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On the two variable distance enumerator of the Shi hyperplane arrangement

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Abstract

We give an interpretation for the coefficients of the two variable refinement $D_{\mathcal{S}_n}(q,t)$ of the distance enumerator of the Shi hyperplane arrangement $\mathcal{S}_n$ in n dimensions. This two variable refinement was defined by Stanley in [R.P. Stanley, Hyperplane arrangements, parking functions and tree inversions, in: B. Sagan, R. Stanley (Eds.), Mathematical Essays in Honor of Gian-Carlo Rota, Birkhauser, Boston, Basel, Berlin, 1998, pp. 359–375] for the general r-extended Shi hyperplane arrangements.

For the Shi hyperplane arrangement, we define three natural partitions of the number $(n+1)^{n-1}$. The first arises from parking functions of length n, the second from geometric considerations and the third from inversions on rooted spanning forests on n vertices. We call the three partitions as the *parking partition*, the *geometric partition* and the *inversion partition* respectively. We show that one of the parts of the parking partition is identical to the number of edge-labelled trees with label set $\{1,2,\ldots,n\}$ on $n+1$ unlabelled vertices. We prove that the parking partition majorises the geometric partition and conjecture that the inversion partition also majorises the geometric partition.

1. Introduction

Let $r \geq 1$ and $n \geq 2$. The r-extended Shi hyperplane arrangement in n dimensions is denoted $\mathcal{S}_n^r$. It is given by the following hyperplanes in $\mathbb{R}^n$.

$$x_i - x_j = -r+1, -r+2, \ldots r, \quad \text{for } 1 \leq i < j \leq n.$$

1.1. The Shi hyperplane arrangement

When $r = 1$, the arrangement is called the Shi hyperplane arrangement in n dimensions and denoted $\mathcal{S}_n$.

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Its distance enumerator is defined with respect to a base region B as follows. Let $\mathcal{R}(\mathcal{S}_n)$ be the set of regions of the Shi hyperplane arrangement. Each region $R \in \mathcal{R}(\mathcal{S}_n)$ is separated from B by a set H_R of hyperplanes. Let $h_R = |H_R|$ and define the distance polynomial as $\mathsf{Dist}(\mathcal{S}_n, q) = \sum_{R \in \mathcal{R}(\mathcal{S}_n)} q^{h_R}$. The base region B is the region bounded by the hyperplanes $x_n > x_{n-1} > \cdots > x_2 > x_1$ and $x_1 - x_n < 1$.

It is known (see [5], Corollary 5.11) that $|\mathcal{R}(\mathcal{S}_n)| = (n+1)^{n-1}$. This is also the number of n-length *Parking Functions*. We recall the definition of an n-length parking function. There are n parking spaces $0, 1, \ldots, n-1$ in a one-way street. n cars $C_1, C_2, \ldots C_n$ enter the street in that order. C_i has a preferred space a_i and proceeds directly to slot a_i. If slot a_i is occupied, it will try to park in the next available space. If a car leaves the street without parking then the process fails. $\overline{a} = (a_1, a_2, \ldots, a_n)$ is an n-length parking function if all cars can park with a_i being their respective choices. The set of all parking functions of length n is denoted PF_n. It is known that $\overline{a} = (a_1, a_2, \ldots, a_n)$ is a parking function iff the weakly increasing permutation $\overline{b} = (b_1, b_2, \ldots, b_n)$ of $\overline{a}$ satisfies $b_i < i$ (see [3], Exercise 5.49).

It is also known (see [5], Corollary 6.14) that $\mathsf{Dist}(\mathcal{S}_n, q) = \sum_{\overline{a} \in \mathsf{PF}_n} q^{a_1 + a_2 + \cdots + a_n}$. $\mathsf{Dist}(\mathcal{S}_n, q)$ satisfies the remarkable identity

$$\mathsf{Dist}(\mathcal{S}_n, q) = q^{\binom{n}{2}} I_{n+1}(1/q), \tag{1}$$

where $I_{n+1}(q) = \sum_T q^{\mathrm{inv}(T)}$ is the inversion enumerator, where the sum is over all spanning trees, T on $[n] \cup \{0\}$ (see [5], Theorem 6.22).

Stanley (see [4]) defined a two variable distance enumerator of the Shi hyperplane arrangement with respect to the same base region B. For each region $R \in \mathcal{R}(\mathcal{S}_n)$, let a_R be the number of separating hyperplanes of the form $x_i - x_j = 0$ and b_R be the number of separating hyperplanes of the form $x_i - x_j = 1$. The two variable distance enumerator is defined as $D_{\mathcal{S}_n}(q, t) = \sum_{R \in \mathcal{R}(\mathcal{S}_n)} q^{a_R} t^{b_R}$. We denote the coefficient of $q^k t^\ell$ of $D_{\mathcal{S}_n}(q, t)$ as $\mathsf{Dist}_n(k, \ell)$. We reproduce from [4] the two variable enumerator for $n = 3, 4$ below. The question of an interpretation for these numbers was posed (see [5], page 106). We give an answer in terms of number of ideals of a poset IP_π associated with permutations $\pi \in S_n$.

$k \backslash \ell$	0	1	2	3
0	1	1	2	1
1	2	2	2	
2	2	2		
3	1			

$n = 3$

$k \backslash \ell$	0	1	2	3	4	5	6
0	1	1	2	3	3	3	1
1	3	3	6	7	6	3	
2	5	5	8	9	5		
3	6	7	9	6			
4	5	6	5				
5	3	3					
6	1						

$n = 4$

Fix n and for $0 \le k \le \binom{n}{2}$ let Π_k be the set of permutations on $[n]$ which have exactly k non-inversions. For a permutation $\pi \in \Pi_k$, let IP_π be a poset of its inversions ordered by containment (that is, if $g = (\pi_i, \pi_j)$ where $i < j$, and $h = (\pi_a, \pi_b)$ where $a < b$, are inversions, then $g \le_{\mathsf{IP}_\pi} h$ iff $a \le i < j \le b$). For example, when $\pi = 623415$, the poset IP_{623415} is shown in Fig. 1. For $\pi \in \Pi_k$, let the number of *ideals* of IP_π with $\binom{n}{2} - k - \ell$ elements be $\mathsf{IP}_\pi(\ell)$. We prove the following.

Theorem 1. $\mathsf{Dist}_n(k, \ell) = \sum_{\pi \in \Pi_k} \mathsf{IP}_\pi(\ell)$.

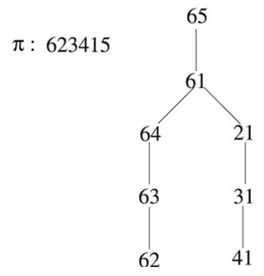

Fig. 1. An example of the poset IP_π.

Theorem 1 gives a two variable generalisation to the equality (see [3], page 96)

$$\sum_{\pi \in S_n} F(J(\mathsf{NIP}_\pi), q) = I_{n+1}(q), \tag{2}$$

where S_n is the set of permutations on n distinct letters, $F(J(\mathsf{NIP}_\pi), q)$ is the rank generating function of the lattice of order ideals of the poset of non-inversion NIP_π which is similar to IP_π, the only difference being that we order *non-inversions* of π instead of its *inversions* and $I_{n+1}(q)$ is the inversion enumerator of rooted spanning forests on $[n]$. The proof of Theorem 1 is given in Section 2.

Please see Section 3 for the definitions and results about the three partitions of the number n^{n-2}.

2. Two variable distance enumerator: An interpretation

In this section, we prove Theorem 1. We recall that the arrangement has hyperplanes $x_i - x_j = 0, 1$ for $1 \le i < j \le n$.

2.1. Representing a region of $\mathcal{R}(\mathcal{S}_n)$

We need the "embroidered permutation" representation (see [5], page 81 or [2]) of an $R \in \mathcal{R}(\mathcal{S}_n)$. An embroidered permutation is a pair $(\pi, \mathcal{C})$ where $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation of $[n]$ and $\mathcal{C}$ is a family of inversions of π such that if $g = (\pi_i, \pi_j) \in \mathcal{C}$ where $i < j$ and $h = (\pi_a, \pi_b) \in \mathcal{C}$ where $a < b$, then it is not the case that $i \le a < b \le j$.

There is a bijection between $\mathcal{R}(\mathcal{S}_n)$ and embroidered permutations. Let $R \in \mathcal{R}(\mathcal{S}_n)$. All points $x = (x_1, x_2, \ldots, x_n) \in R$ can be linearly ordered according to increasing x_i values and this linear ordering gives us a permutation. It is simple to see that this permutation is independent of the point x. We further need to specify which pairs of indices (i, j) satisfy $x_i - x_j < 1$.

Given a pair $(\pi, \mathcal{C})$, consider the region $R \in \mathcal{R}(\mathcal{S}_n)$ bounded by the hyperplanes $x_{\pi_1} < x_{\pi_2} < \cdots < x_{\pi_n}$ and $\forall g = (\pi_i, \pi_j) \in \mathcal{C}$ with $i < j$, $x_{\pi_j} - x_{\pi_i} < 1$. (This is the reason why we need the family $\mathcal{C}$ to contain inversions.) Conversely, given $R \in \mathcal{R}(\mathcal{S}_n)$, all valid points $\overline{a}_R = (a_1, a_2, \ldots, a_n)$ of R will give the same permutation π when the a_i's are listed in increasing order. This gives us the π part. Consider indices i, j where $\pi_i > \pi_j$ such that $a_{\pi_j} - a_{\pi_i} < 1$. Such (π_i, π_j) pairs are as before seen to be inversions and it is simple to see that the containment-wise maximal pairs form a family $\mathcal{C}$ with the non-containment property.

We note that the base region B has the embroidered permutation representation $(\sigma, \mathcal{S})$ where $\sigma = (n, n-1, \ldots, 2, 1)$ and $\mathcal{S} = \{\{n, 1\}\}$.

2.2. *Two parameter distance of* $(\pi, \mathcal{C})$

To each embroidered permutation $(\pi, \mathcal{C})$, we need to give a pair of non-negative integers (k, ℓ) such that $\mathsf{Dist}((\pi, \mathcal{C}), B) = (k, \ell)$, that is, there are k hyperplanes of the type $x_\alpha - x_\beta = 0$ (where $\alpha < \beta$) and ℓ hyperplanes of the type $x_\alpha - x_\beta = 1$ (where $\alpha < \beta$) which separate the regions $(\pi, \mathcal{C})$ and $B = (\sigma, \mathcal{S})$. When the region $(\pi, \mathcal{C})$ is fixed, we call k the number of *zero separating* hyperplanes and ℓ as the number of *one separating* hyperplanes associated with $(\pi, \mathcal{C})$.

2.2.1. *Zero separating hyperplanes*

Lemma 1. *For the region $R = (\pi, \mathcal{C})$, k is the number of non-inversions of π.*

Proof. To see this, we note that if $h = (\pi_i, \pi_j)$ where $i < j$ is a non-inversion of π, then $(\pi, \mathcal{C})$ satisfies $x_{\pi_i} - x_{\pi_j} < 0$ (by definition of the region of the embroidered permutation) while B satisfies $x_{\pi_i} - x_{\pi_j} > 0$. Thus (π_i, π_j) gives rise to a zero separating hyperplane.

Conversely, if $x_\alpha - x_\beta = 0$ (where $\alpha < \beta$) is a hyperplane separating B and $(\pi, \mathcal{C})$, it is easy to see that $x_\alpha - x_\beta > 0$ in the region $(\pi, \mathcal{C})$ and hence that (α, β) is a non-inversion in π. Thus, non-inversions of π correspond to zero separating hyperplanes. ■

2.2.2. *One separating hyperplanes*

We need the notion of the poset IP_π associated with a permutation π. We note that the family $\mathcal{C}$ which occurs as part of the embroidered permutation $(\pi, \mathcal{C})$ is an order ideal of IP_π. The reason for this is geometric: when some inversion has length strictly less than 1, then any inversion contained within it will also have length strictly less than 1. Thus, when we pick the maximal (with respect to IP_π) inversions which have lengths strictly less than 1, we get an ideal (or equivalently, an antichain) of IP_π. Hence, the family $\mathcal{C}$ of any embroidered permutation $(\pi, \mathcal{C})$, can be considered as an order ideal of IP_π.

Below, we connect the size of an ideal represented by the family $\mathcal{C}$ and the number of one separating hyperplanes of $(\pi, \mathcal{C})$.

Lemma 2. *The number of one separating hyperplanes of $(\pi, \mathcal{C})$ is the number of elements of IP_π not contained in the ideal $\mathcal{C}$.*

Proof. We first show that all elements of IP_π not in the order ideal represented by $\mathcal{C}$ are one separating hyperplanes of $(\pi, \mathcal{C})$. Let $g = (\pi_i, \pi_j) \notin \mathcal{C}$ (where $i < j$) be an inversion of π. By definition of the region $(\pi, \mathcal{C})$, since $x_{\pi_i} < x_{\pi_j}$ and since (π_i, π_j) is an inversion, $x_{\pi_j} - x_{\pi_i} > 0$. We claim that in fact $x_{\pi_j} - x_{\pi_i} > 1$. To see this, we note that there are only two choices for the value $x_{\pi_j} - x_{\pi_i}$: < 1 or > 1 and those inversions (π_i, π_j) such that $x_{\pi_j} - x_{\pi_i} < 1$ are those precisely in the ideal of $\mathcal{C}$. Since $g \notin \mathcal{C}$, the claim that $x_{\pi_j} - x_{\pi_i} > 1$ follows.

We note that whereas in B, for all pairs (α, β) with $\alpha < \beta$, $x_\alpha - x_\beta < 1$. The argument is reversible and this gives a bijection between one separating hyperplanes and inversions of π not in the ideal represented by $\mathcal{C}$. ■

Proof (*of Theorem* 1). The theorem follows from Lemmata 1 and 2. ■

Remark 1. Let π have k non-inversions. Let $R = (\pi, \mathcal{C}_1) \in \mathcal{R}(\mathcal{S}_n)$ and $S = (\pi, \mathcal{C}_2) \in \mathcal{R}(\mathcal{S}_n)$ be two regions such that the order ideals corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$ differ by exactly one element of IP_π. By Theorem 1, there is a single hyperplane which separates R and S. Hence, the lattice $J(\mathsf{IP}_\pi)$ when treated as a graph is the subgraph of the distance graph of $\mathcal{R}(\mathcal{S}_n)$ with respect to the base region B consisting of those regions of $\mathcal{R}(\mathcal{S}_n)$ which are within a given region of $\mathcal{B}_n$ (the Braid arrangement).

3. Three partitions of $\mathcal{R}(\mathcal{S}_n)$

3.1. Definitions

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $[n_0] = \{0\} \cup [n]$. Let T be a spanning tree on the set $[n_0]$. We call the vertex 0 as the "root" of T and call such trees 0-rooted spanning trees.

From the bijection between $\mathcal{R}(\mathcal{S}_n)$ and the set of spanning trees on $(n+1)$ vertices $[n_0] = \{0, 1, \ldots, n\}$ (see [5], Theorem 6.23), we can view the regions alternatively as 0-rooted spanning trees on $[n_0]$. Likewise, we can also view the regions as indexed by Parking Functions of length n.

3.1.1. Parking partition

Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathsf{PF}_n$. We partition PF_n into the following three parts: those with $a_1 > a_2$, with $a_1 = a_2$ and with $a_1 < a_2$. We call the number of such n-length parking functions as gt_n, eq_n and lt_n respectively. It is clear that we could have chosen any indices $i \neq j$ and partitioned PF_n into three parts as above depending on the relation between a_i and a_j and still obtained the same numbers. Below we tabulate the numbers gt_n, eq_n and lt_n for small values of n.

3.1.2. Geometric partition

Consider the hyperplane $x_1 - x_2 = \alpha$ for $\alpha = 0, 1$; and let $R \in \mathcal{R}(\mathcal{S}_n)$. Let $\overline{a}_R = (a_1, a_2, \ldots, a_n)$ be any point in R. Clearly, the value $a_1 - a_2$ is either < 0, strictly between 0 and 1, or > 1 and this condition is independent of the point $\overline{a}_R$. Thus each region R with respect to the dimensions x_1 and x_2 satisfies one of the three properties: all points $\overline{a}_R \in R$ either have $a_1 - a_2 < 0$, or $0 < a_1 - a_2 < 1$ or $a_1 - a_2 > 1$.

Let $R_n^{\cdot<0}$, $R_n^{0<\cdot<1}$ and $R_n^{\cdot>1}$ respectively denote the number of regions satisfying the above three conditions. The main reason for this definition is to understand how $|\mathcal{R}(\mathcal{S}_n)|$ gets partitioned by the parallel hyperplanes $x_1 - x_2 = 0, 1$. Below we tabulate the numbers $R_n^{\cdot<0}$, $R_n^{0<\cdot<1}$ and $R_n^{\cdot>1}$ for a few initial values of n.

3.1.3. Inversion partition

Let T be a 0-rooted spanning tree on $[n_0]$. Let $v_1, v_2 \in [n]$, $v_1 < v_2$ be two fixed vertices of T. There are again three possibilities for the following path relation: either v_1 is on the unique v_2-0 path; or v_2 is in the unique v_1-0 path (that is, the pair (v_1, v_2) is an inversion of T); or neither of the two happens. Let T_n^{v1}, T_n^{v2} and T_n^{disj} be the number of 0-rooted spanning trees on $[n_0]$ for each of the above three choices. These numbers are again independent of the choices v_1, v_2. We tabulate the numbers T_n^{disj}, T_n^{v1} and T_n^{v2} for small values of n below.

n	gt_n	lt_n	eq_n
3	6	6	4
4	50	50	25
5	540	540	216
6	7203	7203	2401

n	$R_n^{\cdot<0}$	$R_n^{0<\cdot<1}$	$R_n^{\cdot>1}$
3	6	5	5
4	50	37	38
5	540	366	390
6	7203	4553	5051

n_0	T_n^{disj}	T_n^{v1}	T_n^{v2}
3	6	5	5
4	51	37	37
5	564	366	366
6	7701	4553	4553

From the above tables, we have the following.

Conjecture 1. *The smallest parts of the geometric partition and the inversion partition are equal.*

3.2. Properties of the partitions

We prove some properties about the order of the components of the three partitions.

Lemma 3. *For $n \geq 2$, the parking partition satisfies $gt_n = lt_n \geq eq_n$.*

Proof. Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathsf{PF}_n$ with $a_1 > a_2$. Clearly, $\overline{a}' = (a_2, a_1, \ldots, a_n)$ obtained from $\overline{a}$ by swapping the first two coordinates is also a valid parking function, and has $a'_1 < a'_2$. The argument is reversible and this bijection proves that $gt_n = lt_n$.

We show that $lt_n \geq eq_n$. Let $\overline{a} \in eq_n$. Let $\overline{b} = (a_1, a_2+1, a_3, \ldots, a_n)$ and $\overline{c} = (c_1, c_2, \cdots, c_n)$ be a weakly increasing permutation of $\overline{b}$. We show that $\overline{b} \in lt_n$. We only need to check that $\overline{b} \in \mathsf{PF}_n$. Suppose not, then there is an index k such that $c_k \not< k$ since we changed only one coordinate to obtain $\overline{b}$ from $\overline{a}$, $c_k = a_2 + 1$. But then $a_1 = c_k - 1$ will be c_{k-x} for $x \geq 1$. Hence, there exists an index $k - x$ such that $c_{k-x} \not< k - 1$, which means that $\overline{a} \notin \mathsf{PF}_n$ which is a contradiction. ■

Lemma 4. *For $n \geq 2$, the geometric partition satisfies $R_n^{\cdot<0} \geq \max(R_n^{\cdot>1}, R_n^{0<\cdot<1})$.*

Proof. We first prove that $R_n^{\cdot<0} \geq R_n^{\cdot>1}$. To do this, we note that the regions $R_n^{\cdot<0}$ are those which are separated from B by the hyperplane $x_1 - x_2 = 0$. Hence, they correspond to embroidered permutations $(\pi, \mathcal{C})$ such that 1 precedes 2 in π. Such a permutation contributes $|J(\mathsf{IP}_\pi)|$ elements to $R_n^{\cdot<0}$.

Similarly, regions $R_n^{\cdot>1}$ are those which correspond to embroidered permutations $(\pi', \mathcal{C})$ such that 2 precedes 1 in π' and with $(1, 2) \notin \mathcal{C}$. Consider the ideals of $\mathsf{IP}_{\pi'}$ which do not contain the inversion $\{2, 1\}$ (and hence all elements $X = \{x \geq_{\mathsf{IP}_{\pi'}} \{2, 1\}\}$). Let $\mathsf{IP}_{\pi'}(21)$ denote the subposet $\mathsf{IP}_{\pi'} - X$.

There is a simple bijection between a π with 1 preceding 2 and a π' with 2 preceding 1. We claim a slightly stronger property: For each (π, π') pair, $|J(\mathsf{IP}_\pi)| \geq |J(\mathsf{IP}_{\pi'}(21))|$. To prove this, we note that it is simple to see that the $\mathsf{IP}_{\pi'}(21)$ is a subposet of IP_π. Summing over the (π, π') pairs completes the proof.

An almost identical proof works to show that $R_n^{\cdot<0} \geq R_n^{0<\cdot<1}$. We note that $R_n^{0<\cdot<1}$ is the number of $(\pi', \mathcal{C})$ where π' is a permutation with 2 preceding 1 and $\mathcal{C}$ is an ideal of $\mathsf{IP}_{\pi'}$ such that the inversion $(2, 1) \in \mathcal{C}$. Thus $X = \{x | x <_{\mathsf{IP}_{\pi'}} (2, 1)\} \in \mathcal{I}$ as well. Let $\mathsf{IP}_{\pi'}(2, 1) = \mathsf{IP}_{\pi'} - X$. The remaining argument is identical. ■

Lemma 5. *For $n \geq 2$, the inversion partition satisfies $T_n^{\text{disj}} \geq T_n^{v1} = T_n^{v2}$.*

Proof. We first prove that $T_n^{v1} = T_n^{v2}$. Let $T \in T_n^{v1}$. Thus T is a 0-rooted spanning tree on $[n_0]$ and $v1$ is on the unique $v2 - 0$ path. By swapping the vertices $v2$ and $v1$, we get a tree $T' \in T_n^{v2}$. The equality part of the Lemma is thus proved.

To show that $T_n^{\text{disj}} \geq T_n^{v1}$, let $T \in T_n^{v1}$ as before. Let T'' be obtained from T by swapping $v1$ and 0. Clearly $T'' \in T_n^{\text{disj}}$. ■

3.3. Properties among the partitions

We recall that eq_n is the number of $\overline{a} \in \mathsf{PF}_n$ which satisfies $a_1 = a_2$.

Theorem 2. *For all $n \geq 1$, $eq_n = (n + 1)^{n-2}$.*

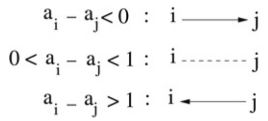

Fig. 2. Representing the three possibilities, where $i < j$.

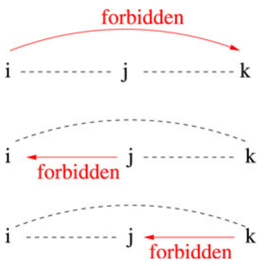

Fig. 3. The three forbidden subposets where $i < j < k$.

Proof (*of Theorem* 2). The proof of Pollack given in [5], page 92 to count the number of n-length parking functions carries over exactly. ■

It would be nice to get a combinatorial proof of Theorem 2. Let $n, k \geq 2$ and let EPF_n^k be the $\overline{a} \in \mathsf{PF}_n$ such that $a_1 = a_2 = \cdots = a_k$. (With this notation, $eq_n = \mathsf{EPF}_n^2$.)

Corollary 1. *For* $n, k \geq 2$, $|\mathsf{EPF}_n^k| = (n+1)^{n-k}$.

Remark 2. For $n \geq 1$, let UT_n be the number of edge-labelled trees with label set $\{1, 2, \ldots, n\}$ on $n+1$ unlabelled vertices. It is known (see [3], Exercise 5.27) that $UT_n = (n+1)^{n-2}$.

Theorem 3. *For* $n \geq 2$, $gt_n = R_n^{\cdot<0}$.

For the proof of Theorem 3, we need the poset representation of a region $R \in \mathcal{R}(\mathcal{S}_n)$. This representation was defined by Athanasiadis [1]. We briefly discuss this below.

Let $\overline{a}_R$ be a point of $R \in \mathcal{R}(\mathcal{S}_n)$. For a pair (i, j) where $i < j$, represent each of the three possibilities $a_i - a_j < 0$, $0 < a_i - a_j < 1$ and $a_i - a_j > 1$ as in Fig. 2 (the dotted lines in the second figure represent an incomparability relation between the vertices i and j). We call arcs of the form (i, j) where $i < j$ as *forward arcs* and those of the form (j, i) where $i < j$ as *backward arcs*.

Athanasiadis showed that this representation yields a poset on $[n]$ and that such posets do not have three forbidden subposets (shown in Fig. 3). Athanasiadis also proved that any poset without these three forbidden subposets arose from a region thereby characterising such posets. We refer to such posets as "tree-posets".

Proof (*of Theorem* 3). For this proof, we need the tree-poset representation of a region $R \in \mathcal{R}(\mathcal{S}_n)$. We use the bijection of Pak and Stanley [4], coupled with the forbidden subposets of Athanasiadis [1]. By Lemma 4, $R_n^{\cdot<0}$ is the largest part of the geometric partition. We recall that the posets P_R of such a region R has a forward arc $(1, 2)$ between vertices 1 and 2. It is straightforward to see that the hyperplane $x_1 - x_2 = 0$ separates all regions of $R_n^{\cdot<0}$ and B.

Thus, in the bijection of Pak and Stanley, we must cross this hyperplane at some point and this crossover contributes a 1 to a_1, the first component of the parking function $\overline{a}$ and 0 to a_2. It

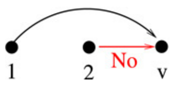

Fig. 4. When $(1, v)$ is a forward arc and $(2, v)$ is not.

is simple to check that the only way to increase a_2 is to cross the hyperplane $x_2 - x_v = 0$ for some $v \in [n] - \{1, 2\}$ on a path from B to R. All such crossovers are recorded by a forward arc $(2, v)$ in the poset representation of R. For such vertices v, since $(2, v)$ and $(1, 2)$ are forward arcs, by transitivity of the poset, $(1, v)$ is also a forward arc and this means we contribute a 1 to a_1 as well. This completes the proof of one-half of the bijection.

For the other half, let $\overline{a} \in gt_n$. We claim that its corresponding region R under the bijection of Pak and Stanley has $(1, 2)$ as a forward arc. As before, if $a_2 = k$, there exists a set S with $|S| = k$ such that for all $v \in S$, $(2, v)$ is a forward arc. Similarly, when $a_1 = k + x$ for $x > 0$, there is a set T such that for all $v \in T$, $(1, v)$ is a forward arc. We claim that $2 \in T$ (and thus $1 \notin S$). Suppose not, then there is a vertex $v \in T - S$, $v \neq 2$ such that $(1, v)$ is a forward arc and $(2, v)$ is not (see Fig. 4). Thus there are two cases for the relation between 2 and v.

- When $(2, v)$ is a backward arc : As $(1, v)$ and $(v, 2)$ are arcs, by transitivity $(1, 2)$ also is, and thus $2 \in T$ contradicting our supposition.
- When $(2, v)$ is an incomparability : If $(1, 2)$ is a backward arc, then transitivity among these three vertices would be violated. If $(1, 2)$ were an incomparability relation, then we would get the first forbidden subposet of Fig. 3 on the vertices $1, 2, v$. Thus again $2 \in T$.

This completes the proof of the theorem. ■

We mention two interesting questions whose answers we do not know.

Conjecture 2. *For fixed n, k, the numbers* $\mathsf{Dist}_n(k, \ell)$ *as ℓ increases are unimodal.*

Question 1. *Is there a recurrence relation or a generating function for the numbers occurring in the inversion partition?*

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