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THE SECOND IMMANANT OF SOME COMBINATORIAL MATRICES

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Abstract. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be an $n \times n$ matrix where $n \geq 2$. Let $\mathsf{det2}(A)$, its second immanant be the immanant corresponding to the partition $\lambda_2 = 2, 1^{n-2}$. Let G be a connected graph with blocks $B_1, B_2, \ldots, B_p$ and with q-exponential distance matrix ED_G. We give an explicit formula for $\mathsf{det2}(\mathsf{ED}_G)$ which shows that $\mathsf{det2}(\mathsf{ED}_G)$ is independent of the manner in which G's blocks are connected. Our result is similar in form to the result of Graham, Hoffman and Hosoya and in spirit to that of Bapat, Lal and Pati who show that $\det \mathsf{ED}_T$ where T is a tree is independent of the structure of T and only dependent on its number of vertices. Our result extends more generally to a product distance matrix associated to a connected graph G. Similar results are shown for the q-analogue of T's laplacian and a suitably defined matrix for arbitrary connected graphs.

1. **Introduction**

We consider the second immanant of $n \times n$ matrices with entries from a commutative ring. We briefly state some needed background from the representation theory of the symmetric group $\mathfrak{S}_n$ on the set $[n] = \{1, 2, \ldots, n\}$ over $\mathbb{C}$, the complex numbers. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be an $n \times n$ matrix with entries from a commutative ring. In this work, all matrices will be either over $\mathbb{Z}$, the integers or over the polynomial ring $\mathbb{R}[q]$ where q is a variable. Let $f : \mathfrak{S}_n \to \mathbb{Z}$ be a function. Define $\det_f(A) = \sum_{\sigma\in\mathfrak{S}_n} f(\sigma)\prod_{i=1}^n a_{i,\sigma(i)}$. We only consider functions $f : \mathfrak{S}_n \mapsto \mathbb{Z}$ that arise as characters of irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$. If f is such a function, we call $\det_f(A)$ as an immanant. When f is the sgn function defined as $f(\pi) = \mathsf{sgn}(\pi)$ for all $\pi \in \mathfrak{S}_n$, then, clearly $\det_{\mathsf{sgn}}(A)$ is the usual determinant of A. That is, we have $\det_{\mathsf{sgn}}(A) = \det A$. If f is the id or all ones function defined as $f(\pi) = 1$ for all $\pi \in \mathfrak{S}_n$, then $\det_{\mathsf{id}}(A) = \mathsf{perm}(A)$, where $\mathsf{perm}(A)$ is the permanent of A.

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The number of distinct irreducible representations of $\mathfrak{S}_n$ is $p(n)$, the number of partitions of the positive integer n (see Sagan's book [10, Proposition 1.10.1]). Thus, if λ is a partition of n, we have functions $\chi_\lambda : \mathfrak{S}_n \to \mathbb{Z}$. As seen, both the determinant and the permanent of a matrix are immanants, with the determinant corresponding to the partition $(1, 1, \ldots, 1)$ and the permanent corresponding to the partition (n).

Let $n \geq 2$ and λ_2 be the partition $(2, 1^{n-2})$ of n. Denote as $\chi_2 : \mathfrak{S}_n \to \mathbb{Z}$, the irreducible character of $\mathfrak{S}_n$ corresponding to the partition λ_2. Define the second immanant of A to be

$$\mathsf{det2}(A) = \sum_{\pi \in \mathfrak{S}_n} \chi_2(\pi) \prod_{i=1}^n a_{i,\pi(i)}.$$

For an $n \times n$ matrix A, $\mathsf{det2}(A)$ can be computed efficiently. Littlewood's book [7, Chapter 6.5] contains a nice exposition of this result. See the work of Merris and Watkins [9] as well. For an $n \times n$ matrix A and for $1 \leq i \leq n$, let $A(i)$ be the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i-th row and its i-th column.

Theorem 1.1. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix. Then,*

$$\mathsf{det2}(A) = \sum_{i=1}^n a_{i,i} \det A(i) - \det A.$$

We now move on to the matrices we consider in this work. Let G be an undirected, connected graph on the vertex set $V(G) = [n]$. A *block* of G is a maximally connected subgraph without a cut-vertex. For a graph G, we will look at functions $\eta : V(G) \times V(G) \to R$ where R is a commutative ring (either $\mathbb{Z}$ or $\mathbb{R}[q]$). A *product distance* on G is a function $\eta : V(G) \times V(G) \to R$, that satisfies the following two conditions:

(1) $\eta(i,i) = 1$ for all $i \in [n]$.
(2) if $i, j \in V(G)$ are vertices such that every path from i to j passes through the cut-vertex k, then $\eta(i,j) = \eta(i,k)\eta(k,j)$.

We emphasize that we start with an undirected, connected graph and for each of its block, assign distances to pairs of vertices within the block. Thus, we essentially have the freedom to assign *distances* within each block subject to diagonal entries being 1. Rule 2 (the product rule) is then used to obtain distances between pairs of vertices from different blocks. Because of this *product rule*, we call such distances as *product distances* and use the terms distance and product distance interchangeably. Since we need not have symmetric distances within each block, it is not necessary that $\eta(i,j) = \eta(j,i)$ for $i, j \in V(G)$. As distances could be asymmetric, we could alternatively phrase our results using the terminology of directed graphs. We prefer to use undirected graph terminology in this work.

We will form a matrix with entries being $\eta(i,j)$ and sometimes write $\eta_{i,j}$ alternatively to denote $\eta(i,j)$. Let G have blocks $H_1, H_2, \ldots, H_r$. Let $\eta(\cdot,\cdot)$ be a product distance on G and let $D_G = (\eta_{i,j})_{1 \leq i,j \leq n}$ be the corresponding distance matrix.

The definition of product-distances is motivated by a concrete example: the exponential distance matrix of a connected graph ED_G. Given a connected graph G on the vertex set $[n]$, let the distance between two vertices $i, j \in V(G)$ be denoted $d_{i,j}$. i.e. $d_{i,j}$ is the length of the minimum length path from i to j in G. Define the $n \times n$ matrix $\mathsf{ED}_G = (q^{d_{i,j}})_{1 \leq i,j \leq n}$ as the exponential distance matrix where q is an indeterminate and $q^0 = 1$. It can be readily checked that $\eta(i,j) = q^{d_{i,j}}$ is a product distance that is symmetric. If $\eta(\cdot,\cdot)$ is a product distance on G and if G has blocks $H_1, H_2, \ldots, H_r$, then, each H_i is a graph in its own right and thus has an induced product distance matrix D_{H_i} obtained by restricting $\eta(\cdot,\cdot)$ to pairs of vertices, both in H_i. If the graph G is clear from the context, we abridge D_G to D.

If D is a matrix whose entries form a product distance on G, Bapat and Sivasubramanian [2] showed that $\det D$ only depends on $\det D_{H_i}$ for individual blocks H_i of G and not on the manner in which the H_i's are connected.

Theorem 1.2. *([2, Theorem 4]) Let G be a connected graph with blocks H_i, $1 \leq i \leq r$ and product distance matrix D_G. For each such i, let the product distance matrix of each H_i be D_{H_i}. Then,*

$$\det D_G = \prod_{i=1}^{r} \det D_{H_i}$$

In particular, $\det D_G$ is independent of the manner in which the blocks H_i of G are connected. In this work, we extend this result to $\mathsf{det2}(D_G)$ by giving an explicit formula for $\mathsf{det2}(D_G)$ in terms of the determinant and the second immanant of the D_{H_i}'s (see Theorem 3.5). Our formula is identical in form to the formula for the determinant of the distance matrix of a connected graph given by Graham, Hoffman and Hosoya. Their formula with the relevant background appears in Subsection 2.2.

As exponential distance matrices are special cases of product distances, considering the case when G is a tree T, we get the result that $\mathsf{det2}(\mathsf{ED}_T)$ is independent of the structure of the tree T. For this special case when G is a tree, our result is true in a more general non-commutative setting with *matrix weights* on the edges e_i. Let T be a tree with vertex set $[n]$. Let the edge e_i of T have a *matrix weight* W_i (for $1 \leq i < n$) and where each W_i is an $s \times s$ matrix over a commutative ring R. For $i, j \in [n]$, clearly, there is a unique path $p_{i,j}$ between i and j given by the sequence of edges $e_1, e_2, \ldots, e_r$ where $i \in e_1, j \in e_r$ and there is a common vertex in the edges e_i, e_{i+1} for $1 \leq i < r$. Define $d_{i,j}$, the distance between vertices i and j as $\prod_{i=1}^{r} W_i$ where the product takes the order of the matrices as they appear in $p_{i,j}$ into account. Note that $d_{i,j}$ is an $s \times s$ matrix. When $i = j$, define $d_{i,j}$ to be the $s \times s$ identity matrix. Consider the $ns \times ns$ matrix $\mathcal{D}_T = (d_{i,j})_{1 \leq i,j \leq n}$ (i.e. we have a block matrix). We call $\mathcal{D}_T$ as the *non-commutative analogue* of the distance matrix of T.When the tree T is clear, we abuse notation and write $\mathcal{D}$ instead of the more precise $\mathcal{D}_T$. Bapat and Sivasubramanian [3, Theorem 3] showed the following.

Theorem 1.3. *Let T be a tree on n vertices and for $1 \leq i < n$, let edge e_i have an $s \times s$ matrix weight W_i. Then, $\det \mathcal{D}_T = \prod_{i=1}^{n-1}(I - W_i^2)$. In particular, $\det \mathcal{D}_T$ is independent of the structure of the tree T and only depends on n and the weights $W_1, W_2, \ldots, W_{n-1}$.*

We show a second-immanant analogue of Theorem 1.3 by giving an explicit formula for $\mathsf{det2}(\mathcal{D}_T)$ (see Theorem 3.8). Our proofs rely on explicit inverse results found by Bapat and Sivasubramanian and on Theorem 1.1.

2. Second immanant of combinatorial matrices

We consider two families of matrices in this section and give relevant background on their second-immanants. We recall that all graphs in this work are connected.

2.1. **Laplacian matrices.** Let G be a connected graph with adjacency matrix A and diagonal matrix D with $D(i,i) = \deg(i)$, where $\deg(i)$ is the degree of vertex i. The laplacian of G is the matrix $L = D - A$. The following result of Merris [8] is immediate when we combine Theorem 1.1 with the Matrix Tree Theorem (see West's book [15, Page 86]).

Corollary 2.1. *Let G be a connected graph with n vertices, m edges and κ spanning trees. Let L be its laplacian matrix. Then,* $\mathsf{det2}(L) = 2m\kappa$.

Below, we mention a special case of Corollary 2.1, when the graphs are trees. We mention this as we generalise this case when G is a tree to the q-analogue of the laplacian of a tree (see Corollary 4.5).

Corollary 2.2. *Let T be a tree on n vertices and let L be its laplacian matrix. Then,* $\mathsf{det2}(L) = 2(n-1)$. *Hence,* $\mathsf{det2}(L)$ *only depends on n and is independent of the structure of the tree T.*

2.2. **Distance matrices.** Let G be a connected graph with vertex set $[n]$ and with distance matrix $D = (d_{i,j})_{1 \le i,j \le n}$. Thus $d_{i,j}$ is the length of the shortest path between i and j in G and $d_{i,i} = 0$ for all $i \in [n]$. We begin with the following result of Grone and Merris (see [6, Page 590]).

Lemma 2.3. *If $A = (a_{i,j})_{1 \le i,j \le n}$ is an $n \times n$ matrix such that $a_{i,i} = 0$ for all $1 \le i \le n$, then* $\mathsf{det2}(A) = -\det A$.

Thus, for distance matrices D of connected graphs G, up to sign $\det D = \mathsf{det2}(D)$. The following is a well known result of Graham and Pollak [5].

Theorem 2.4 (Graham and Pollak)**.** *Let T be a tree with vertex set $[n]$ with distance matrix D_T. Then,* $\det D_T = (-1)^{n-1}(n-1)2^{n-2}$. *Thus,* $\det D_T$ *only depends on n and is independent of the structure of the tree T.*

Later, Graham, Hoffman and Hosoya [4] proved a more general and more attractive theorem about the determinant of the distance matrix D_G of a strongly connected digraph G as a function of the distance matrix of its *2-connected blocks* (also called blocks). Denote the sum of the cofactors of a matrix A as $\mathsf{cofsum}(A)$. Graham, Hoffman and Hosoya (see [4]) showed the following.

Theorem 2.5 (Graham, Hoffman and Hosoya)**.** *Let G be a strongly connected digraph with 2-connected blocks $G_1, G_2, \ldots, G_r$. Then,*

(1) $\mathsf{cofsum}(D_G) = \prod_{i=1}^{r} \mathsf{cofsum}(D_{G_i})$ *and*
(2) $\det D_G = \sum_{i=1}^{r} \det D_{G_i} \prod_{j \neq i} \mathsf{cofsum}(D_{G_j})$.

Graham, Hoffman and Hosoya's theorem implies that $\det D_G$ is independent of the manner in which the blocks of G are connected. It is also easy to recover Theorem 2.4 from Theorem 2.5 (as all blocks of T are K_2, the complete graph on 2 vertices). Since D_T and D_G are distance matrices, all their diagonal elements are zero. Further, q-analogues of Theorems 2.4 and 2.5 were given by Bapat, Lal and Pati [1] and by Sivasubramanian [12] respectively. Here, each positive integer n is replaced by the polynomial $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, and $[0]_q = 0$. Denote the q-analogues of D_T and D_G as qD_T and qD_G, respectively. Thus, by Lemma 2.3, we have the following simple corollary.

Corollary 2.6. *Let G be a connected graph on the vertex set $[n]$ with distance matrix D_G and with qD_G being the q-analogue of D_G. Then, both $\mathsf{det2}(D_G)$ and $\mathsf{det2}(qD_G)$ are independent of the tree-like manner of connection of its blocks.*

Consider the case now when D_G is the product distance matrix of a graph G. Since all the diagonal elements of D_G are 1, Lemma 2.3 is not applicable. It is these matrices whose second immanant we find in this work.

3. The second immanant of D_G and $\mathcal{D}_T$

Let G be a graph with blocks $B_1, B_2, \ldots, B_p$ and with a product distance $\eta(\cdot, \cdot)$. Let $D_G = (\eta_{i,j})$ be the matrix of product distances on G. Let the restriction of D_G to vertices of B_i be denoted D_{B_i} for all $1 \leq i \leq p$. Assume that for all $1 \leq i \leq p$, we have $\det D_{B_i} \neq 0$. That is, assume all the matrices D_{B_i} are invertible. For each $1 \leq i \leq p$, let $D_{B_i}^{-1} = N_i$. If $|V(B_i)| = n_i$, then N_i has dimension $n_i \times n_i$. Let M_i be the matrix N_i enlarged to have dimension $n \times n$ with zeroes added for all entries outside $V(B_i)$.

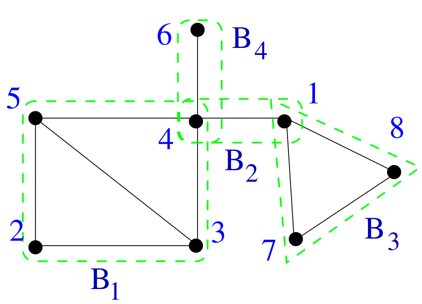

FIGURE 1. Decomposing G into blocks.

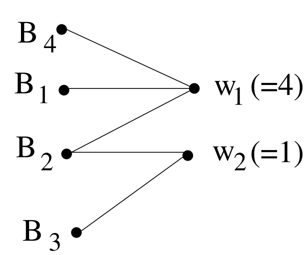

FIGURE 2. The block cut-point graph of G of Figure 1.

We illustrate getting the matrix M_i from N_i on an example. For the graph given in Figure 1, clearly, there are four blocks. These are marked as B_1, B_2, B_3 and B_4 respectively. We show how to get M_2 from N_2. As shown in the figure, B_2 is the block consisting of the edge $\{1, 4\}$. If B_2 has N_2 as given below, then we obtain M_2 by padding zeroes outside all vertices of B_2. As the two vertices of B_2 are 1 and 4, M_2 will be as given below.

$$N_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad M_2 = \begin{pmatrix} A & 0 & 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let G have ℓ cut vertices $w_1, w_2, \ldots, w_\ell$ in between its blocks. In the block cutpoint graph of G (see West's book [15, Page 156]), let the degree of w_i be d_i. Let R be the diagonal matrix with its (w_i, w_i)-th entry being $d_i - 1$ for each $1 \leq i \leq \ell$. For non cut-vertices a, define the (a, a)-th entry of R to be 0. For the graph given in Figure 1, we have its block cutpoint graph given in Figure 2. For this graph, we have $p = 4$ (G has four blocks) and $\ell = 2$ (G has two cut vertices). Let $w_1 = 4$ and $w_2 = 1$ be the cut vertices (w_i is the index of the i-th cut-vertex). From the block-cutpoint graph, we see that $d_1 = 3, d_2 = 2$ (number of blocks incident on the w_i's). The matrix R for the graph of Figure 1 is given above.

Form an $n \times n$ matrix $K = \sum_{i=1}^p M_i - R$. We reiterate below the steps required to obtain K.

(1) For each block B_i, where $1 \leq i \leq p$, form M_i by padding zeroes to the inverse matrix $N_i = D_{B_i}^{-1}$ at all indices outside B_i.
(2) Form the diagonal matrix R from the block-cutpoint graph of G.
(3) Set $K = \sum_{i=1}^p M_i - R$.

With the above definitions, we have the following result of Bapat and Sivasubramanian.

Theorem 3.1. *([2, Theorem 6]) Let G be a connected graph with blocks $B_1, B_2, \ldots, B_p$. Let D_G be the product distance matrix of G and for $1 \leq i \leq p$, let D_{B_i} be the restriction of D to the vertices in B_i. If $\det D_{B_i} \neq 0$ for all $1 \leq i \leq p$, then, $D_G^{-1} = K$.*

We begin with the following simple lemma. For a matrix A, denote its trace as $\mathsf{Trace}(A)$.

Lemma 3.2. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an invertible $n \times n$ matrix with $a_{i,i} = 1$ for all i. Then,*

$$\mathsf{Trace}(A^{-1}) = 1 + \frac{\mathsf{det2}(A)}{\det A}.$$

Proof 3.3. *We use Theorem 1.1. Since $a_{i,i} = 1$ for all i, $\mathsf{det2}(A) = \det A \cdot \mathsf{Trace}(A^{-1}) - \det A$, completing the proof.*

Remark 3.4. *Recall the matrices N_i and M_i given at the beginning of this section. We will apply Lemma 3.2 to the matrices M_i. Since $\mathsf{Trace}(M_i) = \mathsf{Trace}(N_i)$ and the matrix $N_i = D_{B_i}^{-1}$, we get $\mathsf{Trace}(M_i) = 1 + \dfrac{\mathsf{det2}(D_{B_i})}{\det D_{B_i}}$.*

We are now ready to prove our first main result.

Theorem 3.5. *Let D_G be the product distance matrix of a connected graph G with blocks $B_1, B_2, \ldots, B_p$, with the property that $\det D_{B_i} \neq 0$ for all $1 \leq i \leq p$. Then,*

$$\mathsf{det2}(D_G) = \sum_{i=1}^{p} \left[\mathsf{det2}(D_{B_i}) \prod_{j\neq i} \det D_{B_j}\right].$$

In particular, $\mathsf{det2}(D_G)$ is independent of the tree-like manner of connection of G's blocks.

Proof 3.6. *Since the (i,i)-th element of D_G is 1 for all i, using Theorems 1.1 and 3.1, we have*

$$\mathsf{det2}(D_G) = \det D_G \cdot \mathsf{Trace}(K) - \det D_G = \det D_G\,(\mathsf{Trace}(K) - 1).$$

By definition, we have $\mathsf{Trace}(K) = \sum_{i=1}^{p} \mathsf{Trace}(M_i) - \mathsf{Trace}(R)$. It is easy to see by induction on the number of blocks in G that $\mathsf{Trace}(R) = p - 1$. Thus,

$$\begin{aligned}\mathsf{det2}(D_G) &= \det D_G \left(\sum_{i=1}^{p} \mathsf{Trace}(M_i) - p\right) = \det D_G \left(\sum_{i=1}^{p} \left\{\frac{\mathsf{det2}(D_{B_i})}{\det D_{B_i}} + 1\right\} - p\right)\\ &= \det D_G \left(\sum_{i=1}^{p} \frac{\mathsf{det2}(D_{B_i})}{\det D_{B_i}}\right) = \sum_{i=1}^{p} \mathsf{det2}(D_{B_i}) \prod_{j\neq i} \det D_{B_j}.\end{aligned}$$

We have used Remark 3.4 in the second line above and Theorem 1.2 in the third line. This completes the proof.

We note that in Theorem 3.1, we do not need the product distance to be symmetric. i.e. we do not require $\eta_{i,j} = \eta_{j,i}$. It is simple to see that symmetry of the product distance is not required for Theorem 3.5 either. Our earlier proof crucially uses the fact that each diagonal entry of D_G is 1 and our explicit inverse result.

We note that Theorem 3.5 and Theorem 1.2 are for product distance matrices. They are counterparts of Theorem 2.5 for ordinary distance matrices with det and $\mathsf{det2}()$ playing the roles of $\mathsf{cofsum}()$ and det respectively.

3.1. $\mathsf{det2}(\mathcal{D}_T)$ for a tree T.

A similar proof gives us a non-commutative analogue of Theorem 3.5 for trees as described in Section 1. Recall $\mathcal{D}_T$ defined for a tree T on n vertices with edge e_i bearing a matrix weight W_i for $1 \leq i < n$. As each W_i is an $s \times s$ matrix, $\mathcal{D}_T$ is an $ns \times ns$ matrix. Bapat and Sivasubramanian showed that if for each $1 \leq i < n$, the $s \times s$ matrix $(I - W_i^2)$ is invertible, then the inverse of $\mathcal{D}_T$ can be written explicitly. To describe it, recall that if $A = (a_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ is an $m \times n$ matrix and B is a $p \times q$ matrix, then their Kronecker product $A \otimes B$ is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{pmatrix}$$

Let the degree sequence of T be $d_1, d_2, \ldots, d_n$ and define an $n \times n$ diagonal matrix Deg by $\mathsf{Deg} = \mathsf{Diag}(d_1, d_2, \ldots, d_n)$. Define the $ns \times ns$ matrix $\Delta = \mathsf{Deg} \otimes I_s$ (thus Δ has n non-zero diagonal blocks, each of size $s \times s$). For an $s \times s$ matrix P, if the matrix $I - P^2$ is invertible, then it is clear that

$$(3.1) \qquad \begin{pmatrix} I & P \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-P^2)^{-1} & -P(I-P^2)^{-1} \\ -P(I-P^2)^{-1} & (I-P^2)^{-1} \end{pmatrix}.$$

Recall that edge e_k has an $s \times s$ matrix weight W_k attached to it. For each edge $e_k = \{i, j\}$ of T, consider the following $ns \times ns$ matrix M_{e_k} which we describe in terms of $s \times s$ blocks as follows. The (i,i)-th and (j,j)-th blocks of M_{e_k} are $(I - W_k^2)^{-1}$ and the (i,j)-th and (j,i)-th blocks are $-W_k(I - W_k^2)^{-1}$. For other indices (a, b), define the (a, b)-block to be the $s \times s$ zero block. We have $n - 1$ such matrices M_{e_k} one for each edge e_k where $1 \le k \le n - 1$. With these definitions, the following inverse result was proved by Bapat and Sivasubramanian [3, Theorem 4].

Theorem 3.7. *Let $\mathcal{D}_T$ be the non-commutative analogue of the distance matrix of a tree T on n vertices with edge e_k having matrix weight W_k for $1 \le k \le n-1$. Then, $\mathcal{D}_T^{-1} = I - \Delta + \sum_{k=1}^{n-1} M_{e_k}$.*

With this background, we can now show our next main result.

Theorem 3.8. *Let T be a tree on n vertices with edge e_i having an $s \times s$ matrix weight W_i and let $\mathcal{D}_T$ be the non-commutative analogue of its distance matrix. Recall I_s is the $s \times s$ identity matrix and for $1 \le i < n$, define the $2s \times 2s$ matrix $L_i = \begin{pmatrix} I_s & W_i \\ W_i & I_s \end{pmatrix}$. Then,*

$$\mathsf{det2}(\mathcal{D}_T) = \left(\sum_{i=1}^{n-1} \mathsf{det2}(L_i) \prod_{j \ne i} \det L_j \right) - (n-2)(s-1) \det \mathcal{D}_T.$$

In particular, $\mathsf{det2}(\mathcal{D}_T)$ is independent of the structure of T and only depends on n and the matrices W_i for $1 \le i < n$.

Proof 3.9. *Using Theorem 1.1 and the fact that all diagonal entries are 1, we get,* $\mathsf{det2}(\mathcal{D}_T) = \det \mathcal{D}_T \cdot \mathsf{Trace}(\mathcal{D}_T^{-1}) - \det \mathcal{D}_T$. *By Theorem 3.7,* $\mathsf{Trace}(\mathcal{D}_T^{-1}) = \mathsf{Trace}(I - \Delta + \sum_{k=1}^{n-1} M_{e_k})$, *where all matrices have dimension $ns \times ns$. Breaking this up into two terms, we get* $\mathsf{Trace}(I - \Delta) = s(n - (2n-2)) = s(2-n)$ *and the term* $\mathsf{Trace}(\sum_{k=1}^{n-1} M_{e_k})$. *Recall the $2s \times 2s$ matrix $L_k = \begin{pmatrix} I_s & W_k \\ W_k & I_s \end{pmatrix}$, where all four block matrices are of dimension $s \times s$. For all $1 \le k < n$, since* $\mathsf{Trace}(M_{e_k}) = \mathsf{Trace}(L_k^{-1})$ *and since L_k has all diagonal entries 1, by Lemma 3.2, we get that* $\mathsf{Trace}(L_k^{-1}) = \left(1 + \frac{\mathsf{det2}(L_k)}{\det L_k}\right)$. *Thus,*

$$\begin{aligned}
\mathsf{det2}(\mathcal{D}_T) &= \det \mathcal{D}_T \mathsf{Trace}(\mathcal{D}_T^{-1}) - \det \mathcal{D}_T \\
&= \det \mathcal{D}_T \left(s(2-n) + \sum_{k=1}^{n-1} \left\{ 1 + \frac{\mathsf{det2}(L_k)}{\det L_k} \right\} - 1 \right) \\
&= \det \mathcal{D}_T \left(\sum_{k=1}^{n-1} \frac{\mathsf{det2}(L_k)}{\det L_k} \right) + \det(\mathcal{D}_T)(s-1)(2-n) \\
&= \left(\sum_{i=1}^{n-1} \mathsf{det2}(L_i) \prod_{j \ne i} \det L_j \right) - (n-2)(s-1) \det \mathcal{D}_T.
\end{aligned}$$

In the last line, we have used Theorem 1.3. The proof is complete.

We note that when each W_i is the 1×1 indeterminate w_i, then $L_i = \begin{pmatrix} 1 & w_i \\ w_i & 1 \end{pmatrix}$. That is, L_i is a matrix with $L_i = D_{B_i}$. It is easy to check that $\det L_i = 1 - w_i^2$, $\mathsf{det2}(L_i) = 1 + w_i^2$. In this case, we recover a special case of Theorem 3.5.

3.2. **Monomial immanant corresponding to $\lambda = 2, 1^{n-2}$.** In this subsection, we show that our results can be stated in the language of a monomial immanant. To describe monomial immanants, we need a few preliminaries from the theory of symmetric functions. We refer the reader to Stanley [13, Chapter 7] for relevant background. Given any symmetric function f of degree n in infinitely many variables $x_1, x_2, \cdots$, we can get a function $\psi_f : \mathfrak{S}_n \rightarrow \mathbb{Z}$ as follows.

Recall that each permutation $\pi \in \mathfrak{S}_n$ can be written in cycle notation. Let π have ℓ-cycles $C_1, C_2, \ldots, C_\ell$ with $t_i = |C_i|$ for $1 \leq i \leq \ell$. Since there is no order among the cycles C_i of π, we assume that $t_1 \geq t_2 \geq \cdots \geq t_\ell$. Thus, $w(\pi) = (t_1, t_2, \ldots, t_\ell)$ is a partition of the positive integer n and hence for each $\pi \in \mathfrak{S}_n$, we get a partition $w(\pi)$ of the integer n. We write this as $w(\pi) \vdash n$.

Symmetric functions of degree n with rational coefficients form a vector space denoted $\Lambda_{\mathbb{Q}}^n$ which is equipped with a standard inner product. To define the inner product, consider the basis, $\{m_\lambda\}$ for $\lambda \vdash n$, of monomial immanants and the basis $\{h_\mu\}$ for $\mu \vdash n$, of complete homogenous symmetric functions. That is, both bases are indexed by partitions $\lambda \vdash n$. If $f, g \in \Lambda_{\mathbb{Q}}^n$, write $f = \sum_{\lambda \vdash n} A_\lambda m_\lambda$ and $g = \sum_{\mu \vdash n} B_\mu h_\mu$. Their inner-product denoted $\langle f, g \rangle$ is defined as $\langle f, g \rangle = \sum_{\lambda \vdash n} A_\lambda B_\lambda$.

Another basis for degree n symmetric functions are the power sum symmetric functions p_λ for $\lambda \vdash n$. We are now in a position to define the map $\psi_f : \mathfrak{S}_n \rightarrow \mathbb{Z}$. Define $\psi_f : \mathfrak{S}_n \rightarrow \mathbb{Z}$ by $\psi_f(\pi) = \langle f, p_{w(\pi)} \rangle$.

Consider the partition $\lambda_2 = (2, 1^{n-2})$ and the symmetric function m_{λ_2}. Define $\psi_2 = \psi_{m_{\lambda_2}}$. Thus $\psi_2 : \mathfrak{S}_n \rightarrow \mathbb{Z}$ is the function obtained from the monomial symmetric function m_{λ_2}. For an $n \times n$ matrix A, consider the immanant $\det_{\psi_2}(A)$ defined with respect to ψ_2. That is, $\det_{\psi_2}(A)$ is the immanant defined with respect to the monomial symmetric function m_{λ_2}. Such immanants are referred to as monomial immanants. We need the following result (see Stembridge, proof of Theorem 2.7 [14]).

Theorem 3.10. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix. Then,*

$$\det_{\psi_2}(A) = \sum_{i=1}^n a_{i,i} \det A(i) - n \det A.$$

We get the following simple corollary from Theorems 3.10 and 1.1.

Corollary 3.11. *Let A be an $n \times n$ matrix. Then,* $\mathsf{det2}(A) = \det_{\psi_2}(A) + (n-1) \det A$.

With this result, we get the following corollary of Theorem 3.5.

Corollary 3.12. *Let D_G be the product distance matrix of a connected graph G with blocks $B_1, B_2, \ldots, B_p$, with the property that $\det D_{B_i} \neq 0$ for all $1 \leq i \leq p$. Then,*

$$\det_{\psi_2}(D_G) = \sum_{i=1}^p \left[\mathsf{det2}(D_{B_i}) \prod_{j \neq i} \det D_{B_j}\right] - (n-1) \prod_{i=1}^p \det D_{B_i}.$$

In particular, $\det_{\psi_2}(D_G)$ is independent of the tree-like manner of connection of G's blocks.

4. **Corollaries**

In this section, we derive some corollaries for the exponential distance matrix of a tree T and the q-analogue of T's laplacian. We then give a q-analogue again for trees of Schur's dominance theorem. Lastly, we find $\mathsf{det2}(K)$ where K is the matrix appearing in Theorem 3.1.

4.1. **The q-analogue of T's laplacian.** Let $G = T$ be a tree with exponential distance matrix ED_T. Let A be the adjacency matrix of T and D be a diagonal matrix with $d_{v,v} = \deg(v)$ where $\deg(v)$ is the degree of vertex v in T. Define the q-analogue of T's laplacian as $\mathcal{L}_q = I - qA + q^2(D - I)$ where q is an indeterminate. It is easy to see that when $q = 1$, $\mathcal{L}_q = L$, where L is the laplacian matrix of T.

Bapat, Lal and Pati (see [1, Proposition 3.3]) showed the following surprising inverse result for trees. Their result can alternatively be proved using Theorem 3.1.

Theorem 4.1. *Let T be a tree with exponential distance matrix ED_T. Then $\mathsf{ED}_T^{-1} = \frac{1}{1-q^2}\mathcal{L}_q$.*

Theorem 4.1 gives us the following corollary.

Corollary 4.2. *Let T be a tree on $n \geq 2$ vertices and let ED_T be its exponential distance matrix. Then, $\mathsf{det2}(\mathsf{ED}_T) = (n-1)(1-q^2)^{n-2}(1+q^2)$.*

Proof 4.3. *There are several ways to prove this. We give two of them. From Theorem 4.1, we see that $\det \mathsf{ED}_T(i) = (1-q^2)^{n-2}[1 + q^2(\deg(i) - 1)]$. It is known that $\det \mathsf{ED}_T = (1-q^2)^{n-1}$. Plugging both of these in Theorem 1.1, we get $\mathsf{det2}(\mathsf{ED}_T) = (n-1)(1-q^2)^{n-2}(1+q^2)$.*

Alternatively, we use Theorem 3.5. Each block of T is K_2, the complete graph on 2 vertices and there are $n-1$ such blocks. Thus, for all $1 \leq i < n$, $D_{B_i} = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}$. It is simple to see that $\det D_{B_i} = 1 - q^2$ and that $\mathsf{det2}(D_{B_i}) = 1 + q^2$. The proof is complete by applying Theorem 3.5.

The second immanant $\mathsf{det2}(\mathcal{L}_q)$ can also be found using the above result. We recall the following striking theorem of Merris (see Merris and Watkins [9, Page 239]).

Theorem 4.4 (Merris)**.** *Let A be an invertible matrix. Then, $\det A \cdot \mathsf{det2}(A^{-1}) = \det A^{-1} \cdot \mathsf{det2}(A)$.*

Let T be a tree with ED_T as its exponential distance matrix and let $\mathcal{L}_q$ be the q-analogue of its laplacian. Below, we present a q-analogue of Corollary 2.2.

Corollary 4.5. *If T is a tree on n vertices and $\mathcal{L}_q$ is the q-analogue of its laplacian, then, $\mathsf{det2}(\mathcal{L}_q) = (n-1)(q^2+1)$. Hence, $\mathsf{det2}(\mathcal{L}_q)$ only depends on n, and is independent of the structure of the tree T.*

Proof 4.6. *It is known (see [1]), that $\det \mathsf{ED}_T = (1-q^2)^{n-1}$, $\det \mathcal{L}_q = (1-q^2)$. Using Theorems 4.4, 4.1 and combining this with Corollary 4.2 yields $\mathsf{det2}(\mathcal{L}_q) = (n-1)(q^2+1)$. Alternatively, one can just use Theorems 4.1 and 1.1.*

We recall the following notation used in q-series theory. Let q be an indeterminate and for a positive integer i, let $[i]_q = 1 + q + \cdots + q^{i-1}$ with $[0]_q = 0$. Then, the above corollary can be alternatively written as $\mathsf{det2}(\mathcal{L}_q) = (n-1)[2]_{q^2}$ for trees. Clearly, setting $q = 1$ gives us $[2]_{q^2} = 2$ and $\mathcal{L}_q = L$. In this case, we recover Corollary 2.2.

4.2. **Schur's Dominance Theorem.** Schur [11] showed the following "Dominance Theorem" for positive semidefinite matrices $A = (a_{i,j})_{1 \le i,j \le n}$. Recall that if λ is an irreducible representation of $\mathfrak{S}_n$ over the complex numbers $\mathbb{C}$ with character χ_λ, then, $\det_{\chi_\lambda}(A) = \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) \prod_{i=1}^n a_{i,\sigma_i}$. Also recall that $\chi_\lambda(\mathsf{id})$ is the degree of the representation λ, where id is the identity permutation in $\mathfrak{S}_n$. With these, Schur's result can be stated as follows.

Theorem 4.7. *Let A be an $n \times n$ positive semidefinite matrix and let λ be an irreducible representation of $\mathfrak{S}_n$ with character χ_λ. Then, $\det_{\chi_\lambda}(A) \ge \chi_\lambda(\mathsf{id}) \det A$.*

It is well known that the laplacian matrix L of any graph is positive semidefinite (see [15]). Similarly, the $n \times n$ matrix J_n with all entries being 1 is also positive semidefinite. If λ_2 is the irreducible representation of $\mathfrak{S}_n$ indexed by the partition $2, 1^{n-2}$, then by the Hook-length formula, it follows that $\chi_2(\mathsf{id}) = n - 1$ where χ_2 denotes χ_{λ_2}, see Sagan [10]. Thus, if T is a tree on n vertices, then it follows that $\mathsf{det2}(L) \ge (n-1)\det L \ge 0$. As seen earlier for all graphs, when $q = 1$, $\mathcal{L}_q = L$. Similarly for trees T on n vertices, when $q = 1$, then $\mathsf{ED}_T = J_n$. Thus, when $q = 1$, we have $\mathsf{det2}(\mathcal{L}_q) \ge 0$ for all graphs (and hence for trees) and for trees T, we have $\mathsf{det2}(\mathsf{ED}_T) \ge 0$. We show that for a tree, both ED_T and $\mathcal{L}_q$ satisfy Theorem 4.7 for all $q \in \mathbb{R}$.

Corollary 4.8. *Let T be a tree on n vertices. Let $\mathcal{L}_q$ be the q-analogue of its laplacian and let ED_T be its exponential distance matrix. Then, for all $q \in \mathbb{R}$, $\mathsf{det2}(\mathcal{L}_q) \ge \chi_2(\mathsf{id}) \det \mathcal{L}_q$, and $\mathsf{det2}(\mathsf{ED}_T) \ge \chi_2(\mathsf{id}) \det \mathsf{ED}_T$.*

Proof 4.9. *From the proof of Corollary 4.5, we have $\mathsf{det2}(\mathcal{L}_q) = (n-1)(1+q^2)$, $\det \mathcal{L}_q = (1-q^2)$ and $\chi_2(\mathsf{id}) = (n-1)$. Similarly, Corollary 4.2 gives us $\mathsf{det2}(\mathsf{ED}_T) = (n-1)(1-q^2)^{n-2}(1+q^2)$ and $\det \mathsf{ED}_T = (1-q^2)^{n-1}$. Plugging in these values completes the proof.*

4.3. $\mathsf{det2}(\mathcal{D}_T^{-1})$**.** In this subsection, we find $\mathsf{det2}(\mathcal{D}_T^{-1})$, where $\mathcal{D}_T$ is the non-commutative analogue of the distance matrix of T. Using Theorem 4.4 and Theorem 1.3, we get the following corollary. Recall the following notation from Theorem 3.8. If W_i is the $s \times s$ "weight matrix" on edge e_i, then $L_i = \begin{pmatrix} I & W_i \\ W_i & I \end{pmatrix}$. It is simple to see that $\det L_i = \det(I - W_i^2)$.

Corollary 4.10. *For a tree T, let $\mathcal{D}_T$ be the non-commutative analogue of its distance matrix and let $K = \mathcal{D}_T^{-1}$ be its inverse. Then,*

$$\mathsf{det2}(K) = \frac{\sum_{i=1}^{n-1} \frac{\mathsf{det2}(L_i)}{\det L_i} - (n-2)(s-1)}{\det \mathcal{D}_T} \tag{4.1}$$

Proof 4.11. *From Theorem 4.4, we get* $\mathsf{det2}(A^{-1}) = \dfrac{\mathsf{det2}(A)}{(\det A)^2}$*. Theorem 1.3 gives a product rule that* $\det \mathcal{D}_T = \prod_{i=1}^{n-1} \det L_i$ *and Theorem 3.8 gives us*

$$\mathsf{det2}(\mathcal{D}_T) = \left(\sum_{i=1}^{n-1} \mathsf{det2}(L_i) \prod_{j \neq i} \det L_j \right) - (n-2)(s-1) \det \mathcal{D}_T.$$

Combining these completes the proof.

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