & \mathcal{G}_{k-1}^{e-1} \end{array}$$

Now switching back to our earlier notation, the above commutative diagram can be rewritten as

$$(13) \quad \begin{array}{ccccc} \mathcal{G}_k^{(r)} & \xrightarrow[\cong]{M(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_k^{(r)} \\ 2n\partial_E + \partial_H \downarrow & & \delta_E^* \downarrow & & \partial_E \downarrow \\ \mathcal{G}_{k-1}^{(r)} & \xrightarrow[\cong]{M(n)} & (\mathcal{G}_{k-1}^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_{k-1}^{(r)} \end{array}$$

We point out that $M(n)$ is used to denote both the pairing and the map it induces.

Corollary. *Let k and r be fixed. Then for n large enough*

$$H_k(\mathcal{G}^{(r)}, \partial_n) = H_k(\mathcal{G}^{(r)}, \partial_E).$$

We will call the map $G^{(r)} \rightarrow G^{(r)}$, which induces the above isomorphism, the deformation map $D(n)$. It is the map $M(n)$ followed by A^{-1} . We note that for a graph Γ , its image $D(n)\Gamma$ is a polynomial in $2n$ of degree e with coefficients in \mathcal{G} . Furthermore, the coefficient of $(2n)^e$ is exactly Γ . Here $e = |E(\Gamma)|$, the number of edges in Γ . The deformation map is a key step in the proof of the main theorem. We will also study it briefly in Appendix B.

11. THE MAIN THEOREM

In this section, we give a precise statement of the main theorem. We will also discuss the classical case briefly. First recall some facts from Section 5.

Let P be a reversible operad and Q be its mated species. There are two algebraic objects PA and QA associated to P and Q respectively. One thinks of QA as ‘‘Hamiltonian functions on the symplectic P -manifold’’. It has the structure of a Lie algebra. The Lie algebra QA depends on the dimension of the symplectic P -manifold. So we write QA_n when the dimension is $2n$. The dimension is necessarily even. We then have a sequence of Lie algebra inclusions

$$QA_1 \subset \dots \subset QA_n \subset QA_{n+1} \subset \dots$$

We denote the direct limit by QA_∞ . And QA_n always contains the symplectic Lie algebra $\mathfrak{sp}(2n)$ as an anti-subalgebra. We write $\mathfrak{sp}(2\infty)$ for the corresponding direct limit. The rational homology of QA_∞ , which we denote $H_*(QA_\infty)$, has the structure of a Hopf algebra. We write $PH_*(QA_\infty)$ for the subspace of primitive elements.

11.1. Statement of the main theorem. Kontsevich’s result can now be stated as

Theorem 1. $PH_*(QA_\infty) = H_*(Q\mathcal{G}, \partial_E) \oplus PH_*(\mathfrak{sp}(2\infty))$.

The term $H_*(Q\mathcal{G}, \partial_E)$ is the graph homology of the mated species Q . It is the homology of the chain complex $(Q\mathcal{G}, \partial_E)$ of graphs that was defined in Section 8.

There are two conditions that we require in the theorem. We assume that the operad P is based on sets rather than vector spaces. This is because the proof involves a pairing on graphs (10.2), which we know how to define only in the former case. We hope that eventually this restriction would not be necessary. Hence, as of now, the theorem cannot be applied to the Lie operad, which is one of the cases claimed in [22].

Secondly, we assume that $P[1]$ is a singleton consisting of the unit element u , see (2.2). This is done mainly for simplicity. If we drop this assumption then the summand $PH_*(\mathfrak{sp}(2\infty))$ has to be replaced by the homology of the bivalent graph complex $(\mathcal{B}, \partial_E)$ defined in Section 8.

Remark. The primitive homology of $\mathfrak{sp}(2\infty)$ is known (9.3.1). It will be computed in the course of proving the theorem.

11.2. The classical case. This is the commutative case, $P = c$ and $Q = cc$, see (4.2.2). In this case, the ‘‘symplectic operad manifold’’ actually exists and is simply $(\mathbb{R}^{2n}, \omega_0)$. The Lie algebra QA_n , which we write as ccA_n , consists of polynomial functions in $2n$ variables with no constant or linear terms. The Lie structure is given by the usual Poisson bracket (equation (2) in Section 4.2). There is an evident subalgebra uuA_n consisting of all homogeneous polynomials of degree 2.

The reason for this notation is that uuA_n can also be seen as an example of the theory with $P = u$ and $Q = uu$, see (5.4). The Lie algebra uuA_n is anti-isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n)$.

It is also clear that we have a sequence of Lie algebra inclusions

$$ccA_1 \subset \dots \subset ccA_n \subset ccA_{n+1} \subset \dots$$

The direct limit ccA_∞ consists of finite polynomials in infinitely many variables $p_1, p_2, \dots, q_1, q_2, \dots$

12. PROOF OF THE MAIN THEOREM-PART I

We now begin the proof of the main theorem. It will be done in three steps. In this section, we take the first step of relating the homology of a Lie algebra to graph homology. We will prove the following theorem.

Theorem 2. $H_*(QA_n) = H_*(\mathcal{G}, \partial_n)$.

For definitions of the above terms, see (5.1-5.3) and (8.3-8.4). For the commutative case, the Lie algebra QA_n is easy to define, see (11.2). The main ideas of the proof are already present in the commutative case. Hence the reader may specialise to this case on a first reading.

Corollary. $H_*(\mathfrak{sp}(2n)) = H_*(\mathcal{F}, \partial_n)$.

Proof. We apply the theorem to the unit species, i.e. $Q = uu$. In this case, $QA_n = uuA_n$, which is anti-isomorphic to $\mathfrak{sp}(2n)$, see (5.4). Due to the trivial nature of the species, the graphs have only fake (bivalent) vertices. Hence the chain complex \mathcal{G} , in this case, is simply the fake bivalent complex \mathcal{F} , see (8.6). \square

We now start the proof of Theorem 2. It is best summarised in the following Kontsevich sentence.

The spirit of the (quite simple) computations is somewhere between
Gelfand-Fuks computations (see [11] and [9]) and cyclic homology.

12.1. Lie algebra homology. A good introduction to Lie algebra homology can be found in Weibel's book [39, Chapter 7]. Recall that the homology of the Lie algebra QA_n can be computed using the Chevalley-Eilenberg or standard complex

$$\dots \longrightarrow \mathcal{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathcal{C}_k \xrightarrow{\partial_k} \mathcal{C}_{k-1} \xrightarrow{\partial_{k-1}} \dots$$

with $\mathcal{C}_k = \Lambda^k(QA_n)$ and

$$(14) \quad \partial_k(F_1 \wedge \dots \wedge F_k) = \sum_{1 \leq s < t \leq k} (-1)^{s+t-1} \{F_s, F_t\} \wedge F_1 \dots \hat{F}_s \dots \hat{F}_t \dots \wedge F_k.$$

The boundary operator commutes with the action of the Lie algebra QA_n on the exterior powers $\Lambda^k(QA_n)$.

We have learnt to think of the Lie bracket $\{ , \}$ on QA_n as a mating. For the commutative case, this was discussed in (4.2.2). In the formula (14), we take k elements of QA_n , say F_1, \dots, F_k , and to apply the boundary map we do pairwise matings. Now let us think for a moment about graphs. The boundary operators on a graph are all defined using quasi-edge contractions, which are again matings. Hence to relate the two notions, all one needs to do is imagine the elements F_1, \dots, F_k as being the vertices of a graph. We now work towards making this precise.

12.2. Passing to the subcomplex of $\mathfrak{sp}(2n)$ invariants. Recall from (5.1) that

$$(15) \quad QA_n = \bigoplus_{j \geq 2} (Q[j] \otimes V^{\otimes j})_{\Sigma_j} = \bigoplus_{j \geq 2} QA_n^j.$$

In the commutative case, $QA_n^j = S^j(V)$ is the j th symmetric tensor power of V . This is just the space of commuting polynomials of degree j in a basis of V .

The space QA_n^j is a left $\mathfrak{sp}(2n)$ module with the action induced by the usual action on V and the trivial action on $Q[j]$, see (5.4). Thus, the Lie algebra QA_n is a direct sum of finite dimensional $\mathfrak{sp}(2n)$ modules. Since the Lie algebra $\mathfrak{sp}(2n)$ is simple, it follows that QA_n and $\Lambda^k(QA_n) = \mathcal{C}_k$ are semisimple $\mathfrak{sp}(2n)$ modules. This allows us to write $\mathcal{C}_k = (\mathcal{C}_k)^{\mathfrak{sp}(2n)} \oplus \mathfrak{sp}(2n) \cdot \mathcal{C}_k$. A standard argument now shows that the subcomplex $\mathfrak{sp}(2n) \cdot \mathcal{C}$ is exact. Hence the homology of the standard complex is the same as the homology of the subcomplex of $\mathfrak{sp}(2n)$ -invariants:

$$\dots \longrightarrow (\mathcal{C}_{k+1})^{\mathfrak{sp}(2n)} \xrightarrow{\partial_{k+1}} (\mathcal{C}_k)^{\mathfrak{sp}(2n)} \xrightarrow{\partial_k} (\mathcal{C}_{k-1})^{\mathfrak{sp}(2n)} \xrightarrow{\partial_{k-1}} \dots$$

Remark. The standard argument is as under. Let \mathfrak{g} be a Lie algebra and \mathcal{C} be its standard complex. Then for $\xi \in \mathfrak{g}$, define Lie derivative L_ξ (action of ξ) and contraction operator i_ξ (wedge with ξ) as operators on \mathcal{C} . The relations of Lemma 1 (Section 6) hold in this case also. Cartan's formula implies that the Lie derivative is zero on $H_*(\mathcal{C})$. Now let \mathfrak{h} be a simple subalgebra of \mathfrak{g} such that \mathcal{C} is a semisimple \mathfrak{h} module. Then the subcomplex $\mathfrak{h} \cdot \mathcal{C}$ is exact.

For the semisimplicity of $\Lambda^k(QA_n)$, we used two facts. Any finite dimensional module of a simple Lie algebra is semisimple. The tensor product of two finite dimensional semisimple modules over any Lie algebra is again semisimple [18, pg 83].

12.3. Passing from invariants to oriented graphs. We want to relate the above subcomplex of $\mathfrak{sp}(2n)$ -invariants to the graph complex $(\mathcal{G}, \partial_n)$. To this end, we first try to get a better understanding of the chain groups $(\Lambda^k QA_n)^{\mathfrak{sp}(2n)}$.

The description of QA_n given by equation (15) gives us the following formula:

$$\Lambda^k QA_n = \bigoplus_{\substack{k_1 + k_2 + \dots + k_r = k \\ 2 \leq j_1 < j_2 < \dots < j_r}} (\Lambda^{k_1} QA^{j_1} \otimes \dots \otimes \Lambda^{k_r} QA^{j_r}).$$

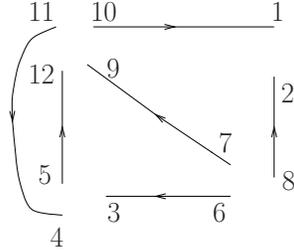
We try to understand the $\mathfrak{sp}(2n)$ invariants in each summand. This will be done in three stages. As a concrete example, we will consider the summand $\Lambda^1 QA^2 \otimes \Lambda^2 QA^3 \otimes \Lambda^1 QA^4$ and see how the analysis works on it at every stage. We will illustrate it with the tree species $(Q = tt)$.

The term $\Lambda^{k_1} QA^{j_1}$ is a quotient of the tensor power $(Q[j_1] \otimes V^{\otimes j_1})^{\otimes k_1}$. Hence each summand of $\Lambda^k QA_n$ is a quotient of the tensor power $\otimes_{t=1}^r (Q[j_t] \otimes V^{\otimes j_t})^{\otimes k_t}$. In the first two stages, we will figure out the $\mathfrak{sp}(2n)$ -invariants in this tensor power. In the third stage, we will mod out by appropriate actions of the symmetric groups.

12.3.1. The first stage. We find the invariants in $V^{\otimes \sum_{t=1}^r j_t k_t}$. By the invariant theory of $\mathfrak{sp}(2n)$, we know that a base for the invariants in $V^{\otimes \sum_{t=1}^r k_t j_t}$ is given by oriented chord diagrams on $\sum_{t=1}^r k_t j_t$ vertices, if n is sufficiently large; see [10, Appendix F].

For the definition of a chord diagram, see (2.3.5). By an oriented chord diagram, we mean that each chord is oriented and reversing the orientation of a single chord

incurs a minus sign.

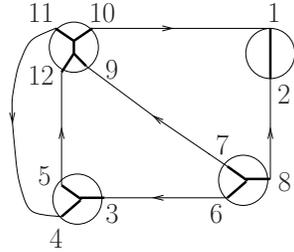


An oriented chord diagram on 12 vertices. It gives a $\mathfrak{sp}(2n)$ invariant in $V^{\otimes 12} = (V^{\otimes 2})^{\otimes 1} \otimes (V^{\otimes 3})^{\otimes 2} \otimes (V^{\otimes 4})^{\otimes 1}$.

12.3.2. *On how an oriented chord diagram gives an invariant.* Each vertex of the diagram represents a tensor factor, in the order given by the vertex labelling. For each edge, we put a p_i at the tail of the arrow and a q_i at the head or we put a q_i at the tail of the arrow and a p_i at the head, incurring a minus sign as a result. We then sum over all possible choices to get the invariant.

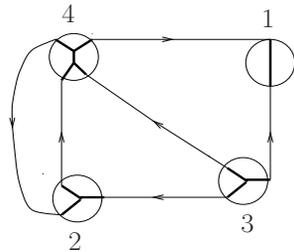
The smallest $\mathfrak{sp}(2n)$ invariant lies in $V \otimes V$. It is given by $\sum_{i=1}^n p_i \otimes q_i - q_i \otimes p_i$, which we represent by the chord diagram $1 \rightarrow 2$.

12.3.3. *The second stage.* To get the invariants in $\otimes_{t=1}^r (Q[j_t] \otimes V^{\otimes j_t})^{\otimes k_t}$, we tensor the space of invariants obtained above by $\otimes_{t=1}^r Q[j_t]^{\otimes k_t}$. This is alright because the $\mathfrak{sp}(2n)$ action is trivial on the $Q[j]$'s. This has the following effect on our picture.



Remark. We have drawn trees inside the circles because we are illustrating with the tree species. In the commutative case, we will instead get ordinary graphs with directed edges and an order on the set of half-edges. And in the associative case, we get ribbon graphs with similar data.

12.3.4. *The third stage.* Modding out the actions of the symmetric groups has the following effect on our picture. Again, we are illustrating with the tree case.



Firstly, we have removed all the labels on the half-edges and instead given an ordering to the vertices. This is because we are modding out the action of the Σ_j 's. Secondly, we must now interpret the order on the vertices of the graph in the sense

of orientation. That is, if we interchange the order of two consecutive vertices then we pick a minus sign. This is due to the presence of the wedges. What we are left with is precisely an oriented Q -graph, see (8.2). Note that the valence of the vertices is at least 2, since the grading on QA begins at 2. Thus we get an isomorphism of chain groups $(\mathcal{C}_k)^{\mathfrak{sp}(2n)} \cong \mathcal{G}_k$, see (8.3).

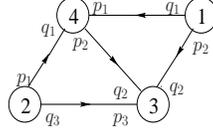
12.4. Comparison of the boundary maps of the two complexes. To complete the proof of Theorem 2, we need to show that the following diagram commutes.

$$(16) \quad \begin{array}{ccc} \mathcal{G}_k & \xrightarrow{\cong} & (\mathcal{C}_k)^{\mathfrak{sp}(2n)} \\ 2n\partial_E + \partial_H \downarrow & & \partial \downarrow \\ \mathcal{G}_{k-1} & \xrightarrow{\cong} & (\mathcal{C}_{k-1})^{\mathfrak{sp}(2n)}. \end{array}$$

Define $\partial' : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$ as the composite of three maps in the above diagram (the middle map is ∂). We want to show that $\partial' = 2n\partial_E + \partial_H$. For this, we first understand the map $\mathcal{G}_k \rightarrow (\mathcal{C}_k)^{\mathfrak{sp}(2n)}$ better.

12.4.1. On how an oriented graph gives an invariant. Starting with an oriented graph with k vertices, we want to construct a $\mathfrak{sp}(2n)$ invariant in $\mathcal{C}_k = \Lambda^k(QA_n)$. The description that we give follows directly from the one that we gave for an oriented chord diagram (12.3.2).

Let (Γ, σ) be an oriented graph. Choose a representative for σ . This means that the vertices of Γ are ordered and the edges are oriented. Each vertex of the graph represents a tensor factor, in the order given by the vertex labelling. For each edge, we put a p_i at the tail of the arrow and a q_i at the head or we put a q_i at the tail of the arrow and a p_i at the head, but picking a minus sign. This is called a state of the edge. And a *state* of the graph is a choice of a state for every edge. Summing over all states of Γ and passing to the wedge product gives an $\mathfrak{sp}(2n)$ invariant in $\Lambda^*(QA_n)$.



An example of a state is shown above. It gives us the term

$$q_1 \text{ --- } \bullet \text{ --- } p_2 \quad \wedge \quad p_1 \text{ --- } \bullet \text{ --- } q_3 \quad \wedge \quad p_3 \text{ --- } \bigcirc \begin{array}{l} / q_2 \\ \backslash q_2 \end{array} \quad \wedge \quad p_1 \text{ --- } \bigcirc \begin{array}{l} / p_2 \\ \backslash q_1 \end{array}$$

in $\Lambda^4(QA_n)$. Since there are an even number of negative signs, the net sign is positive. To get the invariant, we sum over all states.

12.4.2. Comparing the matings. Now we show that $\partial' = 2n\partial_E + \partial_H$. Let $(\Gamma, \sigma) \in \mathcal{G}_k$. And let I be the invariant in $(\mathcal{C}_k)^{\mathfrak{sp}(2n)}$ corresponding to Γ obtained by the procedure described above.

In order to compute $\partial(I)$, we apply the formula for ∂ given by equation (14) to each state of I . So for each state, we must compute the Lie bracket of the data at each pair of vertices. However, recall from (5.3) that the Lie bracket is simply a sum of matings. Hence $\partial(I)$ can be computed as a sum over all states of I , where

for each state, we mate a p_i on a half-edge at a vertex with a q_i on another half-edge at a different vertex in all possible ways. Another way to say this is that a mating occurs along a quasi-edge (which is a pair of distinct half-edges) that is not a quasi-loop (8.4).

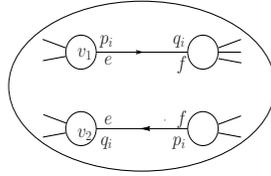
Now we rewrite $\partial(I)$ as a sum indexed by quasi-edges e of Γ . The summand for a quasi-edge e is the sum of those terms for which the mating occurs along e . It is obtained in two steps.

1. Fix a state for e . Then sum over all possible states of the other edges.
2. Do this for each of the $2n$ states of e .

We explain a little of how the orientation will work out. We fix a representative σ for Γ . Let v_1 and v_2 with labels $s < t$ be the vertices of the quasi-edge e that we want to contract. In order to get a representative for σ/e , we first reorder the vertices so that v_1 and v_2 have labels 1 and 2 respectively. In doing so, we pick up a factor of $(-1)^{s+t-1}$. This is precisely the factor that appears in formula (14) for $\partial(I)$.

Now for a quasi-edge e , we perform the two steps above. The analysis splits into two cases. In the first case, assume that e is an actual edge, i.e. $e \in E(\Gamma)$. Then the first step itself gives an invariant that comes from $(\Gamma/e, \sigma/e)$. Hence, after the second step, the net contribution is $2n(\Gamma/e, \sigma/e)$.

In the second case, assume that e is a quasi-edge that is not an edge, i.e. $e \in E_q(\Gamma) \setminus E(\Gamma)$. So the first step does not give an invariant. This is because, if the state of e is fixed then the state of its partner, the quasi-edge f , is also fixed. This is shown in the figure below.



However, after the second step, we do get the invariant $(\Gamma/e, \sigma/e)$.

Hence putting the two cases together, we obtain

$$\partial'(\Gamma, \sigma) = 2n \sum_{e \in E(\Gamma)} (\Gamma/e, \sigma/e) + \sum_{e \in E_q(\Gamma) \setminus E(\Gamma)} (\Gamma/e, \sigma/e).$$

The right hand side is precisely $2n\partial_E + \partial_H$. This shows the commutativity of diagram (16) and completes the proof of Theorem 2.

13. PROOF OF THE MAIN THEOREM-PART II

In the previous section, the Lie algebra homology $H_*(QA_n)$ was related to graph homology. In this section, we show that the homology of the Lie algebra QA_n , as n varies, is “stable”. Recall that $QA_\infty = \lim_{n \rightarrow \infty} QA_n$, see (5.5). We will prove the following theorem.

Theorem 3. $H_*(QA_\infty) = H_*(\mathcal{G}, \partial_E)$.

As in the previous section, we apply the theorem to the unit species uu and obtain the following corollary.

Corollary. $H_*(\mathfrak{sp}(2\infty)) = H_*(\mathcal{F}, \partial_E)$.

Let us try to prove this theorem. We have

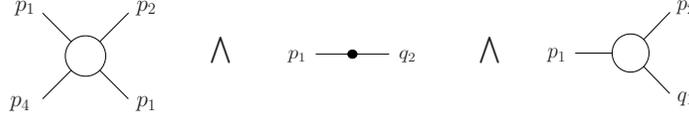
$$\begin{aligned} H_k(QA_\infty) &= \lim_{n \rightarrow \infty} H_k(QA_n). \\ &= \lim_{n \rightarrow \infty} H_k(\mathcal{G}, \partial_n). \\ &= \bigoplus_r \lim_{n \rightarrow \infty} H_k(\mathcal{G}^{(r)}, \partial_n). \end{aligned}$$

The first equality says that homology commutes with direct limits. And the second equality is the content of Theorem 2. The bonding maps $H_k(\mathcal{G}, \partial_n) \rightarrow H_k(\mathcal{G}, \partial_{n+1})$ for the direct limit are defined using the isomorphism $H_k(QA_n) \cong H_k(\mathcal{G}, \partial_n)$. For the third equality, see the discussion in (8.5). There is a subtle point here. We need to know that the bonding maps restrict to $H_k(\mathcal{G}^{(r)}, \partial_n) \rightarrow H_k(\mathcal{G}^{(r)}, \partial_{n+1})$. Even assuming this, we are stuck. To complete the argument, one needs the following proposition.

Proposition 7. $\lim_{n \rightarrow \infty} H_k(\mathcal{G}^{(r)}, \partial_n) = H_k(\mathcal{G}^{(r)}, \partial_E)$.

It will be proved using the ideas of Section 10. The proof will be completed by the end of this section. The subtle point about the definition of the LHS raised above will be dealt in the course of the proof.

13.1. A pairing on $\Lambda^*(QA_n)$. We have fixed a basis $p_1, \dots, p_n, q_1, \dots, q_n$ for V . This gives us a basis for the Lie algebra QA_n and its exterior algebra $\Lambda^*(QA_n)$. To be explicit, a basis for QA_n is given by distinct monomials. And for $\Lambda^*(QA_n)$ is given by taking wedge products of such monomials.



A basis element of $\Lambda^3(QA_4)$.

Define a pairing $M'(n) : \Lambda^*(QA_n) \otimes \Lambda^*(QA_n) \rightarrow \mathbb{Q}$ by $M'(n)(\psi_i, \psi_j) = \delta_{ij}$, where $\{\psi_i\}$ is the basis of the exterior algebra that was chosen above. The subspaces $\Lambda^k(QA_n)$, as k varies, are clearly orthogonal with respect to $M'(n)$.

To be more explicit, let $\varphi_1 = F_1 \wedge \dots \wedge F_k$ and $\varphi_2 = H_1 \wedge \dots \wedge H_k$ be wedges of monomials. Then $M'(n)(\varphi_1, \varphi_2)$ is nonzero only if $\varphi_1 = \varphi_2$ or $\varphi_1 = -\varphi_2$. This happens if $F_i = H_{\pi(i)}$ for a permutation π , i.e. there is a matching of the F_i 's with the H_j 's. This way of saying it brings us closer to the pairing $M(n)$, that was defined on graphs (10.2). We say this more precisely.

The isomorphism $\mathcal{G}_k \xrightarrow{\cong} (\Lambda^k(QA_n))^{\text{sp}(2n)}$, explained in (12.4.1), gives us an inclusion map $\mathcal{G} \hookrightarrow \Lambda^*(QA_n)$. Under this inclusion, the pairing $M'(n)$ has two important restrictions, namely, $M'(n) : \Lambda^*(QA_n) \otimes \mathcal{G} \rightarrow \mathbb{Q}$ and $M'(n) : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathbb{Q}$. We will see in Proposition 8 that the second pairing is just the pairing $M(n)$, that was defined in (10.2).

13.2. Connection between the pairings $M'(n)$ and $M(n)$. We will now see how formula (10) for the pairing $M(n)$ emerges naturally by analysing $M'(n)$.

Proposition 8. *The pairing $M'(n) : \Lambda^*(QA_n) \otimes \Lambda^*(QA_n) \rightarrow \mathbb{Q}$ restricts to the pairing $M(n) : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathbb{Q}$ under the inclusion map $\mathcal{G} \hookrightarrow \Lambda^*(QA_n)$.*

Proof. Let $(\Gamma_1, \sigma_1), (\Gamma_2, \sigma_2) \in \mathcal{G}$. We will suppress the inclusion map and orientations from the notation. We want to show $M'(n)(\Gamma_1, \Gamma_2) = M(n)(\Gamma_1, \Gamma_2)$. We compute the LHS by summing over all pairs of states of Γ_1 and Γ_2 . Recall that a state of a graph is a choice of $p_i \text{---} q_i$ for every edge (12.4.1). We write,

$$M'(n)(\Gamma_1, \Gamma_2) = \sum_{S_1, S_2} M'(n)(S_1, S_2),$$

where S_i is a state of Γ_i . A term in this sum is nonzero only if one can match S_1 and S_2 in the following sense.

There is a matching $m : \Gamma_1 \rightarrow \Gamma_2$ (Definition 13) with the following additional piece of data. For each component in the matching (Definition 14), there is an index i such that the vertices of the polygon are alternately labelled p_i and q_i .



Hence we can group terms and sum over all matchings $m : \Gamma_1 \rightarrow \Gamma_2$. For each matching m , there are $c(m)$ number of polygons. And for each polygon, there are $2n$ choices of the index i . This gives a factor of $(2n)^{c(m)}$ with a sign which can be checked to be $\text{sign}(m)$. Hence $M'(n)(\Gamma_1, \Gamma_2) = \sum_{m: \Gamma_1 \rightarrow \Gamma_2} \text{sign}(m)(2n)^{c(m)}$, which by definition is $M(n)(\Gamma_1, \Gamma_2)$. This completes the proof. \square

13.3. A commutative diagram. Consider the restriction of the pairing $M'(n) : \Lambda^k(QA_n) \otimes \mathcal{G}_k^{(r)} \rightarrow \mathbb{Q}$. This then defines a map $\Lambda^k(QA_n) \rightarrow (\mathcal{G}_k^{(r)})^*$, which we again denote by $M'(n)$. From Proposition 8, it is clear that this map is an extension of the map $\mathcal{G}_k^{(r)} \rightarrow (\mathcal{G}_k^{(r)})^*$ defined using the pairing $M(n)$. We now claim that the following diagram commutes.

$$\begin{array}{ccccccc} \mathcal{G}_k^{(r)} & \longrightarrow & \Lambda^k(QA_n) & \xrightarrow{M'(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_k^{(r)} \\ 2n\partial_E + \partial_H \downarrow & & \partial \downarrow & & \delta_E^* \downarrow & & \partial_E \downarrow \\ \mathcal{G}_{k-1}^{(r)} & \longrightarrow & \Lambda^{k-1}(QA_n) & \xrightarrow{M'(n)} & (\mathcal{G}_{k-1}^{(r)})^* & \xleftarrow[A]{\cong} & \mathcal{G}_{k-1}^{(r)}. \end{array}$$

This is a fattening of the diagram (13) at the end of Section 10, with the Chevalley-Eilenberg complex added in.

The commutativity of the first square follows from diagram (16) in (12.4). Saying that the second square commutes is equivalent to saying that the boundary map $\partial : \Lambda^*(QA_n) \rightarrow \Lambda^*(QA_n)$ and the coboundary map $\delta_E : \mathcal{G} \rightarrow \mathcal{G}$ are adjoints with respect to the pairing $M'(n)$. This follows by generalising the proof of Proposition 6. This proves the claim.

13.4. Invariance of the pairing. The pairing $M'(n) : \Lambda^*(QA_n) \otimes \Lambda^*(QA_n) \rightarrow \mathbb{Q}$ is $\mathfrak{sp}(2n)$ invariant in a precise sense. It is more convenient to express it using the Lie algebra uuA_n , see (5.4), which is anti-isomorphic to $\mathfrak{sp}(2n)$.

Proposition 9. *We have $M'(n)(\varphi_1, \varphi_2 \cdot H) + M'(n)(\varphi_1 \cdot JH, \varphi_2) = 0$, where $H \in uuA_n$ and J is the element of the symplectic group defined by $J(p_i) = q_i$ and $J(q_i) = -p_i$.*

Proof. We may assume that φ_1, φ_2 are wedges of monomials, i.e. they are two basis elements (upto sign) of $\Lambda^k(QA_n)$. We may also assume that H is a monomial in uuA_n . Say, for definiteness, that $H = q_1q_2$ and so $JH = p_1p_2$. Note that $\varphi_2 \cdot H$ involves replacing an occurrence of p_2 (resp. p_1) in φ_2 by q_1 (resp. q_2), see (5.4). And $\varphi_1 \cdot JH$ has exactly the opposite effect. It involves replacing an occurrence of q_1 (resp. q_2) in φ_1 by p_2 (resp. p_1) and picking a minus sign. It is now fairly clear how terms from the two pairing expressions would cancel. \square

Corollary. *The subspaces $(\Lambda^k(QA_n))^{\mathfrak{sp}(2n)}$ and $\mathfrak{sp}(2n) \cdot (\Lambda^k(QA_n))$ are orthogonal with respect to the pairing $M'(n)$.*

13.5. Stability of the pairings. Consider the Lie algebra QA_{n+1} . The underlying vector space V_{n+1} has dimension $2(n+1)$. The basis of V_{n+1} is an extension of the one on V to $p_1, \dots, p_{n+1}, q_1, \dots, q_{n+1}$. Hence the basis for QA_n and $\Lambda^*(QA_n)$ includes into the basis for QA_{n+1} and $\Lambda^*(QA_{n+1})$ respectively.

We now show that the restricted pairings $M'(n) : \Lambda^*(QA_n) \otimes \mathcal{G} \rightarrow \mathbb{Q}$ are stable, that is, for $\varphi \in \Lambda^*(QA_n)$ and $\Gamma \in \mathcal{G}$,

$$(17) \quad M'(n)(\varphi, \Gamma) = M'(n+1)(\varphi, \Gamma).$$

The φ in the RHS is obtained by using the inclusion of $\Lambda^*(QA_n)$ in $\Lambda^*(QA_{n+1})$.

Express both sides as sums over the states of Γ , see (12.4.1). Though the graph Γ is the same on both sides, it has more states for “ $n+1$ ” than “ n ” because there are more variables available. The extra states of Γ in the RHS are the ones that involve at least one p_{n+1} or q_{n+1} . However, they do not contribute anything to the pairing $M'(n+1)$, since by assumption, φ does not involve either p_{n+1} or q_{n+1} . This proves that equation (17) holds.

13.6. The stability commutative diagram. The chain group $\mathcal{C}_k = \Lambda^k(QA_n)$ can be written as $\mathcal{C}_k = (\mathcal{C}_k)^{\mathfrak{sp}(2n)} \oplus \mathfrak{sp}(2n) \cdot \mathcal{C}_k$, see (12.2). It depends on n but we will suppress that in our notation. The stability of the pairings $M'(n)$ (equation (17)) says that the following diagram commutes.

$$\begin{array}{ccccc} (\mathcal{C}_k)^{\mathfrak{sp}(2n)} \oplus \mathfrak{sp}(2n) \cdot \mathcal{C}_k & \xrightarrow{M'(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_k^{(r)} \\ \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\ (\mathcal{C}_k)^{\mathfrak{sp}(2n+2)} \oplus \mathfrak{sp}(2n+2) \cdot \mathcal{C}_k & \xrightarrow{M'(n+1)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[A]{\cong} & \mathcal{G}_k^{(r)}. \end{array}$$

The first vertical map does not restrict to a map on the invariants. However, by projecting on the first factor, we obtain maps $(\mathcal{C}_k)^{\mathfrak{sp}(2n)} \rightarrow (\mathcal{C}_k)^{\mathfrak{sp}(2n+2)}$, one for each n . And these are the maps that one can use for computation of stable homology $\lim_{n \rightarrow \infty} H_k(QA_n)$. This gives us the diagram

$$\begin{array}{ccccc} \mathcal{G}_k \cong (\mathcal{C}_k)^{\mathfrak{sp}(2n)} & \xrightarrow{M'(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_k^{(r)} \\ \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\ \mathcal{G}_k \cong (\mathcal{C}_k)^{\mathfrak{sp}(2n+2)} & \xrightarrow{M'(n+1)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[A]{\cong} & \mathcal{G}_k^{(r)}. \end{array}$$

By the Corollary to Proposition 9, the map $\mathfrak{sp}(2n) \cdot \mathcal{C}_k \rightarrow (\mathcal{G}_k)^*$ is zero. Hence the above diagram still commutes. Furthermore, if one thinks of the map $(\mathcal{C}_k)^{\mathfrak{sp}(2n)} \rightarrow (\mathcal{C}_k)^{\mathfrak{sp}(2n+2)}$ in terms of graphs, then it is clear that it preserves the Euler characteristic. This gives a map $\mathcal{G}_k^{(r)} \rightarrow \mathcal{G}_k^{(r)}$ for each n . The direct limit $\lim_{n \rightarrow \infty} H_k(\mathcal{G}^{(r)}, \partial_n)$ is taken with respect to these maps. Consider the stability commutative diagram

$$\begin{array}{ccccccc} \mathcal{G}_k^{(r)} & \longrightarrow & (\mathcal{C}_k)^{\mathfrak{sp}(2n)} & \xrightarrow{M'(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[\cong]{A} & \mathcal{G}_k^{(r)} \\ \downarrow & & \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\ \mathcal{G}_k^{(r)} & \longrightarrow & (\mathcal{C}_k)^{\mathfrak{sp}(2n)+2} & \xrightarrow{M'(n)} & (\mathcal{G}_k^{(r)})^* & \xleftarrow[A]{\cong} & \mathcal{G}_k^{(r)}. \end{array}$$

We know that the map $M(n) : \mathcal{G}_k^{(r)} \rightarrow (\mathcal{G}_k^{(r)})^*$ is an isomorphism for all n large enough (10.4). In the above diagram, it is the composite of the first two horizontal maps. Proposition 7 now follows by the isomorphism between the leftmost and rightmost columns.

Remark. In [22], Kontsevich conjectured that the stable homology groups $H_k(QA_\infty)$ are finite dimensional. In terms of graphs, this says that $H_k(\mathcal{G}^{(r)}, \partial_E) = 0$ for fixed k and sufficiently large r .

14. PROOF OF THE MAIN THEOREM-PART III

The limit Lie algebra, $QA_\infty = \lim_{n \rightarrow \infty} QA_n$, has the structure of a Hopf algebra on its homology (5.5). Let $PH_*(QA_\infty)$ denote the subspace of primitive elements of $H_*(QA_\infty)$. Also let $(\mathcal{C}, \partial_E)$ be the subcomplex of $(\mathcal{G}, \partial_E)$ spanned by oriented connected graphs. We will prove the following theorem.

Theorem 4. $PH_*(QA_\infty) = H_*(\mathcal{C}, \partial_E)$.

Proof. The Hopf algebra $H_*(QA_\infty)$ is commutative and cocommutative in the graded sense. Hence by the structure theorem of Milnor and Moore [33], $H_*(QA_\infty)$ is a graded polynomial algebra generated by the primitive elements. By Theorem 3, we know $H_*(QA_\infty) = H_*(\mathcal{G}, \partial_E)$. This along with the identification given by Proposition 10 (which we are going to prove) shows that the subspace of primitive elements is precisely $H_*(\mathcal{C}, \partial_E)$. \square

As always, we apply the above theorem to the unit species uu and obtain the following corollary.

Corollary. $PH_*(\mathfrak{sp}(2\infty)) = H_*(\mathcal{P}, \partial_E)$.

Proposition 10. *The product in the stable homology $H_*(QA_\infty)$ can be defined on the chain complex $(\mathcal{G}, \partial_E)$ as disjoint union of graphs.*

Proof. We make use of the deformation map $D(n) : \Lambda^k(QA_n) \rightarrow \mathcal{G}_k^{(r)}$. This is a composition of the maps $\Lambda^k(QA_n) \rightarrow (\mathcal{G}_k^{(r)})^* \rightarrow \mathcal{G}_k^{(r)}$ that we had in the previous section (13.3). It is an extension of the deformation map $D(n) : \mathcal{G}_k^{(r)} \rightarrow \mathcal{G}_k^{(r)}$ defined at the end of (10.5). For simplicity, we will suppress the dependence on n and use the notation $D(\varphi) = \sum_\Gamma D(\varphi, \Gamma)$ Γ for $\varphi \in \Lambda^k(QA_n)$ and $\Gamma \in \mathcal{G}_k^{(r)}$. The usage of D for $D(n)$ is local to this proposition. Later in Appendix B, the letter D will be used for something slightly different.

The map $QA_n \oplus QA_m \hookrightarrow QA_{n+m}$ induces a map $\Lambda^i(QA_n) \otimes \Lambda^{k-i}(QA_m) \rightarrow \Lambda^k(QA_{n+m})$. More explicitly, $\varphi_1 \otimes \varphi_2 \mapsto \varphi_1 \wedge \varphi_2$, with the indices of φ_2 shifted up by n . In order to complete the proof, it is enough to show that the following diagram commutes.

$$\begin{array}{ccc} \Lambda^i(QA_n) \otimes \Lambda^{k-i}(QA_m) & \xrightarrow{D \otimes D} & \mathcal{G}_i \otimes \mathcal{G}_{k-i} \\ \downarrow & & \mu \downarrow \\ \Lambda^k(QA_{n+m}) & \xrightarrow{D} & \mathcal{G}_k. \end{array}$$

Here the map μ stands for disjoint union of graphs. The commutativity is equivalent to showing the identity

$$D(\varphi_1 \wedge \varphi_2, \Gamma) = \sum_{\Gamma_1 \sqcup \Gamma_2 = \Gamma} D(\varphi_1, \Gamma_1) D(\varphi_2, \Gamma_2).$$

To calculate the LHS, we match the i pieces of φ_1 and $k - i$ pieces of φ_2 with the k vertices of Γ and then contract indices. And we sum over all matchings.

The key observation is that φ_1 and φ_2 must land on disconnected parts of Γ , say Γ_1 and Γ_2 , in order to get a non-zero contraction. This is because the indices that occur in φ_1 are disjoint from those in φ_2 . The rest of the argument is now fairly clear and we omit it. \square

Remark. In fact, the chain complex $(\mathcal{G}, \partial_E)$ is a differential graded Hopf algebra with product μ given by disjoint union and the coproduct Δ defined for a connected graph Γ by $\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1$ and extended to \mathcal{G} as a morphism of algebras. Here 1 stands for the unit in \mathcal{G} and may be thought of as the empty graph. The boundary map ∂_E is a derivation with respect to the product μ and a coderivation with respect to the coproduct Δ . And this induces the Hopf algebra structure on $H_*(\mathcal{G}, \partial_E)$.

Proof of the main theorem concluded. Our main theorem (Theorem 1) now follows by putting together Theorem 4 and its corollary and the corollary to Proposition 5.

APPENDIX A. DEFORMATION QUANTISATION

For a good review of the problem of deformation quantisation, see the notes by Voronov [38]. The main object of interest is a Poisson algebra. The algebra of functions on a symplectic, or more generally, a Poisson manifold form a Poisson algebra. This is the classical case. In (A.1-A.3), we provide some background on the classical case. This would also be useful for some of the ideas in Appendix B. The material in A.1 is taken from [38].

In (A.5), we speculate on the form of this problem for operads. The main question is: What is a ‘‘Poisson operad manifold’’? Unlike the classical case, a symplectic operad manifold is not automatically a Poisson operad manifold. The relation between our viewpoint and the standard deformation theory of operads considered by Balavoine in [28] is not clear.

A.1. The problem. Let A be a commutative algebra over a field k of characteristic zero. A formal deformation of A is a $k[[t]]$ bilinear product (which we denote \star) on the space $A[[t]]$ of formal power series in a variable t satisfying:

$$F \star H = F \cdot H + \mu_1(F, H)t + \mu_2(F, H)t^2 + \dots \quad \text{for } F, H \in A,$$

where $F \cdot H$ is the original product on A and the star product \star is associative. The deformation is called *trivial* if all the higher products μ_1, μ_2, \dots are zero.

Suppose that \star is a deformation of \cdot , the original product. Then define $\{F, H\} = \frac{1}{2}(\mu_1(F, H) - \mu_1(H, F))$. One can check that $\{, \}$ is a Lie bracket on A and further that the triple $(A, \cdot, \{, \})$ is a Poisson algebra. Recall that a Poisson algebra is a space with compatible commutative and Lie structures. The compatibility relation is $\{F \cdot G, H\} = F \cdot \{G, H\} + G \cdot \{F, H\}$.

In physical terms, one regards the Poisson algebra A as the quasi-classical limit of the associative algebra $A[[t]]$, and the algebra $A[[t]]$ as a deformation quantisation of the Poisson algebra A . The deformation quantisation problem is the inverse problem: given a Poisson algebra $(A, \cdot, \{, \})$, find a formal deformation \star returning the original Poisson algebra structure on A in the quasi-classical limit.

A.2. Gauge equivalence. There is a natural gauge group acting on star products. This group consists of automorphisms D of $A[[t]]$ which are $k[[t]]$ linear. They have the form $D = D_0 + tD_1 + t^2D_2 + \dots$, where $D_i : A \rightarrow A$ are operators with D_0 being the identity. This means that

$$F \mapsto F + tD_1(F) + t^2D_2(F) + \dots \quad \text{for } F \in A,$$

and for a general element in $A[[t]]$, one uses $k[[t]]$ linearity.

Two star products \star and \star' are gauge equivalent if there is an automorphism D as above so that the following diagram commutes.

$$\begin{array}{ccc} A[[t]] \otimes A[[t]] & \xrightarrow{D \otimes D} & A[[t]] \otimes A[[t]] \\ \star \downarrow & & \star' \downarrow \\ A[[t]] & \xrightarrow{D} & A[[t]]. \end{array}$$

A.3. The simplest example. Let A be the space of all polynomial functions on \mathbb{R}^{2n} , i.e. polynomials in the $2n$ variables $p_1, \dots, p_n, q_1, \dots, q_n$. Then the triple $(A, \cdot, \{, \})$ is a Poisson algebra, where \cdot is the usual product in A and $\{, \}$ is the Poisson bracket defined by equation (2) in (4.2).

In this case, the solution to the deformation quantisation problem is given by the Moyal \star product. In fact, we will have $F \star H = F \cdot H + \{F, H\}t + \text{higher terms}$; that is, $\mu_1(F, H) = \{F, H\}$.

A.3.1. The Moyal product. Let B be any operator on $A \otimes A$. For $F, H \in A$, define $B(F, H)$ to be the element of A obtained by applying the product in A to $B(F \otimes H)$. Namely, if $B = \sum B_{(1)} \otimes B_{(2)}$ then $B(F, H) = \sum B_{(1)}F \cdot B_{(2)}H$.

Now let $B = \sum_{i=1}^n \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i}$ be a bi-differential operator on $A \otimes A$. By our notation,

$$B(F, H) = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i}.$$

In other words, $B(F, H) = \{F, H\}$ is just the Poisson bracket. Now define the Moyal \star product by

$$(18) \quad F \star H = e^{tB}(F, H) = \sum_{n \geq 0} t^n \frac{B^n}{n!}(F, H).$$

Observe that $F \star H = F \cdot H + \{F, H\}t +$ higher terms, as claimed earlier. To show that the Moyal product solves the deformation quantisation problem, one must prove that it is associative. We will do this by interpreting equation (18) using pictures.

A.3.2. *Associativity of the Moyal product.* If $F \in A$ is a monomial, say $F = p_1^2 p_2 q_2$,

then we represent it as $F = \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. And if F is a polynomial rather than a monomial then we represent it as a formal sum of pictures. It is clear that the product of two monomials in A is defined as the disjoint union of pictures.

Our goal is to understand the operator $\frac{B^n}{n!}$ via pictures. As a start, we understand the term $\frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i}$. This is a sum of pictures of the form $\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$, where the

edge goes from a point labelled p_i to a point labelled q_i . The other points in F and H are not shown in the picture. We will have the following interpretation. A picture

of the form $\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$ means that we delete the edge and its endpoints and

then take disjoint union of the two parts. Earlier in the paper, such an operation was called a mating.

Now we claim that

$$(19) \quad \frac{B^n}{n!}(F, H) = \sum_P (-1)^{\text{sign}(P)} \left(\begin{array}{|c|} \hline \bullet \text{---} \bullet \\ \hline \end{array} \right),$$

where P is a picture as shown. There are n edges between F and H each connecting a p_i in F with a q_i in H or vice-versa. And $\text{sign}(P)$ is the number of edges that are out of order; i.e. they connect a q_i in F to a p_i in H . The interpretation for P is that we delete the n edges and their endpoints and take disjoint union of what remains.

Note that the factor $n!$ is no longer necessary in the pictorial description. Each of the n B 's in B^n contributes to one of the n edges in the picture. So there are $n!$ ways to obtain the same picture. After dividing by $n!$, we get every picture exactly once, giving us equation (19).

The associativity of the Moyal product can be seen from the formula

$$(20) \quad F \star G \star H = \sum_{\substack{n \geq 0 \\ n=k+l+m}} t^n \sum_P (-1)^{\text{sign}(P)} \left(\begin{array}{|c|} \hline \bullet \text{---} \bullet \\ \hline \end{array} \right),$$

where the picture P means that delete all edges and their endpoints and then take disjoint union of what remains. And $\text{sign}(P)$ is the number of edges that connect a q_i to a p_i . (The edges go from F to G , G to H and F to H .) The letters k , l and m refer to the number of connecting edges.

$\frac{B^n}{n!} = \sum B_{(1)} \otimes B_{(2)}$, where $B_{(1)}$ and $B_{(2)}$ are n th order differential operators, we obtain $\frac{B^n}{n!}(F, H) = \sum B_{(1)}F \otimes B_{(2)}H$. One needs to make sense of this. We first explain how a higher order derivative works by showing it on a schematic example.

$$(22) \quad \begin{array}{ccc} \begin{array}{c} x_2 \quad x_1 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ x_4 \quad x_1 \end{array} & \xrightarrow{\frac{\partial^2}{\partial x_1^2}} & 2 \left(\begin{array}{c} x_2 \quad x_1 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ x_4 \quad x_1 \end{array} \right) \end{array}.$$

Thus a n th order differential operator acting on F produces an algebraic object with n outputs (7.1). We now interpret the tensor sign as a n th order mating. We write

$$\frac{B^n}{n!}(F, H) = \sum_P (-1)^{\text{sign}(P)} \left(\begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad / \\ \circ \quad \circ \\ / \quad \diagdown \\ \text{---} \quad \text{---} \\ F \quad H \end{array} \right),$$

where the picture P shows a higher order mating. There are n ideal edges between F and H each connecting a p_i with a q_i or vice-versa. And $\text{sign}(P)$ is the number of edges that connect a q_i on F to a p_i on H . For the definition of an ideal edge, see (3.1).

The examples discussed in Section 7 have associated PROPs. In those cases, we understand the meaning of higher order derivatives, i.e. the right hand side of equation (22) makes sense. Similarly, higher order matings have a natural meaning in those examples. The example of the surface species was illustrated in (A.4). The deformation quantisation for the Poisson algebra A is again given by the formula in equation (18). The Moyal \star product is associative in these cases for the same reason as in the commutative case ($Q = cc$). The associativity can be seen from the formula

$$F \star G \star H = \sum_{\substack{n \geq 0 \\ n=k+l+m}} t^n \sum_P (-1)^{\text{sign}(P)} \left(\begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad / \\ \circ \quad \circ \quad \circ \\ / \quad \diagdown \quad / \quad \diagdown \\ \text{---} \quad \text{---} \quad \text{---} \\ F \quad G \quad H \end{array} \right),$$

where $\text{sign}(P)$ is the number of ideal edges that connect a q_i to a p_i . (The edges go from F to G , G to H and F to H .) The letters k, l, m stand for the number of edges.

The interested reader can work out the details for the examples based on graphs (7.2-7.3). Another nice example is obtained by exponentiating the associative species.

APPENDIX B. THE DEFORMATION MAP ON GRAPHS

Recall that the chain complex $(\mathcal{G}, \partial_E)$ is a differential graded Hopf algebra with product μ given by disjoint union. For a definition of $(\mathcal{G}, \partial_E)$, see Section 8 and for the Hopf algebra structure, see the remark at the end of Section 14. In addition, recall that we defined another boundary operator ∂_H on \mathcal{G} that satisfied $\partial_E \partial_H = -\partial_H \partial_E$, see (8.4). So ∂_H induces a map on homology $H_*(\mathcal{G}, \partial_E) \rightarrow H_*(\mathcal{G}, \partial_E)$.

The boundary map ∂_E is a derivation with respect to the product μ . This can be written as $\partial_E \mu = \mu \partial_E$. However ∂_H is not a derivation with respect to μ . Following [7], define a bracket $[\ , \] : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathcal{G}$ by

$$[\ , \] = \partial_H \mu - \mu \partial_H.$$

In other words, $[\ , \]$ measures the failure of ∂_H to be a derivation with respect to μ . As a formal consequence of the previous three identities, it follows that $\partial_E[\ , \] = -[\ , \]\partial_E$. This gives us an induced map on homology $[\ , \] : H_*(\mathcal{G}, \partial_E) \otimes H_*(\mathcal{G}, \partial_E) \rightarrow H_*(\mathcal{G}, \partial_E)$.

In this section, we show the following.

Proposition 11. *The maps ∂_H and $[\ , \]$ are trivial on homology.*

This answers some of the questions raised in [7]. The proof is largely a matter of putting together things we already know. The main step is to use the pairing $M(n)$ on \mathcal{G} and its adjoint property proved in Proposition 6.

The coproduct on \mathcal{G} is irrelevant to Proposition 11. So we write the differential algebra \mathcal{G} as a triple $(\mathcal{G}, \partial_E, \mu)$. Extending ∂_E and μ by $\mathbb{Q}[[t]]$ linearity, we get a differential algebra $(\mathcal{G}[[t]], \partial_E, \mu)$. In other words, this is just the trivial deformation of the algebra \mathcal{G} , see (A.1). We now consider another deformation of \mathcal{G} that is gauge equivalent to this one (A.2).

B.1. The deformation map. Let $D(n) : \mathcal{G} \rightarrow \mathcal{G}$ be the deformation map defined at the end of (10.4). It was defined by $D(n) = A^{-1} \circ M(n)$, where the map $M(n)$ was induced by a pairing on \mathcal{G} (equation (10)), while the map A scaled a graph Γ by the factor $|\text{Aut}(\Gamma)|$.

Recall that $D(n)\Gamma$ is a polynomial in $2n$ of degree e with coefficients in \mathcal{G} . Furthermore, the coefficient of $(2n)^e$ is exactly Γ . Here $e = |E(\Gamma)|$, the number of edges in Γ . Write $D(n)\Gamma = (2n)^e(D_0 + \frac{1}{2n}D_1 + \dots)(\Gamma)$. This defines maps $D_i : \mathcal{G} \rightarrow \mathcal{G}$ which do not depend on n . Also D_0 is the identity.

Consider the $\mathbb{Q}[[t]]$ linear automorphism of $\mathcal{G}[[t]]$ given by $D = D_0 + tD_1 + t^2D_2 + \dots$, where $D_i : \mathcal{G} \rightarrow \mathcal{G}$ are as above. The map D defines an element of the gauge group (A.2).

B.2. Comparing the two gauge equivalent situations. Let $(\mathcal{G}[[t]], \partial_t, \mu_t)$ be the differential algebra got by applying D to the trivial deformation $(\mathcal{G}[[t]], \partial_E, \mu)$. Stated differently, one has two commutative diagrams.

$$(23) \quad \begin{array}{ccc} \mathcal{G}[[t]] & \xrightarrow{D} & \mathcal{G}[[t]] & & \mathcal{G}[[t]] \otimes \mathcal{G}[[t]] & \xrightarrow{D \otimes D} & \mathcal{G}[[t]] \otimes \mathcal{G}[[t]] \\ \partial_E \downarrow & & \partial_t \downarrow & & \mu \downarrow & & \mu_t \downarrow \\ \mathcal{G}[[t]] & \xrightarrow{D} & \mathcal{G}[[t]] & & \mathcal{G}[[t]] & \xrightarrow{D} & \mathcal{G}[[t]]. \end{array}$$

Since ∂_E is a derivation with respect to the product μ , we have $\partial_E\mu = \mu\partial_E$. Hence we know $\partial_t\mu_t = \mu_t\partial_t$. Write $\partial_t = \partial_0 + t\partial_1 + t^2\partial_2 + \dots$ and $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$. Then the commutativity of the diagrams in (23) implies $\partial_0 = \partial_E$ and $\mu_0 = \mu$.

Lemma 5. *The following relations hold.*

- (1) $-\partial_1 = -D_1\partial_0 + \partial_0D_1$.
- (2) $\mu_1 = D_1\mu_0 - \mu_0D_1$.
- (3) $-\partial_1\mu_0 + \mu_0\partial_1 = \partial_0\mu_1 - \mu_1\partial_0$.

Proof. The first two items follow by looking at the coefficient of t in the diagrams in (23) and the third item follows by looking at the coefficient of t in $\partial_t\mu_t = \mu_t\partial_t$. \square

Lemma 6. $\partial_1 = -\partial_H$.

Proof. The commutative diagram (13) in Section 10 can be redrawn as under

$$\begin{array}{ccc} \mathcal{G}[[t]] & \xrightarrow{D} & \mathcal{G}[[t]] \\ \partial_E + t\partial_H \downarrow & & \partial_E \downarrow \\ \mathcal{G}[[t]] & \xrightarrow{D} & \mathcal{G}[[t]]. \end{array}$$

Comparing the coefficient of t gives $\partial_H = -D_1\partial_0 + \partial_0D_1$. This together with Lemma 5 (item (1)) shows that $\partial_1 = -\partial_H$. \square

Proof of Proposition 11. Items (1) and (3) in Lemma 5 can now be rewritten as $\partial_H = -D_1\partial_E + \partial_E D_1$ and $[\ , \] = \partial_E\mu_1 - \mu_1\partial_E$. Thus ∂_H and $[\ , \]$ induce the zero map on homology, with D_1 and μ_1 providing the respective chain homotopies. This completes the proof of Proposition 11.

Remark. It is possible to just give an explicit definition of μ_1 and check the relation $[\ , \] = \partial_E\mu_1 - \mu_1\partial_E$ directly. This will give us that $[\ , \]$ is zero on homology. And similarly for D_1 . But we prefer the more conceptual approach via deformation theory. It would be interesting to also describe explicitly the higher products μ_2, μ_3, \dots and $\partial_2, \partial_3, \dots$, etc.

Recall that \mathcal{G} also has a coproduct Δ and ∂_E is a coderivation with respect to the coproduct Δ . However, ∂_H is not and this failure can be measured by a cobracket $\theta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$ given by $\theta = \partial_H\Delta - \Delta\partial_H$. Similar arguments show that θ also induces the zero map on homology.

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We shall describe a program here relating Feynman diagrams, topology of manifolds, homotopical algebra, non-commutative geometry and several kinds of “topological physics.”

Enjoy!

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