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Generic pole assignability, structurally constrained controllers and unimodular completion ☆

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ABSTRACT

In this paper we assume dynamical systems are represented by linear differential-algebraic equations (DAEs) of order possibly higher than one. We consider a structured system of DAEs for both the to-be-controlled plant and the controller. We model the structure of the plant and the controller as an undirected and bipartite graph and formulate necessary and sufficient conditions on this graph for the structured controller to generically achieve arbitrary pole placement. A special case of this problem also gives new equivalent conditions for structural controllability of a plant. Use of results in matching theory, and in particular, 'admissibility' of edges and 'elementary bipartite graphs', make the problem and the solution very intuitive. Further, our approach requires standard graph algorithms to check the required conditions for generic arbitrary pole placement, thus helping in easily obtaining running time estimates for checking this. When applied to the state space case, for which the literature has running time estimates, our algorithm is faster for sparse state space systems and comparable for general state space systems.
The solution to the above problem also provides a necessary and sufficient condition for the following matrix completion problem. Given a structured rectangular polynomial matrix, when can it be completed to a unimodular matrix such that the additional rows

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that are added during the completion process are constrained to have zeros at certain locations.

1. Introduction

When dealing with large dynamical systems, numerical computation is often not feasible. Further, the precise values of the parameters arising in system equations are rarely known. Due to these reasons, many standard control problems are analyzed in the framework of 'structured' systems. The survey paper Dion et al. [7] gives a detailed account of how various properties of such systems like generic rank of the transfer matrix of the system, generic structure at infinity and disturbance decoupling have been analyzed under the assumption of genericity of parameters. These problems have been addressed primarily for descriptor/regular state space systems, where the structure of the state space system is modeled by a directed graph, and the graph is used further to characterize system properties. A related context where structural control has been investigated is that of sensor network design and decentralized control; here one deals with constraints on the variables that can occur in controller equations. The notion of 'decentralized fixed modes' is considered in Wang and Davison [22]; Corfmat and Morse [5]; Anderson and Clements [2]. As a generalization of decentralized control, Sezer and Šiljak [19] defined and characterized *structurally fixed modes* for arbitrary feedback patterns. Existence of structurally fixed modes, which rules out the possibility of arbitrary pole placement, has been characterized in terms of directed graphs. In Murota [13] a matroid theoretic approach is used to characterize the structurally fixed modes of a system in descriptor state space form. Algorithmic running times of checking graph properties in the context of structural control problems have been investigated in Papadimitriou and Tsitsiklis [16], and Murota [12,13].

In this paper we address the question: under what conditions on the plant structure can one achieve arbitrary pole placement using a controller that has *structural constraints*. Thus we assume the controller equations are constrained by a structure that specifies which system variables occur in each controller equation. While existing techniques to address structural aspects of control and pole placement using structured controllers start from a *state space* representation of the plant, the results in this paper apply to more general models of dynamical systems: linear differential-algebraic equations of order possibly higher than one.

After characterizing arbitrary pole placement with structured controllers in Theorem 2.6, we formulate and prove new results (Theorem 2.7) for structural controllability of higher order dynamical systems governed by DAEs. Use of behavioral theory of systems (see Polderman and Willems [17]) allows relating controllability properties of such a dynamical system directly in terms of the Smith normal form (SNF) of $P(s)$, an associated polynomial matrix. System controllability is equivalent to all the invariant polynomials of P, i.e. the diagonals in SNF, being equal to one. In a generic/structural context, whether all the invariant polynomials are ones or not has been of interest primarily for *matrix pencils*, see Iwata and Shimizu [9] and van der Woude [26]. Higher degree polynomial matrices have been studied in the context of the generic SNF in Murota [14]. Structured matrices where certain nonzero numbers are fixed, while certain other nonzero entries are generic, have been considered in Murota [14] with the constraint that the fixed part of the polynomial matrix satisfies certain 'homogeneity conditions' on its invariant factors: the factors are assumed to be monomials. Using matroid theoretic methods, Murota [14] computes the degrees of the invariant factors of the SNF through a so-called 'combinatorial canonical form'. An absence of separation between nonzero entries into fixed and generic simplifies the problem and allows us to use techniques from matching theory of bipartite graphs. Such techniques have been utilized in van der Woude [25] in the context of generic dimension of the state space of a dynamical system, where the generic degree of the gcd of all maximal minors has been obtained under a more general formulation. Statements 3 and 4 of Theorem 2.7 below provide new equivalent conditions for structural controllability.

Finally, this paper also solves the problem of completion[1] to a unimodular polynomial matrix $U(s)$, i.e. determinant of $U(s)$ is a nonzero constant. Given a sparse, generic polynomial matrix $P(s)$ with n rows and m columns, $n < m$, and just locations of unspecified entries in $C(s)$ that has $(m-n)$ rows and m columns, we provide necessary and sufficient conditions on the locations of the specified entries in $P(s)$ and unspecified entries in $C(s)$ such that by a suitable choice of polynomials in the allowed locations in $C(s)$, the matrix $\begin{bmatrix} P(s) \\ C(s) \end{bmatrix}$ has determinant equal to a nonzero constant. In the literature on matrix completion problems, a separation of nonzero entries into specified and unspecified entries, and the investigation into the associated graph of specified, unspecified and zero entries' locations, has been primarily in the context of *constant* matrices with additional properties, like positive definiteness, M-matrix: see Hogben [8]. In the context of *polynomial* matrix completion, the structural/generic aspects have not received any interest: for example, the focus in Amidou and Yengui [1] is on constructing the *precise* values of the unspecified entries (from a multivariate Laurent polynomial ring) for unimodular completion.

We summarize the contribution in this paper.

(a) We obtain equivalent graph-theoretic conditions on the plant and controller structures for generic arbitrary pole placement. Equivalently, we obtain conditions for checking whether a rectangular polynomial matrix can be completed to a unimodular matrix with nonzero entries chosen during completion only at prespecified locations. See Theorem 2.6.
(b) We obtain new graph conditions for structural controllability of a plant, see Theorem 2.7.
(c) We specialize the above situation for the state space case (see Theorem 2.9).
(d) We provide algorithms that check conditions listed in Theorems 2.6 and 2.7 and also obtain their running time estimates (Lemmas 6.2 and 6.4). Our algorithmic running time is lower than existing algorithms for sparse system equations, and comparable for general systems.
(e) Inadmissible edges, removal of which is very central to all the graph conditions above, are shown to never occur when building a large system from SISO (Single Input Single Output) subsystems using just the series, parallel and feedback interconnection (see Theorem 2.8).

The paper is organized as follows. Section 2 contains brief preliminaries, our formulation of the problem and our main results. Section 3 explains the remaining preliminaries: polynomial matrices, genericity and bipartite graphs. The results on structural controllability of a plant are proved in Section 4. Section 5 is concerned with proving Theorem 2.6 on conditions on the plant and controller structures for arbitrary pole placement. The running time of the algorithms involved in checking the graph conditions is addressed in Section 6. Section 7 summarizes some concluding remarks of this paper.

2. Problem formulation and main results

Section 2.1 has preliminaries essential for just this section. An unfamiliar reader is urged to read more details in Section 3 in case some terms in Theorems 2.6–2.9 are unclear. The sets $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of real numbers and complex numbers respectively, while $\mathbb{R}^n$ is the vector space of n-tuples having entries from $\mathbb{R}$. The set of polynomials in the indeterminate s with real coefficients is denoted by $\mathbb{R}[s]$. Matrices with n rows and m columns having entries from $\mathbb{R}[s]$ are denoted by $\mathbb{R}^{n\times m}[s]$. A polynomial matrix $P(s) \in \mathbb{R}^{n\times m}[s]$ is called *left-prime* if $P(\lambda)$ has full row rank for each $\lambda \in \mathbb{C}$. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^m$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$.

2.1. Preliminaries in brief

We consider systems which are described by a set of ordinary linear differential equations with constant coefficients. The system behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ consisting of all solutions to the system equations: let $P(s) \in \mathbb{R}^{n\times m}[s]$.

[1] We deal with a specialized case where the specified entries are in the top few rows and the unspecified entries in the remaining rows.

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^m) \;\middle|\; P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}. \tag{1}$$

This representation is called a *kernel representation* of $\mathfrak{B}$.

Since we seek only 'generic' results, we consider just the structural aspects of the system. In this context, we associate a weighted bipartite graph to the given system. A graph $G = (V, E)$ with vertex set V and edge set E is said to be bipartite if V can be partitioned into two subsets $\mathcal{R}$ and $\mathcal{C}$ such that no two vertices from the same subset have an edge between them. We associate an edge-weighted bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ to a polynomial matrix $P(s) \in \mathbb{R}^{n\times m}[s]$ as follows. The sets $\mathcal{R}$ and $\mathcal{C}$ denote the rows and columns of the polynomial matrix and are the two disjoint vertex sets of the bipartite graph G, i.e. $|\mathcal{R}| = n$, $|\mathcal{C}| = m$. By definition of G, an edge exists in the bipartite graph between vertex $u_i \in \mathcal{R}$ and $v_j \in \mathcal{C}$ if the (i, j)th entry of the matrix P is nonzero. We will see that all nonconstant polynomial entries in $P(s)$ contribute to the results in the same manner irrespective of the degree. The constant polynomial entries are different since their roots are an empty set, and therefore these entries need to satisfy milder conditions compared to the nonconstant polynomial entries. Hence we distinguish only between constant and nonconstant entries of the matrix. In this regard the edge set is classified into two types.

- Constant edge: if the entry in $P(s)$ corresponding to this edge is a *nonzero constant*.
- Nonconstant edge: if the entry in $P(s)$ corresponding to this edge is a polynomial of degree one or more.

This classification is analogous to having weights on the edges except that there are only two types of weights: a) zero-weight (the constant entries, i.e. degree zero), and b) weight of one or more (nonconstant polynomial entries, i.e. degree $\geqslant 1$). Note that there is no edge between two vertices u_i and v_j if and only if the (i, j)th entry of $P(s)$ is 0; this is different from an edge of weight zero. An edge of weight zero is an entry which is a nonzero constant. We will not use degree of the polynomial entry anymore, but rather refer to the entries as 'constant' and 'nonconstant'. The degree of a *vertex* v in the graph is the number of edges incident on v, equivalently, the number of *neighbors* of v.

For a bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$, and a subset of edges $E_1 \subseteq E$, define the bipartite graph $G[E_1]$ *induced* by E_1 as the subgraph of G consisting of edges from E_1 and their endpoint vertices. A set of edges M in a graph G is called a *matching* if every vertex of the subgraph of G induced by M has degree at most 1. The cardinality of a matching M, denoted by $|M|$, is defined as the number of edges in M. A maximum matching is a matching with maximum number of edges. A graph can have more than one maximum matching. An edge e in G is called *admissible* if e occurs in some maximum matching of G and an inadmissible edge is one which occurs in none of the maximum matchings: see Asratian et al. [3, Section 10.3]. In this paper, we consider polynomial matrices $P(s) \in \mathbb{R}^{n\times m}[s]$ with $n \leqslant m$, i.e. the graph G satisfies $|\mathcal{R}| \leqslant |\mathcal{C}|$. A matching M is said to be $\mathcal{R}$*-saturating* if $|M| = |\mathcal{R}|$. In the special case when G satisfies $|\mathcal{R}| = |\mathcal{C}|$, an $\mathcal{R}$-saturating matching is also called a *perfect matching*. A detailed exposition of matching theory can be found in Lovász and Plummer [11].

A path in a graph G is a sequence of distinct vertices and distinct edges, $v_0 e_1 v_1 \ldots e_n v_n$, where edge e_i connects vertices v_{i-1} and v_i, for each $1 \leqslant i \leqslant n$. We rule out $n = 1$, i.e. a path cannot be a single edge. A cycle in a graph G is a path with the exception that $v_0 = v_n$ and all other vertices and edges remain distinct. Our formulation results in *simple* graphs, i.e. graphs with no parallel edges between any pair of nodes, and no self-loops.

Since every system of linear time-invariant (LTI) ODEs can be written as a kernel representation, and hence can be associated to a polynomial matrix $P(s)$ we use the zero/nonzero structure of the polynomial matrix $P(s)$ and the constant/nonconstant property of the nonzero entries to define a 'structured system'.

Definition 2.1. Consider a system of LTI ODEs $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $P \in \mathbb{R}^{n\times m}[s]$. Classify the nonzero entries in $P(s)$ as constant and nonconstant and then associate the graph $G(\mathcal{R}, \mathcal{C}; E)$ to the polynomial matrix $P(s)$. Such association partitions the set of all polynomial matrices into equivalence classes and each class is identified by its corresponding graph. Further, we say the graph $G(\mathcal{R}, \mathcal{C}; E)$ captures the *structure of the LTI system*.

For a given structure $G(\mathcal{R}, \mathcal{C}; E)$, with $|\mathcal{R}| < |\mathcal{C}|$, the set of left-prime polynomial matrices in this equivalence class is a generic set. See Definition 3.3 for a formal definition of genericity. If *almost all* polynomial matrices in this equivalence class are left-prime, we say the polynomial matrices are 'generically' left-prime: it is well known that left-primeness of almost all polynomial matrices in this class follows from existence of *one* left-prime element in this class, see text after Definition 3.3. The following definition of structural controllability is justified since left-primeness of a polynomial matrix $P(s)$ is equivalent to behavioral controllability of the system defined by $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ (see Proposition 3.1 below).

Definition 2.2. A structured system $G(\mathcal{R}, \mathcal{C}; E)$ is said to be *structurally controllable* if the polynomial matrices in the equivalence class represented by G are *generically left-prime*.

Example 2.3. Consider a system described by the three differential equations in the variables w_1, w_2, w_3 and w_4 presented in Fig. 1. The corresponding polynomial matrix $P(s) \in \mathbb{R}^{3\times 4}[s]$ and the associated bipartite graph are shown in Fig. 1. System parameters a_{ij} and b_{ij} are arbitrary real numbers.

$$a_{11}\dot{w}_1 + b_{11}w_1 + a_{12}\dot{w}_2 + b_{12}w_2 = 0$$

$$a_{21}\dot{w}_1 + b_{21}w_1 + b_{22}w_2 = 0$$

$$b_{31}w_1 + a_{32}\dot{w}_2 + b_{32}w_2 + a_{33}\dot{w}_3 + b_{33}w_3 + a_{34}\dot{w}_4 + b_{34}w_4 = 0$$

$$P(s) = \begin{bmatrix} a_{11}s + b_{11} & a_{12}s + b_{12} & 0 & 0 \\ a_{21}s + b_{21} & b_{22} & 0 & 0 \\ b_{31} & a_{32}s + b_{32} & a_{33}s + b_{33} & a_{34}s + b_{34} \end{bmatrix}$$

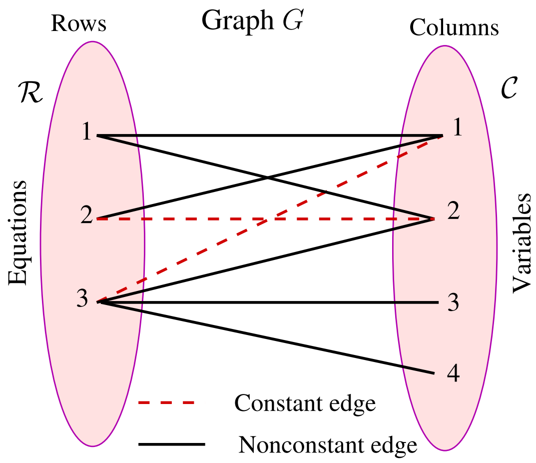

Fig. 1. Graph for $P(s)$.

We show later after Theorem 2.7 that the above system is structurally uncontrollable, i.e. $P(s)$ is not generically left-prime. This means that for almost every value of a_{ij} and $b_{ij} \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that $P(\lambda)$ loses rank at λ.

2.2. Problem formulation and main results

The first important problem we address is whether one can generically place the poles of a given structured system using some controller with a prespecified controller structure. In the behavioral framework, when a plant $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is interconnected with a controller $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, the system variables w have to satisfy

$$\begin{bmatrix} P(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} w = 0.$$

Autonomy of the interconnected/controlled system, together with so-called regularity of the controller, translates to the property that the polynomial matrix $\left[\begin{smallmatrix} P(s) \\ K(s) \end{smallmatrix}\right]$ is square and has determinant not identically zero; see Willems [24, Sections 7 and 8] and an elaboration in Footnote 7 in Section 3.2 below. The roots of the determinant polynomial are the closed loop poles. Ability to achieve arbitrary pole placement of the closed loop is just the ability of choosing K to obtain any desired polynomial as this determinant.

In the framework of structured equations, let $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$ denote the specified controller structure, i.e. $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$ captures the zero/nonzero structure of K. Note that the vertex set $\mathcal{C}$ remains the same as the variables in the plant and controller equations are the same. The edges of $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$ are termed as controller edges and they are *not* classified into constant/nonconstant, unlike the plant edges: this is because the controller is to be *designed* and entries are to be chosen to meet a control objective, for example, arbitrary pole placement.

Problem 2.4. Consider a plant structure $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$ and a controller structure $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$. Find necessary and sufficient conditions on graphs G^P and G^K under which for almost any plant with structure G^P there exists a controller with structure G^K that achieves arbitrarily specified poles for the interconnected system. Develop an algorithm to check these conditions and estimate its running time.

A special case of the above problem is when there is *no structure* specified for the controller, i.e. each controller equation can access *all* system variables. It is well known that controllability of a plant is a necessary and sufficient condition for arbitrary pole placement (see Willems [24] and also Proposition 3.2 below). This brings us to the next main problem of this paper.

Problem 2.5. Consider the plant structure $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$. Find necessary and sufficient conditions on G^P such that the system is structurally controllable. Develop an algorithm to check these conditions and estimate its running time.

Two of the main results are stated below; they are solutions to the above problems. The following theorem is about pole placement for structured system with constraints on controller structure.

Theorem 2.6. *Let $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$ represent the structure of the plant and assume G^P contains an $\mathcal{R}_P$-saturating matching.*[2] *Consider a controller structure $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$. Suppose $\mathfrak{L}_P$ and $\mathfrak{L}_K$ denote the equivalence classes of polynomial matrices with graphs G^P and G^K respectively. Define $\mathcal{R} := \mathcal{R}_P \cup \mathcal{R}_K$ and $E := E_P \cup E_K$ and construct $G^{\text{aut}}(\mathcal{R}, \mathcal{C}; E)$, the graph of the* interconnection *of the plant and the controller. Remove the inadmissible edges from G^{aut} to get G_a^{aut}. Let $\chi_{PK}(s)$ refer to the determinant of $A(s) := \begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$, for $P \in \mathfrak{L}_P$ and $K \in \mathfrak{L}_K$. Then the following are equivalent.*

1. *Arbitrary pole placement is possible generically using controllers having structure G^K.*
2. *$\mathfrak{L}_P$ generically satisfies the following property:*
   for each $P \in \mathfrak{L}_P$, $\bigcap_{K \in \mathfrak{L}_K}$ roots of $\chi_{PK} = \phi$.
3. *There do not exist subsets $r \subseteq \mathcal{R}_P$ and $c \subset \mathcal{C}$ that satisfy the following three conditions:*
   (a) $|r| = |c|$,
   (b) *there is a nonconstant plant edge in G_a^{aut} incident on r,*
   (c) *every perfect matching M of G_a^{aut} matches r and c.*
4. *Every nonconstant plant edge in G_a^{aut} is in some cycle containing controller edges in G_a^{aut}.*

The proof of the above theorem requires development of other results. See Section 5 for its proof. Arbitrary pole placement with a specified controller structure which is in output feedback has been addressed for state space systems in Šiljak [21]. The inability to achieve arbitrary pole placement has been shown there to be related to so-called 'structurally fixed modes'. Condition 2 of Theorem 2.6 is about non-existence of structurally fixed modes. Statement 4 explains that a nonconstant plant edge e_p that is admissible in G^{aut} but not in any cycle involving edges from G^K contributes to a factor in the characteristic polynomial which is independent of the controller. The inadmissible plant edges do not contribute to any term in the determinant expansion anyway. Notice that Statement 3 in

[2] This corresponds to the kernel representation of the plant being minimal, i.e. the associated structured polynomial matrix is generically full row rank.

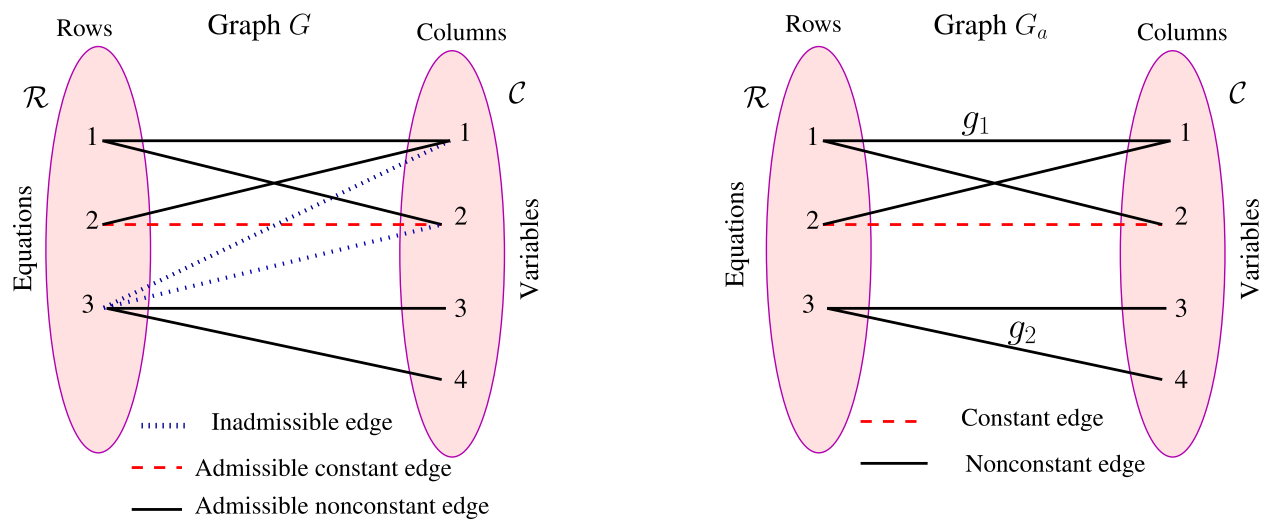

Fig. 2. Bipartite graph for $P(s)$ with and without inadmissible edges.

our result above involves a check over all subsets, thus suggesting a potentially non-efficient algorithm to verify Statement 1. A key aspect of our paper is the equivalence of Statements 3 and 4, the latter can be checked more efficiently: see Algorithm 6.3 and Lemma 6.4 below for the algorithm and its running time estimate.

The next theorem solves Problem 2.5 on structural controllability.

Theorem 2.7. *Let $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$, $|\mathcal{R}_P| < |\mathcal{C}|$ represent the structure of the plant and assume G^P contains an $\mathcal{R}_P$-saturating matching. Suppose all inadmissible edges in G^P are removed to obtain G_a^P. Let $g_1, g_2, \ldots, g_t$ be the connected components of G_a^P. Then the following are equivalent.*

1. *The plant is structurally controllable.*
2. *The graph G^P represents an equivalence class of generically left-prime polynomial matrices.*
3. *Each component g_i that contains a nonconstant plant edge satisfies $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$.*
4. *For each nonconstant plant edge e_p in G_a^P, there exist $\mathcal{R}_P$-saturating matchings M and N such that e_p is in a* path[3] *in $G_a^P[M \Delta N]$, the subgraph of G_a^P on the symmetric difference*[4] *between M and N.*

The proof of this theorem also requires more preliminaries and hence is given in Section 4. The fourth condition requires a check on nonconstant and admissible plant edges while the third condition involves checking the sizes of various components of G_a^P. According to Condition 3, lack of structural controllability is equivalent to the existence of a component, say g, in the bipartite graph G_a^P being such that g contains a nonconstant edge and also contains a perfect matching.

Consider the system in Example 2.3 above and Fig. 1. The determinant of the top 2×2 block in the polynomial matrix $P(s)$ is generically a polynomial of degree 2 and rank of $P(\lambda)$ falls when λ is a root of this polynomial. This verifies generic non-left-primeness of P, equivalently, the structural uncontrollability of the corresponding system. Consider Fig. 2 that shows $G(\mathcal{R}, \mathcal{C}; E)$ and also the one after removal[5] of inadmissible edges. Condition 3 of Theorem 2.7 is not satisfied, thus verifying generic non-left-primeness of $P(s)$. The significance of removal of inadmissible edges can be seen by noting that polynomials b_{31} and $a_{32}s + b_{32}$ do not affect any 3×3 minors of $P(s)$, but result in connectedness of G. Theorem 2.7 is essentially about permuting the rows and columns and first bringing a

[3] Note that, by definition (see Section 2.1), a path has length at least two. In fact, the proof of Theorem 2.7 below concludes that this path is of even length.

[4] The symmetric difference between two sets A and B, denoted as $A \Delta B$, is defined as $(A \cup B) \backslash (A \cap B)$.

[5] There exists a maximum matching of size 3, but none of the two edges corresponding to polynomials b_{31} and $a_{32}s + b_{32}$ occur in any matching of size three: these are both inadmissible.

polynomial matrix $P(s)$ into block lower triangular form. The diagonal blocks are the connected components after removal of inadmissible edges. Square blocks, if any, are required to generically evaluate to a constant polynomial as their determinants: this forces square blocks to have all admissible edges as constant.

2.3. Series, parallel and feedback interconnections

In this subsection we bring out a control-theoretic significance of the notion of admissibility of an edge. The following theorem states that each of series, parallel and feedback interconnection of two systems retains structural controllability, and moreover, there are no inadmissible edges in the resulting bipartite graph.

Theorem 2.8. *Let S_1 and S_2 be two Single Input Single Output (SISO) systems. Consider the system S_3 obtained by any one of the following interconnection procedures:*
1) *series,* 2) *parallel,* 3) *feedback.*
Then S_3 is structurally controllable and the bipartite graph constructed from the equations describing S_3 has no inadmissible edges.

Proof. Let systems S_1 and S_2 have transfer functions $\frac{q_1(s)}{p_1(s)}$ and $\frac{q_2(s)}{p_2(s)}$ respectively. Consider first their interconnection in the feedback configuration. Denote respectively the input and output variables of S_1 by e and y and those of S_2 by y and v. The differential equations describing systems S_1 and S_2 are $p_1(\frac{d}{dt})y = q_1(\frac{d}{dt})e$ and $p_2(\frac{d}{dt})v = q_2(\frac{d}{dt})y$. Feedback interconnection results in the additional equation: $e = r - v$. These three equations constitute matrix M_{fdb}:

$$M_{\mathrm{fdb}} = \begin{bmatrix} & 1 & 1 & -1 \\ p_1 & -q_1 & & \\ q_2 & & -p_2 & \end{bmatrix} \quad \text{and} \quad w_{\mathrm{fdb}} = \begin{bmatrix} y \\ e \\ v \\ r \end{bmatrix},$$

with $M_{\mathrm{fdb}} w_{\mathrm{fdb}} = 0$ as the system equations. The blank entries in the polynomial matrix M_{fdb} are all zero. It is straightforward to see that each nonzero entry in $M_{\mathrm{fdb}}(s)$ occurs in some term of a suitable 3×3 minor of M_{fdb}. This means that the bipartite graph constructed from M_{fdb} has no inadmissible edges, thus proving the theorem for this interconnection configuration.

For S_1 and S_2 connected in series and in parallel, write the two systems of equations in matrix form as follows:

$$M_{\mathrm{ser}} = \begin{bmatrix} q_1 & -p_1 & \\ & q_2 & -p_2 \end{bmatrix}, \qquad w_{\mathrm{ser}} = \begin{bmatrix} r \\ v \\ y \end{bmatrix} \quad \text{and}$$

$$M_{\mathrm{par}} = \begin{bmatrix} p_1 & & -q_1 & \\ & -p_2 & q_2 & \\ 1 & 1 & & -1 \end{bmatrix}, \qquad w_{\mathrm{par}} = \begin{bmatrix} u \\ v \\ r \\ y \end{bmatrix}.$$

Like the feedback interconnection case, admissibility of every edge is verified in these interconnections too. Finally, the structural controllability claim is verified using Theorem 2.7 for M_{fdb}, M_{ser} and M_{par}. □

The significance of the above result is in the following sense as far as larger interconnections are concerned. If a connected bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ comprises of only admissible edges, and contains a perfect matching, then further arbitrary 'addition' of new edges ensures admissibility of both the old and new edges (see text following Definition 3.5 in Section 3.4). Addition of edges, without adding vertices, means that an existing equation now involves additional existing variables. Due to guaranteed admissibility of the new edges, these newly introduced entries generically do affect the closed loop characteristic polynomial. This modification of equations however involves more complex/meshed interconnection, and not necessarily within the series/parallel/feedback building blocks.

An algorithmic significance of the absence of inadmissible edges is that there is a significant improvement in the running time estimates of our algorithms to solve Problems 2.4 and 2.5: we revisit this in Section 6 below.

2.4. State space systems

In this subsection we assume the plant is in a regular state space form and we formulate conditions for structural controllability. Though the state space case is well studied, the results in the literature involve, unlike our approach, directed and non-bipartite graphs. Let $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$ be the graph corresponding to $P(s) = [\,sI - A \quad B\,] \in \mathbb{R}^{n\times(n+m)}[s]$. Since the column indices of $P(s)$ correspond to variables x_i and u_j, call the vertices $v_1, \ldots, v_n \in \mathcal{C}$ as state vertices and $v_{n+1}, \ldots, v_m \in \mathcal{C}$ as input vertices. The following theorem states that structural controllability of (A, B) is equivalent to each state in G^P_a being connected to some input vertex and an additional condition that is relevant to only the state space case.

Theorem 2.9. *Let $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$ denote the structure of the state space system (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Obtain G^P_a by removal of all inadmissible edges. The system (A, B) is generically controllable if and only if the following two conditions are satisfied.*

1. *Each state is connected to some input vertex in the graph G^P_a.*
2. *The bipartite graph constructed from the constant matrix $[A \mid B]$ contains an $\mathcal{R}_P$-saturating matching.*

Proof.

To ease the proof, we introduce some notation. We index the $\mathcal{R}_P$-vertex set of the graph by $\dot{x}_i$ for $i = 1, \ldots, n$, while the $\mathcal{C}$-vertex is indexed by x_i for $i = 1, \ldots, n$ and u_j for $j = 1, \ldots, m$ (see Fig. 3). The only nonconstant edges are those that connect $\dot{x}_i$ to x_i and the remaining edges are constant. There are exactly n nonconstant edges, and these form one $\mathcal{R}_P$-saturating matching in G^P, and hence all the nonconstant edges are also in G^P_a. Due to all nonconstant edges being admissible, and due to each $\dot{x}_i$ being connected to x_i, we first infer that vertices $\dot{x}_i$ and x_i lie in the same connected component g. Hence, for each connected component g of G^P_a, the condition $|\mathcal{R}(g)| = |\mathcal{C}(g)|$ is equivalent to the absence of any input vertex u in $\mathcal{C}(g)$.

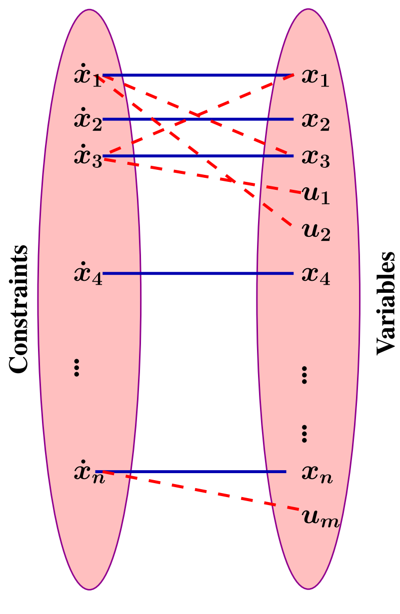

Fig. 3. State space system.

(Only if part): We first assume that there exists a state x_i such that x_i is *not* connected to any input vertex in G^P_a, and show that (A, B) is not controllable. Consider g, the connected component of G^P_a which contains x_i. Due to the fact that each $\dot{x}_j$ of $\mathcal{R}_P(G^P_a)$ is connected to x_j (of $\mathcal{C}(G^P_a)$), the assumption on x_i implies that there is no input vertex in $\mathcal{C}(g)$. This implies that for g, we have $|\mathcal{R}(g)| = |\mathcal{C}(g)|$. This means every $\mathcal{R}_P$-saturating matching in G^P_a matches $\mathcal{R}_P(g)$ and $\mathcal{C}(g)$. Since g has at least one nonconstant edge, Condition 3 in Theorem 2.7 is not satisfied. Thus (A, B) is structurally uncontrollable. This proves necessity of Condition 1 of Theorem 2.9.

We next assume that Condition 2 of Theorem 2.9 is not satisfied, and show that the system is not controllable. Suppose the bipartite graph corresponding to $[A \mid B]$ does not have an $\mathcal{R}_P$-saturating matching. This implies that at $\lambda = 0$, the matrix $[\lambda I - A \mid B]$ does not have full row rank. Using the Popov–Belevitch–Hautus (PBH) test, it follows that the system is uncontrollable.

(If part): We now show that if every state x_i in G_a^P is connected to some input vertex u_j, then every connected component g of G_a^P satisfies the condition $|\mathcal{R}_P(g)| < |\mathcal{C}(g)|$; from Theorem 2.7 above, it then follows that the system (A, B) is generically controllable for all nonzero complex numbers $\lambda \in \mathbb{C}$. Condition 2, which concerns controllability at $\lambda = 0$, is used while inferring structural controllability: this is addressed within Footnote 6.

Consider the connected components of G_a^P. We noted above that since the n nonconstant edges in G_a^P that connect each state x_j and $\dot{x}_j$ form an $\mathcal{R}_P$-saturating matching, each nonconstant edge is admissible in G^P, and hence that edge is retained in G_a^P. Further, this also causes $|\mathcal{R}_P(g)| \leqslant |\mathcal{C}(g)|$ for each component g of G_a^P. Note that $|\mathcal{C}(g)| - |\mathcal{R}_P(g)|$ is precisely the number of input vertices in g. Thus a state x_i is connected to some input vertex if and only if $|\mathcal{R}_P(g)| < |\mathcal{C}(g)|$ for the component g that contains x_i. Hence each state being connected to some input vertex G_a^P is equivalent to $|\mathcal{R}_P(g)| < |\mathcal{C}(g)|$ for every connected component of G_a^P. Using[6] Theorem 2.7 above, we conclude that the system is structurally controllable. □

Remark 2.10. Conditions 1 and 2 of Theorem 2.9 can be easily seen to be related to respectively ruling out uncontrollability of Form I and Form II: see Šiljak [21, p. 23]. See also Murota [15, Theorem 6.4.7], for example, about the interpretation of Condition 2 as generic controllability at $\lambda = 0$.

Example 2.11. Consider $A = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix}$, for real numbers α, β, γ and δ. This example has been shown to be structurally uncontrollable in Reinschke [18] using the method developed there. The bipartite graph constructed from $[A \mid B]$ (shown in Fig. 4) does not contain a perfect matching (see Condition 2 of Theorem 2.9). Due to loss of controllability at $\lambda = 0$, this system is not structurally controllable.

Example 2.12. Consider $A = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & \beta & 0 \\ \gamma & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix}$, for real numbers α, β, δ and γ: a slightly modified version of the previous example. The corresponding graph G_a^P is now such that each state is connected to the input vertex. See Fig. 5. Moreover, $[A \mid B]$ too has full row rank generically. Hence the pair (A, B) is structurally controllable.

Example 2.13. Consider $A = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & \alpha_5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ \alpha_6 \end{bmatrix}$, for real numbers α_i. This system is structurally uncontrollable, in particular, has Form I (see Šiljak [21, p. 22]). Check that in the corresponding bipartite graph, there exists a path between the input and every state vertex, however, after removal of inadmissible edges, the state x_1 is no longer connected. This makes α_1 an uncontrollable pole of the system. The system is structurally uncontrollable using Condition 1 of Theorem 2.9 above.

[6] Strictly speaking, Theorem 2.7 refers to the case when, once a polynomial is nonconstant, *all* the coefficients are arbitrary real numbers, and hence generically nonzero, while $[sI - A \mid B]$ has its degree-one coefficients equal to one and diagonal constant entries possibly zero: this is where Condition 2 plays a crucial role as follows. Since scaling of each row by a nonzero real number has no effect on the zero set of the polynomial matrix, the diagonal entries having their degree-one coefficients each equal to one is not the issue. However, the constant along the diagonals being assumed to be nonzero due to the formulation of Theorem 2.7 causes A to be generically nonsingular and hence $[A \mid B]$ is generically controllable at the origin. Since this may not be the actual case for structured matrices A and B, Condition 2 of Theorem 2.9 is required to conclude generic controllability at the origin too.

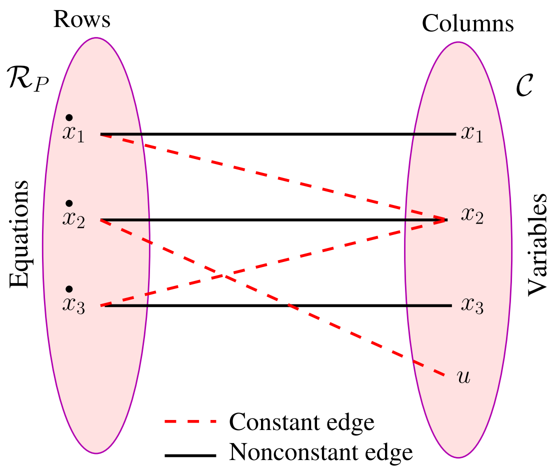

Fig. 4. Graph for Example 2.11.

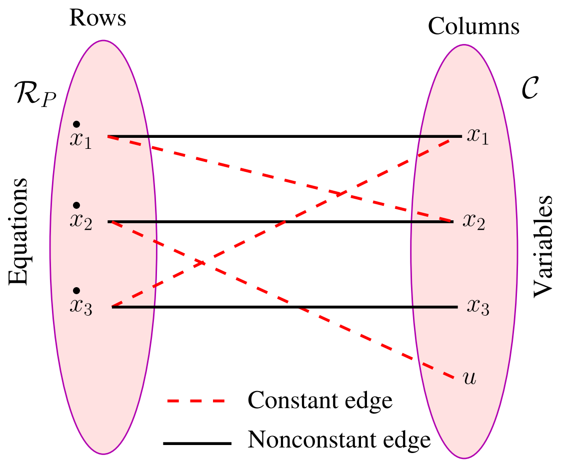

Fig. 5. Graph for Example 2.12.

3. Preliminaries

In this section we elaborate on the preliminaries required in this paper. Section 3.1 contains essentials of the behavioral approach and Section 3.2 describes pole placement in this approach. Genericity of parameters and properties of polynomial matrices in the generic context are covered in Section 3.3, while Section 3.4 elaborates on the notion of an admissible edge: this notion plays a central role in the results and the proofs.

3.1. Behavioral approach

In Section 2.1, the concept of system behavior and its kernel representation was introduced. A behavior $\mathfrak{B}$ is called *controllable* if for any two trajectories $w_1, w_2 \in \mathfrak{B}$ there exists $T \geqslant 0$ and a trajectory $w \in \mathfrak{B}$ with the property

$$w(t) = \begin{cases} w_1(t) & \text{for } t \leqslant 0, \\ w_2(t) & \text{for } t \geqslant T. \end{cases} \tag{2}$$

In other words, if $\mathfrak{B}$ is controllable it is possible to patch from any past trajectory to any other desired trajectory using a suitable w that satisfies the system laws, perhaps with some finite delay. A behavior $\mathfrak{B}$ is called *autonomous* if $w_1 = w_2$ whenever $w_1, w_2 \in \mathfrak{B}$ satisfy $w_1(t) = w_2(t)$ for all $t \leqslant 0$. We state the required results from behavioral literature in the following proposition for easy reference: see Polderman and Willems [17, Theorems 2.5.23, 3.2.16 and 5.2.5].

Proposition 3.1. *Consider $P \in \mathbb{R}^{n\times m}[s]$ and let behavior $\mathfrak{B}$ be described by the kernel representation $P(\frac{d}{dt})w = 0$. Then:*

1. *$\mathfrak{B}$ is autonomous if and only if the polynomial matrix P has full column rank.*
2. *$\mathfrak{B}$ is controllable if and only if $P(\lambda)$ has constant row rank for every complex number $\lambda \in \mathbb{C}$.*

A kernel representation $P(\frac{d}{dt})w = 0$ is called minimal if the polynomial matrix P has full row rank. Without loss of generality, a kernel representation can be assumed to be minimal: see Polderman and Willems [17, Theorem 2.5.23]. Consider a polynomial matrix $P \in \mathbb{R}^{n\times m}[s]$. Define the *zeros* of P to be the set of complex numbers where P loses its rank:

$$\text{zeros}(P) := \{\lambda \in \mathbb{C} \mid \text{rank}(P(\lambda)) < \text{rank}(P(s))\}. \tag{3}$$

Note that 'rank' has slightly different meanings on the two sides of inequality in (3): in one case rank is of a *constant* matrix $P(\lambda)$ and in the other case rank is of a *polynomial* matrix $P(s)$. The polynomial matrix $P(s)$ is said to be full rank if $\text{rank}(P) = \min(n, m)$. If P is a full rank polynomial matrix, zeros(P) are the roots of the gcd of all the maximal minors of P. For a detailed exposition of these notions, we refer to Kailath [10, Section 6.3]. A square polynomial matrix U is called *unimodular* if $\det(U)$ is a nonzero constant. These are precisely the square nonsingular polynomial matrices whose zero set is empty. In the context of controllability and completion to a unimodular matrix, we need coprimeness of polynomials and Bézout's identity. Polynomials $p_1, \ldots, p_n \in \mathbb{R}[s]$ are said to be coprime if their gcd is 1. Polynomials $p_1, \ldots, p_n \in \mathbb{R}[s]$ are coprime if and only if there exist polynomials $c_1, \ldots, c_n \in \mathbb{R}[s]$ such that $c_1 p_1 + \cdots + c_n p_n = 1$: see Polderman and Willems [17, Corollary 2.5.12], for example.

Using the definition of zeros(P) as in (3) above, we see that a behavior described by $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is controllable if and only if the zero set of P is empty. We use this characterization of controllability and give equivalent graph-theoretic conditions under the assumption of genericity of parameters.

3.2. Pole placement

Let $A(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a kernel representation of an autonomous behavior $\mathfrak{B}$. The determinant of A is called the characteristic polynomial (assumed monic, without loss of generality) of the system, and is denoted by $\chi(\mathfrak{B})$. The roots of $\chi(\mathfrak{B})$ counted with multiplicities are called the poles of the behavior $\mathfrak{B}$. The following proposition gives a necessary and sufficient condition for pole placement using the behavioral approach.

Proposition 3.2. *(See Willems [24, Theorem 7].) Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, $P(s) \in \mathbb{R}^{n\times m}[s]$ denote a minimal kernel representation of the plant. Then the following are equivalent.*

- *For any monic $d(s) \in \mathbb{R}[s]$, there exists a regular[7] controller $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ such that the corresponding closed system has characteristic polynomial $d(s)$, i.e.* $\det \begin{bmatrix} P(s) \\ K(s) \end{bmatrix} = d(s)$.
- *The plant is controllable.*

In particular, if we choose d as 1, the matrix $\begin{bmatrix} P \\ K \end{bmatrix}$ is unimodular thus relating controllability, left-primeness and unimodular completion. Since we focus on generic arbitrary pole placement, which is nothing but assigning the roots of χ counted with multiplicity, we ignore the 'monic' aspect of χ for the rest of this paper.

3.3. Generic properties of polynomial matrices

This paper concerns studying controllability property in a structural sense, i.e. for almost all values of system parameters. This brings us to the notion of genericity.

Definition 3.3. A property P in terms of variables $a_1, \ldots, a_n$ is said to be satisfied *generically* if the set $A \subseteq \mathbb{R}^n$ of values that do *not* satisfy property P is contained in the zero set of some nonzero polynomial in $a_1, \ldots, a_n$.

A property P is true generically in $\mathbb{R}^n$ if and only if P is satisfied for *almost all* values in $\mathbb{R}^n$. A generic set is measure exhausting, i.e. the complement of the set has Lebesgue measure zero: see Willems [23, p. 344], for example. When n is clear from the context, we skip specifying it explicitly. For example, one can check that two nonzero polynomials $a(s)$ and $b(s) \in \mathbb{R}[s]$ are generically coprime; this is verified as follows. In this case, $n := \deg a(s) + \deg b(s) + 2$ is the number of real coefficients. Generic coprimeness follows since the set of coefficients have to satisfy a nontrivial algebraic

[7] The interconnection is said to be *regular* if $\operatorname{rank} \begin{bmatrix} P(s) \\ K(s) \end{bmatrix} = \operatorname{rank} P(s) + \operatorname{rank} K(s)$. Regularity of interconnection is closely related to implementation of the controller in the feedback configuration: see Willems [24]. In this paper, we consider only regular interconnections. Given a plant system, a *controller* is called *regular* if the interconnection between the plant and that controller is regular.

relation for the two polynomials to have a common factor: the algebraic relation is nothing but the resultant of $a(s)$ and $b(s)$: see Kailath [10, Section 2.4.4].

For a square polynomial matrix P, we review the relation between the determinant of P and the perfect matchings of the bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ associated to P. Let M be a perfect matching in G. Then M corresponds to a nonzero term in the determinant expansion of P: the term consists of the product of all entries corresponding to the edges in M. The determinant of P is just the sum of the terms over all perfect matchings in G, with suitable signs. P is nonsingular generically if and only if G contains a perfect matching (see Murota [15]; Babai and Frankl [4]). In addition to generic nonsingularity, the determinant being generically a nonzero constant, i.e. unimodularity of P, is important too. The following proposition (specialized from van der Woude [25, Theorem 5.2]) formulates equivalent graph conditions for this.

Proposition 3.4. *Consider the edge-weighted bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ corresponding to a structured polynomial matrix $P \in \mathbb{R}^{n\times n}[s]$. Then P is generically unimodular if and only if there exists a perfect matching and every perfect matching in G comprises of only constant edges.*

The above proposition is easier to state using the notion of an admissible edge: a necessary and sufficient condition for generic unimodularity is that there exists a perfect matching and *all admissible* edges are constant. The following subsection delves further into this notion and graphs comprising of just admissible edges.

3.4. Inadmissible edges

In the graph $G(\mathcal{R}, \mathcal{C}; E)$ constructed from $P \in \mathbb{R}^{n\times m}[s]$ an edge e which does not occur in any maximum matching is called an *inadmissible* edge of G. Consequently, the entry in P corresponding to this edge e does not play a role in any maximal minor of P; this means e does not affect the zero set of the polynomial matrix P. After removing the inadmissible edges from G the resulting subgraph, denoted as G_a, is such that every edge is *admissible*.[8] Clearly, G has an $\mathcal{R}$-saturating matching if and only if G_a has one. Due to the genericity assumption on P, and since the nonzero entries in P_a corresponding to G_a are also in P, we have the genericity property for entries in P_a also.

Definition 3.5. A bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ with $|\mathcal{R}| = |\mathcal{C}|$ is called *elementary* if its admissible edges form a connected subgraph of G.

An interesting property is that a bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ with $|\mathcal{R}| = |\mathcal{C}|$ is elementary if and only if G is connected, contains a perfect matching and has no inadmissible edges. Very relevant to our paper is the following proposition dealing with cycles.

Proposition 3.6. *(See Lovász and Plummer [11, Corollary 4.2.10].) Any two edges of an elementary bipartite graph are contained in a cycle.*

Next we review the Dulmage and Mendelsohn (DM) decomposition of a bipartite graph G, where G is decomposed into edge disjoint subgraphs, called its DM components. Each of the subgraphs has its edges either all admissible in G or all inadmissible in G. In this paper, we deal with a simpler situation: the case when all inadmissible edges have been removed to get G_a and further G_a has a perfect matching. We review the DM decomposition for just this case: it turns out that the connected components of G_a are its DM components. We use this decomposition in Proposition 5.1 while factorizing the determinant of matrix A in the proof of Theorem 2.6. The following proposition is about an important property of the DM components in this special case.

[8] An admissible edge e has also been referred to as 'allowed' and 'maximally-matchable' in the literature: see Lovász and Plummer [11] and Tassa [20], for example. There are minor differences depending on whether the maximum matching containing e should also be perfect.

Proposition 3.7. *(See Asratian et al. [3, p. 187].) Let $G(\mathcal{R}, \mathcal{C}; E)$ with $|\mathcal{R}| = |\mathcal{C}|$ be a bipartite graph with a perfect matching. Assume G_a is the subgraph induced by the admissible edges of G. Then each component of G_a is an elementary bipartite graph.*

In the situation under consideration, the DM decomposition is said to be nontrivial if G_a is connected, i.e. G_a is an elementary bipartite graph.

4. Proof of Theorem 2.7: structural controllability

In this section we prove the equivalence between the conditions listed in Theorem 2.7 for structural controllability. We state some definitions and preliminary lemmas that are required to prove this main result. The following two propositions relate cycles and paths to the components of G obtained from the symmetric difference of two matchings in G.

Proposition 4.1. *(See Asratian et al. [3, p. 57].) Let M and N be matchings in a graph G. Then each component of $G[M\Delta N]$ is exactly one of the following:*

1. *an even cycle with edges alternating in $M\backslash N$ and $N\backslash M$, or*
2. *a path whose edges are alternating in $M\backslash N$ and $N\backslash M$.*

Proposition 4.2. *(See Asratian et al. [3, p. 58].) If a graph G has two perfect matchings M and N then all components of $G[M\Delta N]$ are even cycles.*

The following lemma relates components in the symmetric difference of two $\mathcal{R}$-saturating matchings in a graph to being a path/cycle.

Lemma 4.3. *Consider a bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$, with $|\mathcal{R}| < |\mathcal{C}|$. Let M and N be two $\mathcal{R}$-saturating matchings in G and let $C_M, C_N \subset \mathcal{C}$ denote the vertices in $\mathcal{C}$ of the edges in the matchings M and N respectively. Then the following are true.*

1. *As sets, $C_M = C_N \Leftrightarrow$ each component of $G[M\Delta N]$ is a cycle.*
2. *As sets, $C_M \neq C_N \Leftrightarrow$ there exists a component of $G[M\Delta N]$ which is a path of even length.*

Proof. Using Propositions 4.1 and 4.2 we note that Statements 1 and 2 are the same. Hence it is enough to prove just the '$\Rightarrow$ part' of each statement.

1. ($\Rightarrow$): Since the two $\mathcal{R}$-saturating matchings M and N satisfy $C_M = C_N$, they are two perfect matchings on a set of $2|\mathcal{R}|$ vertices. Hence from Proposition 4.2 we have that each component of $G[M\Delta N]$ is a cycle.

2. ($\Rightarrow$): In the graph $G[M\Delta N]$, let $r_{MN} \subseteq \mathcal{R}$, $c_{MN} \subseteq \mathcal{C}$ denote the set of vertices on which the edges of $G[M\Delta N]$ are incident. If $C_M \neq C_N$, then $|r_{MN}| < |c_{MN}|$, which implies all the components of $G[M\Delta N]$ cannot be cycles. Hence from Proposition 4.1 there exists a path in $G[M\Delta N]$. Since both M and N are $\mathcal{R}$-saturating matchings, in $G[M\Delta N]$, the incidence degree of each vertex in $\mathcal{R}$ is either two or zero. Hence a path in $G[M\Delta N]$ has its start and end vertex only in $\mathcal{C}$. This implies that the path is of even length. $\square$

Using the above results, we prove Theorem 2.7: our main result on structural controllability.

Proof of Theorem 2.7. Some simplifications are helpful for proving the implications. We remove the inadmissible edges in G^P to get G_a^P. Let P_a be the corresponding structured matrix. We permute the rows and columns of P_a such that each connected component of G_a^P corresponds to consecutive rows/columns. Suppose the components of G_a^P are $g_1, \ldots, g_c$. Thus P_a is now in the form:

$$P_a = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_c \end{bmatrix}$$

with P_i the submatrices of P_a corresponding to the connected components g_i. Moreover, P_i is square if and only if $|\mathcal{R}_P(g_i)| = |\mathcal{C}(g_i)|$. Since there exists an $\mathcal{R}_P$-saturating matching in G^P, and therefore in G_a^P, each g_i satisfies $|\mathcal{R}_P(g_i)| \leqslant |\mathcal{C}(g_i)|$, and there exists at least one row-saturating matching for each component g_i. Further,

$$\text{zeros}(P_a) = \bigcup_{i=1,\dots,c} \text{zeros}(P_i).$$

With this simplification, we proceed to the proof. Statements 1 and 2 are equivalent by Definition 2.2. We prove $2 \Leftrightarrow 3$ and then $3 \Leftrightarrow 4$.

($2 \Rightarrow 3$): Suppose the polynomial matrices represented by G^P are generically left-prime, i.e. the zero set of P is empty generically. We show that for every component g_i of G_a^P satisfying $|\mathcal{R}_P(g_i)| = |\mathcal{C}(g_i)|$, all edges in g_i are constant edges. On the contrary, suppose g_i is such that $|\mathcal{R}_P(g_i)| = |\mathcal{C}(g_i)|$, and g_i contains a nonconstant edge. Since each edge is admissible, the determinant of P_i is generically a nonconstant polynomial. Hence the zero set of P has to be non-empty, thus proving the necessity of Condition 3.

($3 \Rightarrow 2$): For this part, we need to show that each of the P_i is such that its zero set is empty. There are two cases.

Case 1: P_i is such that $|R_P(g_i)| = |C(g_i)|$ in G_a^P, or
Case 2: P_i is such that $|R_P(g_i)| < |C(g_i)|$ in G_a^P.

In Case 1, by assumption, all edges in g_i are constant edges, and hence determinant of P_i is generically a nonzero constant: see van der Woude [25, Theorem 5.2] and also Proposition 3.4. This proves that the zero set of that P_i is empty. For Case 2, connectedness of g_i and admissibility of all its edges make the zero set of the corresponding P_i empty. This is proved exactly along the lines of the proofs of Murota [15, Theorems 6.3.8 and 6.3.4]: such a component g_i has been termed there as the 'horizontal tail' of the DM decomposition.

($3 \Rightarrow 4$): We assume a nonconstant plant edge e_p in G_a^P belongs to a connected component, say g_i, satisfying $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$. We prove that there exist two $\mathcal{R}_P$-saturating matchings M and N such that e_p is in a path in a component of $G_a^P[M \Delta N]$. Using the absence of inadmissible edges, the connectedness of g_i and $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$ together, we construct two $\mathcal{R}_P(g_i)$-saturating matchings m and n in g_i as follows. Choose an $\mathcal{R}_P(g_i)$-saturating matching, say m, which contains e_p. Let $\mathcal{C}_m \subset \mathcal{C}(g_i)$ denote the vertices of the edges in the matching m. The other matching n is chosen to construct the required path. Let $v_c(e_p)$ denote the vertex of e_p in $\mathcal{C}(g_i)$. Consider a vertex v, such that $v \in \mathcal{C}(g_i) \backslash \mathcal{C}_m$. Such a vertex v exists because $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$. Since g_i is connected there exists a path p between $v_c(e_p)$ and v. Moreover, edges in p alternate between m and $E_P(g_i) \backslash m$. Obtain a matching n by "transferring[9] from m along p". Further, n does not contain e_p. Notice that $G_a^P[m \Delta n]$ is exactly p. Use matchings m and n in g_i to conclude that there exist two $\mathcal{R}_P$-saturating matchings M and N in G such that the edge $e_p \in M$ and $e_p \notin N$. This proves that e_p is in a path in $G_a^P[M \Delta N]$. Finally, since both the end vertices of p are in $\mathcal{C}(g_i)$, it follows that length of p is even. This proves Condition 4.

($4 \Rightarrow 3$): We assume that every nonconstant plant edge e_p is in a path in a component of $G_a^P[M \Delta N]$. We show that any component g_i of G_a^P containing a nonconstant plant edge satisfies $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$. Consider a nonconstant plant edge e_p which belongs to a connected component g_i of G_a^P. Our assumption implies that there are two $\mathcal{R}_P(g_i)$-saturating matchings $m \subset M$ and $n \subset N$

[9] Refer Asratian et al. [3, Theorem 5.1.7] which explains that it is possible to obtain a new maximum matching from an existing maximum matching by a "sequence of transfers along alternating paths".

such that the edge $e_p \in G_a^P[m\Delta n]$ and is in a path. Then from Lemma 4.3 we have that $C_m \neq C_n$. This implies that the connected component, g_i which contains e_p satisfies $|\mathcal{R}_P(g_i)| < |\mathcal{C}(g_i)|$. This proves Condition 3 and completes the proof of Theorem 2.7. □

5. Pole placement with constraints on controller structure

In this section we prove Theorem 2.6. The key idea in the proof is to ensure that there are no structurally fixed modes, and the cycle condition for each nonconstant and admissible plant edge then guarantees absence of common factors generically: thus allowing the use of Bézout's identity.

The following proposition plays an important role in the proof. This is about factorizing the determinant of a nonsingular matrix in accordance with a canonical decomposition (DM decomposition) of an associated graph. Note that the factorization is defined over a specific ring which we explain as follows. Assume we have a nonsingular matrix A whose nonzero entries are replaced by distinct indeterminates from the set $X := \{x_1, \ldots, x_p\}$. Let $\mathbb{R}[X]$ denote $\mathbb{R}[x_1, x_2, \ldots, x_p]$, i.e. the ring of polynomials with indeterminates from X and coefficients from the field $\mathbb{R}$. The subgraphs in the DM decomposition play a role in the proper[10] factorization of $\det(A)$ in $\mathbb{R}[X]$.

Proposition 5.1. *(See Yamada [27].) Let A be a nonsingular matrix whose nonzero entries are distinct elements from X. Assume G is the bipartite graph corresponding to A. Then* $\det(A)$ *has a proper factorization in* $\mathbb{R}[X]$ *if and only if the DM decomposition of G is nontrivial. Furthermore, the irreducible factors of* $\det(A)$ *correspond to the components in the DM decomposition.*

Thus the determinant of the submatrix corresponding to each component does not admit a proper factorization in $\mathbb{R}[X]$. Further, if G is an elementary bipartite graph, then $\det(A)$ does not admit a proper factorization in $\mathbb{R}[X]$. A graph G whose DM decomposition is trivial is called *DM-irreducible*.

Proof of Theorem 2.6. Assume the DM decomposition (Proposition 3.7) on G_a^{aut} results in components $H_1, \ldots, H_k$. Let A_a be the structured polynomial matrix corresponding to G_a^{aut}. Using Proposition 5.1, each irreducible factor of $\det(A_a)$ corresponds to a component H_i. Corresponding to this decomposition of G_a^{aut}, decompose A_a as

$$A_a = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}. \tag{4}$$

Block A_i corresponds to the graph H_i and A_i is square, as each H_i is an elementary bipartite graph. Note that determinant of A_a is $\prod_{i=1}^{k} \det(A_i)$. We use this to prove Theorem 2.6: we show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

($1 \Rightarrow 2$): Since arbitrary pole placement is possible, the intersection of closed loop poles obtained using different controllers is empty. This proves Statement 2.

($2 \Rightarrow 3$): We prove this by contradiction. We assume there exist subsets $r \subseteq \mathcal{R}_P$ and $c \subseteq \mathcal{C}$ in G_a^{aut} such that $|r| = |c|$ and every perfect matching matches r to c with at least one nonconstant plant edge incident on r. We prove that for each $P \in \mathfrak{L}_P$, the intersection $\bigcap_{K \in \mathfrak{L}_K}$ roots of $\chi_{PK} \neq \phi$. Subsets $r \subseteq \mathcal{R}$ and $c \subset \mathcal{C}$ with $|r| = |c|$ such that every perfect matching of G_a^{aut} that matches r to c corresponds to components H_i in G_a^{aut}. Since the subset in our case has $r \subseteq \mathcal{R}_P$, the component H_i does not have any vertex from $\mathcal{R}_K$. Hence for a given $P \in \mathfrak{L}_P$, the determinant of the corresponding block A_i is a polynomial that is independent of $K \in \mathfrak{L}_K$. Therefore the $\det(A_a)$ also has $\det(A_i)$ as a factor which cannot be modified by the entries corresponding to controller edges. Since there is a

[10] A factorization of a polynomial p in the ring $\mathbb{R}[X]$ into factors q and r is called *proper* if neither q nor r is invertible in $\mathbb{R}[X]$. In such a case, we say p is properly factorizable. If p is not properly factorizable, then p is called irreducible.

nonconstant plant edge incident on r, $\det(A_i)$ generically has degree at least one. This proves that for each $P \in \mathfrak{L}_P$, the intersection $\bigcap_{K \in \mathfrak{L}_K}$ roots of $\chi_{PK} \neq \phi$.

$(3 \Rightarrow 4)$: A component H_i in G_a^{aut} corresponds to subsets $r \subseteq \mathcal{R}$ and $c \subseteq \mathcal{C}$ with $|r| = |c|$ such that every perfect matching of G_a^{aut} matches r to c. Condition 3 implies that if there is at least one nonconstant plant edge in a component H_i then there exists also one $\mathcal{R}_K$ vertex in this H_i. Since each H_i is an elementary bipartite graph, each edge in H_i is in a cycle with some controller edges. (Refer Proposition 3.6.) Hence every admissible nonconstant plant edge in G^{aut} is in a cycle with controller edges.

$(4 \Rightarrow 1)$: Assume that every nonconstant plant edge which is admissible in G_a^{aut} is in a cycle involving controller edges in G_a^{aut}. Let H_i be a component in G_a^{aut} with a nonconstant plant edge and let A_i be the block in A_a corresponding to this H_i. This means H_i is an elementary bipartite graph and $\det(A_i)$ is not properly factorizable in $\mathbb{R}[X]$: see text after Proposition 5.1. Factorize the determinant of A_a into $\prod_{i=1}^{k} \det(A_i)$. From our assumption, H_i has a controller vertex. Analogously for a given $P \in \mathfrak{L}_P$, the coefficients of the determinant polynomial of A_i depend nontrivially on entries of both P and K. We need to prove that $\det(A_i)$ can be assigned arbitrary by appropriately choosing the entries corresponding to the controller edges.

Consider a component H_i with nonconstant plant edges and choose a vertex $r_k \in \mathcal{R}_K$. Suppose n controller edges are incident on r_k. Since H_i is elementary, using Proposition 3.6, we first infer that $n \geqslant 2$. Corresponding to these edges a row of A_i has the entries $c_1, \ldots, c_n$ at the appropriate positions. In case there is more than one controller vertex, then some more rows of A_i have controller edge entries. Choose these entries generically from $\mathbb{R}$. Expand the determinant along the row r_k to get

$$\det(A_i) = \sum_{i=1}^{n} c_i p_i$$

where polynomials p_i are defined appropriately for each term in the determinant expansion.

The next claim is that at least two polynomials p_i are nonzero and that all the nonzero polynomials are generically coprime for all $P \in \mathfrak{L}_P$ and the generically chosen real numbers for the controller edges incident on other controller vertices in H_i. We prove this claim by contradiction. Suppose they are not generically coprime, then there exists a factor $f(s)$, of degree at least one, which divides each p_i generically for all $P \in \mathfrak{L}_P$. Hence $\det(A_i) = f(s)\sum_{i=1}^{n} c_i \tilde{p}_i$. Since this is true for all polynomials p_i allowed by the plant structure, Proposition 5.1 helps conclude that $\det(A_i)$, viewed now in the notation of that proposition, admits a proper factorization. This contradicts the irreducibility of $\det(A_i)$ and hence contradicts that H_i is an elementary bipartite graph. This proves that the polynomials p_i's are generically coprime. Admissibility of all edges in H_i, along with the existence of a nonconstant plant edge, implies that at least one polynomial p_i is generically nonconstant. The coprimeness of all p_i's ensures that there exists one more polynomial p_j which is nonzero. Finally, using Bézout's identity, we conclude that c_i's can be chosen such that $\det(A_i)$ is any desired polynomial. Hence $\det(A_a)$ can also be assigned arbitrarily, thus proving Statement 1. This completes the proof of Theorem 2.6. □

6. Algorithm and running time estimates

In this section we provide algorithms comprising of standard graph tests to solve Problems 2.4 and 2.5. We also estimate the running time of these algorithms.

6.1. *Structural controllability: algorithm and running time*

This subsection focuses on structural controllability.

Algorithm 6.1. The algorithm is based on Statement 3 of Theorem 2.7.
Input: A bipartite graph $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$, $|\mathcal{R}_P| < |\mathcal{C}|$ with each edge of E_P classified as constant/nonconstant.
Output: "Structurally controllable" if so, and "Structurally uncontrollable" otherwise.

1: Remove the inadmissible edges of G^P to get G_a^P.
2: Let $A_1, A_2, \ldots, A_t$ be the connected components of G_a^P.
{Comment: Let A_i be a graph with vertex set $V(A_i)$ and edge set $E_P(A_i)$.}
{Comment: Let $\mathcal{R}_P(A_i) = \mathcal{R}_P \cap V(A_i)$ and $\mathcal{C}(A_i) = \mathcal{C} \cap V(A_i)$.}
3: **if** every component A_i with $|\mathcal{R}_P(A_i)| = |\mathcal{C}(A_i)|$ has each edge in $E_P(A_i)$ as constant edge **then**
4: **print** `"System structurally controllable"`
5: **else**
6: **print** `"System structurally uncontrollable"`
7: **end if**
8: **if** no component A_i satisfies $|\mathcal{R}_P(A_i)| = |\mathcal{C}(A_i)|$ **then**
9: **print** `"System structurally controllable"`
10: **end if**

We estimate the running time of each step of the above algorithm.

Step 1: *Removal of inadmissible edges*: Using recent results from Tassa [20], the running time for classifying the edges in G^P as admissible or inadmissible is $O(|E_P|\sqrt{|V|})$. Removal of inadmissible edges takes as much time too.

Step 2: *Decomposition of G_a^P into its connected components*: Once all inadmissible edges in G^P have been removed, the algorithm for decomposing G_a^P (with edges E_P^a and $V = \mathcal{R}_P \cup \mathcal{C}$) into its connected components can be done in $O(|E_P^a| \log^*(|\mathcal{R}_P| + |\mathcal{C}|))$ and is again standard, see Cormen et al. [6, p. 522].

Steps 3–7: *Connected component checking*: For each connected component A_i satisfying $|\mathcal{R}_P(A_i)| = |\mathcal{C}(A_i)|$, it takes $|E_P^a(A_i)|$ operations to check if all edges are constant edges. In other words, in at most $|E_P^a(A_i)|$ operations, one can determine whether A_i corresponds generically to a unimodular submatrix or not. (The components A_i satisfying $|\mathcal{R}_P(A_i)| \neq |\mathcal{C}(A_i)|$ require no further check.)

Lemma 6.2. *Consider a plant structure $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$. Let $V := \mathcal{R}_P \cup \mathcal{C}$. Then, Algorithm 6.1 takes $O(|E_P|\sqrt{|V|})$ time to check if the plant is structurally controllable or not.*

Proof. Using the steps listed above and the running time involved for each step, the total running time of the algorithm is at most $O(|E_P|\sqrt{|V|}) + O(|E_P^a| \log^*(|V|)) + O(|E_P^a|)$ which is $O(|E_P|\sqrt{|V|})$. □

6.2. *Arbitrary pole placement with structured controller: algorithm and running time*

This subsection focuses on arbitrary pole placement using structured controllers.

Algorithm 6.3. This algorithm is based on Statement 4 of Theorem 2.6.
Input: The structure of plant $G^P(\mathcal{R}_P, \mathcal{C}; E_P)$ with edges classified as constant/nonconstant and structure of controller; $G^K(\mathcal{R}_K, \mathcal{C}; E_K)$.
Output: "Arbitrary pole placement feasible" if so, and "Arbitrary pole placement infeasible" otherwise.
Let $G^{\text{aut}}(\mathcal{R}, \mathcal{C}; E)$; $\mathcal{R} := \mathcal{R}_P \cup \mathcal{R}_K$, $E := E_P \cup E_K$ denote the combined graph of plant and controller.

1: Remove the inadmissible edges of G^{aut} to get G_a^{aut}.
{Let $A_1, A_2, \ldots, A_t$ be the connected components of G_a^{aut}.}
2: **if** each component A_i containing a nonconstant plant edge contains an edge from G^K **then**
3: **print** `"Arbitrary pole placement feasible"`
4: **else**
5: **print** `"Arbitrary pole placement infeasible"`
6: **end if**

Here $V = \mathcal{R}_P \cup \mathcal{R}_K \cup \mathcal{C}$.

Step 1: *Removal of inadmissible edges in G_a^{aut}*: This step requires $O(|E|\sqrt{|V|})$ time as explained in Step 1 of Algorithm 6.1.

Step 2: *Decomposition of* G_a^{aut} *into its connected components*: This requires $O(|E^a|\log^*(|\mathcal{R}|+|\mathcal{C}|))$. This is explained in Step 2 of Algorithm 6.1.

Steps 3–6: *Check within each connected component*: By Proposition 3.7, each A_i is an elementary bipartite graph. If the number of edges in A_i is more than one, then any two edges in A_i are contained in a cycle: see Proposition 3.6. Checking Condition 4, Theorem 2.6 boils down to checking if each component A_i containing a nonconstant plant edge has a controller edge, equivalently, a vertex in $\mathcal{R}_K$. For each component this requires at most $|E^a(A_i)|$ operations.

Lemma 6.4. *Let* $G^P(\mathcal{R}_P,\mathcal{C};E_P)$ *and* $G^K(\mathcal{R}_K,\mathcal{C};E_K)$ *denote the plant and controller structures respectively. Define* $V:=\mathcal{R}_P\cup\mathcal{R}_K\cup\mathcal{C}$ *and* $E:=E_P\cup E_K$. *Then, Algorithm 6.3 takes* $O(|E|\sqrt{|V|})$ *time to check if arbitrary pole placement is possible with the controller structure or not.*

Proof. Proceeding as before, it follows that the running time of the algorithm is at most $O(|E|\sqrt{|V|})+O(|E^a|\log^*(|V|))+O(|E^a|)$ which is $O(|E|\sqrt{|V|})$. □

Note that the classification of edges into admissible and inadmissible is the most intensive of the operations within the two algorithms above. The classification (and the removal) operations are inessential for the construction of a large system by interconnection of SISO subsystems using one or more of series/parallel and/or feedback interconnections. This was described above in Section 2.3 and stated in Theorem 2.8.

We compare the above running time estimates with that of similar algorithms in the literature. In order to compare, we deal with the state space case, though our approach applies to the higher order case and also to the unimodular completion problem. Consider a state space system with n states, m inputs and p outputs and with output feedback. The running time of the algorithm in Murota [12] that checks structural controllability is $O(n^2(n+m)\log n)$. Similarly, in Papadimitriou and Tsitsiklis [16], an $O(n^{2.5})$ running time algorithm for checking structurally fixed modes has been described, while Murota [13, p. 1394] describes one that is $O(d^3\log d)$, with $d:=n+m+p$. Of course, both [12] and [13] deal with more general so-called 'mixed' representations. In our case the algorithms for both structural controllability and arbitrary pole placement require at most $O(|E|\sqrt{|V|})$ running time as removal of inadmissible edges is the dominant term among all the steps. Assuming $|E|=O(|V|^2)$, our algorithm takes $O(|V|^{2.5})$ time and is thus comparable to those in Papadimitriou and Tsitsiklis [16]. Of course, both structural controllability and generic pole assignability are more relevant for *sparse* systems for which it is reasonable to assume $|E|=o(|V|^2)$ (for example, $|E|$ could be $O(|V|\log(|V|))$ or $O(|V|^{1.5})$). In the case when $|E|=o(|V|^2)$, our algorithm requires at most $o(|V|^{2.5})$.

7. Concluding remarks

We considered the problem where the plant equations structure is given, i.e. which variable occurs in which equation is specified. This structure was translated to an equivalence class of polynomial matrices, with the zero and nonzero entries' locations specified. Amongst the nonzero entries, we distinguished between entries that are constant, and polynomials of degree at least one. Considering 'open' plants meant considering an under-determined system of plant equations, i.e. a *rectangular* polynomial matrix. We studied structural controllability of the plant as ability to generically complete a polynomial matrix from this equivalence class to a unimodular matrix. This amounts to potentially using nonzero entries everywhere during the completion (Theorem 2.7). Instead of using nonzero entries *everywhere*, a more refined problem is that of generic controllability using a controller with structural constraints, i.e. controller equations having prespecified constraints about which variable can occur in which equation. This is of obvious practical relevance to sensor network design problems. We obtained necessary and sufficient conditions for such generic arbitrary pole placement using a structured controller (Theorem 2.6).

Our solutions used techniques from matching theory, in particular, admissibility of edges and results from elementary bipartite graphs. Recall that an edge is called admissible if it occurs in some maximum matching. We showed that arbitrary pole placement using a structured controller was possible if and only if every nonconstant admissible plant edge occurs in some cycle involving admissible controller edges. The conditions we obtained for the absence of structurally fixed modes is a natural extension of the notion of such modes from the state space case to higher order systems.

The problem formulation and solution became easy by getting rid of inadmissible edges. This omission was justified since they do not occur in any maximum matching, equivalently, these entries do not contribute to the zero set. Absence of inadmissible edges allowed use of results about elementary bipartite graphs for solving the problems. In addition to playing a central role in our results, admissibility of an edge turned out to also be guaranteed by the three basic interconnection procedures of SISO systems: series, parallel and feedback.

We developed algorithms and obtained its running time estimates for solving Problems 2.4 and 2.5. This was easily possible due to the necessary and sufficient conditions being stated in terms of standard graph algorithms. Comparing with the state space case, for which algorithms have been estimated for their running time in the literature, our algorithm is comparable for the general case and significantly faster for the *sparse* state space system case.

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