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Inverse of the distance matrix of a block graph

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A connected graph G, whose 2-connected blocks are all cliques (of possibly varying sizes) is called a *block graph*. Let D be its distance matrix. By a theorem of Graham, Hoffman and Hosoya, we have $\det(D) \neq 0$. We give a formula for both the determinant and the inverse, D^{-1} of D.

Keywords: distance matrix; determinant; Laplacian matrix

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1. Introduction

Graham et al. [3] proved a very attractive theorem about the determinant of the distance matrix D_G of a connected graph G as a function of the distance matrix of its 2-*connected blocks*. In a connected graph, the distance between two vertices $d(u, v)$ is the length of the shortest path between them. Let A be an $n \times n$ matrix. Recall that for $1 \leq i, j \leq n$, the cofactor $c_{i,j}$ is defined as $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j of A. For a matrix A, let $\#(A) = \sum_{i,j} c_{i,j}$ be the sum of its cofactors. Graham et al. [3] showed the following theorem.

THEOREM 1 *If G is a connected graph with* 2-*connected blocks* $G_1, G_2, \ldots, G_r$, *then* $\#(D_G) = \prod_{i=1}^r \#(D_{G_i})$ *and* $\det(D_G) = \sum_{i=1}^r \det(D_{G_i}) \prod_{j \neq i} \#(D_{G_j})$.

Let D be the distance matrix of a connected graph, all of whose blocks are cliques. Such graphs are called *block graphs* in [2] and let G_i denote the blocks of G (for $1 \leq i \leq r$). See Figure 1 for an example.

Further, we give a formula for $\det(D)$ for the distance matrix D of a block graph G in terms of its block sizes and n, its number of vertices.

From the formula it will be clear that $\det(D) \neq 0$. Hence, we are interested in finding D^{-1}. For the case when all blocks are K_2's (i.e. the graph G is a tree) it is known [1,4] that $D^{-1} = -\frac{L}{2} + \frac{1}{2(n-1)} \tau\tau^t$, where L is the Laplacian matrix of G and τ is the $n \times 1$ column vector with $\tau_v = 2 - \deg_v$. Similarly, it is known that when all blocks are K_3's [5], we have $D^{-1} = -\frac{L}{3} + \frac{1}{3(n-1)} \mu\mu^t$ where L is again the Laplacian of G and

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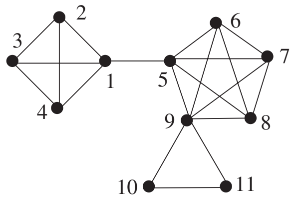

Figure 1. An example of a block graph.

μ is the column vector with $\mu_v = 3 - \deg_v$. Thus, D^{-1} is a constant times L plus a multiple of a rank one matrix. We show a similar statement for block graphs.

2. Determinant and inverse of *D*

Let G be a block graph on n vertices with blocks G_i, $1 \le i \le r$, where each G_i is a p_i-clique. Denote by λ_G the non-zero constant

$$\lambda_G = \sum_{i=1}^{r} \frac{p_i - 1}{p_i}. \tag{1}$$

The following theorem is easily derived from Theorem 1.

THEOREM 2 *Let G be a block graph on n vertices with blocks G_i, $1 \le i \le r$, where each G_i is a p_i-clique. Let D be its distance matrix. Then,* $\det(D) = (-1)^{n-1} \lambda_G \prod_{j=1}^{r} p_j$.

Proof As each D_{G_i} is the matrix $J - I$ where J is the all ones matrix and I is the identity matrix of dimension $p_i \times p_i$, it is easy to see that $\det(D_{G_i}) = (-1)^{p_i - 1}(p_i - 1)$ and $\#(D_{G_i}) = (-1)^{p_i - 1} p_i$ (the $\#(D_{G_i})$ calculation is immediate if we use [3, Lemma 1]).

Since $\sum_{i=1}^{r} p_i = n + r - 1$, the equality of the theorem follows from Theorem 1. ■

For a block graph G, consider the $|V(G)|$-dimensional column vector β defined as follows. Let a vertex $v \in V$ be in $k \ge 1$ cliques of sizes $p_1, p_2, \ldots, p_k$ (where each $p_i > 1$). Let

$$\beta_v = \left[\sum_{i=1}^{k} \frac{1}{p_i} \right] - (k - 1). \tag{2}$$

For the block graph given in Figure 1, we have $\lambda_G = \frac{163}{60}$, and $\beta^t = (-1/4, 1/4, 1/4, 1/4, -3/10, 1/5, 1/5, 1/5, -7/15, 1/3, 1/3)$.

LEMMA 1 *Let G be a block graph and let β be the vector defined above. Then,* $\sum_{v \in V(G)} \beta_v = 1$.

Proof By induction on b, we have the number of blocks of G, with the case $b = 1$ being clear. When G has more than one block, let H be any leaf block (i.e. a block whose deletion does not disconnect G) connected through cut-vertex c. Clearly, a leaf block H exists and let $F = G - \{H - c\}$ be the smaller graph obtained by deleting $H - c$ from G. Let H be a p-clique (i.e. $H = K_p$). By induction, for the graph F, we know $S = \sum_{v \in V(F)} \beta_v = 1$. It is simple to note that when we move to G from F, the vector β is different from that for F only for the vertices of H. The change in β for G is easily seen to be $(1/p-1)$ for c and $1/p$ for the other $p-1$ vertices of H. Thus the sum of the changes is zero, completing the proof. ■

LEMMA 2 *Let D be the distance matrix of a block graph G. Let $|V|=n$ and β be the vector defined by Equation* (2). *Let $\mathbb{1}$ be the n-dimensional vector with all components equal to* 1. *Then $D\beta=\lambda_G\mathbb{1}$, where λ_G is as given in Equation* (1).

Proof We again induct on b, the number of blocks of G with the case $b=1$ being simple. Delete a leaf block H connected to G through c and let $F=G-\{H-c\}$. Let H be a p-clique and let D_F be the distance matrix of F. Let $\mathbb{1}_F$ be the vector $\mathbb{1}$ restricted to vertices of F. Let α be the column of D_F corresponding to the vertex c. The v-th component of α is $\alpha_v=d_{v,c}$ where $d_{u,v}$ is the distance between vertices $u,v\in F$. It is simple to note that

$$D=\begin{pmatrix} D_F & \alpha+\mathbb{1}_F & \cdots & \alpha+\mathbb{1}_F \\ (\alpha+\mathbb{1}_F)^t & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha+\mathbb{1}_F)^t & 1 & \cdots & 0 \end{pmatrix}.$$

If β_F is the restriction of β to F, then by induction we have $D_F\beta_F=\lambda_F\mathbb{1}_F$. Here λ_F is the vector λ as in Equation (1) for the graph F. Let $t=D\beta$ and for $v\in F$ and let $R_v(D_F)$ be the v-th row of D_F. For vectors a, b with identical dimension, $\langle a,b\rangle$ denotes the usual (real) inner product of two vectors. For a vertex $v\in F$, the v-th component of t is $t_v=\langle R_v(D_F),\beta_F\rangle+(\frac{1}{p}-1)\alpha_v+(\alpha_v+1)\cdot\frac{p-1}{p}$. Hence, $t_v=\lambda_F+\frac{p-1}{p}$. Thus for all vertices in F, we have $\lambda_G=\lambda_F+\frac{p-1}{p}$. For vertices $u\in H-\{c\}$, we have

$$\begin{aligned} t_u &= \langle(\alpha+\mathbb{1}_F)^t,\beta_F\rangle+\frac{1}{p}-1+\frac{p-2}{p} \\ &= \lambda_F+\sum_{v\in F}(\beta_F)_v-1+\frac{p-1}{p}=\lambda_F+\frac{p-1}{p}, \end{aligned}$$

where in the first line we have used the fact that $\alpha_c=0$ and in the second line we have used Lemma 1. Thus, for all vertices $u\in V(G)$, $t_u=\lambda_F+\frac{p-1}{p}$. Since $\lambda_G=\lambda_F+\frac{p-1}{p}$, the proof is complete. ■

Let G have vertex set $[n]$ and blocks G_i where $1\le i\le r$. Each G_i is also considered as a graph on $[n]$ with perhaps isolated vertices and let its edge set be E_i (i.e. G_i is a clique on say p_i vertices, but consider it as a graph on $[n]$). Let L_i be the Laplacian of $G_i=([n],E_i)$. Let I be the $|V|\times|V|$ identity matrix. Define

$$\hat{L}=\sum_{i=1}^{r}\frac{1}{p_i}L_i.$$

LEMMA 3 *With the above notation, $\hat{L}D+I=\beta\mathbb{1}^t$.*

Proof We again induct on b, the number of blocks of G with the base case $b=1$ being simple. Let H, F, c be as in the proof of Lemma 1 and let H be a p-clique. Let $\hat{L}_F$ be the combination of the Laplacian as before, but only for the blocks of F and let D_F be the distance matrix of F. Similarly, let I_F be the identity matrix of order $|F|\times|F|$.

Let e_c be the $|F|$-dimensional column vector with a 1 in position c and zero elsewhere and let $\alpha = D_F e_c$. Let $\mathbb{1}_{H-c}$ be a $(|H|-1)$-dimensional all ones column vector. H is a leaf-block, but considering it as a graph on $[n]$, let its Laplacian be denoted by L_H. Let L_{H-c} be L_H restricted to the set of vertices $V(H)-\{c\}$ and D_{H-c} the distance matrix of G, restricted to the set of vertices $V(H)-\{c\}$. We clearly have

$$\hat{L} = \begin{pmatrix} \hat{L}_F + \dfrac{p-1}{p}(e_c \times e_c^t) & \dfrac{-1}{p}\cdot(e_c \times \mathbb{1}_{H-c}^t) \\ \dfrac{-1}{p}\cdot(\mathbb{1}_{H-c} \times e_c^t) & \dfrac{1}{p}L_{H-c} \end{pmatrix},$$

$$D = \begin{pmatrix} D_F & (\alpha + \mathbb{1}_F) \times (\mathbb{1}_H)^t \\ \mathbb{1}_H \times (\alpha + \mathbb{1}_F)^t & D_{H-c} \end{pmatrix}.$$

We need to show that for all i, j, $(\hat{L}D + I)_{i,j} = \beta_i$.

For rows $i \in F - \{c\}$: For such a row i and for columns $j \in F$, we have by induction, $\hat{L}_F D_F + I_F = (\beta_F)_i$. We denote the i-th row (and j-th column) of matrix M as $R_i(M)$ (and $C_j(M)$, respectively). Since $\beta_i = (\beta_F)_i$ for $i \in F - c$, we are done for all columns in F. For columns $j \in H - \{c\}$, we note that $(\hat{L}D + I)_{i,j} = \langle R_i(\hat{L}_F), \alpha + \mathbb{1}_F\rangle$. Since the row-sum of a linear combination of Laplacians is zero, $\langle R_i(\hat{L}_F), \mathbb{1}_F\rangle = 0$. Thus $(\hat{L}D + I)_{i,j} = (\hat{L}D + I)_{i,c} = (\beta_F)_i = \beta_i$.

For rows $i \in H - \{c\}$: For such rows i, it is easy to see that $\beta_i = \frac{1}{p}$. For all columns $j \in H$, since the entries $\hat{L}_{i,x} \neq 0$ only if $x \in H$, it is simple to see that $(\hat{L}D + I)_{i,j} = \frac{1}{p}$. In the above result, in case $i = j$, since the diagonal entry $D_{i,i} = 0$, we get a 1 from the identity matrix to get $(\hat{L}D + I)_{i,j} = 1 + (p-1)\frac{-1}{p} = \frac{1}{p}$. For columns $j \notin H$, using the matrices L_{H-c} and D_{H-c}, we see that $(\hat{L}D + I)_{i,j} = \langle R_i(L_{H-c}), C_j(D_{H-c})\rangle$. Since $C_j(D_{H-c}) = C_c(D_{H-c}) + d_{j,c}\mathbb{1}_H$ and since the column sum of a Laplacian is zero, we get $(\hat{L}D + I)_{i,j} = (\hat{L}D + I)_{i,c} = \frac{1}{p} = \beta_i$.

For the row c: We need to show that for any column $v \neq c$, $\langle R_c(\hat{L}), C_v(D)\rangle = \beta_c$ and that for column c, $\langle R_c(\hat{L}), C_c(D)\rangle + 1 = \beta_c$. We first show that $\langle R_c(\hat{L}), C_c(D)\rangle + 1 = \beta_c$. By induction, we know that $\langle R_c(\hat{L}_F), C_c(D_F)\rangle + 1 = (\beta_F)_c$. Since $d_{c,c} = 0$, the required product is easily seen to be $\langle R_c(\hat{L}_F), C_c(D_F)\rangle + \frac{-(p-1)}{p}$, which is β_c. We now show for $v \neq c$, $\langle R_c(\hat{L}), C_v(D)\rangle = \beta_c$. First, consider columns $v \in F - \{c\}$. By induction, we know that $\langle R_c(\hat{L}_F), C_v(D_F)\rangle = (\beta_F)_c$. Since $\hat{L}_{c,c} = (\hat{L}_F)_{c,c} + \frac{p-1}{p}$ and for all $u \in H$, $\hat{L}_{c,u} = \frac{-1}{p}$, we get $\langle R_c(\hat{L}), C_v(D)\rangle = (\beta_F)_c + d_{v,c}\cdot 0 - \frac{p-1}{p}$. Since $\beta_c = (\beta_F)_c + \frac{1-p}{p}$, we are done. Next consider columns $v \in H - \{c\}$. We have just shown that $\langle R_c(\hat{L}), C_c(D)\rangle = \beta_c - 1$. The column vectors $C_v(D)$ and $C_c(D)$ only differ in the entries corresponding to row c and v, when restricted to rows in H and differ for all entries in F: each entry of $C_v(D)$ is larger than the corresponding entry of $C_c(D)$ by 1. Since a linear combination of the Laplacian has zero row-sum, we have $\langle R_c(\hat{L}), C_v(D)\rangle = \langle R_c(\hat{L}), C_c(D)\rangle + \frac{p-1}{p} + \frac{1}{p}$, where the term $\frac{p-1}{p}$ arises as $\hat{L}_{c,c} = (\hat{L}_F)_{c,c} + \frac{p-1}{p}$ and $D_{c,v} = 1$ and the term $\frac{1}{p}$ arises as $\hat{L}_{v,c}\cdot D_{c,c} = -\frac{1}{p}$ is to be subtracted from $\langle R_c(\hat{L}), C_c(D)\rangle$. Thus, we have $\langle R_c(\hat{L}), C_v(D)\rangle = \beta_c - 1 + \frac{p-1}{p} + \frac{1}{p} = \beta_c$, completing the proof. ■

THEOREM 3 *Let $\hat{L}, D, \lambda_G$ and β be as above. Then $D^{-1} = -\hat{L} + \frac{1}{\lambda_G}\beta\beta^t$.*

Proof By Lemma 3, we see that $\hat{L}D + I = \beta\mathbb{1}^t$. By Lemma 2, we get $\beta^t D = \lambda_G \mathbb{1}^t$ or $\beta\beta^t D = \lambda_G \beta\mathbb{1}^t$ where clearly $\lambda_G \neq 0$. Thus, $\hat{L}D + I = \frac{1}{\lambda_G}\beta\beta^t D$. i.e. $I = (-\hat{L} + \frac{1}{\lambda_G}\beta\beta^t)D$, completing the proof. ■

Theorem 3 says that even if all the blocks of G are arbitrary sized cliques, D^{-1} is a scalar multiple of a kind of Laplacian matrix plus a constant multiple of a rank one matrix. The following known corollaries are easily derived from Theorem 3.

COROLLARY 1 [4] *Let D be the distance matrix of a tree T on n vertices and let L be its Laplacian matrix. Let τ be the n-dimensional column vector with $\tau_u = 2 - \deg_u$, where $\deg_u$ is the degree of vertex u in T. Then $D^{-1} = \frac{-L}{2} + \frac{1}{2(n-1)}\tau\tau^t$.*

COROLLARY 2 [5] *Let D be the distance matrix of a graph G on n vertices, all of whose blocks are K_3's and let L be its Laplacian matrix. Let μ be the n-dimensional column vector with $\mu_u = 3 - \deg_u$, where $\deg_u$ is the degree of vertex u in G. Then $D^{-1} = \frac{-L}{3} + \frac{1}{3(n-1)}\mu\mu^t$.*

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