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Gamma Positivity of the Excedance-Based Eulerian Polynomial in Positive Elements of Classical Weyl Groups

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Abstract. The Eulerian polynomial $\mathrm{AExc}_n(t)$ enumerating excedances in the symmetric group $\mathfrak{S}_n$ is known to be gamma positive for all n. When enumeration is done over the type B and type D Coxeter groups, the type B and type D Eulerian polynomials are also gamma positive for all n. We consider $\mathrm{AExc}_n^+(t)$ and $\mathrm{AExc}_n^-(t)$, the polynomials which enumerate excedance in the alternating group $\mathcal{A}_n$ and in $\mathfrak{S}_n - \mathcal{A}_n$, respectively. We show that $\mathrm{AExc}_n^+(t)$ is gamma positive iff $n \geq 5$ is odd. When $n \geq 4$ is even, $\mathrm{AExc}_n^+(t)$ is not even palindromic, but we show that it is the sum of two gamma positive summands. An identical statement is true about $\mathrm{AExc}_n^-(t)$. We extend similar results to the excedance based Eulerian polynomial when enumeration is done over the positive elements in both type B and type D Coxeter groups. Gamma positivity results are known when excedance is enumerated over derangements in $\mathfrak{S}_n$. We extend some of these to the case when enumeration is done over even and odd derangements in $\mathfrak{S}_n$.

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1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations on $[n]$. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, define its excedance set as $\mathrm{EXC}(\pi) = \{i \in [n] : \pi_i > i\}$ and its number of excedances as $\mathrm{exc}(\pi) = |\mathrm{EXC}(\pi)|$. Define its number of non-excedances as $\mathrm{nexc}(\pi) = |\{i \in [n] : \pi_i \leq i\}|$. For $\pi \in \mathfrak{S}_n$, define its number of inversions as $\mathrm{inv}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|$. Let $\mathrm{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ and $\mathrm{ASC}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$ be its set of descents and ascents, respectively. Let $\mathrm{des}(\pi) = |\mathrm{DES}(\pi)|$ be its number of descents and $\mathrm{asc}(\pi) = |\mathrm{ASC}(\pi)|$ be its number of ascents. The polynomial $A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$ is the classical Eulerian polynomial. Let $\mathcal{A}_n \subseteq \mathfrak{S}_n$ be the subset of even permutations. Define

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} \text{ and } A_n(s,t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} s^{n-1-\mathrm{des}(\pi)}, \tag{1}$$

$$\mathrm{AExc}_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)} \text{ and } \mathrm{AExc}_n(s,t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1}, \tag{2}$$

$$\mathrm{AExc}_n^+(t) = \sum_{\pi \in \mathcal{A}_n} t^{\mathrm{exc}(\pi)} \text{ and } \mathrm{AExc}_n^+(s,t) = \sum_{\pi \in \mathcal{A}_n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1}, \tag{3}$$

$$\mathrm{AExc}_n^-(t) = \sum_{\pi \in \mathfrak{S}_n - \mathcal{A}_n} t^{\mathrm{exc}(\pi)} \text{ and } \mathrm{AExc}_n^-(s,t) = \sum_{\pi \in \mathfrak{S}_n - \mathcal{A}_n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1}. \tag{4}$$

It is a well-known result of MacMahon [11] that both descents and excedances are equidistributed over $\mathfrak{S}_n$. That is, for all positive integers n, $A_n(t) = \mathrm{AExc}_n(t)$.

Let $f(t) \in \mathbb{Q}[t]$ be a degree n univariate polynomial with $f(t) = \sum_{i=0}^n a_i t^i$ where $a_n \neq 0$. Let r be the least non-negative integer such that $a_r \neq 0$. Define $\mathrm{len}(f) = n - r$. The polynomial $f(t)$ is said to be *palindromic* if $a_{r+i} = a_{n-i}$ for $0 \leq i \leq \lfloor (n-r)/2 \rfloor$. Define the *center of symmetry* of $f(t)$ to be $(n+r)/2$. Note that for a palindromic polynomial $f(t)$, its center of symmetry could be half integral.

Let $\mathrm{PalindPoly}_{(n+r)/2,r}(t)$ denote the set of palindromic univariate polynomials $f(t) = \sum_{i=0}^n a_i t^i$ with r being the least non-negative integer such that $a_r > 0$ and having center of symmetry $(n+r)/2$. Let $\Gamma = \{t^{r+i}(1+t)^{n-r-2i} : 0 \leq i \leq \lfloor (n-r)/2 \rfloor\}$. It is easy to see that if $f(t) \in \mathrm{PalindPoly}_{(n+r)/2,r}(t)$, then we can write $f(t) = \sum_{i=0}^{\lfloor (n-r)/2 \rfloor} \gamma_{n,i} t^{r+i}(1+t)^{n-r-2i}$. The polynomial $f(t)$ is said to be *gamma positive* if $\gamma_{n,i} \geq 0$ for all i (that is, if $f(t)$ has nonnegative coefficients when expressed as a linear combination of elements of Γ).

It is well known that the Eulerian polynomials $\mathrm{AExc}_n(t)$ are palindromic (see Graham et al. [9]). Gamma positivity of $\mathrm{AExc}_n(t)$ was first proved by Foata and Schützenberger in [7]. Foata and Strehl [8] later used a group action based proof which has been termed as "valley hopping" by Shapiro et al. [17]. This approach gives a combinatorial interpretation for $\gamma_{n,i}$, the gamma coefficients. Several refinements of the gamma positivity of $\mathrm{AExc}_n(t)$ are known when enumeration is done both with respect to excedances and with respect to descents. Shareshian and Wachs [19] have shown the following.

Theorem 1 (Shareshian and Wachs). *For natural numbers n and for statistics* $\mathrm{des}^*, \mathrm{maj} : \mathfrak{S}_n \mapsto \mathbb{Z}_{\geq 0}$ *define the polynomial* $\mathrm{AExc}_n(p,q,t) = \sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)-\mathrm{exc}(\pi)} t^{\mathrm{exc}(\pi)}$. *Then,* $\mathrm{AExc}_n(p,q,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i}(p,q) t^i (1+t)^{n-2i}$ *where the $\gamma_{n,i}(p,q)$'s are polynomials with positive coefficients.*

In some results, we have not defined those statistics which are not needed for this paper. For the definition of these statistics that appear in Theorem 1, we refer the reader to the survey paper [1, Theorem 2.3] by Athanasiadis. Similarly, for two statistics $2\text{-}13, 31\text{-}2 : \mathfrak{S}_n \rightarrow \mathbb{Z}_{\geq 0}$, Brändén [2] and Shin-and-Zeng [20,21] have shown a p, q-refinement. We refer the reader to [1, Theorem 2.2] by Athanasiadis for definitions of these statistics.

Theorem 2 (Brändén, Shin and Zeng). *For natural numbers n define the polynomial $A_n(p,q,t) = \sum_{\pi \in \mathfrak{S}_n} p^{2\text{-}13(\pi)} q^{31\text{-}2(\pi)} t^{\mathrm{des}(\pi)}$. Then, $A_n(p,q,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,i}(p,q) t^i (1+t)^{n-2i}$ where the $a_{n,i}(p,q)$'s are polynomials with positive coefficients.*

Dey and Sivasubramanian in [6] have recently given gamma positivity results when one sums descents over $\mathcal{A}_n$. In this paper, we consider the case when we sum excedances over $\mathcal{A}_n$. It is simple to note that when $n \geq 3$, these two polynomials are different. Further, it is not hard to see that the usual *"valley hopping"* gamma positivity proof of Foata and Strehl [8] does not respect sign and hence does not help in getting results for $\mathcal{A}_n$ or $\mathfrak{S}_n - \mathcal{A}_n$. In Sect. 3, we show the following results when we enumerate excedances in $\mathcal{A}_n$. In Theorem 9, we show that $\mathrm{AExc}_n^+(s,t)$ is palindromic iff $n \equiv 1 \pmod 2$. Our first main result is the following.

Theorem 3. *For odd positive integers $n \geq 5$, both $\mathrm{AExc}_n^+(s,t)$ and $\mathrm{AExc}_n^-(s,t)$ are gamma positive and have center of symmetry $(n-1)/2$.*

As mentioned, when $n \equiv 0 \pmod 2$ the polynomial $\mathrm{AExc}_n^+(s,t)$ is not palindromic. We ask for the minimum number of gamma positive polynomials which add up to give $\mathrm{AExc}_n^+(t) = \mathrm{AExc}_n^+(s,t)\Big|_{s=1}$. Note that when a polynomial is not palindromic, the minimum number of palindromic summands needed is at least two. In the case of $\mathrm{AExc}_n^+(t)$, we show that one can choose two palindromic polynomials that are gamma positive as well. Our next main result is the following.

Theorem 4. *For even positive integers $n \geq 4$, both $\mathrm{AExc}_n^+(t)$ and $\mathrm{AExc}_n^-(t)$ can be written as a sum of two gamma positive polynomials. The two summands have centers of symmetry that differ by one.*

Thus, Theorem 3 refines the existing gamma positivity results for $A_n(t)$ when n is odd. When n is even, Theorem 4 refines the gamma positivity results for $A_n(t)$ by giving several gamma positive summands which when added give $A_n(t)$.

We generalize our results to the case when excedances are summed over the elements with positive sign in classical Weyl groups. Let $\mathfrak{B}_n$ denote the group of signed permutations on $[\pm n] = \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}$, that is $\sigma \in \mathfrak{B}_n$ consists of all permutations of $[\pm n]$ that satisfy $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$. We represent $\sigma \in \mathfrak{B}_n$ as $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$ where $\sigma_i = \sigma(i)$. Let $\mathfrak{D}_n \subseteq \mathfrak{B}_n$ denote the subset consisting of those elements of $\mathfrak{B}_n$ which have an even number of negative entries. We use Brenti's definition of excedance from [3] for elements of both $\mathfrak{B}_n$ and $\mathfrak{D}_n$. Thus, similar to the excedance based Eulerian polynomials, we have excedance based Eulerian polynomials of type B and type D. There is also a natural notion of length in these groups and we get similar results when summation is restricted to elements with even length. In Sect. 4, we consider Type-B Weyl groups where our main results are Theorems 14 and 15. In Sect. 5 we cover Type-D Weyl groups, where our main result is Theorem 19. To the best of our knowledge, there were no results on gamma positivity of the type D excedance based Eulerian polynomial.

Gamma positivity results are also known when excedances are summed over derangements in $\mathfrak{S}_n$. Let $\mathfrak{SD}_n = \{\pi \in \mathfrak{S}_n : \pi_i \neq i \text{ for all } i\}$ denote the set of derangements in $\mathfrak{S}_n$. Let $\mathfrak{SD}_n^+ = \mathfrak{SD}_n \cap \mathcal{A}_n$ and $\mathfrak{SD}_n^- = \mathfrak{SD}_n \cap (\mathfrak{S}_n - \mathcal{A}_n)$. Define

$$\mathrm{ADerExc}_n(t) = \sum_{\pi \in \mathfrak{SD}_n} t^{\mathrm{exc}(\pi)}, \tag{5}$$

$$\mathrm{ADerExc}_n^+(t) = \sum_{\pi \in \mathfrak{SD}_n^+} t^{\mathrm{exc}(\pi)} \text{ and } \mathrm{ADerExc}_n^-(t) = \sum_{\pi \in \mathfrak{SD}_n^-} t^{\mathrm{exc}(\pi)}. \tag{6}$$

The polynomial $\mathrm{ADerExc}_n(t)$ is known to be real rooted (see Zhang [26]) and palindromic. Hence $\mathrm{ADerExc}_n(t)$ is gamma positive (see for example Petersens book [16, Page 82]). Shin and Zeng in [20,21] have proved the following refinement with respect to the number of inversions $\mathrm{inv}(\pi)$, number of cycles $\mathrm{cyc}(\pi)$ and the nesting number $\mathrm{nest}(\pi)$ (see [1] for the definition of these statistics).

Theorem 5. (Shin and Zeng) *For positive integers n, and for $\mathrm{stat}(\pi) \in \{\mathrm{cyc}(\pi), \mathrm{inv}(\pi), \mathrm{nest}(\pi)\}$, the polynomial $\sum_{\pi \in \mathfrak{SD}_n} q^{\mathrm{stat}(\pi)} t^{\mathrm{exc}(\pi)}$ equals $\sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i}(q) t^i (1+t)^{n-2i}$ where $b_{n,i}(q)$ is a polynomial with positive integral coefficients.*

Setting $q = 1$ in Theorem 5 shows that the polynomial $\mathrm{ADerExc}_n(t)$ is gamma positive. Sun and Wang in [25] have given an alternative proof of this fact. Their proof actually shows that the polynomials $\mathrm{ADerExc}_n^+(t)$ and $\mathrm{ADerExc}_n^-(t)$ are also gamma positive, though they do not explicitly mention it. We move on to the bivariate polynomials appearing in Theorem 5. In Sect. 6, we prove the following refinement of Theorem 5.

Theorem 6. *For positive integers n, and for $\mathrm{stat}(\pi) \in \{\mathrm{cyc}(\pi), \mathrm{inv}(\pi)\}$, we have the following: both the polynomials $\sum_{\pi \in \mathfrak{SD}_n^+} q^{\mathrm{stat}(\pi)} t^{\mathrm{exc}(\pi)}$ and $\sum_{\pi \in \mathfrak{SD}_n^-} q^{\mathrm{stat}(\pi)} t^{\mathrm{exc}(\pi)}$ equal $\sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i}^+(q) t^i (1+t)^{n-2i}$ and $\sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i}^-(q) t^i (1+t)^{n-2i}$ respectively. Here, both $b_{n,i}^+(q)$ and $b_{n,i}^-(q)$ are polynomials with positive integral coefficients.*

The proof of Theorem 6 essentially follows from the proof of Theorem 5. This is true for all results in Sect. 6. In that section, our contribution lies more in making the refining statements for $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$ explicit, than in their proofs.

2. Preliminaries on Gamma Positive Polynomials

Most of the gamma positive polynomials in this work have homogeneous bivariate counterparts with the following slightly more general definition of gamma positivity. Let $f(s,t) = \sum_{i=0}^{n} a_i s^{n-i} t^i$ be a degree n homogenous bivariate polynomial. Define $f(s,t)$ to be palindromic if $a_i = a_{n-i}$ for all i. More generally, we consider palindromic, degree n homogenous bivariate polynomials which have $n-r$ as the highest exponent of s with non-zero coefficient. Thus, $a_{n-r} \neq 0$ and as f is palindromic $a_r = a_{n-r} \neq 0$. Define the length $\mathrm{len}(f)$ of f to be $n-2r$. Define the center of symmetry of a degree n homogenous bivariate palindromic polynomial to be $n/2$. If $f(s,t)$ is palindromic, it is said to be bivariate gamma positive if it can be written as $f(s,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i} (st)^i (s+t)^{n-2i}$ with $\gamma_{n,i} \geq 0$ for all i.

Clearly, if $f(s,t)$ is a bivariate gamma positive polynomial, then $f(t) = f(s,t)|_{s=1}$ is clearly a univariate gamma positive polynomial with the same center of symmetry. We need the following lemmas. All of them are simple or proved in [6]. For the sake of completeness, we include their proofs.

Lemma 1. *Let $f_1(s,t)$ and $f_2(s,t)$ be two bivariate gamma positive polynomials with respective centers of symmetry m_1 and m_2. Then, $f_1(s,t)f_2(s,t)$ is gamma positive with center of symmetry $m_1 + m_2$.*

Let D be the operator $\left(\dfrac{\partial}{\partial s} + \dfrac{\partial}{\partial t}\right)$ acting on polynomials in $\mathbb{Q}[s,t]$.

Lemma 2. *Let $f(s,t)$ be a bivariate gamma positive polynomial with center of symmetry $n/2$. Then, $Df(s,t)$ is gamma positive with center of symmetry $(n-1)/2$.*

Proof. Let,

$$f(s,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i} (st)^i (s+t)^{n-2i} \text{ with } \gamma_{n,i} \geq 0. \text{ Then,}$$

$$\begin{aligned} Df(s,t) &= \sum_{i=0}^{\lfloor n/2 \rfloor} i\gamma_{n,i} (st)^{i-1} (s+t)^{n-2i+1} + \sum_{i=0}^{\lfloor n/2 \rfloor} 2(n-2i)\gamma_{n,i} (st)^i (s+t)^{n-2i-1} \\ &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \beta_{n,i} (st)^i (s+t)^{n-2i-1} \end{aligned}$$

where $\beta_{n,i} = (i+1)\gamma_{n,i+1} + 2(n-2i)\gamma_{n,i}$. As $n-2i \geq 0$ for all i, each $\beta_{n,i} \geq 0$. It is also easy to see that $Df(s,t)$ has center of symmetry $(n-1)/2$, completing the proof. □

The following are easy corollaries.

Corollary 1. *Let $f(s,t)$ be a bivariate gamma positive polynomial with center of symmetry $n/2$.*

1. *Then, for a natural number $\ell \leq n$, $D^\ell f(s,t)$ is gamma positive with center of symmetry $(n-\ell)/2$.*

2. *Then,* $(st)^i f(s,t)$ *is gamma positive with center of symmetry* $i + n/2$.
3. *Then,* $(s+t)f(s,t)$ *is gamma positive with center of symmetry* $(n+1)/2$.

Our next important result is about univariate polynomials with odd length.

Lemma 3. *Let* $f(t)$ *be a gamma positive polynomial with center of symmetry* $n/2$ *and with odd* $\mathrm{len}(f(t))$. *Then,* $f(t)$ *is the sum of two gamma positive polynomials* $p_1(t)$ *and* $p_2(t)$ *with centers of symmetry* $(n-1)/2$ *and* $(n+1)/2$ *respectively. In this way of writing, we have* $p_2(t) = tp_1(t)$.

Proof. Assume that $f(t)$ has degree d. As $f(t)$ has center of symmetry $n/2$ and degree d, the lowest exponent of t with non-zero coefficient is $n/2-(d-n/2) = n-d$. Thus, $\mathrm{len}(f) = d-(n-d) = 2d-n$. Since $f(t)$ is gamma positive and has center of symmetry $n/2$, we have $f(t) = \sum_{i=n-d}^{\lfloor n/2 \rfloor} \gamma_i t^i (1+t)^{n-2i}$. We also have

$$\begin{aligned} f(t) &= \sum_{i=n-d}^{\lfloor n/2 \rfloor} \gamma_i t^i (1+t)^{n-2i-1}(1+t) \\ &= \left(\sum_{i=n-d}^{\lfloor n/2 \rfloor} \gamma_i t^i (1+t)^{n-1-2i} \right) + \left(\sum_{i=n-d}^{\lfloor n/2 \rfloor} \gamma_i t^{i+1} (1+t)^{n-1-2i} \right) = p_1(t) + p_2(t). \end{aligned}$$

It is easy to see that $p_1(t)$ and $p_2(t)$ are gamma positive with respective centers of symmetry $(n-1)/2$ and $(n+1)/2$. The argument requires the exponent $n-2i$ to be odd to enable us to pull out a $(1+t)$ factor from each term $\gamma_i t^i (1+t)^{n-2i}$ especially when $i = \lfloor n/2 \rfloor$. As $\mathrm{len}(f(t)) = 2d-n$ is odd, hence $n-2i$ will be odd for all i. It is easy to see that $p_2(t) = tp_1(t)$ and that $p_1(t)$ and $p_2(t)$ have even length, completing the proof. $\square$

3. Type A Coxeter Groups

For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, define $\mathrm{pos}_n(\pi) = \pi^{-1}(n)$ to be the index i such that $\pi(i) = n$. Let $\mathfrak{S}_n^i = \{\pi \in \mathfrak{S}_n : \mathrm{pos}_n(\pi) = i\}$, $\mathcal{A}_n^i = \{\pi \in \mathcal{A}_n : \mathrm{pos}_n(\pi) = i\}$, and $\mathfrak{S}_n^i - \mathcal{A}_n^i = \{\pi \in \mathfrak{S}_n - \mathcal{A}_n : \mathrm{pos}_n(\pi) = i\}$. We need Foata's First Fundamental Transformation which is a well known bijection that maps excedances to descents (see Lothaire [10, Section 10.2]).

Theorem 7 (Foata's First Fundamental Transformation). *For positive integers* $n \geq 2$, *there exist a bijection* $\mathrm{FT}_n : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ *such that* $\mathrm{des}(\mathrm{FT}_n(\pi)) = \mathrm{exc}(\pi)$. *Hence, the statistics* des *and* exc *are equidistributed over* $\mathfrak{S}_n$.

For $\pi \in \mathfrak{S}_n$, define π' to be π restricted to $[n-1]$. We use the bijection FT_n of Theorem 7 to prove the following.

Lemma 4. *For positive integers* $n \geq 2$, *the following holds:*

$$\sum_{\pi \in \mathfrak{S}_n^{n-1}} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \sum_{\pi \in \mathfrak{S}_n^1} t^{\mathrm{des}(\pi)} s^{\mathrm{asc}(\pi)}. \tag{7}$$

Proof. Define $f_n : \mathfrak{S}_n^{n-1} \mapsto \mathfrak{S}_n^1$ as follows: let $\pi = \pi_1, \pi_2, \ldots, \pi_{n-2}, n, \pi_n \in \mathfrak{S}_n^{n-1}$. Let $\mathrm{FT}_{n-1}(\pi') = a_1, a_2, \ldots, a_{n-1}$. Then, define $f_n(\pi) = n, a_1, a_2, \ldots, a_{n-1}$. Clearly, f_n is well-defined and is a bijection. Further, for $\pi \in \mathfrak{S}_n^{n-1}$, we have $\mathrm{exc}(\pi) = 1 + \mathrm{exc}(\pi') = 1 + \mathrm{des}(\mathrm{FT}_{n-1}(\pi')) = \mathrm{des}(f_n(\pi))$. Hence, $\mathrm{nexc}(\pi) = \mathrm{asc}(f_n(\pi)) + 1$. The proof is complete. $\square$

Lemma 5. *For positive integers $n \geq 3$ and r with $1 \leq r \leq n-2$, the following holds:*

$$\sum_{\pi \in \mathcal{A}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \sum_{\pi \in \mathfrak{S}_n^r - \mathcal{A}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \frac{1}{2} \sum_{\pi \in \mathfrak{S}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1}.$$

Proof. Define $g : \mathcal{A}_n^r \mapsto \mathfrak{S}_n^r - \mathcal{A}_n^r$ by $g(\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n) = \pi_1, \pi_2, \ldots, \pi_n, \pi_{n-1}$. That is, g swaps the last two elements of π. Clearly, g is a bijection with $\mathrm{inv}(\pi) \not\equiv \mathrm{inv}(g(\pi)) \pmod 2$. Moreover, as $\mathrm{pos}_n(\pi) \leq n-2$, we have $\mathrm{exc}(\pi) = \mathrm{exc}(g(\pi))$, completing the proof. $\square$

3.1. Recurrences for $\mathbf{AExc}_n^+(s,t)$ and $\mathbf{AExc}_n^-(s,t)$

The main result of this subsection is the following recurrence for $\mathrm{AExc}_n^+(s,t)$ and $\mathrm{AExc}_n^-(s,t)$.

Theorem 8. *For positive integers n, the polynomials $\mathrm{AExc}_n^+(s,t)$ and $\mathrm{AExc}_n^-(s,t)$ satisfy the following recurrence:*

$$\mathrm{AExc}_n^+(s,t) = s\mathrm{AExc}_{n-1}^+(s,t) + t\mathrm{AExc}_{n-1}^-(s,t) + \frac{1}{2} stDA_{n-1}(s,t), \tag{8}$$

$$\mathrm{AExc}_n^-(s,t) = s\mathrm{AExc}_{n-1}^-(s,t) + t\mathrm{AExc}_{n-1}^+(s,t) + \frac{1}{2} stDA_{n-1}(s,t). \tag{9}$$

Proof. Foata's First Fundamental transformation gives us:

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} s^{\mathrm{asc}(\pi)}, \tag{10}$$

$$\sum_{\pi \in \mathfrak{S}_n^n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \sum_{\pi \in \mathfrak{S}_n^n} t^{\mathrm{des}(\pi)} s^{\mathrm{asc}(\pi)}. \tag{11}$$

In the second line, we have identified permutations in $\mathfrak{S}_{n-1}$ with $\sigma \in \mathfrak{S}_n$ having $\mathrm{pos}_n(\sigma) = n$. Subtracting (7) and (11) from (10), we get

$$\sum_{r=1}^{n-2} \sum_{\pi \in \mathfrak{S}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} = \sum_{r=2}^{n-1} \sum_{\pi \in \mathfrak{S}_n^r} t^{\mathrm{des}(\pi)} s^{\mathrm{asc}(\pi)}. \tag{12}$$

We have

$$\begin{aligned}
\mathrm{AExc}_n^+(s,t) &= \sum_{r=1}^{n-2} \sum_{\pi \in \mathcal{A}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} + \sum_{\pi \in \mathcal{A}_n^{n-1}} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} \\
&\quad + \sum_{\pi \in \mathcal{A}_n^n} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} \\
&= \frac{1}{2} \sum_{r=1}^{n-2} \sum_{\pi \in \mathfrak{S}_n^r} t^{\mathrm{exc}(\pi)} s^{\mathrm{nexc}(\pi)-1} + t\mathrm{AExc}_{n-1}^-(s,t) + s\mathrm{AExc}_{n-1}^+(s,t)
\end{aligned}$$

$$= \frac{1}{2}\sum_{r=2}^{n-1}\sum_{\pi\in\mathfrak{S}_n^r} t^{\mathrm{des}(\pi)}s^{\mathrm{asc}(\pi)} + t\mathrm{AExc}_{n-1}^-(s,t) + s\mathrm{AExc}_{n-1}^+(s,t)$$
$$= \frac{1}{2}stDA_{n-1}(s,t) + t\mathrm{AExc}_{n-1}^-(s,t) + s\mathrm{AExc}_{n-1}^+(s,t).$$

The second line follows using Lemma 5. The third line follows from (12). The last line follows using the fact that $stDA_{n-1}(s,t)$ is the contribution of all the permutations where n is not in the initial or the final position. This is implicit in the recurrence for the bivariate Eulerian polynomial $A_n(s,t)$ given by Foata and Schützenberger. This point is also elaborated in [6, Theorem 8]. The proof of (9) is identical and is hence omitted. □

A simple corollary of Theorem 8 is the following result of Mantaci (see [13]). Sivasubramanian in [22, Theorem 1] has given an alternate proof of Mantaci's result by evaluating the determinant of an appropriately defined $n \times n$ matrix.

Corollary 2. *For positive integers $n \geq 2$, $\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}t^{\mathrm{exc}(\pi)}s^{\mathrm{nexc}(\pi)-1} = (s-t)^{n-1}$. In particular, $\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}t^{\mathrm{exc}(\pi)} = (1-t)^{n-1}$.*

Proof. We use induction on n with $n=2$ being the base case. When $n=2$, it is simple to note that $\sum_{\pi\in\mathfrak{S}_2}(-1)^{\mathrm{inv}(\pi)}t^{\mathrm{exc}(\pi)}s^{\mathrm{nexc}(\pi)-1} = s-t$. Subtracting (9) from (8) we get

$$\begin{aligned}\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}t^{\mathrm{exc}(\pi)}s^{\mathrm{nexc}(\pi)-1} &= \mathrm{AExc}_n^+(s,t) - \mathrm{AExc}_n^-(s,t)\\ &= (s-t)\left[\mathrm{AExc}_{n-1}^+(s,t) - \mathrm{AExc}_{n-1}^-(s,t)\right]\\ &= (s-t)^{n-1}.\end{aligned}$$

The proof is complete. □

We need one more lemma. Recall that D is the operator $\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)$.

Lemma 6. *For positive integers $n \geq 2$, $D\mathrm{AExc}_n^+(s,t) = D\mathrm{AExc}_n^-(s,t) = \frac{1}{2}D\mathrm{AExc}_n(s,t) = \frac{1}{2}DA_n(s,t)$.*

Proof. We use induction on n. Clearly, when $n=2$, $\mathrm{AExc}_2^+(s,t) = s$, $\mathrm{AExc}_2^-(s,t) = t$ while $\mathrm{AExc}_2(s,t) = s+t$. Thus, $D\mathrm{AExc}_2^+(s,t) = D\mathrm{AExc}_2^-(s,t) = \frac{1}{2}D\mathrm{AExc}_2(s,t)$. By induction, let $D\mathrm{AExc}_{n-1}^+(s,t) = D\mathrm{AExc}_{n-1}^-(s,t) = \frac{1}{2}D\mathrm{AExc}_{n-1}(s,t)$. By Theorem 8,

$$\begin{aligned}D\mathrm{AExc}_n^+(s,t) &= D\left(s\mathrm{AExc}_{n-1}^+(s,t) + t\mathrm{AExc}_{n-1}^-(s,t) + \frac{1}{2}stD\mathrm{AExc}_{n-1}(s,t)\right)\\ &= sD\mathrm{AExc}_{n-1}^+(s,t) + \mathrm{AExc}_{n-1}^+(s,t) + tD\mathrm{AExc}_{n-1}^-(s,t)\\ &\quad + \mathrm{AExc}_{n-1}^-(s,t)\\ &\quad + \frac{1}{2}D\Big(stD\mathrm{AExc}_{n-1}(s,t)\Big)\end{aligned}$$

$$
\begin{aligned}
&= sD\text{AExc}_{n-1}^{-}(s,t) + \text{AExc}_{n-1}^{+}(s,t) \\
&\quad +tD\text{AExc}_{n-1}^{+}(s,t) + \text{AExc}_{n-1}^{-}(s,t) \\
&\quad +\frac{1}{2}D\Big(stD\text{AExc}_{n-1}(s,t)\Big) \\
&= D\Big(s\text{AExc}_{n-1}^{-}(s,t) + t\text{AExc}_{n-1}^{+}(s,t) + \frac{1}{2}stD\text{AExc}_{n-1}(s,t)\Big) \\
&= D\text{AExc}_{n}^{-}(s,t).
\end{aligned}
$$

Here, the third line follows by induction. The proof is complete. □

Remark 1. Using Lemma 6, we rewrite Theorem 8 as follows. We will need this modified version in our proof of Theorem 4. When we wish to denote either one of $\text{AExc}_n^{+}(s,t)$ or $\text{AExc}_n^{-}(s,t)$, we write $\text{AExc}_n^{\pm}(s,t)$. Theorem 8 is clearly equivalent to the following:

$$\text{AExc}_n^{+}(s,t) = s\text{AExc}_{n-1}^{+}(s,t) + t\text{AExc}_{n-1}^{-}(s,t) + stD\text{AExc}_{n-1}^{\pm}(s,t), \quad (13)$$

$$\text{AExc}_n^{-}(s,t) = s\text{AExc}_{n-1}^{-}(s,t) + t\text{AExc}_{n-1}^{+}(s,t) + stD\text{AExc}_{n-1}^{\pm}(s,t). \quad (14)$$

Let $\text{AExc}_n^{+}(s,t) = \sum_{k=0}^{n-1} a_{n,k}^{+} t^k s^{n-1-k}$ and let $\text{AExc}_n^{-}(s,t) = \sum_{k=0}^{n-1} a_{n,k}^{-} t^k s^{n-1-k}$. The following recurrences for the numbers $a_{n,k}^{+}$ and $a_{n,k}^{-}$ were proved by Mantaci in [12,13]. It is easy to derive them using Remark 1.

Corollary 3. *For positive integers $n \geq 2$, we have*

$$
\begin{aligned}
a_{n,k}^{+} &= ka_{n-1,k}^{-} + (n-k)a_{n-1,k-1}^{-} + a_{n-1,k}^{+} \text{ and} \\
a_{n,k}^{-} &= ka_{n-1,k}^{+} + (n-k)a_{n-1,k-1}^{+} + a_{n-1,k}^{-}.
\end{aligned}
$$

Proof. We only prove one of the recurrences. Let $[t^k s^{n-k}]f(s,t)$ denote the coefficient of $t^k s^{n-k}$ in the polynomial $f(s,t)$.

$$
\begin{aligned}
a_{n,k}^{+} &= [t^k s^{n-1-k}]\text{AExc}_n^{+}(s,t) \\
&= [t^k s^{n-1-k}]\Big(s\text{AExc}_{n-1}^{+}(s,t) + t\text{AExc}_{n-1}^{-}(s,t) + stD\text{AExc}_{n-1}^{-}(s,t)\Big) \\
&= [t^k s^{n-2-k}]\text{AExc}_{n-1}^{+}(s,t) + [t^{k-1} s^{n-1-k}]\text{AExc}_{n-1}^{-}(s,t) \\
&\quad +[t^{k-1} s^{n-2-k}]D\text{AExc}_{n-1}^{-}(s,t) \\
&= ka_{n-1,k}^{-} + (n-k)a_{n-1,k-1}^{-} + a_{n-1,k}^{+}
\end{aligned}
$$

Here, the second line follows with $stD\text{AExc}_{n-1}^{-}(s,t)$ chosen in the right hand side of (13) (as part of Remark 1). The proof of the other part is similar and is hence omitted. □

3.2. Palindromicity of $\text{AExc}_n^{+}(s,t)$ and $\text{AExc}_n^{-}(s,t)$

We consider palindromicity of the polynomials $\text{AExc}_n^{+}(s,t)$ and $\text{AExc}_n^{-}(s,t)$.

Theorem 9. *For natural numbers n, $\text{AExc}_n^{+}(s,t)$ and $\text{AExc}_n^{-}(s,t)$ are palindromic if and only if $n \equiv 1 \mod 2$.*

Proof. By Corollary 2,

$$\mathrm{AExc}_n^+(s,t) = \frac{1}{2}\Big(\mathrm{AExc}_n(s,t) + (s-t)^{n-1}\Big) \text{ and}$$
$$\mathrm{AExc}_n^-(s,t) = \frac{1}{2}\Big(\mathrm{AExc}_n(s,t) - (s-t)^{n-1}\Big).$$

As mentioned earlier, it is well known that $\mathrm{AExc}_n(s,t)$ is palindromic for all n with center of symmetry $(n-1)/2$ while it is clear that $(s-t)^{n-1}$ is palindromic with the same center of symmetry if and only if n is odd. Hence, $\mathrm{AExc}_n^+(s,t)$ and $\mathrm{AExc}_n^-(s,t)$ are palindromic iff $n \equiv 1 \mod 2$. □

Using Theorem 8, multiple times, we get the following.

Theorem 10. *For positive integers n, we have*

$$\mathrm{AExc}_{n+4}^+(s,t) = L_1(s,t)\mathrm{AExc}_n^+(s,t) + L_2(s,t)\mathrm{AExc}_n^-(s,t) + L_3(s,t)D\mathrm{AExc}_n^+(s,t) + \Big[L_4(s,t)D^2 + L_5(s,t)D^3 + L_6(s,t)D^4\Big]\mathrm{AExc}_n^+(s,t), \tag{15}$$

$$\mathrm{AExc}_{n+4}^-(s,t) = L_1(s,t)\mathrm{AExc}_n^-(s,t) + L_2(s,t)\mathrm{AExc}_n^+(s,t) + L_3(s,t)D\mathrm{AExc}_n^-(s,t) + \Big[L_4(s,t)D^2 + L_5(s,t)D^3 + L_6(s,t)D^4\Big]\mathrm{AExc}_n^-(s,t). \tag{16}$$

where the $L_i(s,t)$ with their centers of symmetry are as follows. Further, each $L_i(s,t)$ is also gamma positive.

$f(s,t)$	*center of symmetry of $f(s,t)$)*
$L_1(s,t) = (s+t)^4 + 7st(s+t)^2 + 16(st)^2$	2
$L_2(s,t) = 15st(s+t)^2$	2
$L_3(s,t) = 15st(s+t)^3 + 60(st)^2(s+t)$	5/2
$L_4(s,t) = 25(st)^2(s+t)^2 + 20(st)^3$	3
$L_5(s,t) = 10(st)^3(s+t)$	7/2
$L_6(s,t) = (st)^4$	4

Proof. Apply Theorem 8 twice to get

$$\begin{aligned}\mathrm{AExc}_{n+2}^+(s,t) = {} & (s^2+st+t^2)\mathrm{AExc}_n^+(s,t) \\ & +3st\mathrm{AExc}_n^-(s,t) + 3st(s+t)D\mathrm{AExc}_n^+(s,t) \\ & +(st)^2D^2\mathrm{AExc}_n^+(s,t),\end{aligned} \tag{17}$$

$$\begin{aligned}\mathrm{AExc}_{n+2}^-(s,t) = {} & (s^2+st+t^2)\mathrm{AExc}_n^-(s,t) \\ & +3st\mathrm{AExc}_n^+(s,t) + 3st(s+t)D\mathrm{AExc}_n^-(s,t) \\ & +(st)^2D^2\mathrm{AExc}_n^-(s,t).\end{aligned} \tag{18}$$

Thus, we have

$$\begin{aligned}D\mathrm{AExc}_{n+2}^+(s,t) = {} & 3(s+t)\mathrm{AExc}_n(s,t) \\ & +4(s^2+4st+t^2)D\mathrm{AExc}_n^+(s,t) \\ & +5st(s+t)D^2\mathrm{AExc}_n^+(s,t) + (st)^2D^3\mathrm{AExc}_n^+(s,t),\end{aligned} \tag{19}$$

$$D^2\mathrm{AExc}_{n+2}^+(s,t) = 6\mathrm{AExc}_n(s,t)$$

$$+30(s+t)D\mathrm{AExc}_n^+(s,t)+9(s^2+4st+t^2)D^2\mathrm{AExc}_n^+(s,t)$$
$$+7st(s+t)D^3\mathrm{AExc}_n^+(s,t)+(st)^2D^4\mathrm{AExc}_n^+(s,t). \qquad (20)$$

(17) is equivalent to

$$\begin{aligned}\mathrm{AExc}_{n+4}^+(s,t) = {} & (s^2+st+t^2)\mathrm{AExc}_{n+2}^+(s,t)+3st\mathrm{AExc}_{n+2}^-(s,t)\\ & +3st(s+t)D\mathrm{AExc}_{n+2}^+(s,t)\\ & +(st)^2D^2\mathrm{AExc}_{n+2}^+(s,t). \end{aligned} \qquad (21)$$

We get (15) by plugging (17), (18), (19) and (20) in (21). It is routine to check that each $L_i(s,t)$ is in addition, gamma positive. The proof of (16) makes identical moves and is thus omitted. This completes the proof. □

3.3. Proof of Theorem 3

Our proof of Theorem 3 needs us to jump down from $n+4$ to n rather than from $n+2$ to n. This is because the factors in each summand turns out to be gamma positive only when the jump size is 4.

Proof of Theorem 3. When $n=5$ and $n=7$ one can check that

$$\begin{aligned}
\mathrm{AExc}_5^+(s,t) &= s^4+11s^3t+36s^2t^2+11st^3+t^4=(s+t)^4+7st(s+t)^2+16(st)^2,\\
\mathrm{AExc}_5^-(s,t) &= 15s^3t+30s^2t^2+15st^3=15st(s+t)^2,\\
\mathrm{AExc}_7^+(s,t) &= s^6+57s^5t+603s^4t^2+1198s^3t^3+603s^2t^4+57st^5+t^6\\
&= (s+t)^6+51st(s+t)^4+384(st)^2(s+t)^2+104(st)^3,\\
\mathrm{AExc}_7^-(s,t) &= 63s^5t+588s^4t^2+1218s^3t^3+588s^2t^4+63st^5,\\
&= 63st(s+t)^4+336(st)^2(s+t)^2+168(st)^3.
\end{aligned}$$

Let $n>7$ be odd positive integer and let $m=n-4$. By induction, both $\mathrm{AExc}_m^+(s,t)$ and $\mathrm{AExc}_m^-(s,t)$ are gamma positive with centers of symmetry $\frac{1}{2}(m-1)$. Further, the $L_i(s,t)$'s for $1\le i\le 6$ that appear in Theorem 10 are gamma positive. Moreover, all the six terms in (15) have the same center of symmetry $\frac{1}{2}(m+3)=\frac{1}{2}(n-1)$. Thus by Theorem 10, $\mathrm{AExc}_n^+(s,t)$ is gamma positive. In an identical manner, one can prove that $\mathrm{AExc}_n^-(s,t)$ is gamma positive with center of symmetry $\frac{1}{2}(n-1)$, completing the proof. □

3.4. Proof of Theorem 4

When $n=2m$ for a natural number m, Theorem 9, shows that the polynomials $\mathrm{AExc}_n^+(t)$ and $\mathrm{AExc}_n^-(t)$ are not palindromic. Thus, they cannot be written in the gamma basis. Nonetheless, we show in this case that both $\mathrm{AExc}_n^+(t)$ and $\mathrm{AExc}_n^-(t)$ can be written as the sum of two gamma positive polynomials.

Proof of Theorem 4. We use induction on n. Our base case is when $n=2m=4$, one can check that $\mathrm{AExc}_4^+(t)=1+4t+7t^2=(1+4t+t^2)+6t^2$ and $\mathrm{AExc}_4^-(t)=t^3+4t^2+7t=t(1+4t+t^2)+6t$. When $n\ge 6$, by Remark 1 we have

$$\begin{aligned}\mathrm{AExc}^+_{2m+2}(s,t) &= s\mathrm{AExc}^+_{2m+1}(s,t) + t\mathrm{AExc}^-_{2m+1}(s,t) + stD\mathrm{AExc}^+_{2m+1}(s,t)\\ \mathrm{AExc}^+_{2m+2}(t) &= \mathrm{AExc}^+_{2m+1}(t) + t\mathrm{AExc}^-_{2m+1}(t) + \left(stD\mathrm{AExc}^+_{2m+1}(s,t)\right)\Big|_{s=1}\end{aligned} \tag{22}$$

The polynomial $p(s,t) = stD\mathrm{AExc}^+_{2m+1}(s,t)$ is gamma positive with center of symmetry $m - 1/2 + 1$. Hence, $p(t) = p(s,t)|_{s=1}$ is a univariate gamma positive polynomial with center of symmetry $m + 1/2$. Further, it is simple to see that $p(t)$ has odd length and thus by Lemma 3, it can be written as $p(t) = p_1(t) + p_2(t)$ with respective centers of symmetry m and $m + 1$. Thus, (22) becomes

$$\mathrm{AExc}^+_{2m+2}(t) = \mathrm{AExc}^+_{2m+1}(t) + t\mathrm{AExc}^-_{2m+1}(t) + p_1(t) + p_2(t) \tag{23}$$

$$= w_1(t) + w_2(t) \tag{24}$$

where $w_1(t) = \mathrm{AExc}^+_{2m+1}(t) + p_1(t)$ and $w_2(t) = t\mathrm{AExc}^-_{2m+1}(t) + p_2(t)$. By Theorem 3, $\mathrm{AExc}^+_{2m+1}(t)$ and $t\mathrm{AExc}^-_{2m+1}(t)$ are gamma positive with respective centers of symmetry m and $m + 1$. Thus, $w_1(t)$ and $w_2(t)$ are gamma positive with the required centers of symmetry. As the proof for $\mathrm{AExc}^-_{2m+2}(t)$ is identical, we omit it. $\square$

4. Type B Coxeter Groups

Let $\mathfrak{B}_n$ be the set of permutations π of $[\pm n]$ satisfying $\pi(-i) = -\pi(i)$. We denote $-k$ as $\overline{k}$ as well. $\mathfrak{B}_n$ is referred to as the hyperoctahedral group or the group of signed permutations on $[\pm n]$. For $\pi \in \mathfrak{B}_n$ we alternatively denote $\pi(i)$ as π_i. For $\pi \in \mathfrak{B}_n$, define $\mathrm{Negs}(\pi) = \{i : i > 0, \pi_i < 0\}$ to be the set of elements which occur with a negative sign. As defined by Brenti in [3], let $\mathrm{inv}_B(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| - \sum_{i \in \mathrm{Negs}(\pi)} \pi_i$. Let $\mathfrak{B}^+_n \subseteq \mathfrak{B}_n$ denote the subset of elements having even $\mathrm{inv}_B()$ value and let $\mathfrak{B}^-_n = \mathfrak{B}_n - \mathfrak{B}^+_n$.

Following Brenti's definition of excedance from [3], let $\mathrm{exc}_B(\pi) = |\{i \in [n] : \pi_{|\pi(i)|} > \pi_i\}| + |\{i \in [n] : \pi_i = -i\}|$ and let $\mathrm{nexc}_B(\pi) = n - \mathrm{exc}_B(\pi)$. Let $\mathrm{wkexc}_B(\pi) = |\{i \in [n] : \pi_{|\pi(i)|} > \pi_i\}| + |\{i \in [n] : \pi_i = i\}|$. For $\pi \in \mathfrak{B}_n$, let $\pi_0 = 0$. As defined in Petersen's book [16, Chapter 13], we give the following definition of type B descents: $\mathrm{des}_B(\pi) = |\{i \in [0, 1, 2 \ldots, n-1] : \pi_i > \pi_{i+1}\}|$ and let $\mathrm{asc}_B(\pi) = n - \mathrm{des}_B(\pi)$. Let

$$B_n(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)} \text{ and } B_n(s,t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)} s^{\mathrm{asc}_B(\pi)}, \tag{25}$$

$$B^+_n(s,t) = \sum_{\pi \in \mathfrak{B}^+_n} t^{\mathrm{des}_B(\pi)} s^{\mathrm{asc}_B(\pi)} \text{ and } B^-_n(s,t) = \sum_{\pi \in \mathfrak{B}^-_n} t^{\mathrm{des}_B(\pi)} s^{\mathrm{asc}_B(\pi)}, \tag{26}$$

$$\mathrm{BExc}_n(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{exc}_B(\pi)} \text{ and } \mathrm{BExc}_n(s,t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}, \tag{27}$$

$$\mathrm{BExc}_n^+(t) = \sum_{\pi \in \mathfrak{B}_n^+} t^{\mathrm{exc}_B(\pi)} \text{ and } \mathrm{BExc}_n^+(s,t) = \sum_{\pi \in \mathfrak{B}_n^+} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}, \tag{28}$$

$$\mathrm{BExc}_n^-(t) = \sum_{\pi \in \mathfrak{B}_n^-} t^{\mathrm{exc}_B(\pi)} \text{ and } \mathrm{BExc}_n^-(s,t) = \sum_{\pi \in \mathfrak{B}_n^-} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}, \tag{29}$$

$$\begin{aligned} \mathrm{SgnBExc}_n(t) &= \sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{exc}_B(\pi)} \text{ and} \\ \mathrm{SgnBExc}_n(s,t) &= \sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}. \end{aligned} \tag{30}$$

$B_n(s,t)$ is called the type B Eulerian polynomial and Brenti in [3, Theorem 3.15] proved a type B counterpart of MacMahon's theorem. Thus, we have $B_n(s,t) = \mathrm{BExc}_n(s,t)$. Brenti proved his result by showing the following.

Theorem 11 (Brenti). *For positive integers n, there exists a bijection $h_n : \mathfrak{B}_n \mapsto \mathfrak{B}_n$ such that $\mathrm{asc}_B(h_n(\pi)) = \mathrm{wkexc}_B(\pi)$ and $|\mathrm{Negs}((h_n(\pi))| = |\mathrm{Negs}(\pi)|$.*

Gamma positivity of the type B Eulerian polynomial was shown by Chow [5] and Petersen [15]. It can be checked that both proofs do not go through for showing gamma positivity of $B_n^+(s,t)$ and $B_n^-(s,t)$. Dey and Sivasubramanian in [6, Equation 39] gave the following recurrence for $B_n(s,t)$.

Theorem 12. *For positive integers $n \geq 2$, the following recurrence holds:*

$$B_{n+1}(s,t) = (s+t)B_n(s,t) + 2stDB_n(s,t). \tag{31}$$

Two points about this recurrence will be used later on in this work. The first is the following explanation for the terms of the recurrence. This can be seen from the proof of [6, Theorem 23].

Corollary 4. *The term $sB_n(s,t)$ is the contribution from all $\pi \in \mathfrak{B}_{n+1}$ with $\pi(n+1) = n+1$. The term $tB_n(s,t)$ is the contribution of $\pi \in \mathfrak{B}_{n+1}$ with $\pi(n+1) = \overline{n+1}$. The term $2stDB_n(s,t)$ is the contribution of all $\pi \in \mathfrak{B}_{n+1}$ with $\pi(n+1) \notin \{n+1, \overline{n+1}\}$.*

The second point is that the following recurrence for the gamma coefficients $\gamma_{n,i}^B$, which was given by Chow [5, Proposition 4.9], also follows from Theorem 12. As we will need this in the proof of Theorem 20, we record it here.

Corollary 5. *If $B_n(s,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i}^B (st)^i (s+t)^{n-2i}$, then for $i > 0$, the gamma coefficients satisfy the following recurrence $\gamma_{n+1,i}^B = (2i+1)\gamma_{n,i}^B + 4(n-2i+2)\gamma_{n,i-1}^B$. Further $\gamma_{n,0}^B = 1$ for all n. From this, we get that for all n, $\gamma_{n,i}^B$ is even when $i > 0$.*

We consider palindromicity of $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{BExc}_n^-(s,t)$. Using sign reversing involutions, Sivasubramanian in [23, Theorem 8] proved

$$\mathrm{SgnBExc}_n(t) = (1-t)^n. \tag{32}$$

We claim that the same sign reversing involution gives a bivariate version of (32). We note that each term in the bivariate polynomial $\mathrm{SgnBExc}_n(s,t)$ has degree n. If a $\pi \in \mathfrak{B}_n$ has $\mathrm{exc}_B(\pi) = k$, then clearly, $\mathrm{nexc}_B(\pi) = n-k$. Thus, the coefficient of t^k in $\mathrm{SgnBExc}_n(t)$ is the same as the coefficient of $t^k s^{n-k}$ in $\mathrm{SgnBExc}_n(s,t)$. Thus, we get the following bivariate version of (32):

$$\mathrm{SgnBExc}_n(s,t) = (s-t)^n. \tag{33}$$

We start by proving the following palindromicity result.

Lemma 7. *Let n be a positive integer.* $\mathrm{BExc}_n^+(s,t)$ *and* $\mathrm{BExc}_n^-(s,t)$ *are palindromic iff $n \equiv 0$ (mod 2).*

Proof. It is clear that $\mathrm{BExc}_n^+(s,t) = \frac{1}{2}\left(\mathrm{BExc}_n(s,t) + (s-t)^n\right)$. As $\mathrm{BExc}_n(s,t) = B_n(s,t)$, it is palindromic for all n. The polynomial $(s-t)^n$ is palindromic iff the exponent $n \equiv 0 \pmod 2$. Thus, $\mathrm{BExc}_n^+(s,t)$ is palindromic iff $n \equiv 0 \pmod 2$. A similar proof works for $\mathrm{BExc}_n^-(s,t)$. □

We will need the following two lemmas to deduce the recurrences for $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{BExc}_n^-(s,t)$.

Lemma 8. *For positive integers n,*

$$\sum_{\pi \in \mathfrak{B}_n^+, \mathrm{pos}_n(\pi) \neq n} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)} = \sum_{\pi \in \mathfrak{B}_n^-, \mathrm{pos}_n(\pi) \neq n} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}$$
$$= \frac{1}{2} \sum_{\pi \in \mathfrak{B}_n, \mathrm{pos}_n(\pi) \neq n} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)}. \tag{34}$$

Proof. We need (33) for both n and $n-1$. We have

$$\mathrm{SgnBExc}_{n-1}(s,t) = (s-t)^{n-1}. \tag{35}$$

Let $\pi \in \mathfrak{B}_{n-1}$ and put n at the last position in all such π. Such permutations contribute $s(s-t)^{n-1}$ to $\mathrm{SgnBExc}_n(s,t)$. Next, put $\overline{n}$ at the last position in all $\pi \in \mathfrak{B}_{n-1}$. Such permutations contribute $-t(s-t)^{n-1}$ to $\mathrm{SgnBExc}_n(s,t)$. Hence, we get

$$\sum_{\pi \in \mathfrak{B}_n, \mathrm{pos}_n(\pi) = n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)} = (s-t)(s-t)^{n-1}. \tag{36}$$

Subtracting (36) from (33) we get

$$\sum_{\pi \in \mathfrak{B}_n, \mathrm{pos}_n(\pi) \neq n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{exc}_B(\pi)} s^{\mathrm{nexc}_B(\pi)} = 0. \tag{37}$$

Thus, we get (34), thereby completing the proof. □

Lemma 9. *For positive integers n,*

$$\sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)\neq n} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)} = \sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)\neq n} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}. \tag{38}$$

Proof. From Brenti's Theorem 11, we get the following

$$\sum_{\pi\in\mathfrak{B}_n} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)} = \sum_{\pi\in\mathfrak{B}_n} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}, \tag{39}$$

and

$$\sum_{\pi\in\mathfrak{B}_{n-1}} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)} = \sum_{\pi\in\mathfrak{B}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}. \tag{40}$$

As in the earlier proof, let $\pi\in\mathfrak{B}_{n-1}$ and put n at the last position of π. Such permutations have neither an extra excedance nor an extra descent. Next, put $\overline{n}$ at the last position of all permutations in $\mathfrak{B}_{n-1}$. Each such permutation will have one extra excedance and one extra descent. This gives

$$\sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)=n} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)} = \sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)=n} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}. \tag{41}$$

Subtracting (41) from (39), we get (38). The proof is complete. □

We can now give recurrences for $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{BExc}_n^-(s,t)$.

Theorem 13. *For positive integers $n\geq 2$, $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{BExc}_n^-(s,t)$ satisfy the following recurrence relations:*

$$\mathrm{BExc}_n^+(s,t) = s\mathrm{BExc}_{n-1}^+(s,t) + t\mathrm{BExc}_{n-1}^-(s,t) + stD\mathrm{BExc}_{n-1}(s,t), \tag{42}$$

$$\mathrm{BExc}_n^-(s,t) = s\mathrm{BExc}_{n-1}^-(s,t) + t\mathrm{BExc}_{n-1}^+(s,t) + stD\mathrm{BExc}_{n-1}(s,t). \tag{43}$$

Proof. We consider (42) first. Recall $\mathrm{BExc}_n^+(s,t) = \sum_{\pi\in\mathfrak{B}_n^+} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)}$. Consider the contribution to the right hand side from $\pi\in\mathfrak{B}_n^+$ with 'n' or '$\overline{n}$' occurring in position k for all possible choices of k. We claim that

$$\mathrm{BExc}_n^+(s,t) = s\mathrm{BExc}_{n-1}^+(s,t) + t\mathrm{BExc}_{n-1}^-(s,t) + \sum_{\pi\in\mathfrak{B}_n^+,\mathrm{pos}_n(\pi)\neq n} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)}.$$

Here, $s\mathrm{BExc}_{n-1}^+(s,t)$ accounts for the contribution of all $\pi\in\mathfrak{B}_n^+$ in which the letter n appears in the last position and $t\mathrm{BExc}_{n-1}^-(s,t)$ is the contribution of all $\pi\in\mathfrak{B}_n^+$ in which the letter $\overline{n}$ appears in the last position. Further, by Lemma 8 we get

$$\begin{aligned}\mathrm{BExc}_n^+(s,t) &= s\mathrm{BExc}_{n-1}^+(s,t) + t\mathrm{BExc}_{n-1}^-(s,t) + \frac{1}{2}\sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)\neq n} t^{\mathrm{exc}_B(\pi)}s^{\mathrm{nexc}_B(\pi)}\\ &= s\mathrm{BExc}_{n-1}^+(s,t) + t\mathrm{BExc}_{n-1}^-(s,t) + \frac{1}{2}\sum_{\pi\in\mathfrak{B}_n,\mathrm{pos}_n(\pi)\neq n} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}\\ &= s\mathrm{BExc}_{n-1}^+(s,t) + t\mathrm{BExc}_{n-1}^-(s,t) + stDB_{n-1}(s,t)\end{aligned}$$

$$= s\mathrm{BExc}^+_{n-1}(s,t) + t\mathrm{BExc}^-_{n-1}(s,t) + stD\mathrm{BExc}_{n-1}(s,t).$$

The second line follows from Lemma 9. The third line follows from Corollary 4. The last line follows as $B_{n-1}(s,t) = \mathrm{BExc}_{n-1}(s,t)$. These complete the proof of (42). The proof of (43) follows identical steps and is hence omitted. $\square$

With these lemmas, we now prove the two main results of this Section.

Theorem 14. *For even positive integers $n \geq 2$, both $\mathrm{BExc}^+_n(s,t)$ and $\mathrm{BExc}^-_n(s,t)$ are gamma-positive with the same center of symmetry $\frac{n}{2}$.*

Proof. By Lemma 7, we need $n \equiv 0 \pmod 2$ for palindromicity. We use induction on n with $n = 2$ being the base case. Clearly, $\mathrm{BExc}^+_2(s,t) = s^2 + 2st + t^2$ and $\mathrm{BExc}^-_2(s,t) = 4st$, both of which are gamma positive with center of symmetry 1. Let $n = 2m$ and by induction, assume that for $n-2 = 2m-2$, both $\mathrm{BExc}^+_{n-2}(s,t)$ and $B\mathrm{BExc}^-_{n-2}(s,t)$ are gamma positive with center of symmetry $m-1$. Using the recurrence relations (42) and (43) twice, we get

$$\begin{aligned}\mathrm{BExc}^+_n(s,t) &= (s^2+2st+t^2)\mathrm{BExc}^+_{n-2}(s,t) + 4st\mathrm{BExc}^-_{n-2}(s,t)\\ &\quad + 4st(s+t)D\mathrm{BExc}_{n-2}(s,t) + 2s^2t^2D^2\mathrm{BExc}_{n-2}(s,t), \quad (44)\\ \mathrm{BExc}^-_n(s,t) &= (s^2+2st+t^2)\mathrm{BExc}^-_{n-2}(s,t) + 4st\mathrm{BExc}^+_{n-2}(s,t)\\ &\quad + 4st(s+t)D\mathrm{BExc}_{n-2}(s,t) + 2s^2t^2D^2\mathrm{BExc}_{n-2}(s,t). \quad (45)\end{aligned}$$

We note that all the factors $s^2+2st+t^2$, $4st$, $4st(s+t)$ and $2s^2t^2$ are gamma positive. By induction and Corollary 1, each of the four individual terms in (44) and (45) are gamma positive with center of symmetry m. Thus, both $\mathrm{BExc}^+_{2m}(s,t)$ and $\mathrm{BExc}^-_{2m}(s,t)$ are gamma positive with center of symmetry m, completing the proof. $\square$

Theorem 15. *For odd positive integers $n \geq 3$, $\mathrm{BExc}^+_n(t)$ and $\mathrm{BExc}^-_n(t)$ can be written as the sum of two gamma positive polynomials.*

Proof. Let $n \geq 3$ be an odd positive integer. We first prove that $\mathrm{BExc}^+_n(t)$ can be written as a sum of two gamma positive polynomials. Setting $s = 1$ in (42), we get

$$\mathrm{BExc}^+_n(t) = \mathrm{BExc}^+_{n-1}(t) + t\mathrm{BExc}^-_{n-1}(t) + \left(stD\mathrm{BExc}_{n-1}(s,t)\right)\Big|_{s=1}. \quad (46)$$

The polynomial $r(s,t) = stD\mathrm{BExc}_{n-1}(s,t)$ is gamma positive with center of symmetry $\frac{1}{2}(n-2)+1$. Hence, $r(t) = r(s,t)|_{s=1}$ is a univariate gamma positive polynomial with center of symmetry $n/2$. Further, as n is odd, it is simple to see that $r(t)$ has odd length and hence by Lemma 3 it can be written as $r(t) = r_1(t) + r_2(t)$ with respective centers of symmetry $(n-1)/2$ and $(n+1)/2$. Thus, (46) becomes

$$\mathrm{BExc}^+_n(t) = \mathrm{BExc}^+_{n-1}(t) + t\mathrm{BExc}^-_{n-1}(t) + r_1(t) + r_2(t) = w_1(t) + w_2(t).$$

where $w_1(t) = \mathrm{BExc}^+_{n-1}(t) + r_1(t)$ has center of symmetry $(n-1)/2$ and $w_2(t) = t\mathrm{BExc}^-_{n-1}(t) + r_2(t)$ has center of symmetry $(n+1)/2$. The proof of the second part that $\mathrm{BExc}^-_n(t)$ can be written as a sum of two gamma positive polynomials is identical and hence is omitted. $\square$

Remark 2. When summation is over $\mathfrak{B}_n^+$, the bivariate excedance based Eulerian polynomials are not gamma positive. When $n = 3$, our proof gives

$$\mathrm{BExc}_3^+(s,t) = s^3 + 10s^2t + 13st^2 \text{ with } r_1(s,t) = s(s+t)^2 + 8s^2t \text{ and } r_2(s,t) = 12st^2.$$

Thus, though $\mathrm{BExc}_3^+(t) = \mathrm{BExc}_3^+(s,t)|_{s=1}$ is a sum of two gamma positive polynomials, $\mathrm{BExc}_3^+(s,t)$ is not.

5. Type D Coxeter Groups

Let $\mathfrak{D}_n = \{\sigma \in \mathfrak{B}_n : |\mathrm{Negs}(\sigma)| \text{ is even}\}$ denote the type D Coxeter group. For $\pi \in \mathfrak{D}_n$, define as before, $\mathrm{Negs}(\pi) = \{i : i > 0, \pi_i < 0\}$ to be the set of elements which occur with a negative sign. As defined in Petersen's book [16, Chapter 13], let $\mathrm{inv}_D(\pi) = \mathrm{inv}(\pi) + |\{1 \le i < j \le n : -\pi_i > \pi_j\}|$, where $\mathrm{inv}(\pi)$ is the usual number of type A inversions of π with respect to the standard order on $\mathbb{Z}$. For $\pi \in \mathfrak{D}_n$, $\mathrm{inv}_D(\pi)$ is also termed as the length of π. In $\mathfrak{D}_n$, we have the same definition of excedance as in $\mathfrak{B}_n$. Hence, $\mathrm{exc}_D(\pi) = \mathrm{exc}_B(\pi) = |\{i \in [n] : \pi_{|\pi(i)|} > \pi_i\}| + |\{i \in [n] : \pi_i = -i\}|$, and let $\mathrm{nexc}_D(\pi) = n - \mathrm{exc}_D(\pi)$. Let $\mathfrak{D}_n^+ \subseteq \mathfrak{D}_n$ denote the subset of even length elements of $\mathfrak{D}_n$ and let $\mathfrak{D}_n^- = \mathfrak{D}_n - \mathfrak{D}_n^+$. Define

$$\begin{aligned} \mathrm{DExc}_n(t) &= \sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{exc}_D(\pi)} \text{ and} \\ \mathrm{DExc}_n(s,t) &= \sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}, \end{aligned} \tag{47}$$

$$\text{(B-D)Exc}_n(t) = \sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} t^{\mathrm{exc}_D(\pi)} \text{ and } \text{(B-D)Exc}_n(s,t) = \sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}, \tag{48}$$

$$\mathrm{DExc}_n^+(t) = \sum_{\pi \in \mathfrak{D}_n^+} t^{\mathrm{exc}_D(\pi)} \text{ and } \mathrm{DExc}_n^+(s,t) = \sum_{\pi \in \mathfrak{D}_n^+} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}, \tag{49}$$

$$\mathrm{DExc}_n^-(t) = \sum_{\pi \in \mathfrak{D}_n^-} t^{\mathrm{exc}_D(\pi)} \text{ and } \mathrm{DExc}_n^-(s,t) = \sum_{\pi \in \mathfrak{D}_n^-} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}, \tag{50}$$

$$\begin{aligned} \mathrm{SgnDExc}_n(t) &= \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}_D(\pi)} t^{\mathrm{exc}_D(\pi)} \text{ and} \\ \mathrm{SgnDExc}_n(s,t) &= \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}_D(\pi)} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}. \end{aligned} \tag{51}$$

Recall that Brenti showed that type B excedances and type B ascents are equidistributed in $\mathfrak{B}_n$ (see Theorem 11).

Remark 3. It is simple to give a similar bijection $T_n : \mathfrak{B}_n \mapsto \mathfrak{B}_n$ such that $\mathrm{des}_B(T_n(\pi)) = \mathrm{exc}_B(\pi)$ and $|\mathrm{Negs}((T_n(\pi))| = |\mathrm{Negs}(\pi)|$. The bijection is very similar to the proof of [10, Theorem 10.2.3] and [3, Theorem 3.15] and so we only outline it.

Outline of the bijection T_n: Let τ be a cyclic permutation of a finite set $B = \{b_1, b_2, \dots, b_m\}$ of m integers. Define $q(\tau)$ as the following word of length m:

$$q(\tau) = \tau^m(\max B), \tau^{m-1}(\max B), \dots, \tau(\max B).$$

Clearly, $\tau^m(\max B) = \max B$ and $q(\tau)$ is a rearrangement of the m elements of B in some order.

Now, let $\pi \in \mathfrak{B}_n$ and let $\pi = (\pi_{11}, \pi_{12}, \dots, \pi_{1i_1})(\pi_{21}, \pi_{22}, \dots, \pi_{2i_2}) \dots (\pi_{s1}, \pi_{s2}, \dots, \pi_{si_s})$ be the standard disjoint cycle form of π. That is, each cycle has its largest element (in absolute value) first and the cycles are written in increasing order of the absolute values of their first elements. Let $C_1 = (\pi_{11}, \pi_{12}, \dots, \pi_{1i_1})$, $C_2 = (\pi_{21}, \pi_{22}, \dots, \pi_{2i_2})$, $\dots$, $C_s = (\pi_{s1}, \pi_{s2}, \dots, \pi_{si_s})$. As C_i are all cyclic permutations, hence we can form the words $q(C_1)$, $q(C_2)$, $\dots$, $q(C_s)$. Let $T_n(\pi)$ be the juxtaposition $q(C_1)q(C_2)\dots q(C_s)$. Thus we have constructed the bijection $T_n : \mathfrak{B}_n \mapsto \mathfrak{B}_n$. Proceeding along similar lines as in the proof of [10, Theorem 10.2.3], one can check that $\mathrm{des}_B(T_n(\pi)) = \mathrm{exc}_B(\pi)$. By construction, clearly $|\mathrm{Negs}((T_n(\pi))| = |\mathrm{Negs}(\pi)|$. We provide an example of this bijection.

Let $\pi = 3, \overline{8}, \overline{6}, 9, 5, 1, 4, 2, \overline{7}, \overline{10} \in \mathfrak{B}_{10}$. We consider the standard disjoint cycle form of π and this is $(5)(\overline{6}, 1, 3)(\overline{8}, 2)(9, \overline{7}, 4)(\overline{10})$. Here, $C_1 = (5)$, $C_2 = (\overline{6}, 1, 3)$, $C_3 = (\overline{8}, 2)$, $C_4 = (9, \overline{7}, 4)$ and $C_5 = (\overline{10})$ are the cyclic permutations. It can be checked that $q(C_1) = 5$, $q(C_2) = \overline{6}, 3, 1$, $q(C_3) = \overline{8}, 2$, $q(C_4) = 9, 4, \overline{7}$ and $q(C_5) = \overline{10}$. Thus, for $\pi = 3, \overline{8}, \overline{6}, 9, 5, 1, 4, 2, \overline{7}, \overline{10}$, we have $T_{10}(\pi) = 5, \overline{6}, 3, 1, \overline{8}, 2, 9, 4, \overline{7}, \overline{10}$. It can be checked that $\mathrm{exc}_B(\pi) = 6$ and $\mathrm{des}_B(T_{10}(\pi)) = 6$.

Remark 4. By Remark 3, as the number of negative entries is preserved by the bijection T_n, enumerating type B descents over $\mathfrak{D}_n$ is equivalent to enumerating type B excedances over $\mathfrak{D}_n$ which is equivalent to enumerating Type D excedances over $\mathfrak{D}_n$. That is, for positive integers n, we have

$$\sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{des}_B(\pi)} s^{\mathrm{asc}_B(\pi)} = \sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{exc}_D(\pi)} s^{\mathrm{nexc}_D(\pi)}.$$

Recall the operator $D = \left(\dfrac{\partial}{\partial s} + \dfrac{\partial}{\partial t} \right)$. We start by proving the following recurrence relations.

Lemma 10. *For positive integers n, the polynomials $\mathrm{DExc}_n(s,t)$ satisfy the following recurrence relations.*

$$\mathrm{DExc}_n(s,t) = s\mathrm{DExc}_{n-1}(s,t) + t(\text{B-D})\mathrm{Exc}_{n-1}(s,t) + stD\mathrm{BExc}_{n-1}(s,t), \tag{52}$$

$$(\text{B-D})\mathrm{Exc}_n(s,t) = s(\text{B-D})\mathrm{Exc}_{n-1}(s,t) + t\mathrm{DExc}_{n-1}(s,t) + stD\mathrm{BExc}_{n-1}(s,t). \tag{53}$$

Proof. We consider (52) first. By Remark 4, we evaluate $\sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{des}_B(\pi)} s^{\mathrm{asc}_B(\pi)}$. Similar to Corollary 4, we explain where each term comes from. The contribution from $\pi \in \mathfrak{D}_n$ in which the letter 'n' appears in the last position is clearly $s\sum_{\pi\in\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}$. Likewise, $t\sum_{\pi\in\mathfrak{B}_{n-1}-\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}$ is the contribution from $\pi\in\mathfrak{D}_n$ in which the letter $\overline{n}$ appears in the last position. The term $stD\sum_{\pi\in\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}$ accounts for all those permutations $\pi\in\mathfrak{D}_n$ in which the letter n appears in all positions except the last position. The term $stD\sum_{\pi\in\mathfrak{B}_{n-1}-\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}$ accounts for all those permutations $\pi\in\mathfrak{D}_n$ in which the letter $\overline{n}$ appears at all positions except the last. Summing all these, we get

$$\begin{aligned}\sum_{\pi\in\mathfrak{D}_n} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)} = {} & s\sum_{\pi\in\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)} + t\sum_{\pi\in\mathfrak{B}_{n-1}-\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}\\ &+stD\sum_{\pi\in\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}\\ &+stD\sum_{\pi\in\mathfrak{B}_{n-1}-\mathfrak{D}_{n-1}} t^{\mathrm{des}_B(\pi)}s^{\mathrm{asc}_B(\pi)}.\end{aligned}$$

The sum of the terms in the last line equals $stD\mathrm{BExc}_{n-1}(s,t)$. In an identical manner one can prove (53). □

Using Lemma 10, we next show that the polynomials $\mathrm{DExc}_n(s,t)$ and $\mathrm{(B\text{-}D)Exc}_n(s,t)$ are gamma positive. We first show that the polynomials $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{DExc}_n(s,t)$ are identical.

Theorem 16. *For integers $n\geq 2$, $\mathrm{BExc}_n^+(s,t)=\mathrm{DExc}_n(s,t)$ and $\mathrm{BExc}_n^-(s,t)=\mathrm{(B\text{-}D)Exc}_n(s,t)$. Hence, for even positive integers $n\geq 2$, $\mathrm{DExc}_n(s,t)$ and $\mathrm{(B\text{-}D)Exc}_n(s,t)$ are gamma positive with the same center of symmetry $n/2$. For odd positive integers $n\geq 3$, the univariate polynomials $\mathrm{DExc}_n(t)$ and $\mathrm{(B\text{-}D)Exc}_n(t)$ can be written as a sum of 2 gamma positive polynomials.*

Proof. It is simple to see that $\mathrm{BExc}_2^+(s,t)=s^2+2st+t^2=\mathrm{DExc}_2(s,t)$ and $\mathrm{BExc}_2^-(s,t)=4st=\mathrm{(B\text{-}D)Exc}_2(s,t)$. Further, recurrences (52) and (53) for $\mathrm{DExc}_n(s,t)$ and $\mathrm{(B\text{-}D)Exc}_n(s,t)$ respectively are identical to the recurrences (42) and (43). Thus, both the pairs of polynomials $\mathrm{BExc}_n^+(s,t)$ and $\mathrm{DExc}_n(s,t)$, and $\mathrm{BExc}_n^-(s,t)$ and $\mathrm{(B\text{-}D)Exc}_n(s,t)$ are identical, completing the proof. □

Though we have Lemma 10, to show gamma positivity of $\mathrm{DExc}_n^+(s,t)$ and $\mathrm{DExc}_n^-(s,t)$, we need the following alternate recurrence relation satisfied by $\mathrm{DExc}_n(s,t)$. This recurrence has a different form than the recurrence given in Lemma 10. In particular, not every term involves n. We give this for ease of later arguments.

Lemma 11. *For positive integers $n\geq 2$,*

$$\begin{aligned}\mathrm{DExc}_{n+4}(s,t) = {} & R_1(s,t)\mathrm{DExc}_n(s,t)+R_2(s,t)\mathrm{(B\text{-}D)Exc}_n(s,t)\\ &+R_3(s,t)D\mathrm{BExc}_n(s,t)\\ &+R_4(s,t)D^2\mathrm{BExc}_n(s,t)+R_5(s,t)D\mathrm{BExc}_{n+1}(s,t)\\ &+R_6(s,t)D^2\mathrm{BExc}_{n+1}(s,t)+R_7(s,t)D^2\mathrm{BExc}_{n+2}(s,t) \qquad (54)\end{aligned}$$

where the following table lists $R_i(s,t)$ and its center of symmetry. Further, the $R_i(s,t)$ are gamma positive.

$R_i(s,t)$	*center of symmetry of $R_i(s,t)$)*
$R_1(s,t) = (s+t)^4 + 8st(s+t)^2 + 16(st)^2$	2
$R_2(s,t) = 16st(s+t)^2$	2
$R_3(s,t) = 4st(s+t)^3 + 32(st)^2(s+t)$	5/2
$R_4(s,t) = 2(st)^2(s+t)^2 + 8(st)^3$	3
$R_5(s,t) = 12(st)(s+t)^2$	2
$R_6(s,t) = 8(st)^2(s+t)$	5/2
$R_7(s,t) = 2(st)^2$	2

Define

$$S_{n+4}(s,t) = R_2(s,t)(\text{B-D})\text{Exc}_n(s,t) + \Big(R_3(s,t)D + R_4(s,t)D^2\Big)\text{BExc}_n(s,t) + \Big(R_5(s,t)D + R_6(s,t)D^2\Big)\text{BExc}_{n+1}(s,t) + R_7(s,t)D^2\text{BExc}_{n+2}(s,t). \tag{55}$$

Then, $S_{n+4}(s,t)$ is gamma positive with center of symmetry $\frac{1}{2}(n+4)$ and each gamma coefficient of $S_{n+4}(s,t)$ is even.

Proof. Applying Lemma 10 twice, we get

$$\begin{aligned}\text{DExc}_{n+2}(s,t) &= (s+t)^2\text{DExc}_n(s,t) + 4st(\text{B-D})\text{Exc}_n(s,t)\\ &\quad +4st(s+t)D\text{BExc}_n(s,t) + 2(st)^2D^2\text{BExc}_n(s,t),\end{aligned} \tag{56}$$

$$\begin{aligned}(\text{B-D})\text{Exc}_{n+2}(s,t) &= 4st\text{DExc}_n(s,t) + (s+t)^2(\text{B-D})\text{Exc}_n(s,t)\\ &\quad +4st(s+t)D\text{BExc}_n(s,t) + 2(st)^2D^2\text{BExc}_n(s,t)\\ &.\end{aligned} \tag{57}$$

From (31) and the fact that $B_n(s,t) = \text{BExc}_n(s,t)$ for all positive integer n, we get

$$\text{BExc}_{n+2}(s,t) = (s+t)\text{BExc}_{n+1}(s,t) + 2stD\text{BExc}_{n+1}(s,t). \tag{58}$$

Using (58) twice we get

$$\begin{aligned}D\text{BExc}_{n+2}(s,t) &= 2(s+t)\text{BExc}_n(s,t) + 4stD\text{BExc}_n(s,t)\\ &\quad +3(s+t)D\text{BExc}_{n+1}(s,t) + 2stD^2\text{BExc}_{n+1}(s,t).\end{aligned} \tag{59}$$

(56) is equivalent to

$$\begin{aligned}\text{DExc}_{n+4}(s,t) &= (s+t)^2\text{DExc}_{n+2}(s,t) + 4st(\text{B-D})\text{Exc}_{n+2}(s,t)\\ &\quad +4st(s+t)D\text{BExc}_{n+2}(s,t) + 2(st)^2D^2\text{BExc}_{n+2}(s,t).\end{aligned} \tag{60}$$

We get (54) by plugging in (56), (57) and (59) in (60). Further, note that $S_{n+4}(s,t)$ is a sum of six gamma positive polynomials, each with center of symmetry $\frac{1}{2}(n+4)$. From the table, $R_i(s,t)$ for $2 \le i \le 7$ clearly have even gamma coefficients and hence, $S_{n+4}(s,t)$ also has even gamma coefficients.

□

Remark 5. In (57), when n is even, by induction, we get the following: (B-D)$\mathrm{Exc}_n(s,t)$ is gamma positive with even gamma coefficients.

Lemma 11 has a few corollaries. By setting $t = 1$ and $q_i = t$ in [24, Theorem 18] of Sivasubramanian, we get the following.

Theorem 17. *For positive integers n,* $\mathrm{SgnDExc}_n(t) = \begin{cases} (1-t)^n & \text{if } n \text{ even} . \\ (1-t)^{n-1} & \text{if } n \text{ odd} . \end{cases}$

A slight modification of this result gives us a bivariate version of Theorem 17. Each term in the bivariate polynomial $\mathrm{SgnDExc}_n(s,t)$ clearly has degree n. Any $\pi \in \mathfrak{D}_n$ with $\mathrm{exc}_D(\pi) = k$ has $\mathrm{nexc}_D(\pi) = n-k$. Thus, the coefficient of t^k in $\mathrm{SgnDExc}_n(t)$ is the same as the coefficient of $t^k s^{n-k}$ in $\mathrm{SgnDExc}_n(s,t)$. Hence, we have the following bivariate version of Theorem 17.

Theorem 18. *For positive integers n,* $\mathrm{SgnDExc}_n(s,t) = \begin{cases} (s-t)^n & \text{if } n \text{ even} . \\ s(s-t)^{n-1} & \text{if } n \text{ odd} . \end{cases}$

We use Theorem 18 twice to get the following.

Corollary 6. *For positive integers n, we have* $\mathrm{SgnDExc}_{n+4}(s,t) = (s-t)^4$ $\mathrm{SgnDExc}_n(s,t)$.

We now come to the main Theorem of this section.

Theorem 19. *For even positive integers $n \ge 4$, $\mathrm{DExc}_n^+(s,t)$ and $\mathrm{DExc}_n^-(s,t)$ are gamma positive with center of symmetry $\frac{1}{2}n$.*

Proof. We use induction on n with the base cases being $n = 4$ and $n = 6$. When $n = 4$ and $n = 6$ one can check the following.

$$
\begin{aligned}
\mathrm{DExc}_4^+(s,t) &= s^4 + 16s^3t + 62s^2t^2 + 16st^3 + t^4 = (s+t)^4 + 12st(s+t)^2 + 32(st)^2, \\
\mathrm{DExc}_4^-(s,t) &= 20s^3t + 56s^2t^2 + 20st^3 = 20st(s+t)^2 + 16(st)^2, \\
\mathrm{DExc}_6^+(s,t) &= s^6 + 176s^5t + 2647s^4t^2 + 5872s^3t^3 + 2647s^2t^4 + 176st^5 + t^6 \\
&= (s+t)^6 + 170st(s+t)^4 + 1952(st)^2(s+t)^2 + 928(st)^3, \\
\mathrm{DExc}_6^-(s,t) &= 182s^5t + 2632s^4t^2 + 5892s^3t^3 + 2632s^2t^4 + 182st^5, \\
&= 182st(s+t)^4 + 1904(st)^2(s+t)^2 + 992(st)^3.
\end{aligned}
$$

By Lemma 11 and Corollary 6,

$$\begin{aligned}
\mathrm{DExc}^+_{n+4}(s,t) &= \frac{1}{2}\Big(\mathrm{DExc}_{n+4}(s,t) + \mathrm{SgnDExc}_{n+4}(s,t)\Big) \\
&= \frac{1}{2}\Big(R_1(s,t)\mathrm{DExc}_n(s,t) + S_{n+4}(s,t) + (s-t)^4\mathrm{SgnDExc}_n(s,t)\Big) \\
&= \frac{1}{2}\Big(\Big[R_1(s,t) + (s-t)^4\Big]\mathrm{DExc}^+_n(s,t)\Big) \\
&\quad + \frac{1}{2}\Big(\Big[R_1(s,t) - (s-t)^4\Big]\mathrm{DExc}^-_n(s,t)\Big) + \frac{1}{2}S_{n+4}(s,t), \qquad (61)
\end{aligned}$$

The last line follows as $\mathrm{DExc}_n(s,t) = \mathrm{DExc}^+_n(s,t) + \mathrm{DExc}^-_n(s,t)$ while $\mathrm{SgnDExc}_n(s,t) = \mathrm{DExc}^+_n(s,t) - \mathrm{DExc}^-_n(s,t)$. Similarly one can see that

$$\begin{aligned}
\mathrm{DExc}^-_{n+4}(s,t) &= \frac{1}{2}\Big(\Big[R_1(s,t) - (s-t)^4\Big]\mathrm{DExc}^+_n(s,t)\Big) \\
&\quad + \frac{1}{2}\Big(\Big[R_1(s,t) + (s-t)^4\Big]\mathrm{DExc}^-_n(s,t)\Big) + \frac{1}{2}S_{n+4}(s,t).
\end{aligned}$$

It is simple to see that $R_1(s,t) + (s-t)^4 = 2(s+t)^4 + 32(st)^2$ and $R_1(s,t) - (s-t)^4 = 16st(s+t)^2$ are both gamma positive with center of symmetry 2 and both have even gamma coefficients. Let n be even with $n > 7$. By induction, both $\mathrm{DExc}^+_n(s,t)$ and $\mathrm{DExc}^-_n(s,t)$ are gamma positive with the same center of symmetry $\frac{1}{2}n$. By Lemma 11, $S_{n+4}(s,t)$ has even gamma coefficients and hence $\frac{1}{2}S_{n+4}(s,t)$ has integral gamma coefficients. Further, each of the three terms in (61) have the same center of symmetry $\frac{1}{2}(n+4)$. Thus by Lemma 11, the polynomial $\mathrm{DExc}^+_{n+4}(s,t)$ is gamma positive with center of symmetry $\frac{1}{2}(n+4)$. In an identical manner, one can prove that $\mathrm{DExc}^-_{n+4}(s,t)$ is gamma positive with center of symmetry $\frac{1}{2}(n+4)$. □

Theorem 20. *For odd positive integers $n \geq 5$, $\mathrm{DExc}^+_n(t)$ and $\mathrm{DExc}^-_n(t)$ can be written as a sum of two gamma positive polynomials.*

Proof. Let $n \geq 5$ be an odd integer. We have

$$\begin{aligned}
\mathrm{DExc}^+_n(s,t) &= \frac{1}{2}\Big(\mathrm{DExc}_n(s,t) + \mathrm{SgnDExc}_n(s,t)\Big) \\
&= \frac{1}{2}\Big(s\mathrm{DExc}_{n-1}(s,t) + t\text{(B-D)}\mathrm{Exc}_{n-1}(s,t) \\
&\quad + stD\mathrm{BExc}_{n-1}(s,t) + \mathrm{SgnDExc}_n(s,t)\Big) \\
&= \frac{1}{2}\Big(s\mathrm{DExc}_{n-1}(s,t) + t\text{(B-D)}\mathrm{Exc}_{n-1}(s,t) \\
&\quad + stD\mathrm{BExc}_{n-1}(s,t) + s\mathrm{SgnDExc}_{n-1}(s,t)\Big).
\end{aligned}$$

Here, the second equality follows from Lemma 10 and the last equality follows from Theorem 18 and the fact that n is odd. As $s\mathrm{DExc}_{n-1}(s,t) + s\mathrm{SgnDExc}_{n-1}(s,t) = 2s\mathrm{DExc}^+_{n-1}(s,t)$, we get

$$\mathrm{DExc}^+_n(t) = \frac{1}{2}\left(2\mathrm{DExc}^+_{n-1}(t) + t(\text{B-D})\mathrm{Exc}_{n-1}(t) + \left[stD\mathrm{BExc}_{n-1}(s,t)\right]\Big|_{s=1}\right). \tag{62}$$

Our proof now follows along similar lines as the proof of Theorem 4. The polynomial $p(s,t) = stD\mathrm{BExc}_{n-1}(s,t)$ is gamma positive with center of symmetry $(n-1)/2 - 1/2 + 1 = n/2$. Hence, $p(t) = p(s,t)|_{s=1}$ is a univariate gamma positive polynomial with center of symmetry $n/2$. Further, it is simple to see that $p(t)$ has odd length and thus by Lemma 3, it can be written as $p(t) = p_1(t) + p_2(t)$ where $p_2(t) = tp_1(t)$. It can be seen that $p_1(t)$ and $p_2(t)$ have respective centers of symmetry $(n-1)/2$ and $(n+1)/2$. Thus, (62) becomes

$$\mathrm{DExc}^+_n(t) = \frac{1}{2}\left(2\mathrm{DExc}^+_{n-1}(t) + t(\text{B-D})\mathrm{Exc}_{n-1}(t) + p_1(t) + p_2(t)\right) = w_1(t) + w_2(t)$$

where $w_1(t) = \frac{1}{2}\left(2\mathrm{DExc}^+_{n-1}(t) + p_1(t)\right)$ and $w_2(t) = \frac{1}{2}\left(t(\text{B-D})\mathrm{Exc}_{n-1}(t) + p_2(t)\right)$. Clearly, $w_1(t)$ has center of symmetry $(n-1)/2$. By Theorem 16, as n is even, $n-1$ is odd and hence $(\text{B-D})\mathrm{Exc}_{n-1}(t)$ is gamma positive polynomial with center of symmetry $(n-1)/2$. Therefore, $w_2(t)$ has center of symmetry $(n+1)/2$. By Remark 5, since n is odd, $(\text{B-D})\mathrm{Exc}_{n-1}(t)$ has even gamma coefficients. Further, if $\mathrm{BExc}_{n-1}(s,t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma^B_{n-1,i}(st)^i(s+t)^{n-1-2i}$, then it can be checked that

$$\begin{aligned} stD\mathrm{BExc}_{n-1}(s,t) = & \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} i\gamma^B_{n-1,i}(st)^i(s+t)^{n-2i} \\ & + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 2(n-1-2i)\gamma^B_{n-1,i}(st)^{i+1}(s+t)^{n-2-2i}. \end{aligned} \tag{63}$$

By Corollary 5, the gamma coefficients $\gamma^B_{n,i}$ when $i > 0$ are even. From (63), it is clear that the gamma coefficients of $stD\mathrm{BExc}_{n-1}(s,t)$ are even. Since $p_2(t) = tp_1(t)$, the coefficients of both $p_1(t)$ and $p_2(t)$ are even. Thus, the coefficients of both $w_1(t)$ and $w_2(t)$ are integral. An identical proof works for $\mathrm{DExc}^-_n(s,t)$. □

Remark 6. Using Theorem 16, it can be seen that Theorems 19 and 20 refine our earlier Theorems 14 and 15, respectively.

6. Excedances in Even and Odd Derangements

Let $\mathfrak{SD}_n \subseteq \mathfrak{S}_n$ be the subset of derangements in $\mathfrak{S}_n$. Recall $\mathrm{ADerExc}_n(t)$ was defined in (5). Sharesian and Wachs in [18] and Shin and Zeng in [20] proved gamma positivity of $\mathrm{ADerExc}_n(t)$. We need one more definition. Let $w = w_1, w_2, \ldots, w_n \in \mathfrak{S}_n$. For $1 \leq i \leq n$, an index i is said to be a double excedance if $w_i > i > (w^{-1})_i$. Their result is as follows.

Theorem 21 (Shin and Zeng). *For positive integers n, $\mathrm{ADerExc}_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \theta_{n,i} t^i (1+t)^{n-2i}$ where $\theta_{n,i}$ is equal to the number of derangements $w \in \mathfrak{SD}_n$ with i excedances and no double excedance. Thus, $\mathrm{ADerExc}_n(t)$ is gamma positive.*

Sun and Wang in [25] gave an alternate proof of Theorem 21 based on a variant of *valley hopping* that we call *cyclic valley hopping*. It is a cyclic version of the Modified Foata Strehl action. They constructed a group action $\theta : \mathbb{Z}_2^n \times \mathfrak{SD}_n \mapsto \mathfrak{SD}_n$ where the group $\mathbb{Z}_2^n$ acts on the set of derangements via the functions $\theta_S(\pi) : \mathfrak{SD}_n \mapsto \mathfrak{SD}_n$, defined by $\theta_S(\pi) = \prod_{x \in S} \theta_x(\pi)$ for any set $S \subseteq [n]$. In the penultimate paragraph on Page 3, while showing that the map θ_x is well-defined, they show that θ_x preserves cycle type, but do not explicitly mention this.

Remark 7. From the above discussion, we get that the cyclic valley hopping proof of Sun and Wang preserves cycle type and hence sign.

Thus, we get the following refinement of Theorem 21.

Theorem 22. *For positive integers n, the polynomial $\mathrm{ADerExc}_n^+(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \theta_{n,i}^+ t^i (1+t)^{n-2i}$ where $\theta_{n,i}^+$ is equal to the number of derangements $w \in \mathfrak{SD}_n \cap \mathcal{A}_n$ with i excedances and no double excedance. An identical statement is true about $\mathrm{ADerExc}_n^-(t)$. Hence both $\mathrm{ADerExc}_n^+(t)$ and $\mathrm{ADerExc}_n^-(t)$ are gamma positive.*

We move on to refinements of this polynomial. Recall that $\mathrm{cyc}(w)$ denotes the number of cycles of a permutation w. We are now ready to prove Theorem 6.

Proof of Theorem 6. We prove separately for the two statistics $\mathrm{inv}(\pi)$ and $\mathrm{cyc}(\pi)$.

$\mathrm{stat}(\pi) = \mathrm{inv}(\pi)$: Shin and Zeng [20]. showed that

$$\sum_{\pi \in \mathfrak{SD}_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{exc}(\pi)} = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i}(q) t^i (1+t)^{n-2i}. \tag{64}$$

where $b_{n,i}(q) = \sum_{\pi \in \mathfrak{SD}_n(i)} q^{\mathrm{inv}(\pi)}$. Here, $\mathfrak{SD}_n(i)$ consists of all elements of $\mathfrak{SD}_n$ with exactly i excedances and no double excedances. Consider terms on either side of the equality as polynomials in $\mathbb{R}[t, q]$. The coefficient of q^{2r} and q^{2r+1} for each $r \geq 0$ on either side are identical. Clearly, $\pi \in \mathfrak{SD}_n^+$ iff $\mathrm{inv}(\pi)$ or equivalently, the exponent of q is even. Thus, (64) factors nicely for $\mathfrak{S}\mathfrak{D}_n^+$ and $\mathfrak{S}\mathfrak{D}_n^-$. The same proof of Shin and Zeng [21, Theorem 2] gives us the following

$$\sum_{\pi \in \mathfrak{S}\mathfrak{D}_n^+} q^{\mathrm{inv}(\pi)} t^{\mathrm{exc}(\pi)} = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i}^+(q) t^i (1+t)^{n-2i} \tag{65}$$

with the following interpretation for the gamma coefficients:

$$b_{n,i}^+(q) = \sum_{\pi \in \mathfrak{S}\mathfrak{D}_n^+(i)} q^{\mathrm{inv}(\pi)} \tag{66}$$

where $\mathfrak{S}\mathfrak{D}_n^+(i) \subseteq \mathcal{A}_n$ is the set of positive derangements with i excedances and no double excedance. An identical proof works when we sum over $\mathfrak{S}\mathfrak{D}_n^-$.

$\mathrm{stat}(\pi) = \mathrm{cyc}(\pi)$: Shin and Zeng in [20, Theorem 11] also showed

$$\sum_{\pi \in \mathfrak{S}\mathfrak{D}_n} q^{\mathrm{cyc}(\pi)} t^{\mathrm{exc}(\pi)} = \sum_{i=0}^{\lfloor n/2 \rfloor} f_{n,i}(q) t^i (1+t)^{n-2i} \tag{67}$$

where $f_{n,i}(q) = \sum_{\pi \in \mathfrak{S}\mathfrak{D}_n(i)} q^{\mathrm{cyc}(\pi)}$. The following alternate definition of sign is known, see for example Nelson [14]: $\mathrm{sign}(\pi) = (-1)^{n-\mathrm{cyc}(\pi)}$. Again, comparing the coefficients of q^{2r} and q^{2r+1} in both sides of (67), taking parity of n into account gives us

$$\sum_{\pi \in \mathfrak{S}\mathfrak{D}_n^+} q^{\mathrm{cyc}(\pi)} t^{\mathrm{exc}(\pi)} = \sum_{i=0}^{\lfloor n/2 \rfloor} f_{n,i}^+(q) t^i (1+t)^{n-2i} \tag{68}$$

with the following interpretation for the gamma coefficients:

$$f_{n,i}^+(q) = \sum_{\pi \in \mathfrak{S}\mathfrak{D}_n^+(i)} q^{\mathrm{cyc}(\pi)} \tag{69}$$

where $\mathfrak{S}\mathfrak{D}_n^+(i) \subseteq \mathcal{A}_n$ is the set of positive derangements with i excedances and no double excedance. An identical proof works when we sum over $\mathfrak{S}\mathfrak{D}_n^-$. $\square$

7. Open Problems

The main open question is to give interpretations of the gamma coefficients as the cardinality of appropriately defined sets. We have several positivity results about the gamma coefficients and interpretations for their gamma coefficients are open.

Problem 1. Find an interpretation for the gamma coefficients that appear in Theorems 3 and 4 for $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$. How do these add up to give the gamma coefficients of $\mathfrak{S}_n$?

Problem 2. Similarly, find an interpretation for the gamma coefficients and how they add up when we sum over even length elements in type-B Coxeter groups (Theorems 14, 15) and type-D Coxeter groups (Theorems 19 and 20).

Problem 3. Theorem 5 has three results while in Theorem 6 we are only able to prove a refinement for only two of them. A bivariate refinement of Theorem 5 with respect to the statistic $\mathrm{nest}(\pi)$ seems true but we are unable to prove it.

7.1. Type B Derangements

Derangements in Type B Coxeter groups are well known and the distribution of Brenti's definitions of excedance in them is well studied (see Chen, Tang and Zhao [4]). It is known that $\mathrm{BDerExc}_n(q)$, the polynomial that enumerates excedances in type-B derangements is not palindromic. We end this work with a conjecture.

Conjecture 1. For all natural numbers $n \geq 2$, $\mathrm{BDerExc}_n(q)$ is the sum of two gamma positive polynomials.

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