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On the alternating runs polynomial in type B and type D Coxeter groups

Hiranya Kishore Dey[1] and Sivaramakrishnan Sivasubramanian[2]

[1] *Department of Mathematics, Indian Institute of Science, Bangalore, Bangalore 560 012, India.*
e-mail: hiranya.dey@gmail.com

[2] *Department of Mathematics, Indian Institute of Technology, Bombay, Mumbai 400 076, India.*
e-mail: krishnan@math.iitb.ac.in

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Abstract. Let $R_n(t)$ denote the polynomial enumerating alternating runs in the symmetric group $\mathfrak{S}_n$. Wilf showed that $(1+t)^m$ divides $R_n(t)$ where $m = \lfloor (n-2)/2 \rfloor$. Bóna recently gave a group-action-based proof of this fact. In this work, we give a group-action-based proof for type B and type D analogues of this result. Interestingly, our proof gives a group action on the positive/negative parts $\mathfrak{B}_n^{\pm}$ and $\mathfrak{D}_n^{\pm}$ and so we get refinements of the result to the case when summation is over $\mathfrak{B}_n^{\pm}$ and $\mathfrak{D}_n^{\pm}$. We are unable to get a group-action-based proof of Wilf's result when summation is over the alternating group $\mathcal{A}_n$ and over $\mathfrak{S}_n - \mathcal{A}_n$, but using other ideas, give a different proof. We give similar results to the polynomial which enumerates *alternating sequences* in $\mathcal{A}_n$, $\mathfrak{S}_n - \mathcal{A}_n$, $\mathfrak{B}_n^{\pm}$ and $\mathfrak{D}_n^{\pm}$. As a corollary, we get moment type identities for coefficients of such polynomials.

1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. For an index i with $2 \leq i \leq n-1$, we say that $\pi \in \mathfrak{S}_n$ changes direction at i, if either $\pi_{i-1} < \pi_i > \pi_{i+1}$ or if $\pi_{i-1} > \pi_i < \pi_{i+1}$.

Definition 1. *We say that π has k alternating runs, denoted by* $\operatorname{altruns}(\pi) = k$*, if π has $k-1$ indices where it changes direction.*

For example, the permutation $\pi = 132465 \in \mathfrak{S}_6$ has 4 alternating runs. Consider the polynomial

$$R_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{altruns}(\pi)} = \sum_{k=1}^{n-1} R_{n,k} t^k.$$

André in [1] started the study of permutations enumerated by its number of alternating runs. Canfield and Wilf in [6], Stanley in [14] and later Ma in [11] give explicit formulae for $R_{n,k}$. Wilf in [15], showed the following result about the power of $(1+t)$ that divides $R_n(t)$.

Theorem 2 (Wilf). *For positive integers $n \geq 4$, the polynomial $R_n(t)$ is divisible by $(1+t)^m$, where $m = \lfloor (n-2)/2 \rfloor$.*

Wilf's proof depends on a relation between the Eulerian polynomial and $R_n(t)$. Later, Bóna and Ehrenborg in [5] gave an inductive proof of Theorem 2. Recently, Bóna in [4] gave a proof of Theorem 2 based on group actions. Inspired by Bóna's proof, in this paper, we give a group action based proof of type B and type D counterparts of Theorem 2.

Let $\mathfrak{B}_n$ be the set of permutations of $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\pi(-i) = -\pi(i)$. $\mathfrak{B}_n$ is known as the *hyperoctahedral* group or the group of signed permutations on $[n]$. For $\pi \in \mathfrak{B}_n$ and an index $1 \leq i \leq n$, we denote $\pi(i)$ alternatively as π_i and for $1 \leq k \leq n$, we denote $-k$ as $\overline{k}$. For $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{B}_n$, define $\operatorname{Negs}(\pi) = \{\pi_i : i > 0, \pi_i < 0\}$

as the set of elements which occur in π with a negative sign. Let $\mathfrak{D}_n \subseteq \mathfrak{B}_n$ denote the subset consisting of those elements of $\mathfrak{B}_n$ which have an even number of negative entries. $\mathfrak{D}_n$ is referred to as the *demihyperoctahedral* group on $[n]$. For $\pi \in \mathfrak{B}_n$, let $\pi_0 = 0$. For an index i with $1 \leq i \leq n-1$, we say that π changes direction at i if either $\pi_{i-1} < \pi_i > \pi_{i+1}$ or if $\pi_{i-1} > \pi_i < \pi_{i+1}$. We say that $\pi \in \mathfrak{B}_n$ has k alternating runs, denoted $\mathrm{altruns}_B(\pi) = k$ if π has $k-1$ indices where it changes direction. For example, the permutation $514\overline{3}\overline{6}2$ has $\mathrm{altruns}_B(\pi) = 5$. Define $\mathfrak{B}_n^> = \{\pi \in \mathfrak{B}_n, \pi_1 > 0\}$, $\mathfrak{D}_n^> = \{\pi \in \mathfrak{D}_n, \pi_1 > 0\}$ and $\mathfrak{B}_n^> - \mathfrak{D}_n^> = \{\pi \in \mathfrak{B}_n - \mathfrak{D}_n, \pi_1 > 0\}$. Further, define $\mathfrak{B}_n^< = \{\pi \in \mathfrak{B}_n, \pi_1 < 0\}$, $\mathfrak{D}_n^< = \{\pi \in \mathfrak{D}_n, \pi_1 < 0\}$ and $\mathfrak{B}_n^< - \mathfrak{D}_n^< = \{\pi \in \mathfrak{B}_n - \mathfrak{D}_n, \pi_1 < 0\}$. Define the following polynomial:

$$R(W,t) = \sum_{\pi \in W} t^{\mathrm{altruns}_B(\pi)}$$

where $W \subseteq \mathfrak{B}_n$.

Zhao in [16, Theorem 4.3.2] proved the following refinement of a type B analogue of Theorem 2. See the paper by Chow and Ma [7] as well. Later, Gao and Sun in [10, Corollary 2.4] considered the two polynomials $R(D_n^>,t)$ and $R(\mathfrak{B}_n^> - \mathfrak{D}_n^>,t)$ and gave the following refinement of a type D analogue. As the results are very similar, we have combined their respective results. Though the form appears to be slightly different, Remark 4 shows that Theorem 3 implies a type B and type D counterpart of Theorem 2. Theorem 3 refines and thus implies a type B and type D counterpart of Theorem 2.

Theorem 3 (Zhao, Gao and Sun). *For positive integers n, the polynomials $R(B_n^>,t)$, $R(D_n^>,t)$ and $R(B_n^> - D_n^>,t)$ are all divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$.*

Remark 4. Using the map $\mathrm{Sgn\_flip}_1$ defined in Section 2, it is easy to see that $R(B_n^>,t) = R(B_n^<,t)$ and therefore we have $R(B_n,t) = 2R(B_n^<,t)$. Thus the polynomial $R(B_n,t)$ is divisible by $2(1+t)^m$.

Using the same map, one can also see that $R(B_n^>,t) = R(B_n^<,t)$ when n is even. When n is odd, the same map gives us $R(D_n^<,t) = R(B_n^> - D_n^>,t)$ and $R(D_n^>,t) = R(B_n^< - D_n^<,t)$ Combining this with Theorem 3, we get that both $R(D_n^<,t)$ and $R(B_n^< - D_n^<,t)$ are divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$. Thus, when n is even, the polynomial $R(D_n,t)$ is divisible by $2(1+t)^m$ and when n is odd, $R(D_n,t)$ is divisible by $(1+t)^m$.

In Section 3, we give a group action based proof of Theorem 3. Since $\mathfrak{S}_n, \mathfrak{B}_n$ and $\mathfrak{D}_n$ are Coxeter groups they have a length function (see the book by Björner and Brenti [2]) which we denote by $\mathrm{inv}, \mathrm{inv}_B$ and inv_D respectively (see definitions in Section 4). Let $\mathfrak{B}_n^+ = \{\pi \in \mathfrak{B}_n : \mathrm{inv}_B(\pi) \text{ is even}\}$ and let $\mathfrak{B}_n^- = \mathfrak{B}_n - \mathfrak{B}_n^+$. Similarly, let $\mathfrak{D}_n^+ = \{\pi \in \mathfrak{D}_n : \mathrm{inv}_D(\pi) \text{ is even}\}$ and let $\mathfrak{D}_n^- = \mathfrak{D}_n - \mathfrak{D}_n^+$. By $\mathfrak{B}_n^\pm$ (respectively, $\mathfrak{D}_n^\pm$), we succintly denote both $\mathfrak{B}_n^+$ and $\mathfrak{B}_n^-$ (respectively, both $\mathfrak{D}_n^+$ and $\mathfrak{D}_n^-$). Define $\mathfrak{B}_n^{>,\pm} = \mathfrak{B}_n^> \cap \mathfrak{B}_n^\pm$, $\mathfrak{D}_n^{>,\pm} = \mathfrak{D}_n^> \cap \mathfrak{D}_n^\pm$ and $(\mathfrak{B}_n - \mathfrak{D}_n)^{>,\pm} = (\mathfrak{B}_n - \mathfrak{D}_n)^> \cap \mathfrak{B}_n^\pm$.

Interestingly, we are able to get a group action based proof of a refinement of Theorem 3 to $\mathfrak{B}_n^{>,\pm}$, $\mathfrak{D}_n^{>,\pm}$ and $(\mathfrak{B}_n - \mathfrak{D}_n)^{>,\pm}$. Our refinements are Theorem 18 and Theorem 24 and both are proved in Section 4. As a corollary, we get moment type identities for the coefficients of appropriate polynomials. These are given in Theorem 20 and Theorem 25 respectively.

Bóna's proof when directly applied to permutations in $\mathfrak{B}_n$ and $\mathfrak{D}_n$ also works for the type B and the type D case. But, this straightforward extension to $\mathfrak{B}_n$, $\mathfrak{D}_n$, $\mathfrak{B}_n^>$ and $\mathfrak{D}_n^>$ does not seem to be an action over $\mathfrak{B}_n^{>,\pm}$. Thus, it does not seem to give a proof of Theorem 18 or Theorem 24. In Remark 23, we elaborate on this point. We have given a refinement of Theorem 2 with summation over the alternating group $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$ in [9, Theorem 3], but our proof there is not group action based. We are unable to get a group action that both preserves sign in $\mathfrak{S}_n$ and gives an exponent of roughly $n/2$ (see Remark 8). It would be very interesting to get a group action based proof of [9, Theorem 3] that preserves sign.

Bóna in [4] also gave similar results about alternating sequences.

Definition 5. *For a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$, an alternating subsequence of π is a subsequence $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_l}$ satisfying $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \cdots \pi_{i_l}$. Denote by $\mathrm{as}(\pi)$ the length of the longest alternating subsequence of π.*

The following remark tells the connection between alternating subsequences and alternating runs.

Remark 6. If $\pi_1 < \pi_2$, we have $\mathrm{as}(\pi) = \mathrm{altruns}(\pi)$ and if $\pi_1 > \pi_2$, we have $\mathrm{as}(\pi) = \mathrm{altruns}(\pi) + 1$.

For $\pi = \pi_1\pi_2\dots\pi_n \in \mathfrak{S}_n$, define its number of inversions as $\mathrm{inv}(\pi) = |\{1 \le i < j \le n : \pi_i > \pi_j\}|$. Define the following polynomials

$$\mathrm{Altseq}_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{as}(\pi)}, \qquad \mathrm{SgnAltseq}_n(t) = \sum_{\pi\in\mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{as}(\pi)},$$

$$\mathrm{Altseq}_n^+(t) = \sum_{\pi\in\mathcal{A}_n} t^{\mathrm{as}(\pi)}, \qquad \mathrm{Altseq}_n^-(t) = \sum_{\pi\in\mathfrak{S}_n-\mathcal{A}_n} t^{\mathrm{as}(\pi)}.$$

In [3, Section 1.3.2], Bóna presents the following.

$$\mathrm{Altseq}_n(t) = \frac{(1+t)R_n(t)}{2}. \tag{1}$$

An immediate consequence of (1) and Theorem 7 is the following.

Theorem 7 (Bóna). *For integers $n \ge 4$, the polynomial $\mathrm{Altseq}_n(t)$ is divisible by $(1+t)^{\lfloor n/2\rfloor}$.*

Bóna in [4] also mentions that one can prove Theorem 7 directly by using the same group action he used to prove Theorem 2. As mentioned above, we are unable to get a group action on $\mathcal{A}_n$ and on $\mathfrak{S}_n - \mathcal{A}_n$. We elaborate on it below.

Remark 8. We recall the group action defined by Bóna. Given $\pi \in \mathfrak{S}_n$ and an index i, consider the sequence $S_i = \pi_i\dots\pi_n$ of elements of π starting from the i-th position. Let $c_i : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ be the map which replaces the p-th smallest element by the p-th largest element for all values of p in the sequence S_i. Let $C_n = \{c_1, c_3, c_5, \dots, c_t\}$ where $t = n-1$ if n is even and $t = n-2$ if n is odd. Bóna shows that C_n acts on $\mathfrak{S}_n$, but this action does not restrict to $\mathcal{A}_n$. For instance, let $n = 8$ and let $\pi = 21436587 \in \mathfrak{S}_8$. It is easy to check that $\mathrm{inv}(\pi) = 4$ and hence $\pi \in \mathcal{A}_8$ is an even permutation. However, it can be checked that $c_3(\pi) = 21785634$ has $\mathrm{inv}(c_3(\pi)) = 13$ and is thus an odd permutation.

Therefore we are unable to prove Theorem 9 by using Bóna's action. We can however get the following weaker version. It is easy to see that for a permutation π, the permutation $c_i(\pi)$ will have the same sign if and only if $n-i+1 \equiv 0, 1 \pmod 4$ (see [8, Lemma 10] for a proof). Moreover, the key requirement of [4, Lemma 2.4] is that indices i's in C_n should be non-consecutive. From these two conditions, we can choose $C_n = \{1, 5, 9, \dots, 4k-3\}$ when $n = 4k$ or $n = 4k+1$ and choose $C_n = \{3, 7, 11, \dots, 4k-1\}$ when $n = 4k+2$ or $n = 4k+3$. One can check that this C_n acts on $\mathcal{A}_n$ and on $\mathfrak{S}_n - \mathcal{A}_n$. From this, we get that where $m = \lfloor n/4\rfloor$.

The proof of our earlier result [9, Theorem 3] gives refined counting which we use to give a counterpart of Theorem 7 for $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$. In Section 5 of this paper, we prove the following.

Theorem 9. *For positive integers n, let $m = \lfloor n/2\rfloor$. The polynomial $\mathrm{Altseq}_n^{\pm}(t)$ is divisible by $(1+t)^{m-1}$.*

Clearly, Theorem 9 is a near refinement of Theorem 7 as the exponent falls short by 1. We move on to type B and type D counterparts of Theorem 9. For $\pi = \pi_0\pi_1\dots\pi_n \in \mathfrak{B}_n$ with $\pi_0 = 0$, an alternating subsequence of π is a subsequence $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \cdots$. By $\mathrm{as}_B(\pi)$, we denote the length of the longest alternating subsequence of π. For $\pi \in \mathfrak{D}_n$, we have the same definition of alternating subsequence as in $\mathfrak{B}_n$ and so, let $\mathrm{as}_D(\pi) = \mathrm{as}_B(\pi)$. Define the following polynomial $\mathrm{Altseq}(W,t) = \sum_{\pi\in W} t^{\mathrm{as}_B(\pi)}$ where $W \subseteq \mathfrak{B}_n$.

In Section 5 we give the following type B and type D counterpart of Theorem 7.

Theorem 10. *For positive integers n, the polynomials $\mathrm{Altseq}(\mathfrak{B}_n,t)$ are divisible by $(1+t)^{\lceil \frac{n}{2}\rceil}$. For positive integers n, the polynomial $\mathrm{Altseq}(\mathfrak{D}_n,t)$ is divisible by $(1+t)^{\lfloor \frac{n}{2}\rfloor}$. Further, for positive integers n, the polynomials $\mathrm{Altseq}(\mathfrak{B}_n^{\pm},t)$ and $\mathrm{Altseq}(\mathfrak{D}_n^{\pm},t)$ are divisible by $(1+t)^{\lfloor \frac{n}{2}\rfloor}$.*

2. The sign-flip map and its properties

Let $\pi = \pi_1\pi_2\dots\pi_n \in \mathfrak{B}_n$. For $1 \le k \le n$, we define the *sign-flip* map as follows:

$$\mathrm{Sgn\_flip}_k(\pi_1\pi_2\dots\pi_n) = \pi_1\pi_2\dots\pi_{k-1}\overline{\pi_k}\dots,\overline{\pi_n}.$$

That is, $\mathrm{Sgn\_flip}_k(\pi)$ flips the sign of all the elements of π from π_k onwards.

Example 11. Let $\pi = 132\overline{64}5 \in \mathfrak{B}_6$. Then, $\text{Sgn\_flip}_1(\pi) = \overline{132}64\overline{5}$. When $k = 4$, we have $\text{Sgn\_flip}_4(\pi) = 13264\overline{5}$ and when $k = 5$, we have $\text{Sgn\_flip}_5(\pi) = 132\overline{6}4\overline{5}$.

We note some basic properties of the map Sgn_flip_i.

Lemma 12. *For positive integers $1 \leq i \leq n$, the map Sgn_flip_i is an involution.*

Lemma 13. *Let $1 \leq i < j \leq n$. Then, the maps Sgn_flip_i and Sgn_flip_j commute. That is, for $\pi \in \mathfrak{B}_n$, we have $\text{Sgn\_flip}_i(\text{Sgn\_flip}_j(\pi)) = \text{Sgn\_flip}_j(\text{Sgn\_flip}_i(\pi))$.*

The proof of the following lemma is along the same lines as Bóna's proof of [4, Proposition 2.3]. We however give a proof for the sake of completeness.

Lemma 14. *Let $n \geq 3$ and $2 \leq i \leq n-1$. Then, one of π and $\text{Sgn\_flip}_i(\pi)$ has exactly one more alternating run than the other.*

Proof. Let $\pi = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1} \ldots \pi_n$. Clearly,

$$\text{Sgn\_flip}_i(\pi) = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}} \ldots \overline{\pi_n}.$$

The number of alternating runs in the string $\pi_i\pi_{i+1} \ldots \pi_n$ and its image under Sgn_flip_i, that is in the string $\overline{\pi_i\pi_{i+1}} \ldots \overline{\pi_n}$ are clearly identical. Therefore, we only need to consider the changes in the four-element string $\pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1}$ and $\pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}}$. We break them into $2^3 = 8$ cases as follows:

1. If $\pi_{i-2} < \pi_{i-1} < \pi_i < \pi_{i+1}$, then either $\pi_{i-2} < \pi_{i-1} < \overline{\pi_i} > \overline{\pi_{i+1}}$ or $\pi_{i-2} < \pi_{i-1} > \overline{\pi_i} > \overline{\pi_{i+1}}$.
2. If $\pi_{i-2} < \pi_{i-1} < \pi_i > \pi_{i+1}$, then either $\pi_{i-2} < \pi_{i-1} < \overline{\pi_i} < \overline{\pi_{i+1}}$ or $\pi_{i-2} < \pi_{i-1} > \overline{\pi_i} < \overline{\pi_{i+1}}$.
3. If $\pi_{i-2} < \pi_{i-1} > \pi_i < \pi_{i+1}$, then either $\pi_{i-2} < \pi_{i-1} > \overline{\pi_i} > \overline{\pi_{i+1}}$ or $\pi_{i-2} < \pi_{i-1} < \overline{\pi_i} > \overline{\pi_{i+1}}$.
4. If $\pi_{i-2} < \pi_{i-1} > \pi_i > \pi_{i+1}$, then either $\pi_{i-2} < \pi_{i-1} < \overline{\pi_i} < \overline{\pi_{i+1}}$ or $\pi_{i-2} < \pi_{i-1} > \overline{\pi_i} < \overline{\pi_{i+1}}$.
5. If $\pi_{i-2} > \pi_{i-1} < \pi_i < \pi_{i+1}$, then either $\pi_{i-2} > \pi_{i-1} < \overline{\pi_i} > \overline{\pi_{i+1}}$ or $\pi_{i-2} > \pi_{i-1} > \overline{\pi_i} > \overline{\pi_{i+1}}$.
6. If $\pi_{i-2} > \pi_{i-1} < \pi_i > \pi_{i+1}$, then either $\pi_{i-2} > \pi_{i-1} < \overline{\pi_i} < \overline{\pi_{i+1}}$ or $\pi_{i-2} > \pi_{i-1} > \overline{\pi_i} < \overline{\pi_{i+1}}$.
7. If $\pi_{i-2} > \pi_{i-1} > \pi_i < \pi_{i+1}$, then either $\pi_{i-2} > \pi_{i-1} > \overline{\pi_i} > \overline{\pi_{i+1}}$ or $\pi_{i-2} > \pi_{i-1} < \overline{\pi_i} > \overline{\pi_{i+1}}$.
8. If $\pi_{i-2} > \pi_{i-1} > \pi_i > \pi_{i+1}$, then either $\pi_{i-2} > \pi_{i-1} < \overline{\pi_i} < \overline{\pi_{i+1}}$ or $\pi_{i-2} > \pi_{i-1} > \overline{\pi_i} < \overline{\pi_{i+1}}$.

In each of the eight cases above, the difference between the number of alternating runs of the four-element string $\pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1}$ and $\pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}}$ is exactly 1. This completes the proof. □

3. Proof of Theorem 3

Let $n \geq 3$ be an integer. When n is even, let $T_n = \{3, 5, \ldots, n-1\}$ and when n is odd, let $T_n = \{2, 4, \ldots, n-1\}$. In both cases, let $M_n = \{\text{Sgn\_flip}_i : i \in T_n\}$. In either case, M_n clearly consists of $m = \lfloor (n-1)/2 \rfloor$ pairwise commuting involutions. Therefore, we get an action of the group $\mathbb{Z}_2^m$ on $\mathfrak{B}_n$ and $\mathfrak{D}_n$. For integers n with $n \geq 3$, since $1 \notin T_n$, we do not flip the sign of the first element and so $\mathbb{Z}_2^m$ also acts on $\mathfrak{B}_n^{>}$. By our definition of T_n, the action of $\mathbb{Z}_2^m$ preserves the parity of the number of negatives in $\pi \in \mathfrak{B}_n$. As such $\mathbb{Z}_2$ acts on D_n as well. With these set of generators, we note the following properties of the maps in M_n.

Lemma 15. *Let $i, j \in T_n$ with $i \neq j$ and suppose $\pi \in \mathfrak{B}_n$ is such that $\text{altruns}_B(\text{Sgn\_flip}_i(\pi)) = \text{altruns}_B(\pi) + 1$. Then, $\text{altruns}_B(\text{Sgn\_flip}_j(\text{Sgn\_flip}_i(\pi))) = \text{altruns}_B(\text{Sgn\_flip}_j(\pi)) + 1$.*

Proof. Without loss of generality, let $i < j$. From the structure of T_n, the indices i and j are not consecutive. Let $\pi = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1} \ldots \pi_j \ldots \pi_n$. We have

$$\text{Sgn\_flip}_i(\pi) = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}} \ldots \overline{\pi_{j-1}\pi_j} \ldots \overline{\pi_n},$$

$$\text{Sgn\_flip}_j(\pi) = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1} \ldots \pi_{j-1}\overline{\pi_j} \ldots \overline{\pi_n},$$

$$\text{Sgn\_flip}_j(\text{Sgn\_flip}_i(\pi)) = \pi_1 \ldots \pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}} \ldots \overline{\pi_{j-1}}\pi_j \ldots \pi_n.$$

As $\text{altruns}_B(\text{Sgn\_flip}_i(\pi)) = \text{altruns}_B(\pi) + 1$, the string $\pi_{i-2}\pi_{i-1}\overline{\pi_i\pi_{i+1}}$ has one more alternating run than the string $\pi_{i-2}\pi_{i-1}\pi_i\pi_{i+1}$. Thus, we get $\text{altruns}_B(\text{Sgn\_flip}_j(\text{Sgn\_flip}_i(\pi))) = \text{altruns}_B(\text{Sgn\_flip}_j(\pi)) + 1$. The proof is complete. □

The action of $\mathbb{Z}_2^m$ on $\mathfrak{B}_n^>$ and $\mathfrak{D}_n^>$ clearly creates orbits of size 2^m.

Lemma 16. *Let O be any orbit of $\mathbb{Z}_2^m$ acting on $\mathfrak{B}_n^>$. Then, for a non-negative integer a, we have*

$$\sum_{\pi \in O} t^{\mathrm{altruns}_B(\pi)} = t^a(1+t)^m.$$

Proof. Let π' be a permutation in the orbit O with the minimum number of alternating runs, say a many. By Lemma 14, for all $r \in T_n$, we have $\mathrm{altruns}_B(\mathrm{Sgn\_flip}_r(\pi')) = \mathrm{altruns}_B(\pi') + 1$. For $i, j \in T_n$ with $i < j$, by Lemma 15, we have $\mathrm{altruns}_B(\mathrm{Sgn\_flip}_i(\mathrm{Sgn\_flip}_j(\pi'))) = \mathrm{altruns}_B(\mathrm{Sgn\_flip}_j(\mathrm{Sgn\_flip}_i(\pi'))) = \mathrm{altruns}_B(\mathrm{Sgn\_flip}_j(\pi')) + 1 = \mathrm{altruns}_B(\pi') + 2$. Continuing this way, we get $\mathrm{altruns}_B(\mathrm{Sgn\_flip}_{i_1}(\mathrm{Sgn\_flip}_{i_2}(\ldots(\mathrm{Sgn\_flip}_{i_k}(\pi'))))) = \mathrm{altruns}_B(\pi') + k$ for $i_1 < i_2 < \cdots < i_k$, where $i_1, i_2, \ldots, i_k \in T_n$. We now show that all those $\binom{m}{k}$ permutations are distinct. Let $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ be with $(i_1, i_2, \ldots, i_k) \neq (j_1, j_2, \ldots, j_k)$. Then there exists $1 \leq r \leq k$ such that $i_r \neq j_r$ and r' be the minimum index such that this happens. Then, clearly the permutations $\mathrm{Sgn\_flip}_{i_1}(\mathrm{Sgn\_flip}_{i_2}(\ldots(\mathrm{Sgn\_flip}_{i_k}(\pi'))))$ and $\mathrm{Sgn\_flip}_{j_1}(\mathrm{Sgn\_flip}_{j_2}(\ldots(\mathrm{Sgn\_flip}_{j_k}(\pi'))))$ must differ at the r'-th entry. Thus there are at least $\binom{m}{k}$ permutations in O with $a+k$ alternating runs.

Now consider $\tau \in O$. By the definition of an orbit, we have

$$\tau = \mathrm{Sgn\_flip}_{i_1}(\mathrm{Sgn\_flip}_{i_2}(\ldots(\mathrm{Sgn\_flip}_{i_k}(\pi'))))$$

for some choices of $i_1, i_2, \ldots, i_k$. By Lemmas 12 and Lemma 13, we may assume $i_1 < i_2 < \cdots < i_k$. Hence, there are exactly $\binom{m}{k}$ many elements with $a+k$ alternating runs. Summing over k, completes the proof. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3. When $n \leq 2$, the exponent of $(1+t)$ is 0 and so there is nothing to prove. Thus, let $n \geq 3$. We first show the result for the type B case. We write $\mathfrak{B}_n^>$ as a disjoint union of its orbits and apply Lemma 16. Doing this, we get $R(\mathfrak{B}_n^>, t) = (1+t)^m \sum_\pi t^{\mathrm{altruns}_B(\pi)}$, where the summation is over permutations $\pi \in \mathfrak{B}_n^>$ that have the minimum $\mathrm{altruns}_B(\pi)$ value in its orbit. This completes the proof for the type B case.

For the type D case, we make the same moves, while making the following observation: by our choice of the set T_n, we assert that for each $i \in T_n$, we have $|\mathrm{Negs}(\pi)| \equiv |\mathrm{Negs}(\mathrm{Sgn\_flip}_i(\pi))| \pmod 2$. Therefore, all permutations π of any orbit have the same value of $|\mathrm{Negs}(\pi)| \pmod 2$. Thus, an orbit lies entirely in $\mathfrak{D}_n^>$ or entirely in $\mathfrak{B}_n^> - \mathfrak{D}_n^>$. Decomposing $\mathfrak{D}_n^>$ and $\mathfrak{B}_n^> - \mathfrak{D}_n^>$ into orbits O and summing $t^{\mathrm{altruns}_B(\pi)}$ over each O completes the proof of the other two results. □

4. Refining Theorem 3 by taking parity into account

As $\mathfrak{B}_n$ and $\mathfrak{D}_n$ are Coxeter groups, there is a natural notion of length, denoted by inv_B and inv_D respectively, associated to them. We first consider the type B case. The following combinatorial definition is from Petersen's book [12, Page 294]:

$$\mathrm{inv}_B(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}| + |\mathrm{Negs}(\pi)|. \tag{2}$$

For $\pi \in \mathfrak{B}_n$, we refer to $\mathrm{inv}_B(\pi)$ alternatively as its *length*. Let $\mathfrak{B}_n^+ \subseteq \mathfrak{B}_n$ denote the subset of even length elements of $\mathfrak{B}_n$ and let $\mathfrak{B}_n^- = \mathfrak{B}_n - \mathfrak{B}_n^+$. Here, we use the notation $\mathfrak{B}_n^\pm$ to succintly denote both $\mathfrak{B}_n^+$ and $\mathfrak{B}_n^-$.

Recall the sets T_n from Section 3. We first show the following.

Lemma 17. *For positive integers n, and $\pi \in \mathfrak{B}_n$, we have* $\mathrm{inv}_B(\mathrm{Sgn\_flip}_k(\pi)) \equiv \mathrm{inv}_B(\pi)$ *(mod 2) whenever $k \in T_n$.*

Proof. Observe that for any positive integer n, our choice of T_n ensures the following: for all $k \in T_n$ and $\pi \in \mathfrak{B}_n$, the map $\mathrm{Sgn\_flip}_k$ flips the sign of an even number of elements π_i's of π. Flipping the sign of a single π_i changes the parity $\mathrm{inv}_B(\pi)$ (see [13, Lemma 3]). Thus, flipping the sign of an even number of π_i's preserve the parity of $\mathrm{inv}_B(\pi)$. This completes the proof. □

Theorem 18. *For positive integers n, the polynomials $R(\mathfrak{B}_n^{>,\pm}, t)$ and $R(\mathfrak{B}_n^{<,\pm}, t)$ are divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$.*

Proof. We first consider the polynomials $R(\mathfrak{B}_n^{>,+},t)$. By Lemma 17, all permutations π in an orbit have the same value of $\mathrm{inv}_B(\pi)$ (mod 2). Thus, any orbit lies entirely in $\mathfrak{B}_n^{>,+}$ or entirely outside $\mathfrak{B}_n^{>,+}$. Decomposing $\mathfrak{B}_n^{>,+}$ as a disjoint union of orbits O and summing $t^{\mathrm{altruns}_B(\pi)}$ over each O proves that $R(\mathfrak{B}_n^{>,+},t)$ is divisible by $(1+t)^m$. In an identical manner, one can prove that $R(\mathfrak{B}_n^{>,-},t)$ and $R(\mathfrak{B}_n^{<,\pm},t)$ □

We need the following lemma which appears in the proof of a result of Chow and Ma [7, Corollary 2]. We have paraphrased their result, but from their proof, this change of form will be clear.

Lemma 19 (Chow and Ma). *If $(1+t)^m$ divides $f(t) = \sum_{i=0}^n f_i t^i$, then for positive integers $k \le m-1$, we have*

$$1^k f_1 + 3^k f_3 + \cdots = 2^k f_2 + 4^k f_4 + \cdots.$$

From Lemma 19 and Theorem 18, we immediately get the following moment-type identity. This refines a similar identity involving $R_{n,k}^B$ and $R_{n,k}^D$ obtained by combining Lemma 19 and Remark 4. Let $R_{n,k}^{B,>,\pm}$ and $R_{n,k}^{B,<,\pm}$ denote the number of permutations with k alternating runs in $\mathfrak{B}_n^{>,\pm}$ and $\mathfrak{B}_n^{<,\pm}$ respectively.

Theorem 20. *For $n \ge 2k+3$, we have*

$$1^k R_{n,1}^{B,>,\pm} + 3^k R_{n,3}^{B,>,\pm} + 5^k R_{n,5}^{B,>,\pm} + \cdots = 2^k R_{n,2}^{B,>,\pm} + 4^k R_{n,4}^{B,>,\pm} + 6^k R_{n,6}^{B,>,\pm} + \cdots,$$
$$1^k R_{n,1}^{B,<,\pm} + 3^k R_{n,3}^{B,<,\pm} + 5^k R_{n,5}^{B,<,\pm} + \cdots = 2^k R_{n,2}^{B,<,\pm} + 4^k R_{n,4}^{B,<,\pm} + 6^k R_{n,6}^{B,<,\pm} + \cdots.$$

We now move on to the type D case. The following definition for inv_D is from Petersen's book [12, Page 302].

$$\mathrm{inv}_D(\pi) = \mathrm{inv}(\pi) + |\{1 \le i < j \le n : -\pi_i > \pi_j\}|, \tag{3}$$

where $\mathrm{inv}(\pi) = |\{1 \le i < j \le n : \pi_i > \pi_j\}|$. Let $\mathfrak{D}_n^+ = \{\pi \in \mathfrak{D}_n : \mathrm{inv}_D(\pi) \text{ is even}\}$ and let $\mathfrak{D}_n^- = \mathfrak{D}_n - \mathfrak{D}_n^+$.

We start by proving the following type D counterpart of Lemma 17.

Lemma 21. *For positive integers n, and $\pi \in \mathfrak{D}_n$, we have* $\mathrm{inv}_D(\mathrm{Sgn\_flip}_k(\pi)) \equiv \mathrm{inv}_D(\pi)$ *(mod 2) whenever $k \in T_n$.*

Proof. Though $\mathrm{inv}_D(\pi)$ is defined only for $\pi \in \mathfrak{D}_n$, $\mathrm{inv}_B(\pi)$ is defined even if $\pi \in \mathfrak{D}_n$ as $\mathfrak{D}_n \subseteq \mathfrak{B}_n$. From (2) and 3, we get $\mathrm{inv}_B(\pi) = \mathrm{inv}_D(\pi) + |\mathrm{Negs}(\pi)|$. Thus, $\mathrm{inv}_B(\pi) \equiv \mathrm{inv}_D(\pi)$ (mod 2) for $\pi \in \mathfrak{D}_n$. The proof now follows from Lemma 17. □

Remark 22. By Lemma 21, under the action of the group $\mathbb{Z}_2^m$ on $\mathfrak{D}_n^>$, all π of any orbit O have the same parity of $\mathrm{inv}_D(\pi)$ (mod 2) and thus, O lies entirely in $\mathfrak{D}_n^{>,+}$ or entirely in $\mathfrak{D}_n^{>,-}$. In a similar manner, it is easy to see that $\mathbb{Z}_2^m$ acts on each of the sets $\mathfrak{D}_n^{>,-}$, $\mathfrak{B}_n^{>,\pm}$ and $\mathfrak{B}_n^{<,\pm}$.

Remark 23. Lemma 21 fails for a straightforward extension of Bóna's group action from [4] to the parity based subsets of the type B and the type D cases. We illustrate this point for the type B case. Similar examples can be given for the type D case as well. Given $\pi \in \mathfrak{B}_n$ and an index i, consider the sequence $S_i = \pi_i \pi_{i+1} \dots \pi_n$ of elements of π. In the sequence S_i, replace the p-th smallest element by the p-th largest element for all values of p. Let $c_i : \mathfrak{B}_n \mapsto \mathfrak{B}_n$ be this operation. Consider the group action generated by c_i with $i \in \{3,5,\dots,t\}$ where $t = n-1$ if n is even and $t = n-2$ if n is odd.

This is a group action on $\mathfrak{B}_n^>$ but all orbits do not contain elements π with the same parity of $\mathrm{inv}_B(\pi)$. To see this, let $\pi = 12\overline{3}\overline{4} \in \mathfrak{B}_4^>$. Then, $c_3(\pi) = 12\overline{4}\overline{3}$. Clearly, $\mathrm{inv}_B(\pi) \not\equiv \mathrm{inv}_B(c_3(\pi))$ (mod 2) and so the orbit of π lies neither within $B_n^{>,+}$ nor $B_n^{>,-}$.

Remark 22 helps us to prove the following refinement of Theorem 3.

Theorem 24. *For positive integers n, the polynomials $R(W,t)$ are divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$ and W is one of the following four sets: $\mathfrak{D}_n^{>,\pm}, \mathfrak{D}_n^{<,\pm}, \mathfrak{B}_n^{>,\pm} - \mathfrak{D}_n^{\pm}$, and $\mathfrak{B}_n^{<,\pm} - \mathfrak{D}_n^{<,\pm}$.*

Proof. We first consider the case when $W = \mathfrak{D}_n^{>,+}$. By Remark 22, $\mathbb{Z}_2^m$ acts on $\mathfrak{D}_n^{>,+}$. As done in the proof of Theorem 3, summing $t^{\mathrm{altruns}_D(\pi)}$ over the orbits, we get that $R(\mathfrak{D}_n^{>,+}, t)$ is divisible by $(1+t)^m$. In an identical manner, one can prove that $(1+t)^m$ divides the polynomials $R(W,t)$ when W is one of $\mathfrak{D}_n^{>,-}, \mathfrak{B}_n^{>,\pm} - \mathfrak{D}_n^{\pm}$, and $\mathfrak{B}_n^{<,\pm} - \mathfrak{D}_n^{<,\pm}$. The proof is complete. □

Lemma 19 and Theorem 3 give a moment-type identity which is implicit in the work of Gao and Sun. Combining Theorem 24 and Lemma 19, we get the following refined moment-type identity. Let $R_{n,k}^{D,>,\pm}$ and $R_{n,k}^{D,<,\pm}$ denote the number of permutations with k alternating runs in $\mathfrak{D}_n^{>,\pm}$ and $\mathfrak{D}_n^{<,\pm}$ respectively. Similarly, let $R_{n,k}^{B-D,>,\pm}$ and $R_{n,k}^{B-D,<,\pm}$ denote the number of permutations with k alternating runs in $\mathfrak{B}_n^{>,\pm} - \mathfrak{D}_n^{>,\pm}$ and $\mathfrak{B}_n^{<,\pm} - \mathfrak{D}_n^{<,\pm}$ respectively.

Theorem 25. *For $n \geq 2k+3$, we have*

$$1^k R_{n,1}^{D,>,\pm} + 3^k R_{n,3}^{D,>,\pm} + 5^k R_{n,5}^{D,>,\pm} + \cdots = 2^k R_{n,2}^{D,>,\pm} + 4^k R_{n,4}^{D,>,\pm} + 6^k R_{n,6}^{D,>,\pm} + \cdots,$$

$$1^k R_{n,1}^{D,<,\pm} + 3^k R_{n,3}^{D,<,\pm} + 5^k R_{n,5}^{D,<,\pm} + \cdots = 2^k R_{n,2}^{D,<,\pm} + 4^k R_{n,4}^{D,<,\pm} + 6^k R_{n,6}^{D,<,\pm} + \cdots,$$

$$1^k R_{n,1}^{B-D,>,\pm} + 3^k R_{n,3}^{B-D,>,\pm} + \cdots = 2^k R_{n,2}^{B-D,>,\pm} + 4^k R_{n,4}^{B-D,>,\pm} + \cdots,$$

$$1^k R_{n,1}^{B-D,<,\pm} + 3^k R_{n,3}^{B-D,<,\pm} + \cdots = 2^k R_{n,2}^{B-D,<,\pm} + 4^k R_{n,4}^{B-D,<,\pm} + \cdots.$$

5. Longest alternating subsequence polynomials

For results involving $\mathfrak{S}_n$, we start by defining some sets and their corresponding alternating run enumerating polynomials. We partition $\mathfrak{S}_n$ into four (disjoint) subsets and compute the signed alternating runs polynomial on each of these subsets. Our partitioning is based on the type of the first and the last pairs. Either pair could be an ascent or a descent. When $n \geq 3$, we get the following four sets:

$$\mathfrak{S}_{n,a,a} = \{\pi \in \mathfrak{S}_n : \pi_1 < \pi_2, \pi_{n-1} < \pi_n\}, \qquad \mathfrak{S}_{n,a,d} = \{\pi \in \mathfrak{S}_n : \pi_1 < \pi_2, \pi_{n-1} > \pi_n\},$$

$$\mathfrak{S}_{n,d,a} = \{\pi \in \mathfrak{S}_n : \pi_1 > \pi_2, \pi_{n-1} < \pi_n\}, \qquad \mathfrak{S}_{n,d,d} = \{\pi \in \mathfrak{S}_n : \pi_1 > \pi_2, \pi_{n-1} > \pi_n\}.$$

Define the following alternating runs enumerator polynomials.

1. $\mathrm{SgnAltrun}_{n,a,a}(t) = \sum_{\pi \in \mathfrak{S}_{n,a,a}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altruns}(\pi)}$,
2. $\mathrm{SgnAltrun}_{n,a,d}(t) = \sum_{\pi \in \mathfrak{S}_{n,a,d}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altruns}(\pi)}$,
3. $\mathrm{SgnAltrun}_{n,d,a}(t) = \sum_{\pi \in \mathfrak{S}_{n,d,a}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altruns}(\pi)}$,
4. $\mathrm{SgnAltrun}_{n,d,d}(t) = \sum_{\pi \in \mathfrak{S}_{n,d,d}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altruns}(\pi)}$,
5. $\mathrm{SgnAltrun}_{n,a,-}(t) = \sum_{\pi \in \mathfrak{S}_{n,a,a} \cup \mathfrak{S}_{n,a,d}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altruns}(\pi)}$.

The following result was proved in [9, Corollary 22].

Theorem 26. *For positive integers k, the following signed enumeration results hold. When $n = 4k$ and $n = 4k+1$, we have:*

$$\mathrm{SgnAltrun}_{n,a,a}(t) = \mathrm{SgnAltrun}_{n,d,d}(t) = t(1+t^2)(1-t^2)^{2k-2},$$

$$\mathrm{SgnAltrun}_{n,a,d}(t) = \mathrm{SgnAltrun}_{n,d,a}(t) = -2t^2(1-t^2)^{2k-2},$$

$$\mathrm{SgnAltrun}_n(t) = 2t(1-t)^{2k}(1+t)^{2k-2}.$$

When $n = 4k+2$ and $n = 4k+3$, we have:

$$\mathrm{SgnAltrun}_{n,a,d}(t) = 0 \textit{ and } \mathrm{SgnAltrun}_{n,d,a}(t) = 0,$$

$$\mathrm{SgnAltrun}_{n,a,a}(t) = -\mathrm{SgnAltrun}_{n,d,d}(t) = t(1-t^2)^{2k},$$

$$\mathrm{SgnAltrun}_n(t) = 0.$$

Using Theorem 26, we get the following version of (1) for $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$.

Theorem 27. *For positive integers $n = 4k$ and $n = 4k+1$, we have*

$$\text{Altseq}_n^{\pm}(t) = \frac{(1+t)R_n^{\pm}(t)}{2}. \tag{4}$$

For positive integers $n = 4k+2$ and $n = 4k+3$, we have

$$\text{Altseq}_n^{\pm}(t) = \frac{(1+t)R_n^{\pm}(t) \pm t(1-t)(1-t^2)^{2k}}{2}. \tag{5}$$

Proof. Consider the map $f : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ defined as

$$f(\pi_1\pi_2\ldots\pi_n) = n+1-\pi_1 n+1-\pi_2\ldots n+1-\pi_n.$$

By Remark 6 and the fact that the map f flips the parity of inv if and only if $n \equiv 0,1 \pmod 4$, we have

$$\text{SgnAltseq}_n(t) = \begin{cases} (1+t)\text{SgnAltrun}_{n,a,-}(t) & \text{when } n \equiv 0,1 \pmod 4 \\ (1-t)\text{SgnAltrun}_{n,a,-}(t) & \text{when } n \equiv 2,3 \pmod 4. \end{cases} \tag{6}$$

Therefore, when $n = 4k$ or $n = 4k+1$, we have

$$\begin{aligned} 2\text{Altseq}_n^{\pm}(t) &= \text{Altseq}_n(t) \pm \text{SgnAltseq}_n(t) \\ &= \frac{1}{2}(1+t)R_n(t) \pm (1+t)\text{SgnAltrun}_{n,a,-}(t) \\ &= \frac{1}{2}(1+t)R_n(t) \pm \frac{1}{2}(1+t)\text{SgnAltrun}_n(t) = (1+t)R_n^{\pm}(t). \end{aligned}$$

The second line above uses (1). Therefore, we are done in this case. When $n = 4k+2$ or $n = 4k+3$, we have

$$\begin{aligned} 2\text{Altseq}_n^{\pm}(t) &= \text{Altseq}_n(t) \pm \text{SgnAltseq}_n(t) \\ &= \frac{1}{2}(1+t)R_n(t) \pm (1-t)\text{SgnAltrun}_{n,a,-}(t) \\ &= (1+t)R_n^{\pm}(t) \pm t(1-t)(1-t^2)^{2k}. \end{aligned}$$

The last line uses [9, Corollary 22]. The proof is complete. □

We need the following result proved in [9, Theorem 3].

Theorem 28. *For positive integers $n \geq 4$, let $m = \lfloor (n-2)/2 \rfloor$. When $n \equiv 0,1$ (mod 4), the polynomials $R_n^{\pm}(t)$ are divisible by $(1+t)^{m-1}$, When $n \equiv 2,3$ (mod 4), the polynomials $R_n^{\pm}(t)$ are divisible by $(1+t)^m$.*

Proof of Theorem 9. Follows immediately from Theorem 27 and Theorem 28. □

We move on to our final proof of this work.

Proof of Theorem 10. Consider the type B case first. Using the map Sgn_flip$_1$, it is easy to see that

$$\text{Altseq}(\mathfrak{B}_n, t) = (1+t)\sum_{\pi \in \mathfrak{B}_n^{>}} t^{\text{as}_B(\pi)} = (1+t)R(\mathfrak{B}_n^{>}, t).$$

The proof is complete by combining with Theorem 3. We give our proof for $\text{Altseq}(\mathfrak{B}_n^{\pm}, t)$ below.

For the type D case, recall the following sets from Section 3: let $T_n = \{1,3,5,\ldots,n-1\}$ when n is even and let $T_n = \{2,4,6,\ldots,n-1\}$ when n is odd. As seen in Section 3, we have a group action on $\mathfrak{D}_n$. We can repeat the same argument to get that $\text{Altseq}(\mathfrak{D}_n, t)$ is divisible by $(1+t)^{|T_n|}$. As $|T_n| = \lfloor n/2 \rfloor$, we are done. Further, by Lemma 17 and Lemma 21, since these are actions on $\mathfrak{D}_n^{\pm}$ and on $\mathfrak{B}_n^{\pm}$, an identical argument shows that the polynomials $\text{Altseq}(\mathfrak{B}_n^{\pm}, t)$ and $\text{Altseq}(\mathfrak{D}_n^{\pm}, t)$ are divisible by $(1+t)^{\lfloor \frac{n}{2} \rfloor}$, completing the proof. □

Remark 29. The difference in the exponent of $(1+t)$ between the whole group and its positive-and-negative parts in Theorem 9 and Theorem 10 exists and our result is the best that one can get. We give examples below. For the $\mathfrak{S}_n$ case, when $n=6$, one can check that $(1+t)^3$ divides $\mathrm{Altseq}_6(t)$, while $(1+t)^2$ divides $\mathrm{Altseq}_6^{\pm}(t)$ as we have

$$\begin{aligned}\mathrm{Altseq}_6(t) &= (t+1)^3(61t^3+28t^2+t),\\ \mathrm{Altseq}_6^{+}(t) &= (t+1)^2t(6t^2+8t+1)(5t+1),\\ \mathrm{Altseq}_6^{-}(t) &= (t+1)^2t^2(31t^2+43t+16).\end{aligned}$$

For the type B case, when $n=3$, one can check that $(1+t)^2$ divides $\mathrm{Altseq}(\mathfrak{B}_3,t)$, while $(1+t)$ divides $\mathrm{Altseq}(\mathfrak{B}_3^{\pm},t)$ as we have

$$\begin{aligned}\mathrm{Altseq}(\mathfrak{B}_3,t) &= 11t^4+23t^3+13t^2+t = (t+1)^2t(11t+1),\\ \mathrm{Altseq}(\mathfrak{B}_3^{+},t) &= 6t^4+11t^3+6t^2+t \quad = (t+1)t(2t+1)(3t+1),\\ \mathrm{Altseq}(\mathfrak{B}_3^{-},t) &= 5t^4+12t^3+7t^2 \qquad\quad = (t+1)t^2(5t+7).\end{aligned}$$

Finally, we also mention that moment identities similar to Theorem 20 and Theorem 25 can be given involving the coefficients of $\mathrm{Altseq}_n^{\pm}(t)$, $\mathrm{Altseq}(\mathfrak{B}_n^{\pm},t)$ and $\mathrm{Altseq}(\mathfrak{D}_n^{\pm},t)$.

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