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Inequalities among two rowed immanants of the q-Laplacian of trees and odd height peaks in generalized Dyck paths

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ABSTRACT

Let T be a tree on n vertices and let $\mathcal{L}_q^T$ be the q-analogue of its Laplacian. For a partition $\lambda \vdash n$, let the normalized immanant of $\mathcal{L}_q^T$ indexed by λ be denoted as $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$. A string of inequalities among $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$ is known when λ varies over hook partitions of n as the size of the first part of λ decreases. In this work, we show a similar sequence of inequalities when λ varies over two row partitions of n as the size of the first part of λ decreases. Our main lemma is an identity involving binomial coefficients and irreducible character values of $\mathfrak{S}_n$ indexed by two row partitions. Our proof can be interpreted using the combinatorics of Riordan paths and our main lemma admits a nice probabilisitic interpretation involving peaks at odd heights in generalized Dyck paths or equivalently involving special descents in Standard Young Tableaux with two rows. As a corollary, we also get inequalities between $\overline{\mathrm{Imm}}_{\lambda_1}(\mathcal{L}_q^{T_1})$ and $\overline{\mathrm{Imm}}_{\lambda_2}(\mathcal{L}_q^{T_2})$ when T_1 and T_2 are comparable trees in the GTS_n poset and when λ_1 and λ_2 are both two rowed partitions of n, with λ_1 having a larger first part than λ_2.

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1. Introduction

Immanants of positive semidefinite matrices and their generalizations have been a topic of interest since Schur [15]. Let $A = (a_{i,j})_{1\le i,j\le n}$ be an $n \times n$ matrix. For a partition $\lambda \vdash n$, let χ_λ denote the character of the irreducible representation of the symmetric group $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by λ. We think of $\chi_\lambda : \mathfrak{S}_n \mapsto \mathbb{C}$ as a function from $\mathfrak{S}_n$ to $\mathbb{C}$ and for $\pi \in \mathfrak{S}_n$ let $\chi_\lambda(\pi)$ denote the value of χ_λ on the permutation π. Let $\mathsf{id} \in \mathfrak{S}_n$ denote the identity permutation in $\mathfrak{S}_n$. For a partition $\lambda \vdash n$, define the normalized immanant of A as

$$\overline{\mathrm{Imm}}_\lambda(A) = \frac{1}{\chi_\lambda(\mathsf{id})} \sum_{\pi\in\mathfrak{S}_n} \chi_\lambda(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}. \tag{1}$$

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The bar in our notation for $\overline{\mathrm{Imm}}_\lambda(A)$ above indicates that a factor $\chi_\lambda(\mathrm{id})$ is used in the denominator to define the *normalized immanant*. Let T be a tree on n vertices and let $\mathcal{L}_q^T$ be the q-analogue of its Laplacian (see Section 2 for definitions). For $\lambda \vdash n$, let the normalized immanant of $\mathcal{L}_q^T$ indexed by λ be denoted as $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$. As done usually, when parts of a partition are repeated, we write such a partition with the multiplicity as an exponent of that part. Thus $\lambda = 1^n$ denotes the partition $\lambda = 1, 1, \ldots, 1$ with the part 1 having multiplicity n. Partitions of the form $\lambda = k, 1^{n-k}$ are called hook partitions and when n is clear, are denoted as hook_k. Denote the normalized immanant of $\mathcal{L}_q^T$ indexed by hook_k as $\overline{\mathrm{HookImm}}_k(\mathcal{L}_q^T)$. For any tree T with n vertices, the following sequence of inequalities involving $\overline{\mathrm{HookImm}}_k(\mathcal{L}_q^T)$'s was shown by Nagar and Sivasubramanian (see [12, Lemma 27 and Theorem 2]).

Theorem 1.1 (Nagar, Sivasubramanian): *Let T be a tree on n vertices with q-Laplacian $\mathcal{L}_q^T$. Then, for $2 \le k \le n$ and for all $q \in \mathbb{R}$, the normalized immanants of $\mathcal{L}_q^T$ satisfy the following:*

$$\overline{\mathrm{HookImm}}_{k-1}(\mathcal{L}_q^T) \le \overline{\mathrm{HookImm}}_k(\mathcal{L}_q^T). \tag{2}$$

The following version (which is stronger than (2) when $|q| > 1$) also holds

$$\overline{\mathrm{HookImm}}_{k-1}(\mathcal{L}_q^T) + \frac{q^2 - 1}{k - 1} \le \frac{k-2}{k-1}\overline{\mathrm{HookImm}}_k(\mathcal{L}_q^T).$$

In this paper, we give a counterpart of inequality (2) to immanants indexed by partitions with at most two rows. For $k \ge 0$, let $\lambda = \mathrm{TwoRow}_k$ be the partition $\lambda = n - k, k$ of n with at most two rows. Since λ is a partition, we must have $k \le \lfloor n/2 \rfloor$ and also have $k \ge 0$. Let $\overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^T)$ denote the normalized immanant of $\mathcal{L}_q^T$ indexed by TwoRow_k. The main result of this paper is the following.

Theorem 1.2: *Let T be a tree on $n \ge 5$ vertices and let $\mathcal{L}_q^T$ be its q-Laplacian. Then for $1 \le k \le \lfloor n/2 \rfloor$ and for all $q \in \mathbb{R}$, the normalized two rowed immanants of $\mathcal{L}_q^T$ satisfy the following:*

$$\overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^T) \ge \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^T).$$

Underlying the proof of Theorem 1.1 from [12], is a string of inequalities involving binomial coefficients and irreducible character values of $\mathfrak{S}_n$ indexed by hook partitions. Counterparts of several relations that are true for irreducible characters indexed by hook partitions have been found for irreducible characters indexed by two row partitions. See for example the paper by Zeilberger and Regev [17] and Bessenrodt's refinement [5]. Inspired by such parallels, we get similar inequalities involving binomial coefficients and irreducible character values indexed by two row partitions in this work (see Lemma 3.16).

Somewhat surprisingly, quantities that appear in the proof of Theorem 1.2 are intimately connected to the combinatorics of Generalized Riordan paths. We outline the background and the general strategy of our proof in Section 2 and then move on to the proof of Theorem 1.2 in Section 3.1. Our inequalities admit a probabilistic interpretation to present which, a little more background is given in Section 4. In Section 5, we give our interpretation involving peaks at odd heights in generalized Dyck paths or equivalently, involving special descents in Standard Young Tableaux with at most two rows.

A poset denoted GTS_n on the set of trees with n vertices was defined by Csikvari (see [8,9]). He showed that several tree parameters are monotonic as one goes up this GTS_n poset. Results involving this poset are usually shown for a fixed tree parameter as the tree varies on GTS_n. Using the above results, we show that one can vary both the normalized two row immanant as the size of the first part decreases (this is the tree parameter) and the tree T. In Corollary 3.18, we give comparability results about $\overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^{T_1})$ and $\overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_2})$ when T_1 and T_2 are comparable trees in GTS_n.

2. Preliminaries

For a graph G on n vertices, we need the following two $n \times n$ matrices. Let A and D denote G's adjacency matrix and the diagonal matrix with degrees on the diagonal respectively. Define the q-Laplacian of G to be $\mathcal{L}_q^G = I + (D-I)q^2 - qA$. On setting $q = 1$, we have $\mathcal{L}_1^G = D - A$ which is the usual Laplacian $L(G)$ of G. Thus, $\mathcal{L}_q^G$ is a more general matrix than the Laplacian and is termed the q-Laplacian of G. The matrix $\mathcal{L}_q^G$ has connections to the Ihara Selberg zeta function of G (see Bass [4], Foata and Zeilberger [10]). When the graph G is a tree, $\mathcal{L}_q^T$ is upto a scalar, the inverse of the exponential distance matrix ED_T of T (see Bapat, Lal and Pati [2]). Several results about $\mathcal{L}_q^T$ have then subsequently been proved, see [1,3] and the references therein. Thus, $\mathcal{L}_q^T$ is a well studied object.

Normalized immanants of the q-Laplacian $\mathcal{L}_q^T$ of a tree T can be computed using the dual and alternative notion of vertex orientations. We refer the reader to [12, Lemmas 5, 17 and Theorem 11] for an introduction to this and terms undefined here. We have not defined them again in this paper as we do not have anything new to say on them. For $\lambda \vdash n$ and $j \leq \lfloor n/2 \rfloor$, denote by $\chi_\lambda(j)$, the irreducible character χ_λ evaluated at a permutation with cycle type $2^j 1^{n-2j}$.

For a tree T, when $i \geq 1$, let $\mathcal{O}_i$ denote the set of vertex orientations with i bidirected arcs. We need the following from [12, Corollary 13]. There exists a statistic Lexaway : $\mathcal{O}_i \mapsto \mathbb{N}$ whose ordinary generating function $a_i^T(q) = \sum_{O \in \mathcal{O}_i} q^{\mathrm{Lexaway}(O)}$ will be used to compute normalized immanants of $\mathcal{L}_q^T$. In this work, we will need the tree T and so we have embedded it in our notation of $a_i^T(q)$. This was not needed in [12] and there the same quantity was denoted $a_i(q)$ as the tree T was implicit. With this slight change in notation, we recall [12, Lemma 17] as a starting point of this work.

Lemma 2.1 (Nagar and Sivasubramanian): *Let T be a tree on n vertices with q-Laplacian $\mathcal{L}_q^T$. For $\lambda \vdash n$, let $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$ denote the normalized immanants of $\mathcal{L}_q^T$ indexed by λ. Then, we have*

$$\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T) = \frac{1}{\chi_\lambda(id)} \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) \left(\sum_{j=0}^{i} \binom{i}{j} \chi_\lambda(j) \right), \tag{3}$$

$$= \frac{1}{\chi_\lambda(id)} \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \alpha_{n,\lambda,i}, \tag{4}$$

where we define

$$2^i \alpha_{n,\lambda,i} = \sum_{j=0}^{i} \binom{i}{j} \chi_\lambda(j). \tag{5}$$

2.1. Two rowed immanants

Fixing the number n of vertices of T, we specialize Lemma 2.1 to the case when $\lambda =$ TwoRow$_k$ and denote the irreducible character indexed by TwoRow$_k$ as TwoRow$\chi_{n,k}$. We denote the normalized immanant of $\mathcal{L}_q^T$ corresponding to the partition TwoRow$_k$ as $\overline{\text{TwoRowImm}}_k(\mathcal{L}_q^T)$. When $\lambda =$ TwoRow$_k$, we also denote the term $\alpha_{n,\lambda,i}$ as $\alpha_{n,k,i}$ to avoid any confusion. With this notation, substituting $\lambda =$ TwoRow$_k$ in Lemma 2.1, gives us:

$$\overline{\text{TwoRowImm}}_k(\mathcal{L}_q^T) = \frac{1}{\text{TwoRow}\chi_{n,k}(\text{id})} \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \alpha_{n,k,i}. \tag{6}$$

Equation (4) and hence Equation (6) shows that one can split the computation of the normalized immanant of $\mathcal{L}_q^T$ into two parts: the first being $a_i^T(q)$ which only depends on the tree T and does not depend on the partition λ (or TwoRow$_k$). The second part is $2^i\alpha_{n,\lambda,i}$ (or $2^i\alpha_{n,k,i}$) which by Lemma 2.1, depends on the character values of λ (or TwoRow$_k$) and does not depend on the tree T.

Chan and Lam in [7] showed that $\alpha_{n,\lambda,i} \geq 0$ for all n, i and $\lambda \vdash n$. As mentioned above, when $i \geq 1$, the $a_i^T(q)$'s are ordinary generating functions of the statistic Lexaway and hence have non negative integral coefficients. Further, by [12, Lemma 16] when $i \geq 1$, $a_i^T(q)$ is actually a polynomial in q^2 with non negative coefficients while $a_0(q) = 1 - q^2$ for all trees (see [12, Corollary 13]). We wish to compare $\overline{\text{TwoRowImm}}_k(\mathcal{L}_q^T)$ with $\overline{\text{TwoRowImm}}_{k+1}(\mathcal{L}_q^T)$ for the same tree T. Thus, we will analyse $\alpha_{n,k,i}$ in more detail.

3. Proof of Theorem 1.2

Our first lemma gives a recurrence between the $\alpha_{n,k,i}$'s. We adopt the convention that $\alpha_{n,k,i} = 0$ if either $i > \lfloor n/2 \rfloor$ or if $k > \lfloor n/2 \rfloor$. We also define $\alpha_{n,k,i} = 0$ when $i < 0$ or $k < 0$ so that the recursion holds.

Lemma 3.1: *For positive integers $n \geq 3$, when $0 \leq i \leq \lfloor (n-1)/2 \rfloor$ and $0 \leq k \leq \lfloor n/2 \rfloor$, we have $\alpha_{n,k,i} = \alpha_{n-1,k,i} + \alpha_{n-1,k-1,i}$.*

Proof: When $n = 3$, the relation is easy to check. Thus, let $n \geq 4$. By definition, we have $2^i\alpha_{n,k,i} = \sum_{j=0}^{i} \text{TwoRow}\chi_{n,k}(j)\binom{i}{j}$. By the Murnaghan-Nakayama lemma (see Sagan's book [14, Theorem 4.10.2]), for $0 \leq j \leq \lfloor (n-1)/2 \rfloor$ we have TwoRow$\chi_{n,k}(j) =$

$\mathrm{TwoRow}\chi_{n-1,k}(j) + \mathrm{TwoRow}\chi_{n-1,k-1}(j)$. Thus, we get

$$\begin{aligned}\alpha_{n,k,i} &= \frac{1}{2^i}\sum_{j=0}^{i} \mathrm{TwoRow}\chi_{n,k}(j)\binom{i}{j} \\ &= \frac{1}{2^i}\sum_{j=0}^{i} \left[\mathrm{TwoRow}\chi_{n-1,k}(j) + \mathrm{TwoRow}\chi_{n-1,k-1}(j)\right]\binom{i}{j} \\ &= \alpha_{n-1,k,i} + \alpha_{n-1,k-1,i}.\end{aligned}$$

The proof is complete. ■

Example 3.2: We illustrate Lemma 3.1 when $n = 6, 7, 8$ below, where we show the tables containing $\alpha_{n,k,i}$'s.

	$\lambda = 6$	$\lambda = 5,1$	$\lambda = 4,2$	$\lambda = 3,3$
$i = 0$	1	5	9	5
$i = 1$	1	4	6	3
$i = 2$	1	3	4	2
$i = 3$	1	2	3	1

	$\lambda = 7$	$\lambda = 6,1$	$\lambda = 5,2$	$\lambda = 4,3$
$i = 0$	1	6	14	14
$i = 1$	1	5	10	9
$i = 2$	1	4	7	6
$i = 3$	1	3	5	4

	$\lambda = 8$	$\lambda = 7,1$	$\lambda = 6,2$	$\lambda = 5,3$	$\lambda = 4,4$
$i = 0$	1	7	20	28	14
$i = 1$	1	6	15	19	9
$i = 2$	1	5	11	13	6
$i = 3$	1	4	8	9	4
$i = 4$	1	3	6	6	3

We have coloured the cells to illustrate Lemma 3.1. The coloured cell in the table when $n = 8$ is the sum of the two identically coloured cells in the table when $n = 7$ and similarly each cell can be computed recursively. When $n = 8$, $k = 4$ and $0 \leq i \leq \lfloor (n-1)/2 \rfloor$, we note that $\alpha_{8,4,i} = \alpha_{7,3,i}$, though this is illustrated only when $i = 3$. Here, instead of $\alpha_{n,k,i}$ being a sum of two entries, we only have one entry as by our convention, we have $\alpha_{n,k,i} = 0$ if $k > \lfloor n/2 \rfloor$.

Since $0 \leq i \leq \lfloor n/2 \rfloor$, we get one extra row (that is, one extra i) and one extra column (that is, one extra k), for each even n as n increases. When $n = 2\ell$ is even, we call this row corresponding to $i = \ell$ as the *last row* and the column corresponding to $k = \ell$ as the *last column*. As can be seen from Example 3.2, when $n = 2\ell$, the *last row* cannot be obtained as a sum of rows when $n = 2\ell - 1$. When $n = 2\ell$, as can also be seen from Lemma 3.1 and Example 3.2, there is no problem in recursively obtaining the last column from the last column corresponding to $n-1$.

Remark 3.3: In (5), when $\lambda = \text{TwoRow}_k$, the dimension of the irreducible representation indexed by TwoRow_k equals $\alpha_{n,k,0}$. That is, $\alpha_{n,k,0} = \text{TwoRow}\chi_{n,k}(\mathsf{id})$. Further, by the Hook-length formula (see Sagan [14]) the dimension of the irreducible representation indexed by TwoRow_k is $\alpha_{n,k,0} = \binom{n}{k} - \binom{n}{k-1} > 0$.

To prove Theorem 1.2, we write both $\overline{\text{TwoRowImm}}_k(\mathcal{L}_q^T)$ and $\overline{\text{TwoRowImm}}_{k+1}(\mathcal{L}_q^T)$ using (6). As the terms $2^i a_i^T(q)$ are common within the summation, using Remark 3.3, as a first attempt, we want to show that

$$\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \geq \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}}. \tag{7}$$

If this holds, then we can show an inequality for each term of the summation in (6). We check with our data for $n = 6$ and tabulate the ratio $\frac{\alpha_{6,k,i}}{\alpha_{6,k,0}}$. Remark 3.3 tells us that $\text{TwoRow}\chi_{n,k}(\mathsf{id})$ is given by the first row (shown in red colour, seen better on a colour monitor). Dividing, we get the following table of ratios.

	$\lambda = 6$	$\lambda = 5,1$	$\lambda = 4,2$	$\lambda = 3,3$
$i = 0$	1/1	5/5	9/9	5/5
$i = 1$	1	4/5	6/9	3/5
$i = 2$	1	3/5	4/9	2/5
$i = 3$	1	2/5	3/9	1/5

As the data seems to agree with (7), we will prove this. We show inequality (7) when $n = 2\ell$ and $i = \ell$ in the next subsection.

3.1. When $n = 2\ell$ and $i = \ell$

Our first lemma shows that the $\alpha_{2\ell,k,\ell}$'s are differences of successive trinomial coefficients. Let $p_{\ell,k}$ denote the coefficient of x^k in $(1 + x + x^2)^\ell$ and to save one subscript, let $\text{last}_{\ell,k} = \alpha_{2\ell,k,\ell}$. Thus, $\text{last}_{\ell,k} = 0$ when $k > \ell$ and when $k < 0$. We need the following result (see [7, Lemma 2.1]) of Chan and Lam before we state our first lemma.

Lemma 3.4 (Chan and Lam): *Fix positive integers n, k, i with $k, i \leq \lfloor n/2 \rfloor$. Then, the following relation among the $\alpha_{n,k,i}$ holds.*

(1) *For a positive integer ℓ, if $n \neq 2\ell$ or if $n = 2\ell$ and $k \neq \ell$, then $\alpha_{n,k,i} = \alpha_{n-2,k,i-1} + \alpha_{n-2,k-1,i-1} + \alpha_{n-2,k-2,i-1}$.*
(2) *For a positive integer ℓ, when $n = 2\ell$ and $k = \ell$, then $\alpha_{2\ell,\ell,i} = \alpha_{2\ell-2,\ell-2,i-1}$*

Consequently, for fixed positive integers ℓ, k with $\ell \geq 2$, we have

(1) $\text{last}_{\ell,\ell} = \text{last}_{\ell-1,\ell-2}$
(2) $\text{last}_{\ell,k} = \text{last}_{\ell-1,k} + \text{last}_{\ell-1,k-1} + \text{last}_{\ell-1,k-2}$ *when* $0 \leq k < \ell$.

Lemma 3.5: *Fix positive integers ℓ, k with $\ell \geq 2$ and with $0 \leq k \leq \ell$. Then, we have*

(1) $p_{\ell,k} = p_{\ell-1,k} + p_{\ell-1,k-1} + p_{\ell-1,k-2}$.

Table 1. The values of $\text{last}_{\ell,k}$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$\ell=7$	1	6	21	49	84	105	91	36		
$\ell=8$	1	7	28	76	154	238	280	232	91	
$\ell=9$	1	8	36	111	258	468	672	750	603	232

(2) $\alpha_{2\ell,k,\ell} = \text{last}_{\ell,k} = p_{\ell,k} - p_{\ell,k-1}$ *is a positive integer.*

Proof: (1) By the definition of $p_{\ell,k}$, we have

$$\begin{aligned} p_{\ell,k} &= \text{Coeff. of } x^k \text{ in } (1+x+x^2)^\ell \\ &= \text{Coeff. of } x^k \text{ in } (1+x+x^2)^{\ell-1}(1+x+x^2) = p_{\ell-1,k} + p_{\ell-1,k-1} + p_{\ell-1,k-2}. \end{aligned}$$

(2) We induct on ℓ. When $\ell = 2, 3$, it is easy to see that $\text{last}_{\ell,k} = p_{\ell,k} - p_{\ell,k-1}$ is a positive integer (also see Table 1). We first consider the case when $k < \ell$. Let the result be true for all positive integers less than ℓ. From part (1) and Lemma 3.4 we have

$$\begin{aligned} \text{last}_{\ell,k} &= \text{last}_{\ell-1,k} + \text{last}_{\ell-1,k-1} + \text{last}_{\ell-1,k-2} \\ &= p_{\ell-1,k} - p_{\ell-1,k-1} + p_{\ell-1,k-1} - p_{\ell-1,k-2} + p_{\ell-1,k-2} - p_{\ell-1,k-3} \\ &= p_{\ell-1,k} + p_{\ell-1,k-1} + p_{\ell-1,k-2} - p_{\ell-1,k-1} - p_{\ell-1,k-2} - p_{\ell-1,k-3} \\ &= p_{\ell,k} - p_{\ell,k-1}. \end{aligned}$$

We move on to the case when $k = \ell$. As the polynomial $(1+x+x^2)^k$ is clearly palindromic, we have $p_{k,k-1} = p_{k,k+1}$. By Lemma 3.4 we get

$$\begin{aligned} \text{last}_{\ell,\ell} &= \text{last}_{\ell-1,\ell-2} = p_{\ell-1,\ell-2} - p_{\ell-1,\ell-3} \\ &= p_{\ell-1,\ell} + p_{\ell-1,\ell-1} + p_{\ell-1,\ell-2} - p_{\ell-1,\ell-1} - p_{\ell-1,\ell-2} - p_{\ell-1,\ell-3} \\ &= p_{\ell,\ell} - p_{\ell,\ell-1} \end{aligned}$$

The proof is complete. ■

We need a couple of inequalities which we see in the next few lemmas. We need the following lemma whose proof is easy and hence omitted.

Lemma 3.6: *For non-negative real numbers a, b, c, d with $c \neq 0 \neq d$,*

$$\min\left(\frac{a}{c}, \frac{b}{d}\right) \le \frac{a+b}{c+d} \le \max\left(\frac{a}{c}, \frac{b}{d}\right).$$

Our next lemma is an extension of Lemma 3.6.

Lemma 3.7: *For $1 \le i \le 4$, let a_i, b_i be positive integers with $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \frac{a_4}{b_4}$. Then,*

$$\frac{a_1+a_2+a_3}{b_1+b_2+b_3} \le \frac{a_2+a_3+a_4}{b_2+b_3+b_4}.$$

Proof: We know $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \frac{a_4}{b_4}$. Lemma 3.6 implies that $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_2+a_3}{b_2+b_3} \le \frac{a_3}{b_3} \le \frac{a_4}{b_4}$. Applying Lemma 3.6 again, we get

$$\frac{a_1}{b_1} \le \frac{a_1+a_2+a_3}{b_1+b_2+b_3} \le \frac{a_2+a_3}{b_2+b_3} \le \frac{a_2+a_3+a_4}{b_2+b_3+b_4} \le \frac{a_4}{b_4}, \text{ completing the proof.} \qquad \blacksquare$$

We compare the ratio $\text{last}_{\ell,k+1}/\text{last}_{\ell-1,k}$ as k decreases. For this, as background, we need a small detour to Riordan numbers.

Remark 3.8: A combinatorial interpretation of $\text{last}_{\ell,k}$ as the number of Riordan paths (see Section 5.1 for more details) was shown by Callan [6]. Callan showed that what we denote as $\text{last}_{\ell,\ell}$ is the Riordan number, denoted as R_ℓ in the literature. It is well known that R_ℓ satisfies the following recurrence relation: $(\ell+2)R_{\ell+1} = 2\ell R_\ell + 3\ell R_{\ell-1}$.

We will show that when $\ell \ge 7$, the numbers R_ℓ are log-convex. That is, we have $R_\ell^2 \le R_{\ell-1}R_{\ell+1}$. We do this in a similar manner as done by Liu and Wang [11]. Consider the following quadratic equation

$$(\ell+2)\lambda^2 - 2\ell\lambda - 3\ell = 0.$$

The above equation has a unique positive root $\lambda_\ell = \frac{\ell+\sqrt{4\ell^2+6\ell}}{\ell+2}$. When $\ell \ge 3$, using Maple as done by Liu and Wang in [11, Corollary 3.3] it is simple to verify that

$$(\ell+2)\lambda_{\ell-1}\lambda_{\ell+1} - 2\ell\lambda_{\ell-1} - 3\ell \ge 0. \tag{8}$$

It can be checked that $R_6 = 15$, $R_7 = 36$, $R_8 = 91$, $R_9 = 232$ are log-convex. That is, we have $R_\ell^2 \le R_{\ell-1}R_{\ell+1}$ when $\ell = 7, 8$. By [11, Lemma 3.1] and (8), we have the following result.

Lemma 3.9: *When $\ell \ge 7$, the sequence R_ℓ of Riordan numbers is log-convex. That is, when $\ell \ge 7$ we have $\text{last}_{\ell,\ell}^2 \le \text{last}_{\ell-1,\ell-1}\text{last}_{\ell+1,\ell+1}$.*

Lemma 3.10: *Fix positive integers ℓ, k with $1 \le k \le \ell - 1$ and $\ell \ge 7$. Then,*

$$\frac{\text{last}_{\ell,k+1}}{\text{last}_{\ell-1,k}} \le \frac{\text{last}_{\ell,k}}{\text{last}_{\ell-1,k-1}}.$$

Proof: We use induction on ℓ. The result is easily verified when $\ell = 7$ (see Table 1 in Example 3.12 below). Let the result be true when $N \le \ell$ and when $1 \le k \le \ell - 1$. We then show that it holds for $\ell + 1$. Thus, we need to show that $\frac{\text{last}_{\ell+1,k+1}}{\text{last}_{\ell,k}} \le \frac{\text{last}_{\ell+1,k}}{\text{last}_{\ell,k-1}}$, where $k \le \ell$. By Lemma 3.5 and Lemma 3.4 this is equivalent to showing that

$$\frac{\text{last}_{\ell,k+1} + \text{last}_{\ell,k} + \text{last}_{\ell,k-1}}{\text{last}_{\ell-1,k} + \text{last}_{\ell-1,k-1} + \text{last}_{\ell-1,k-2}} \le \frac{\text{last}_{\ell,k} + \text{last}_{\ell,k-1} + \text{last}_{\ell,k-2}}{\text{last}_{\ell-1,k-1} + \text{last}_{\ell-1,k-2} + \text{last}_{\ell-1,k-3}},$$

where $k \le \ell - 1$ and $\frac{\text{last}_{\ell,\ell}}{\text{last}_{\ell-1,\ell-1}} \le \frac{\text{last}_{\ell,\ell-1}}{\text{last}_{\ell-1,\ell-2}}$. By the induction hypothesis, when $k \le \ell - 1$, we know that

$$\frac{\text{last}_{\ell,k+1}}{\text{last}_{\ell-1,k}} \le \frac{\text{last}_{\ell,k}}{\text{last}_{\ell-1,k-1}} \le \frac{\text{last}_{\ell,k-1}}{\text{last}_{\ell-1,k-2}} \le \frac{\text{last}_{\ell,k-2}}{\text{last}_{\ell-1,k-3}} \quad \text{and} \quad \text{last}_{\ell,\ell-1} = \text{last}_{\ell+1,\ell+1}.$$

The proof is complete by applying Lemmas 3.7 and 3.9. $\blacksquare$

By Lemma 3.5, $\mathrm{last}_{\ell,k}$ is a positive integer. Rearranging terms a bit, as an immediate consequence of Lemma 3.10, we obtain the following corollary.

Corollary 3.11: *Fix positive integers ℓ, k with $1 \le k \le \ell - 1$ and with $\ell \ge 7$. Then, we have*

$$\frac{\mathrm{last}_{\ell,k+1}}{\mathrm{last}_{\ell,k}} \le \frac{\mathrm{last}_{\ell-1,k}}{\mathrm{last}_{\ell-1,k-1}}.$$

In general, we have

$$\frac{\mathrm{last}_{\ell,k+1}}{\mathrm{last}_{\ell,k}} \le \frac{\mathrm{last}_{\ell-r,k+1-r}}{\mathrm{last}_{\ell-r,k-r}}, \quad \textit{when } 1 \le r \le k.$$

Example 3.12: We illustrate Lemma 3.10 and Corollary 3.11 when $7 \le \ell \le 9$ and $k \in \{0, 1, \ldots, 9\}$ in Table 1 which contains $\mathrm{last}_{\ell,k}$'s.

With this preparation, we can prove the following result.

Lemma 3.13: *Fix positive integers ℓ, k with $1 \le k \le \ell - 1$ and with $\ell \ge 7$. Then, we have*

$$\frac{\mathrm{last}_{\ell,k+1}}{\binom{2\ell}{k+1} - \binom{2\ell}{k}} \le \frac{\mathrm{last}_{\ell,k}}{\binom{2\ell}{k} - \binom{2\ell}{k-1}}. \tag{9}$$

Proof: Note that $\binom{2\ell}{k}$ equals the coefficient of x^k in the expansion of

$$(1+x)^{2\ell} = (1 + x + x^2 + x)^{\ell} = \sum_{r=0}^{\ell} \binom{\ell}{r} x^r (1 + x + x^2)^{\ell - r}.$$

Thus, $\binom{2\ell}{k} = \sum_{r=0}^{\ell} \binom{\ell}{r} p_{\ell-r,k-r}$, and hence

$$\begin{aligned}\binom{2\ell}{k+1} - \binom{2\ell}{k} &= \sum_{r=0}^{\ell} \binom{\ell}{r} p_{\ell-r,k+1-r} - \sum_{r=0}^{\ell} \binom{\ell}{r} p_{\ell-r,k-r} \\ &= \sum_{r=0}^{\ell} \binom{\ell}{r} \mathrm{last}_{\ell-r,k+1-r} = \mathrm{last}_{\ell,k+1} + \sum_{r=1}^{\ell} \binom{\ell}{r} \mathrm{last}_{\ell-r,k+1-r}.\end{aligned}$$

Hence,

$$\frac{\binom{2\ell}{k+1} - \binom{2\ell}{k}}{\mathrm{last}_{\ell,k+1}} = 1 + \sum_{r=1}^{\ell} \binom{\ell}{r} \frac{\mathrm{last}_{\ell-r,k+1-r}}{\mathrm{last}_{\ell,k+1}}. \tag{10}$$

Similarly,

$$\frac{\binom{2\ell}{k} - \binom{2\ell}{k-1}}{\mathrm{last}_{\ell,k}} = 1 + \sum_{r=1}^{\ell} \binom{\ell}{r} \frac{\mathrm{last}_{\ell-r,k-r}}{\mathrm{last}_{\ell,k}}. \tag{11}$$

Thus, to obtain our required result, we need to show that

$$\frac{\mathrm{last}_{\ell-r,k+1-r}}{\mathrm{last}_{\ell,k+1}} \ge \frac{\mathrm{last}_{\ell-r,k-r}}{\mathrm{last}_{\ell,k}} \Leftrightarrow \frac{\mathrm{last}_{\ell,k+1}}{\mathrm{last}_{\ell,k}} \le \frac{\mathrm{last}_{\ell-r,k+1-r}}{\mathrm{last}_{\ell-r,k-r}}.$$

By Corollary 3.11, $\frac{\mathrm{last}_{\ell,k+1}}{\mathrm{last}_{\ell,k}} \le \frac{\mathrm{last}_{n-r,k+1-r}}{\mathrm{last}_{n-r,k-r}}$ for $1 \le r \le k$, completing the proof. ■

More generally, for positive integers r, s, let $a_{\ell,k,s}$ and $a_{\ell,k,r+s}$ denote the coefficient of x^k in $(1+sx+x^2)^\ell$ and in $(1+(r+s)x+x^2)^\ell$ respectively. The proof of Lemma 3.13 actually shows the following more general result. Since the proof is identical, we only mention its statement.

Remark 3.14: With the above notation for $a_{\ell,k,s}$, the following inequality is true for all positive integers ℓ and all k.

$$\frac{a_{\ell,k+1,s}-a_{\ell,k,s}}{a_{\ell,k+1,r+s}-a_{\ell,k,r+s}} \geq \frac{a_{\ell,k,s}-a_{\ell,k-1,s}}{a_{\ell,k,r+s}-a_{\ell,k-1,r+s}}. \tag{12}$$

Indeed, Lemma 3.13 is a special case of this result with $r=s=1$.

Note that Lemma 3.13 proves inequality (7) when $n=2\ell$ and $k=\ell$. With this preparation, we can now show that (7) holds.

Lemma 3.15: *For positive integers $n \geq 5$ and integers $k \leq \lfloor n/2 \rfloor$, $i \leq \lfloor n/2 \rfloor$, we have*

$$\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \geq \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}} \tag{13}$$

Proof: We use induction on n. We will separately show (13) when $5 \leq n \leq 13$ and when $n \geq 14$. When $4 \leq n \leq 14$, we have tabulated the data $\alpha_{n,k,i}/\alpha_{n,k,0}$ in the Appendix. When $n=14$ (this corresponds to $\ell=7$), the statement can be easily checked (see the Appendix). The case when $n=2\ell$ and $i=\ell$ follows from Lemma 3.13. For $n+1$ with $k \leq \lfloor (n+1)/2 \rfloor$ and $i < \lfloor (n+1)/2 \rfloor$, we will show that

$$\frac{\alpha_{n+1,k,i}}{\alpha_{n+1,k,0}} \geq \frac{\alpha_{n+1,k+1,i}}{\alpha_{n+1,k+1,0}}.$$

By Lemma 3.1, we need to show

$$\frac{\alpha_{n,k,i}+\alpha_{n,k-1,i}}{\alpha_{n,k,0}+\alpha_{n,k-1,0}} \geq \frac{\alpha_{n,k+1,i}+\alpha_{n,k,i}}{\alpha_{n,k+1,0}+\alpha_{n,k,0}}.$$

As $\alpha_{n,k,i} \geq 0$, using Lemma 3.6 we have

$$\frac{\alpha_{n,k,i}+\alpha_{n,k-1,i}}{\alpha_{n,k,0}+\alpha_{n,k-1,0}} \geq \min\left(\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}}, \frac{\alpha_{n,k-1,i}}{\alpha_{n,k-1,0}}\right) = \frac{\alpha_{n,k,i}}{\alpha_{n,k,0}}. \tag{14}$$

By induction, as $\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \geq \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}}$, we also get

$$\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} = \max\left(\frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}}, \frac{\alpha_{n,k,i}}{\alpha_{n,k,0}}\right) \geq \frac{\alpha_{n,k+1,i}+\alpha_{n,k,i}}{\alpha_{n,k+1,0}+\alpha_{n,k,0}}. \tag{15}$$

Equations (14) and (15) imply that

$$\frac{\alpha_{n+1,k,i}}{\alpha_{n+1,k,0}} = \frac{\alpha_{n,k-1,i}+\alpha_{n,k,i}}{\alpha_{n,k-1,0}+\alpha_{n,k,0}} \geq \frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \geq \frac{\alpha_{n,k,i}+\alpha_{n,k+1,i}}{\alpha_{n,k,0}+\alpha_{n,k+1,0}} = \frac{\alpha_{n+1,k+1,i}}{\alpha_{n+1,k+1,0}}. \tag{16}$$

The proof is complete. ∎

As Lemma 3.13 extends Lemma 3.15 to the case when $n = 2\ell$ and $i = \ell$, we record this formally below.

Lemma 3.16: *With $\alpha_{n,k,i}$ as defined in* (5), *for positive integers $n \geq 5$ and integers $k, i \leq \lfloor n/2 \rfloor$, we have*

$$\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \geq \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}}. \tag{17}$$

Proof: We again have two separate cases: when $5 \leq n \leq 13$ and when $n \geq 14$. When $5 \leq n \leq 13$, the result follows from Lemma 3.15. When $n \geq 14$, we induct on n with the base case being $n = 14$. When $n = 14$, the inequality is easy to verify and thus we can assume $n \geq 15$. If n is odd, then by Lemma 3.15, we are done. If $n = 2\ell$ is even, then Lemma 3.15 shows the inequality for i from 0 to $\ell - 1$. Lemma 3.13 shows the inequality when $i = \ell$, completing the proof. ∎

Remark 3.17: We mention our reason as to why Lemma 3.16 requires $n \geq 5$. On $n = 4$ vertices, there are two trees: the path tree P_4 and the star tree S_4. It is very easy to check that Theorem 1.2 is true for S_4. However, when $T = P_4$, we have $\overline{\text{TwoRowImm}_1}(\mathcal{L}_q^T) = 1 + 3q^2 + \frac{4}{3}q^4$ and $\overline{\text{TwoRowImm}_2}(\mathcal{L}_q^T) = 1 + 2q^2 + 2q^4$. Hence, when $|q|$ is sufficiently large, the inequality given in Theorem 1.2 is not true for P_4 when $k = 2$. This however is the only aberration.

We are now in a position to prove Theorem 1.2.

***Proof of Theorem* 1.2.:** By (6) and Remark 3.3 we have,

$$\overline{\text{TwoRowImm}_k}(\mathcal{L}_q^T) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} \text{ and}$$

$$\overline{\text{TwoRowImm}_{k+1}}(\mathcal{L}_q^T) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}}. \text{ Thus,}$$

$$\begin{aligned} \overline{\text{TwoRowImm}_k}(\mathcal{L}_q^T) - \overline{\text{TwoRowImm}_{k+1}}(\mathcal{L}_q^T) &= \sum_{i=0}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \left(\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} - \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}} \right) \\ &= \sum_{i=1}^{\lfloor n/2 \rfloor} a_i^T(q) 2^i \left(\frac{\alpha_{n,k,i}}{\alpha_{n,k,0}} - \frac{\alpha_{n,k+1,i}}{\alpha_{n,k+1,0}} \right). \end{aligned}$$

As mentioned earlier, when $i \geq 1$ the polynomial $a_i^T(q)$ is a polynomial in q^2 with non negative coefficients and so the term $2^i a_i^T(q)$ is non negative for all $q \in \mathbb{R}$ and $i \geq 1$. Combining with Lemma 3.16, we get that each term in the summation is non negative, completing the proof. ∎

Recall the poset GTS_n mentioned in Section 1. This poset was defined as several optimization problems on trees attained their maximum and minumum on star trees and path trees respectively or the other way around. More than ten tree properties are known to be monotonic as one goes up the poset GTS_n (see Csikvari [8,9]). One of the monotonic properties is the absolute value of the coefficients of the characteristic polynomial of the Laplacian matrix of T. Generalizing this, Nagar and Sivasubramanian in [13, Theorem 1] showed that going up along GTS_n poset kept the absolute value of the coefficients of the q-Laplacian matrix for all $q \in \mathbb{R}$ and all immanantal polynomials indexed by $\lambda \vdash n$. In their proof, they showed that going up on GTS_n weakly decreases $a_i^T(q)$ for each i and for all $q \in \mathbb{R}$ and hence weakly decreases $\overline{\mathrm{Imm}}_\lambda(\mathcal{L}_q^T)$ for each $\lambda \vdash n$. By combining [13, Lemma 23] with Theorem 1.2 we get the following.

Corollary 3.18: *Consider the* GTS_n *poset on trees with* $n \geq 5$ *vertices. Let* T_1, T_2 *be trees with* T_2 *covering* T_1 *in* GTS_n. *Then, for all* $q \in \mathbb{R}$ *and for* $k = 1, 2 \ldots, \lfloor n/2 \rfloor$, *we have*

(1) $\overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^{T_1}) \geq \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_1}) \geq \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_2})$.
(2) $\overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^{T_1}) \geq \overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^{T_2}) \geq \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_2})$.

Proof: We sketch a proof of (1) above. The proof of (2) is very similar and hence is omitted. Theorem 1.2 gives us $\overline{\mathrm{TwoRowImm}}_{k-1}(\mathcal{L}_q^{T_1}) \geq \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_1})$. When the shape is the same and T_2 covers T_1 in GTS_n, then by [13, Lemma 23], going up along GTS_n poset weakly decreases $a_i^T(q)$ for each i and for all $q \in \mathbb{R}$. Arguing as in the proof of Theorem 1.2 gives us $\overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_1}) \geq \overline{\mathrm{TwoRowImm}}_k(\mathcal{L}_q^{T_2})$, completing the proof. ∎

4. Polynomials and successive differences

Recall the tables containing $\alpha_{n,k,i}$'s for $n = 7, 8$ given in Example 3.2. When $n = 2\ell$, in Lemma 3.5, we saw a relation between the successive difference of coefficients of the polynomial $(1 + x + x^2)^\ell$ and $\alpha_{2\ell,k,\ell}$'s (that is, the entries of the last row). We next give similar identities for other $\alpha_{n,k,i}$'s (that is, for entries of other rows of the table). We need to define a sequence of polynomials. For $0 \leq i \leq \lfloor n/2 \rfloor$, define the following sequence of polynomials:

$$p_{n,i}(x) = (1 + x)^{n-2i}(1 + x + x^2)^i = \sum_{k=0}^{n} p_{n,i,k}x^k.$$

For example, when $n = 8$, we tabulate the polynomials below:

$p_{8,0}(x)$	$(1 + x)^8$
$p_{8,1}(x)$	$(1 + x)^6 \times (1 + x + x^2)$
$p_{8,2}(x)$	$(1 + x)^4 \times (1 + x + x^2)^2$
$p_{8,3}(x)$	$(1 + x)^2 \times (1 + x + x^2)^3$
$p_{8,4}(x)$	$(1 + x + x^2)^4$

It is easy to see that $p_{8,1}(x) = 1 + 7x + 22x^2 + 41x^3 + 50x^4 + 41x^5 + 22x^6 + 7x^7 + x^8$. Taking successive differences of coefficients, we get $1 = 1-0$, $6 = 7-1$, $15 = 22-7$, $19 = 41-22$, $9 = 50-41$ and thus, we get the row corresponding to $i = 1$ in the table for $n = 8$. Similarly, from $p_{8,2}(x)$, we get the row of the table for $n = 8$, corresponding to $i = 2$. When $n = 7$, we tabulate the polynomials below:

$p_{7,0}(x)$	$(1+x)^7$
$p_{7,1}(x)$	$(1+x)^5 \times (1+x+x^2)$
$p_{7,2}(x)$	$(1+x)^3 \times (1+x+x^2)^2$
$p_{7,3}(x)$	$(1+x) \times (1+x+x^2)^3$

One can check that $p_{7,3}(x) = 1 + 4x + 9x^2 + 13x^3 + 13x^4 + 9x^5 + 4x^6 + x^7$. From this polynomial, taking successive difference as done above, we get 1,3,5,4 which is the row corresponding to $i = 3$ in the table when $n = 7$. One can check that other rows are obtained in a similar manner.

4.1. Successive differences

In the following lemma, we give a similar *successive difference* interpretation for other $\alpha_{n,k,i}$'s.

Lemma 4.1: *With the notation above, for $n \geq 1$ we have*

(1) $p_{n,i,k} = p_{n-2,i-1,k} + p_{n-2,i-1,k-1} + p_{n-2,i-1,k-2}$ *when* $1 \leq i \leq \lfloor n/2 \rfloor$.
(2) $\alpha_{n,k,i} = p_{n,i,k} - p_{n,i,k-1}$ *when* $0 \leq i \leq \lfloor n/2 \rfloor$.

Proof: (1) By definition, we have

$$\begin{aligned} p_{n,i,k} &= \text{Coeff. of } x^k \text{ in } (1+x)^{n-2i}(1+x+x^2)^i \\ &= \text{Coeff. of } x^k \text{ in } (1+x)^{n-2-2(i-1)}(1+x+x^2)^{i-1}(1+x+x^2) \\ &= \text{Coeff. of } x^k \text{ in } p_{n-2,i-1}(x)(1+x+x^2) \\ &= p_{n-2,i-1,k} + p_{n-2,i-1,k-1} + p_{n-2,i-1,k-2}. \end{aligned}$$

(2) To prove the second part, we use induction on n as done in the proof of Lemma 3.5. It is easy to verify the statement for small values of n, k and i. We first consider the case when $i = 0$. For positive integers n, k, we get

$$\alpha_{n,k,0} = \binom{n}{k} - \binom{n}{k-1} = p_{n,0,k} - p_{n,0,k-1}.$$

When $i > 0$ assume that the lemma is true for all values less than n. By induction and Lemma 3.4, when n is odd or n is even with $k \neq \lfloor n/2 \rfloor$ we get

$$\begin{aligned} \alpha_{n,k,i} &= \alpha_{n-2,k,i-1} + \alpha_{n-2,k-1,i-1} + \alpha_{n-2,k-2,i-1} \\ &= p_{n-2,i-1,k} - p_{n-2,i-1,k-1} + p_{n-2,i-1,k-1} - p_{n-2,i-1,k-2} \\ &\quad + p_{n-2,i-1,k-2} - p_{n-2,i-1,k-3} \\ &= p_{n,i,k} - p_{n,i,k-1}. \end{aligned}$$

where the last equality follows from first part.

It is easy to check that the polynomial $p_{n,i}(x)$ is palindromic. When $n = 2\ell$ and $k = \ell$ for some $\ell > 0$ by induction and Lemma 3.4

$$\begin{aligned}\alpha_{2\ell,\ell,i} &= \alpha_{2\ell-2,\ell-2,i-1} \\ &= p_{2\ell-2,i-1,\ell-2} - p_{2\ell-2,i-1,\ell-3} \\ &= p_{2\ell-2,i-1,\ell} - p_{2\ell-2,i-1,\ell-1} + p_{2\ell-2,i-1,\ell-1} - p_{2\ell-2,i-1,\ell-2} \\ &\quad + p_{2\ell-2,i-1,\ell-2} - p_{2\ell-2,i-1,\ell-3} \\ &= p_{2\ell,i,\ell} - p_{2\ell,i,\ell-1}.\end{aligned}$$

In the above, the second last equality follows by using $p_{2\ell-2,i-1,\ell-2} = p_{2\ell-2,i-1,\ell}$. The proof is complete. ∎

5. A probabilistic interpretation

In this section, we give a path based interpretation and recast Lemma 3.16 in a probabilistic setting. Recall from Remark 3.3, that $\alpha_{n,k,0} = \text{TwoRow}\chi_{n,k}(\mathsf{id})$, where $\text{TwoRow}\chi_{n,k}(\mathsf{id})$ is the dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by the two row partition $\text{TwoRow}_k = n-k, k$, which by the Hook-length formula equals the number of Standard Young Tableaux (SYT henceforth) of shape $n-k, k$.

Consider non negative lattice paths on the plane from $(0,0)$ to $(n, n-2k)$, consisting of $n-k$ Up steps which go from (x,y) to $(x+1, y+1)$ denoted U and k Down steps which go from (x,y) to $(x+1, y-1)$, denoted D, that stay on or above the x-axis. By definition, all paths we consider stay on or to the right of the y-axis and a lattice path is termed *non negative* if it stays on or above the x-axis. For $k \le \lfloor n/2 \rfloor$, let $\text{NLP}(n, n-2k)$ be the set of such non negative lattice paths from $(0,0)$ to $(n, n-2k)$. Since $\text{NLP}(2n, 0)$ is the set of Dyck paths of length $2n$ (or semi length n), we refer to $\text{NLP}(n, n-2k)$ as *Generalized Dyck paths* with the word *generalized* implying that the number of Up steps is larger than the number of Down steps.

Remark 5.1: The following well known bijection maps Standard Young tableaux of shape $n-k, k$ to $\text{NLP}(n, n-2k)$ as follows. Given an SYT T of shape $n-k, k$, consider the path P_T whose ith step is U if i is in the first row of T and whose ith step is D if i is in the second row of T.

Combining Remark 3.3 with the bijection in Remark 5.1, the denominator terms in Lemma 3.16, are the cardinalities of $\text{NLP}(n, n-2k)$ and $\text{NLP}(n, n-2k-2)$ respectively. Our first aim is to give a similar interpretation for $\alpha_{n,k,i}$ as the cardinality of some set of paths. Our interpretation will depend on the parity of n. Let

$$(1+x+x^2)^n = \sum_{k=0}^{2n} p_{n,k} x^k \quad \text{and} \quad (1+x+1/x)^n = \sum_{k=-n}^{n} q_{n,k} x^k. \tag{18}$$

As $1+x+x^2 = x(1+x+1/x)$, when $-n \le k \le n$, we get $p_{n,n+k} = q_{n,k}$. Thus, the $p_{n,k}$'s are translates of the $q_{n,k}$'s. It is also easy to see that $(1+x+x^2)^n$ is a palindromic polynomial of degree $2n$. That is, for $0 \le k \le 2n$, we have $p_{n,k} = p_{n,2n-k}$.

From (18), a moments reflection gives the following interpretation for $q_{n,k}$: $q_{n,k}$ equals the number of lattice paths from $(0,0)$ to (n,k) where we are allowed the following three types of steps: U from (x,y) to $(x+1,y+1)$, H from (x,y) to $(x+1,y)$ and D from (x,y) to $(x+1,y-1)$. Note that these lattice paths need not be non negative. We call such paths as UHD paths. By translating, we can get a path based interpretation for the $p_{n,k}$'s. We now bifurcate our discussion into two parts depending on the parity of n.

5.1. When $n = 2\ell$ is even

When $n = 2\ell$, for $0 \le k \le \ell$, by Lemma 3.5, we have $\alpha_{2\ell,k,\ell} = p_{\ell,k} - p_{\ell,k-1}$ (where $p_{\ell,-1} = 0$). Callan in [6] showed that the difference between the central trinomial coefficient and its predecessor is the Riordan number R_n which counts the number of non negative UHD paths from $(0,0)$ to $(n,0)$ with no H steps at height 0. Here *non negative UHD paths* are UHD paths which do not go below the x-axis. By Callans result, we get that $\alpha_{2\ell,\ell,\ell} = R_\ell = p_{\ell,\ell} - p_{\ell,\ell-1}$. We will need *Generalized Riordan paths* which are defined as non negative UHD paths with no H step at height 0, but are from $(0,0)$ to (n,k) where k need not be zero. Callan's result is actually more general and gives an interpretation for the numbers $\alpha_{2\ell,k,\ell}$ (which we had denoted as $\text{last}_{\ell,k}$) as the cardinality of a set of *Generalized Riordan paths*. We give a proof for completeness.

Recall for $0 \le k \le \ell$, that $q_{\ell,k}$, the coefficient of x^k in $(x^{-1}+1+x)^\ell$ is the number of UHD paths from $(0,0)$ to (ℓ,k). By translation, for $0 \le k \le \ell$, $p_{\ell,\ell+k}$ is the number of UHD paths from $(0,0)$ to (ℓ,k). Let $\text{UHD}(\ell,\ell-k)$ be the set of UHD paths from $(0,0)$ to $(\ell,\ell-k)$. Since $p_{\ell,k} = p_{\ell,2\ell-k} = |\text{UHD}(\ell,\ell-k)|$, we will give a combinatorial proof that $\alpha_{2\ell,k,\ell} = p_{\ell,k} - p_{\ell,k-1} = |\text{UHD}(\ell,\ell-k)| - |\text{UHD}(\ell,\ell-k+1)|$.

Lemma 5.2 (Callan): *Let ℓ be a positive integer and let $k \le \ell$ be a non negative integer. The number of Generalized Riordan paths from $(0,0)$ to (ℓ,k) equals $p_{\ell,\ell-k} - p_{\ell,\ell-k-1}$. That is, $\alpha_{2\ell,k,\ell} = |\text{UHD}(\ell,\ell-k)| - |\text{UHD}(\ell,\ell-k+1)|$. Thus, $\alpha_{2\ell,k,\ell}$ equals the number of Generalized Riordan paths from $(0,0)$ to $(\ell,\ell-k)$.*

Proof: For $0 \le k \le \ell$ let $\text{GRP}(\ell,\ell-k)$ denote the set of Generalized Riordan paths from $(0,0)$ to $(\ell,\ell-k)$ without a horizontal step at height zero. We will prove the Lemma by giving a bijection f from the set $\text{UHD}(\ell,\ell-k) \setminus \text{GRP}(\ell,\ell-k)$ to the set $\text{UHD}(\ell,\ell-k+1)$.

Suppose $P \in \text{UHD}(\ell,\ell-k)$ has either a H step at ground level or dips strictly below the x axis at some point or both. Denote by R the subpath of P starting from the x-axis after either the last horizontal step at height 0, or after the last time P went below the x-axis, (if both events happen, choose whichever event happens later). Thus, R is the longest Generalized Riordan sub-path that starts somewhere on the x-axis and ends P. Consider the step X in P that precedes R. It is easy to check that X cannot be D. Thus, we have two cases based on X.

Case 1 (when $X = H$): We have $P = SXR$ for some sub-path S of P ending somewhere on the x-axis. Define $f(P)$ as follows: $f(SXR) = \overline{S}UR^{+1}$, where R^{+1} is sub-path obtained by shifting R from the ground level to level 1 and $\overline{S}$ is obtained by flipping S with respect to the x axis.

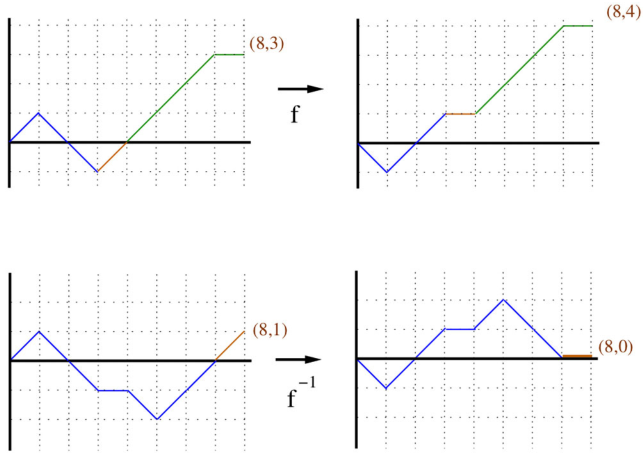

Figure 1. Examples of the bijection f and its inverse f^{-1}.

Case 2 (when $X = U$): We have $P = SXR$ for some sub-path S of P ending at height -1. Define $f(SUR) = \overline{S}HR^{+1}$, where R^{+1} and $\overline{S}$ are as defined in Case 1.

We note that any path with first step U gets mapped under f to a path with first step D and vice-versa. The map f sends paths whose first step is H to paths with first step H itself.

Inverse map f^{-1}: To defined the inverse of f, let $P \in \mathrm{UHD}(\ell, \ell - k + 1)$ be a path from $(0, 0)$ to $(\ell, \ell - k + 1)$. Let R be the largest subpath of P that ends P and does not have a H step at level 1 or goes below level 1. As before, let X be the step in P that precedes R. Note that X cannot be D. Thus we have the following two cases.

Case 1 (when $X = H$): We have $P = SHR$ for some subpath S. Define $f^{-1}(SHR) = \overline{S}UR^{-1}$, where R^{-1} is sub-path obtained by shifting R from the level 1 to ground level and $\overline{S}$ is as defined as in the definition of f.

Case 2 (when $X = U$): Whe have $P = SUR$ for some subpath S. Define $f^{-1}(SUR) = \overline{S}HR^{-1}$.

It is easy to check that $f \odot f^{-1} = \mathrm{id}$, the identity map. The proof is complete. ■

Remark 5.3: The bijection f defined in the proof of Lemma 5.2 is illustrated in Figure 1 where $f(UDDUUUUH) = DUUHUUUH$ and $f^{-1}(UDDHDUUU) = DUUHUDDH$.

Example 5.4: We illustrate Lemma 5.2 when $\ell = 4$. We clearly have $p_{4,-1} = 0$, $p_{4,0} = 1$, $p_{4,1} = 4$, $p_{4,2} = 10$, $p_{4,3} = 16$ and $p_{4,4} = 19$. From Example 3.2, we have the following table of $\alpha_{8,4,k}$. Clearly, $\alpha_{8,4,k} = p_{4,k} - p_{4,k-1}$ and we have the following sets of Generalized Riordan paths.

$i = 4$	$\lambda = 8$ 1	$\lambda = 7, 1$ 3	$\lambda = 6, 2$ 6	$\lambda = 5, 3$ 6	$\lambda = 4, 4$ 3
End point	(4,4)	(4,3)	(4,2)	(4,1)	(4,0)
Path sets	GRP(4, 4)	GRP(4, 3)	GRP(4, 2)	GRP(4, 1)	GRP(4, 0)

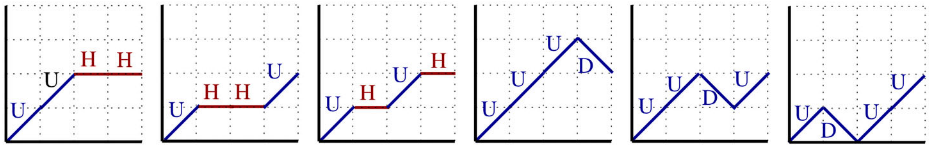

Figure 2. The set GRP(4, 2) of Generalized Riordan paths from (0, 0) to (4, 2).

where $\mathrm{GRP}(4,4) = \{UUUU\}$, $\mathrm{GRP}(4,3) = \{UUUH, UUHU, UHUU\}$, $\mathrm{GRP}(4,2) = \{UUHH, UHHU, UHUH, UUUD, UUDU, UDUU\}$, $\mathrm{GRP}(4,1) = \{UDUH, UUDH, UHDU, UHUD, UUHD, UHHH\}$ and $\mathrm{GRP}(4,0) = \{UDUD, UUDD, UHHD\}$. The set of paths in $\mathrm{GRP}(4,2)$ are drawn in Figure 2.

In our next lemma, we interpret Generalized Riordan paths as generalized Dyck paths with restrictions on the positions of its peaks. We denote paths with only Up and Down steps as UD paths. The following interpretation of Riordan paths is known (see OEIS) and we give a simple proof as we need a version for Generalized Riordan paths as well. Given a UD path P, a peak is a lattice point (p, q) on P such that an up-step ends at (p, q) and a down-step starts at (p, q).

Lemma 5.5: *Let $n = 2\ell$. For $0 \le k \le \ell$, there is a bijection f from $\mathrm{GRP}(\ell, \ell - k)$ to the set $\mathrm{NLP}(2\ell, 2\ell - 2k)$ of generalized Dyck paths from $(0,0)$ to $(2\ell, 2\ell - 2k)$ that have $2\ell - k$ Up steps, k Down steps and have no peaks at any odd height. Thus, $\alpha_{2\ell,k,\ell}$ is the number of generalized Dyck paths from $(0,0)$ to $(2\ell, 2\ell - 2k)$ with $2\ell - k$ Up steps, k Down steps, that have no peaks at any odd height.*

Proof: Let $P \in \mathrm{GRP}(\ell, \ell - k)$ with $P = a_1, a_2, \ldots, a_\ell$ be a Generalized Riordan path from $(0,0)$ to $(\ell, \ell - k)$ where $a_i = U/H/D$, depending on the type of the ith step of P. Perform the following operations: change U to U, U, change D to D, D and change H to D, U. This will convert P to $f(P) = Q = b_1, b_2, \ldots, b_{2\ell}$ where Q is an UD path. We note the following properties of the bijection f.

(Property 1) Q is a non negative path: As $P \in \mathrm{GRP}(\ell, \ell - k)$ and thus has no horizontal steps at height 0. Thus, changing a H step in P to D, U in $f(P)$ will not make the path $f(P)$ go below height 0. Further, since P is non negative, any D step in P is preceded by a U step prior to it. This ensures that while changing D in P to D, D in $f(P)$ we would have earlier changed a U in P to a U, U in $f(P)$ and hence this change will also not make $f(P)$ go below height 0.

(Property 2) Q has no peaks at any odd height: To see this, note that a peak will occur in $f(P)$ iff there is a consecutive U, D pair. Suppose $(b_i, b_{i+1}) = (U, D)$, then it is easy to see that i is even. As i is even, this means that the height at which the peak occurs in $f(P)$ is also even. Thus any peak of $f(P)$ only occurs at an even height. The proof is now complete. ■

Figure 3 shows the generalized Dyck paths output by the bijection f described in Lemma 5.5 on paths $P \in \mathrm{GRP}(4, 2)$. Note that Lemma 5.5 gives an interpretation for $\alpha_{2\ell,k,\ell}$, that is for entries in the last row when $n = 2\ell$. Using this as a building block, we give another expression for $\alpha_{2\ell,k,i}$ in terms of $\alpha_{2m,k,m}$. This will enable us to give an interpretation for $\alpha_{2\ell,k,i}$.

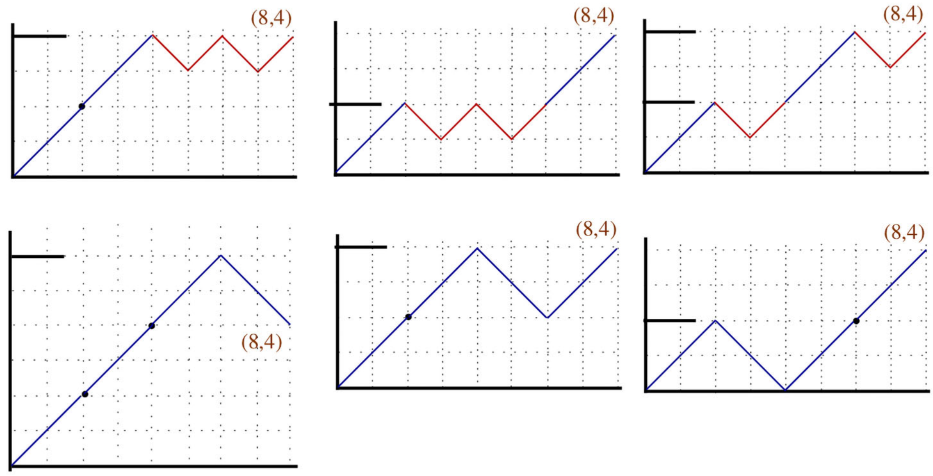

Figure 3. The generalized Dyck paths obtained under the bijection f applied to paths in $\mathrm{GRP}(4,2)$.

Lemma 5.6: *Let $n = 2\ell$ and let $0 \le k, i \le \ell$. Then,*

$$\alpha_{2\ell,k,i} = \sum_{t=0}^{\ell-i} \binom{\ell-i}{t} \alpha_{2\ell-2t,k-t,\ell-t}.$$

Proof: By Lemma 4.1, $\alpha_{2\ell,k,i}$ is the difference of successive coefficients from the polynomial $p_{2\ell,i}(x)$. Set $b = 1 + x + x^2$. Then, for $k \ge 0$, we clearly have $p_{2k,k}(x) = b^k$. It is further clear that

$$\begin{aligned} p_{2\ell,i}(x) &= (x+b)^{\ell-i} p_{2i,i}(x) = (x+b)^{\ell-i} b^i \\ &= \sum_{t=0}^{\ell-i} \binom{\ell-i}{t} x^t b^{\ell-t} = \sum_{t=0}^{\ell-i} \binom{\ell-i}{t} x^t p_{2\ell-2t,\ell-t}(x) \end{aligned}$$

As taking the difference of successive coefficients is a linear operator, we get the desired equation, completing the proof. ■

Example 5.7: We illustrate Lemma 5.6 by getting the last column of the table when $n = 8$. The following data can be easily verified.

ℓ	0	1	2	3	4
$\alpha_{2\ell,\ell,\ell}$	1	0	1	1	3

From the table for $n = 8$ in Example 3.2, one can easily verify the construction of the entries in the last column using the elements $\alpha_{2\ell,\ell,\ell}$ as follows.

$$\alpha_{8,4,3} = 4 = \alpha_{8,4,4} + \alpha_{6,3,3},$$

$$\alpha_{8,4,2} = 6 = \alpha_{8,4,4} + 2\alpha_{6,3,3} + \alpha_{4,2,2},$$

$$\alpha_{8,4,1} = 9 = \alpha_{8,4,4} + 3\alpha_{6,3,3} + 3\alpha_{4,2,2} + \alpha_{2,1,1},$$
$$\alpha_{8,4,0} = 14 = \alpha_{8,4,4} + 4\alpha_{6,3,3} + 6\alpha_{4,2,2} + 4\alpha_{2,1,1} + \alpha_{0,0,0}.$$

Using Lemma 5.6, we give an interpretation for the numbers $\alpha_{2\ell,k,i}$ when $i < \ell$. It will again be the the cardinality of a set of generalized Dyck paths with odd peaks occurring at restricted positions. All our generalized Dyck paths will be from $(0,0)$ to $(2\ell, 2\ell - 2k)$. Divide the 2ℓ steps on the x-axis into ℓ intervals of length 2 each. Thus, we have intervals $s_1 = (0,2)$, $s_2 = (2,4), \ldots, s_\ell = (2\ell - 2, 2\ell)$. For $0 \le i \le \ell$, define the sets $D_i = \{1, 2, \ldots, i\}$. Thus $D_0 = \emptyset$, $D_1 = \{1\}, D_2 = \{1,2\}$ and so on. We will permit peaks to have an odd height at a point (x, y) where $x \in D_i$.

Lemma 5.8: *With the notation described above, $\alpha_{2\ell,k,i}$ is the cardinality of the set of generalized Dyck paths from $(0,0)$ to $(2\ell, 2\ell - 2k)$ with $2\ell - k$ Up steps, k Down steps and odd peaks contained in the set $D_{\ell-i}$.*

Proof: Our proof is inspired by the proof of Lemma 5.6. We construct generalized Dyck paths with peaks at odd height in the set $D_{\ell-i}$ as follows. If there are peaks at odd heights, then as done in the proof of Lemma 5.5, it is clear that any such peak will occur at position (x, y) where both x, y are odd positive integers. Thus, such an odd peak causing 'U, D' pair of steps has to be in positions indexed by s_d for some $d \in \{1, 2, \ldots, \ell\}$.

We claim that the number of generalized Dyck paths with t odd height peaks in the set $D_{\ell-i}$ is $\binom{\ell-i}{t}\alpha_{2\ell-2t,k-t,\ell-t}$. If such a path P is written as a string of U, D's, any peak will have a consecutive 'U, D' substring. Note that if P has t peaks at odd heights, then removing the t 'U, D' pairs will give a generalized Dyck path Q of length $2\ell - 2t$ with no change in the final height of the path (thus having $2\ell - k - t$ Up and $k{-}t$ Down steps) and with t fewer U steps and t fewer D steps. Further, Q has no odd peaks. This argument goes both ways.

Given a generalized Dyck path Q with a total of $2\ell - 2t$ steps from $(0,0)$ to $(2\ell - 2t, 2\ell - 2k)$ that has $2\ell - k - t$ Up steps and $k{-}t$ Down steps with no odd peaks, one can choose a subset T of size t from $D_{\ell-i}$ in $\binom{\ell-i}{t}$ ways and insert a 'U, D' pair at position s_d for $d \in T$. This completes the proof. ■

Example 5.9: We illustrate the bijection described in Lemma 5.8 to get $\alpha_{8,4,0}$. We thus need Dyck paths from $(0,0)$ to $(8,0)$. Since we do not change the height, our building blocks are Dyck paths without peaks at odd heights from $(0,0)$ to $(2m,0)$ for non negative integers m. These are given in Figure 4 with different colours for added clarity. The set of paths formed is given in Figure 5 where the same colours are used and odd peak causing 'U,D' pairs are drawn using dotted lines.

Remark 5.10: Note that when $i = \ell$, Lemma 5.8 gives Lemma 5.5. Further, $\alpha_{2\ell,k,i}$ is a term that occurs in the normalized immanant computation of the partition $TwoRow_k$. Thus, when $\alpha_{n,k,i}$ is viewed as the cardinality of a restricted set of generalized Dyck paths as given in Lemma 5.8, the second parameter k in the subscript of $\alpha_{2\ell,k,i}$ is the number of Down steps while the first parameter is the total number of steps in the generalized Dyck path.

Recall that $\alpha_{2\ell,\ell,0} = C_\ell$, the ℓth Catalan number and as $\alpha_{2\ell-2t,\ell-t,\ell-t} = R_{\ell-t}$, where R_ℓ is the ℓth Riordan number. When $n = 2\ell, k = \ell$ and $i = 0$, Lemma 5.6 gives us the known fact that $C_\ell = \sum_{t=0}^{\ell} \binom{\ell}{t} R_{\ell-t}$.

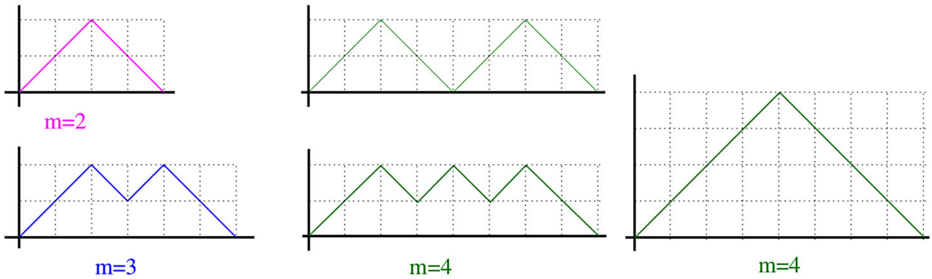

Figure 4. Up down paths with no peaks at an odd height.

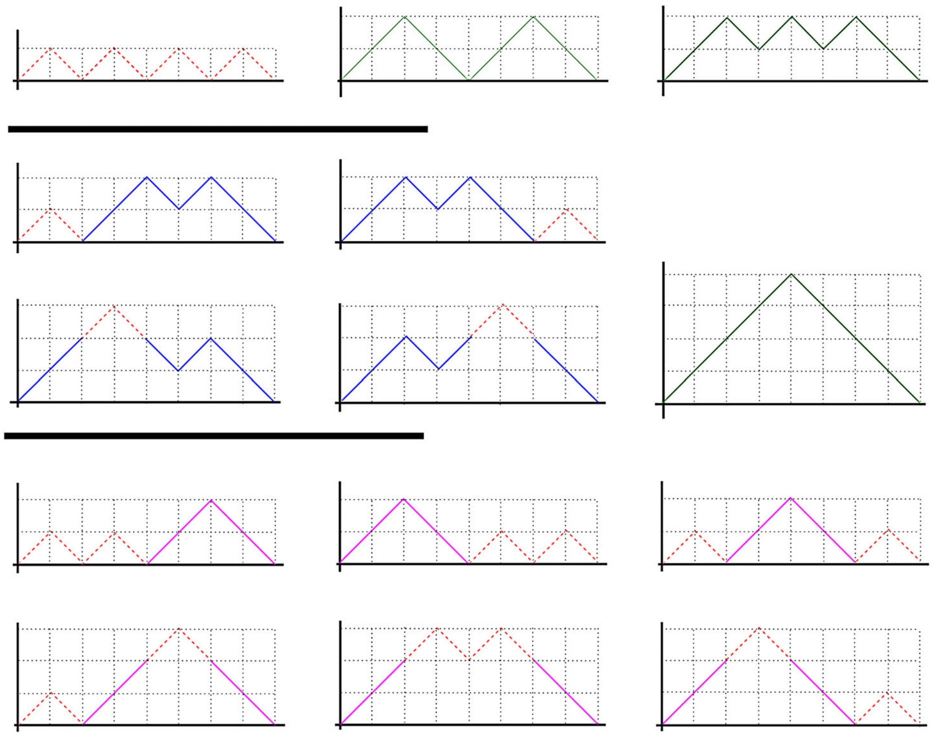

Figure 5. $\alpha_{8,4,0}$ counts all 14 Catalan paths of semi length 4 that have peaks at odd heights in D_4.

5.2. When $n = 2\ell + 1$ is odd

When $n = 2\ell + 1$, we use Lemma 3.1 which states that $\alpha_{2\ell+1,k,i} = \alpha_{2\ell,k,i} + \alpha_{2\ell,k-1,i}$. By Remark 5.10, $\alpha_{2\ell,k,i}$ and $\alpha_{2\ell,k-1,i}$ are the cardinalities of generalized Dyck paths where the first parameter 2ℓ is the total number of steps while the second parameter k is the number of Down steps. Further, these have odd height peaks in the set $D_{\ell-i}$.

The same interpretation works when $n = 2\ell + 1$. Consider generalized Dyck paths with $2\ell + 1$ steps containing k Down steps, which have one more Up step as compared to paths counted by the set with cardinality $\alpha_{2\ell,k,i}$. It is simple to see that there is a bijection between a generalized Dyck path P counted by $\alpha_{2\ell,k,i}$ and the path $P' = P, U$ where we append an Up step at the end of P. That is, if the last step of a path counted by $\alpha_{2\ell+1,k,i}$ is an Up step, then by deleting it, we get a generalized Dyck path counted by $\alpha_{2\ell,k,i}$. However, if the last step is a Down step, then after deletion of this, we get a generalized Dyck path counted by the $\alpha_{2\ell,k-1,i}$. We only need to check that appending a Down step at the end does not create a valley at an odd height. But this follows from the fact that a path with 2ℓ steps and $k-1$ Down steps ends at a point $(2\ell, 2\ell-2k+2)$ and so ends at a point with even y co-ordinate. Adding a Down step to such a path may create a peak but only at an even height. Thus, the set of odd height peaks after addition of a Down step at the end does not change. Thus we get the following counterpart of Lemma 5.8.

Lemma 5.11: *With the notation described above, $\alpha_{2\ell+1,k,i}$ is the cardinality of the set of generalized Dyck paths from $(0,0)$ to $(2\ell+1, 2\ell-2k+1)$ with $2\ell-k+1$ Up steps, k Down steps and odd peaks contained in the set $D_{\ell-i}$.*

5.3. Probabilistic interpretation of Lemma 3.16

From Lemmas 5.8 and 5.11, we get the following probabilistic interpretation of Lemma 3.16 whose straightforward proof we omit.

Lemma 5.12: *Fix a positive integer n and $i \leq \lfloor n/2 \rfloor$. Then, the probability of generalized Dyck path with n total steps and with odd height peaks contained in the set $D_{\lfloor n/2 \rfloor - i}$ decreases as the number k of Down steps increases.*

Recall that Remark 5.1 gives a bijection between generalized Dyck paths with n steps and with k down steps and Standard Young Tableaux of shape $n-k, k$ (denoted $\mathsf{SYT}(n-k,k)$), we can recast Lemma 5.12 in terms of SYTs. We translate the notion of peaks at odd heights to tableaux. For $T \in \mathsf{SYT}(n-k,k)$ define position i to be a peak if i appears in the first row and $i+1$ appears in the second row. This is precisely saying that $i \in \mathrm{DES}(T)$ where $\mathrm{DES}(T)$ is the descent set of T, which is a well studied statistic (see the book by Stanley [16, Chapter 7]). For a descent i, to get the height of its peak under this mapping, consider $T_{|i}$, the restriction of T to the entries $\{1, 2, \ldots, i\}$. Note that $T_{|i}$ is also an SYT. Let $T_{|i} = a_i, b_i$ where a_i and b_i are the number of elements in the first and second row of $T_{|i}$ respectively. Define $\mathrm{RowDiff}(T_{|i}) = a_i - b_i$ to be the difference between the number of elements in the first row and the number of elements in the second row of $T_{|i}$. Define an descent $i \in T$ to have even (or odd) height if $\mathrm{RowDiff}(T_{|i})$ is even (or odd respectively). With these definitions, recalling the set $D_{\ell-i}$, we can give the SYT version of Lemma 5.12.

Lemma 5.13: *Fix a positive integer n and let $i \leq \lfloor n/2 \rfloor$. Then, the probability that an SYT of shape $n-k$, k has all its descents with odd height in $D_{\lfloor n/2 \rfloor - i}$ decreases as the number k increases (and hence the shape of T changes).*

By running the arguments backward, it is clear that an alternate proof of Lemma 5.13 will give us an alternate proof of Theorem 1.2. Thus, it would be interesting to get an alternate proof of Lemma 5.13.

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Appendix

When $4 \leq n \leq 14$, we tabulate the data $\alpha_{n,k,i}/\alpha_{n,k,0}$ below.

Table A1. The values of $\alpha_{4,k,i}/\alpha_{4,k,0}$.

	$\lambda = 4$	$\lambda = 3, 1$	$\lambda = 2, 2$
$i = 0$	1	1	1
$i = 1$	1	0.667	0.500
$i = 2$	1	0.333	0.500

Table A2. The values of $\alpha_{5,k,i}/\alpha_{5,k,0}$.

	$\lambda = 5$	$\lambda = 4, 1$	$\lambda = 3, 2$
$i = 0$	1	1	1
$i = 1$	1	0.750	0.600
$i = 2$	1	0.500	0.400

Table A3. The values of $\alpha_{6,k,i}/\alpha_{6,k,0}$.

	$\lambda = 6$	$\lambda = 5, 1$	$\lambda = 4, 2$	$\lambda = 3, 3$
$i = 0$	1	1	1	1
$i = 1$	1	0.800	0.667	0.600
$i = 2$	1	0.600	0.444	0.400
$i = 3$	1	0.400	0.333	0.200

Table A4. The values of $\alpha_{7,k,i}/\alpha_{7,k,0}$.

	$\lambda = 7$	$\lambda = 6, 1$	$\lambda = 5, 2$	$\lambda = 4, 3$
$i = 0$	1	1	1	1
$i = 1$	1	0.833	0.714	0.643
$i = 2$	1	0.667	0.500	0.429
$i = 3$	1	0.500	0.357	0.286

Table A5. The values of $\alpha_{8,k,i}/\alpha_{8,k,0}$.

	$\lambda = 8$	$\lambda = 7, 1$	$\lambda = 6, 2$	$\lambda = 5, 3$	$\lambda = 4, 4$
$i = 0$	1	1	1	1	1
$i = 1$	1	0.857	0.750	0.679	0.643
$i = 2$	1	0.714	0.550	0.464	0.429
$i = 3$	1	0.571	0.400	0.321	0.286
$i = 4$	1	0.428	0.300	0.214	0.214

Table A6. The values of $\alpha_{9,k,i}/\alpha_{9,k,0}$.

	$\lambda = 9$	$\lambda = 8, 1$	$\lambda = 7, 2$	$\lambda = 6, 3$	$\lambda = 5, 4$
$i = 0$	1	1	1	1	1
$i = 1$	1	0.875	0.778	0.708	0.667
$i = 2$	1	0.750	0.593	0.500	0.452
$i = 3$	1	0.625	0.444	0.354	0.309
$i = 4$	1	0.500	0.333	0.250	0.214

Table A7. The values of $\alpha_{10,k,i}/\alpha_{10,k,0}$.

	$\lambda = 10$	$\lambda = 9,1$	$\lambda = 8,2$	$\lambda = 7,3$	$\lambda = 6,4$	$\lambda = 5,5$
$i = 0$	1	1	1	1	1	1
$i = 1$	1	0.889	0.800	0.733	0.689	0.667
$i = 2$	1	0.778	0.629	0.533	0.478	0.452
$i = 3$	1	0.667	0.486	0.387	0.333	0.309
$i = 4$	1	0.556	0.371	0.280	0.233	0.214
$i = 5$	1	0.444	0.286	0.200	0.167	0.143

Table A8. The values of $\alpha_{11,k,i}/\alpha_{11,k,0}$.

	$\lambda = 11$	$\lambda = 10,1$	$\lambda = 9,2$	$\lambda = 8,3$	$\lambda = 7,4$	$\lambda = 6,5$
$i = 0$	1	1	1	1	1	1
$i = 1$	1	0.900	0.818	0.754	0.709	0.682
$i = 2$	1	0.800	0.659	0.564	0.503	0.470
$i = 3$	1	0.700	0.523	0.418	0.358	0.326
$i = 4$	1	0.600	0.409	0.309	0.254	0.227
$i = 5$	1	0.500	0.318	0.227	0.182	0.159

Table A9. The values of $\alpha_{12,k,i}/\alpha_{12,k,0}$.

	$\lambda = 12$	$\lambda = 11,1$	$\lambda = 10,2$	$\lambda = 9,3$	$\lambda = 8,4$	$\lambda = 7,5$	$\lambda = 6,6$
$i = 0$	1	1	1	1	1	1	1
$i = 1$	1	0.909	0.833	0.773	0.727	0.697	0.682
$i = 2$	1	0.818	0.685	0.591	0.527	0.488	0.670
$i = 3$	1	0.727	0.556	0.488	0.382	0.343	0.326
$i = 4$	1	0.636	0.444	0.338	0.276	0.242	0.227
$i = 5$	1	0.545	0.352	0.253	0.200	0.172	0.159
$i = 6$	1	0.454	0.278	0.188	0.145	0.121	0.113

Table A10. The values of $\alpha_{13,k,i}/\alpha_{13,k,0}$.

	$\lambda = 13$	$\lambda = 12,1$	$\lambda = 11,2$	$\lambda = 10,3$	$\lambda = 9,4$	$\lambda = 8,5$	$\lambda = 7,6$
$i = 0$	1	1	1	1	1	1	1
$i = 1$	1	0.917	0.846	0.788	0.744	0.711	0.692
$i = 2$	1	0.833	0.708	0.615	0.550	0.507	0.482
$i = 3$	1	0.750	0.585	0.476	0.406	0.362	0.338
$i = 4$	1	0.667	0.477	0.365	0.298	0.259	0.238
$i = 5$	1	0.583	0.385	0.279	0.219	0.185	0.168
$i = 6$	1	0.500	0.308	0.211	0.161	0.133	0.119

Table A11. The values of $\alpha_{14,k,i}/\alpha_{14,k,0}$.

	$\lambda = 14$	$\lambda = 13,1$	$\lambda = 12,2$	$\lambda = 11,3$	$\lambda = 10,4$	$\lambda = 9,5$	$\lambda = 8,6$	$\lambda = 7,7$
$i = 0$	1	1	1	1	1	1	1	1
$i = 1$	1	0.923	0.857	0.802	0.758	0.725	0.703	0.692
$i = 2$	1	0.846	0.727	0.637	0.571	0.525	0.496	0.482
$i = 3$	1	0.769	0.610	0.502	0.428	0.381	0.352	0.338
$i = 4$	1	0.692	0.506	0.392	0.320	0.276	0.250	0.238
$i = 5$	1	0.615	0.416	0.304	0.239	0.200	0.178	0.168
$i = 6$	1	0.538	0.338	0.234	0.177	0.145	0.127	0.119
$i = 7$	1	0.461	0.273	0.179	0.132	0.105	0.091	0.084