Linear Algebra and its Applications 435 (2011) 1479–1489

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Identities for minors of the Laplacian, resistance and distance matrices

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ARTICLE INFO

Article history:
Received 1 November 2009
Accepted 14 March 2011
Available online 8 April 2011

Submitted by B. Shader

AMS classification:
15A09
15A15
15A24

Keywords:
Laplacian
Distance matrix
Resistance matrix
Determinant
Partitioned matrix
q-Analogue

ABSTRACT

It is shown that if L and D are the Laplacian and the distance matrix of a tree respectively, then any minor of the Laplacian equals the sum of the cofactors of the complementary submatrix of D, up to sign and a power of 2. An analogous, more general result is proved for the Laplacian and the resistance matrix of any graph. A similar identity is proved for graphs in which each block is a complete graph on r vertices, and for q-analogues of such matrices of a tree. Our main tool is an identity for the minors of a matrix and its inverse.

1. Introduction

We consider simple graphs, that is, graphs which have no loops or parallel edges. For a positive integer n, the set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$. We usually consider a graph with vertex set $[n]$.

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1 The support of the JC Bose Fellowship, Department of Science and Technology, Government of India, is gratefully acknowledged.
2 Supported partially by grant P07IR052.

doi:10.1016/j.laa.2011.03.028

Given a graph, one associates a variety of matrices with the graph. Let G be a graph with vertex set $[n]$. The Laplacian matrix L of the graph is an $n \times n$ matrix defined as follows. For $i, j \in [n]$; $i \neq j$, the (i, j)-element of L is -1 if vertices i and j are adjacent, and zero otherwise. For $i \in [n]$, the (i, i)-element of L is d_i, the degree of the vertex i. The adjacency matrix A of the graph is an $n \times n$ matrix defined as follows. For $i, j \in [n]$; $i \neq j$, the (i, j)-element of A is 1 if vertices i and j are adjacent, and otherwise it is zero. For $i \in [n]$, the (i, i)-element of A is 0. Note that $L = \Delta - A$, where Δ is the diagonal matrix of the vertex degrees.

For $i, j \in [n]$, the distance between i and j is defined as the length (i.e., the number of edges) of the shortest ij-path, if such a path exists. The distance matrix D of a connected graph with vertex set $[n]$ is an $n \times n$ matrix with its (i, j)-element equal to $d(i, j)$ if $i \neq j$, and 0 otherwise.

The interplay between the graph theoretic properties and the algebraic properties comes under the purview of algebraic graph theory and is an area of intense recent research, see [7,10]. There are interesting properties of the distance matrix of a tree, as well as relations between the distance matrix and Laplacian of a tree. In this paper we obtain yet another identity between minors of the Laplacian and the distance matrix of a tree, generalizing some known results.

For graphs which are not trees, the classical shortest path distance is replaced by the resistance distance, motivated by resistance in electrical networks. We obtain an identity involving minors of the Laplacian and the resistance matrix. The main tool in the proofs is a fairly general identity for minors of a matrix and its inverse, proved in Section 2.

We also consider graphs in which each block is a complete graph on r vertices and prove an identity for minors of its Laplacian and distance matrices. Finally, we also consider a q-analogue of the distance matrix of a tree and obtain a determinantal identity

The complement of $S \subseteq [n]$ will be denoted by S^c. Let A be an $n \times n$ matrix and let $S, K \subseteq [n]$. We denote by $A[S, K]$, the matrix obtained by selecting the rows of A indexed by S, and the columns of A indexed by K. By $A(S, K)$, we mean the matrix obtained by deleting the rows of A indexed by S, and the columns of A indexed by K. Note that $A[S, K] = A(S^c, K^c)$. We occasionally use notation such as $A[S, K)$ and $A(S, K]$. Their meaning should be clear. We tacitly assume that these notations do not indicate a vacuous matrix. This amounts to assuming that S, K are nonempty, proper subsets. We do not state these assumptions explicitly. We extend the notation to vectors as well. Thus if x is an $n \times 1$ vector, then $x[S]$ and $x(S)$ will denote the subvector of x, indexed by indices in S and in S^c, respectively. The all ones vector in any (appropriate) dimension is denoted by $\mathbf{1}$.

We state a few preliminaries that will be used. If $S \subseteq [n]$, then $\alpha(S)$ will denote the sum of the integers in S. The transpose of the vector x is denoted by x'. The following determinantal identities are well-known.

Theorem 1 (Jacobi, see [8, Section 4.2])**.** *Let A be an invertible $n \times n$ matrix, let $B = A^{-1}$ and let $S, K \subseteq [n]$ with $|S| = |K|$. Then*

$$\det A[S, K] = (-1)^{\alpha(S)+\alpha(K)} \frac{\det B(K, S)}{\det B}. \tag{1}$$

Theorem 2 (Sherman–Morrison, see [22, Section 14.6])**.** *Let A be an $n \times n$ nonsingular matrix and let u, v be $n \times 1$ vectors. Then*

$$\det(A - uv') = (\det A)(1 - v'A^{-1}u). \tag{2}$$

2. Minors of a matrix and its inverse

One of our main tools will be the following identity involving a partitioned matrix and its inverse, which seems to be of independent interest.

Lemma 3. *Let A be an $n \times n$ invertible matrix, let $B = A^{-1}$ and let x, u be $n \times 1$ vectors. Let $y = Ax$ and $v = A'u$. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then, assuming that the inverses exist,*

$$v(K)'A(S, K)^{-1}y(S) + u[S]'B[K, S]^{-1}x[K] = u'y. \tag{3}$$

Proof. We use the following formula for the inverse of a partitioned matrix from Horn and Johnson's book [14, p. 18].

$$B[K,S]^{-1} = A[S,K] - A[S,K)A(S,K)^{-1}A(S,K].$$

Thus,

$$u[S]'B[K,S]^{-1}x[K] = u[S]'A[S,K]x[K] - u[S]'A[S,K)A(S,K)^{-1}A(S,K]x[K]. \tag{4}$$

It is simple to note that $y(S) = A(S,K)x(K) + A(S,K]x[K]$, and thus

$$A(S,K)^{-1}y(S) = x(K) + A(S,K)^{-1}A(S,K]x[K]. \tag{5}$$

Further, $v(K)' = u(S)'A(S,K) + u[S]'A[S,K)$, and thus it follows from (5) that

$$\begin{aligned} v(K)'A(S,K)^{-1}y(S) &= \{u(S)'A(S,K) + u[S]'A[S,K)\}\{x(K) + A(S,K)^{-1}A(S,K]x[K]\} \qquad (6)\\ &= u(S)'A(S,K)x(K) + u(S)'A(S,K]x[K] + u[S]'A[S,K)x(K)\\ &\quad + u[S]'A[S,K)A(S,K)^{-1}A(S,K]x[K]. \qquad (7) \end{aligned}$$

The proof is complete by adding (4) and (7). $\square$

Special cases of Lemma 3 will be of interest and useful in various situations. We note the following curious consequence of the lemma. If A and $B = A^{-1}$ are $n \times n$ matrices such that the row and column sums of A are all equal to 1, then for any $S, K \subseteq [n]$ with $|S| = |K|$, assuming that the inverses exist, the sum of the elements in $A(S,K)^{-1}$ and $B[K,S]^{-1}$ equals n.

In the next result we present a fairly general identity for minors. The identity will be applied to obtain various consequences in the subsequent sections. For a square matrix A, we denote the sum of all its cofactors by $\operatorname{cofsum} A$.

Theorem 4. *Let X and Y be symmetric $n \times n$ matrices such that X is nonsingular, each row and column sum of Y equals 0, and*

$$X^{-1} = -Y + \delta zz' \tag{8}$$

for some real δ and $n \times 1$ vector z. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then

$$\operatorname{cofsum} X(K,S) = (-1)^{\alpha(S)+\alpha(K)}(-1)^{|S|}\delta(\mathbf{1}'z)^2(\det X)(\det Y[S,K]). \tag{9}$$

Proof. Let $B = X^{-1}$. Since Y has zero row and column sums, observe that, from (8), $B\mathbf{1} = \delta(\mathbf{1}'z)z$. We have $B = -Y + \delta zz'$, and hence $Y = \delta zz' - B$. Therefore

$$Y[S,K] = -\{B[S,K] - \delta z[S]z[K]'\}. \tag{10}$$

It follows from (10) and Theorem 2 that

$$\det Y[S,K] = (-1)^{|S|}\det B[S,K]\{1 - \delta z[K]'(B[S,K])^{-1}z[S]\}. \tag{11}$$

By Theorem 1 we have

$$\begin{aligned} \operatorname{cofsum} X(K,S) &= \det X(K,S)\mathbf{1}'X(K,S)^{-1}\mathbf{1}\\ &= (-1)^{\alpha(S)+\alpha(K)}(\det B[S,K])(\det X)\mathbf{1}'X(K,S)^{-1}\mathbf{1}. \qquad (12) \end{aligned}$$

It follows from (12) that

$$\mathbf{1}'X(K,S)^{-1}\mathbf{1} = (-1)^{\alpha(S)+\alpha(K)}\frac{\operatorname{cofsum} X(K,S)}{\det B[S,K]\det X}. \tag{13}$$

Therefore, setting $A = X, B = X^{-1}, x = u = \delta(\mathbf{1}'z)z$ and $y = v = \mathbf{1}$, we obtain from Theorem 3 that

$$\mathbf{1}'X(S,K)^{-1}\mathbf{1} + \delta^2(\mathbf{1}'z)^2 z[S]'B[K,S]^{-1}z[K] = \delta(\mathbf{1}'z)^2. \tag{14}$$

It follows from (14) and (11) that

$$\begin{aligned}\mathbf{1}'X(K,S)^{-1}\mathbf{1} &= \delta(\mathbf{1}'z)^2(1-\delta z[K]'B[S,K]^{-1}z[S]) \\ &= \delta(\mathbf{1}'z)^2(-1)^{|S|}\frac{\det Y[S,K]}{\det B[S,K]}.\end{aligned} \tag{15}$$

Using (13) and (15) we obtain

$$(-1)^{|S|}\delta(\mathbf{1}'z)^2)\frac{\det Y[S,K]}{\det B[S,K]} = (-1)^{\alpha(S)+\alpha(K)}\frac{\operatorname{cofsum} X(K,S)}{\det B[S,K]\det X},$$

and hence

$$\operatorname{cofsum} X(K,S) = \frac{(-1)^{\alpha(S)+\alpha(K)}}{(-1)^{|S|}}(\det X)(\det Y[S,K]). \tag{16}$$

This completes the proof. □

3. Resistance matrix and its inverse

The distance between two vertices in a graph is traditionally defined as the length (i.e., the number of edges) in a shortest path between the two vertices. In contrast to this notion, the concept of resistance distance arises naturally from several different considerations and is also more amenable to mathematical treatment. We refer to [1,9,15] for more information on the resistance distance and for additional references. Though we restrict ourselves to unweighted graphs, our results easily generalize to edge-weighted graphs. This only requires a small modification in the definition of the Laplacian.

Recall that if A is an $m \times n$ matrix, then an $n \times m$ matrix G is called a g-inverse of A if $AGA = A$. Further, a g-inverse G is called the Moore–Penrose inverse of A if it also satisfies $GAG = G$, $(AG)' = AG$ and $(GA)' = GA$. It is well-known (see Meyer's book [17]) that any real matrix A admits a unique Moore–Penrose inverse which is denoted by A^+.

Let G be a connected graph with vertex set $[n]$. There are several equivalent ways to define the resistance distance between two vertices. We present two of them, both based on the Laplacian matrix. Their equivalence is shown for example in Bapat [1].

Let L be the Laplacian matrix of G and let $L^+ = ((\ell^+_{ij}))$ be the Moore–Penrose inverse of L. Then $r(i,j)$, the resistance distance between vertices $i, j \in [n]$, is given by

$$r(i,j) = \ell^+_{ii} + \ell^+_{jj} - 2\ell^+_{ij}. \tag{17}$$

It may be remarked that we get the same expression if, instead of the Moore–Penrose inverse of L, we use any symmetric g-inverse.

A second definition of $r(i,j)$ can be given in terms of minors of L. Thus for any $i, j \in [n]$, $i \neq j$,

$$r(i,j) = \frac{\det L(\{i,j\},\{i,j\})}{\det L(i,i)}. \tag{18}$$

If $i = j$, then $r(i,j) = 0$. By the Matrix-Tree Theorem, $\det L(i,i)$ is the number of spanning trees of G, which we denote by $\chi(G)$. Thus

$$r(i,j) = \frac{\det L(\{i,j\},\{i,j\})}{\chi(G)}. \tag{19}$$

The resistance matrix R of G is defined as the $n \times n$ matrix with its (i,j)-entry equal to $r(i,j)$. In this section we obtain a generalization of (19) in the form of a minor identity involving L and R. The main tools in the proof are Lemma 3 and some known results for R and its inverse, to be stated next.

We introduce some notation. If i is a vertex of G, then $\mathcal{N}(i)$ will denote the set of vertices adjacent to i. For $i \in [n]$, let

$$\tau_i = 2 - \sum_{j\in\mathcal{N}(i)} r(i,j), \tag{20}$$

and let τ be the $n \times 1$ vector with components $\tau_1, \ldots, \tau_n$. We will use the following result of Bapat [2].

Theorem 5. *Let G be a connected graph with vertex set $[n]$. Let L be its Laplacian matrix and R its resistance matrix. Then the following assertions hold:*

(i) $\mathbf{1}'\tau = \sum_{i=1}^n \tau_i = 2.$
(ii) $\sum_{i=1}^n \sum_{j\in\mathcal{N}(i)} r(i,j) = 2(n-1).$
(iii) *R is nonsingular and*

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R\tau}\tau\tau'. \tag{21}$$

(iv)

$$\det R = (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\chi(G)}.$$

The following result follows easily from Theorem 5 and the fact that the Laplacian has zero row and column sums.

Corollary 6. *Let G be a connected graph with vertex set $[n]$ and let R be its resistance matrix. Then*

$$R^{-1}\mathbf{1} = \frac{2}{\tau' R\tau}\tau. \tag{22}$$

The main result of this section, which is an extension of (19), is presented next.

Theorem 7. *Let G be a connected graph with vertex set $[n]$. Let L be its Laplacian matrix and R its resistance matrix. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then*

$$\operatorname{cofsum} R(K,S) = (-1)^{\alpha(S)+\alpha(K)}(-2)^{n-|S|-1}\frac{\det L[S,K]}{\chi(G)}. \tag{23}$$

Proof. Let $B = R^{-1}$. By Theorem 5, $B = -\frac{1}{2}L + \frac{1}{\tau' R\tau}\tau\tau'$. In view of Corollary 6, an application of Theorem 4 gives

$$\frac{4}{\tau' R\tau}\frac{1}{(-2)^{|S|}}\frac{\det L[S,K]}{\det B[S,K]} = (-1)^{\alpha(S)+\alpha(K)}\frac{\operatorname{cofsum} R(K,S)}{\det B[S,K]\det R}$$

and hence

$$\operatorname{cofsum} R(K,S) = \frac{(-1)^{\alpha(S)+\alpha(K)}}{(-2)^{|S|}}\frac{4}{\tau' R\tau}(\det L[S,K])(\det R). \tag{24}$$

Using (24) and part (iv) of Theorem 5 we get

$$\operatorname{cofsum} R(K,S) = (-1)^{\alpha(S)+\alpha(K)}(-2)^{n-|S|-1}\frac{\det L[S,K]}{\chi(G)}$$

and the proof is complete. □

Let $i,j \in [n]$, $i \neq j$, and set $S = K = [n] \setminus \{i,j\}$ in Theorem 7. Then

$$R(K,S) = \begin{bmatrix} 0 & r(i,j) \\ r(i,j) & 0 \end{bmatrix}$$

and hence $\operatorname{cofsum} R(K,S) = -2r(i,j)$. Hence we see that (19) is a consequence of Theorem 7.

4. Distance matrix and Laplacian of a tree

In this section we only consider trees. Let T be a tree with vertex set $[n]$. Let $d(i,j)$ be the distance, i.e., the length of the unique path, between vertices $i,j \in [n]$. Let D be the distance matrix of T, which is the $n \times n$ matrix with its (i,j)-entry equal to $d(i,j)$. Thus D is a symmetric matrix with zeros on the diagonal. The distance matrix of a tree has been a subject of intensive research, starting with the classical result of Graham and Pollak [13] that $\det D = (-1)^{n-1}(n-1)2^{n-2}$, which shows that the determinant is independent of the structure of the tree. A formula for D^{-1} was given by Graham and Lovasz [12].

It is well-known that the resistance distance between vertices i and j in a graph equals the classical shortest path distance if there is a unique ij-path in the graph. Thus the distance matrix of a tree is the same as its resistance matrix. Hence we may obtain results for the distance matrix of a tree as special cases of the results obtained for the resistance matrix in the previous section.

Let L be the Laplacian of T. It has been observed by several authors that

$$d(i,j) = \det L(\{i,j\},\{i,j\}), \tag{25}$$

which is indeed a special case of (19). The following far-reaching generalization of this identity follows immediately from Theorem 7.

Theorem 8. *Let T be a tree with vertex set $[n]$. Let L be the Laplacian matrix and D the distance matrix of T. Let $S,K \subseteq [n]$ with $|S| = |K|$. Then*

$$\operatorname{cofsum} D(K,S) = (-1)^{\alpha(S)+\alpha(K)}(-2)^{n-|S|-1}\det L[S,K]. \tag{26}$$

A combinatorial interpretation of the minors of the Laplacian matrix of a graph is well-known, see [4,19]. The interpretation is particularly simple for principal minors and is stated next.

Theorem 9. *Let G be a connected graph with vertex set $[n]$, and let L be its Laplacian. Let $S \subseteq [n]$ be a nonempty, proper subset with $|S| = k$. Then $\det L[S,S]$ equals the number of spanning forests of G with $n-k$ components, each component containing a vertex in S^c.*

It is tempting to attempt a combinatorial proof of Theorem 8. We present below such a proof for the case of principal submatrices, i.e., the case when $S = K$. First we state the following simple result of Graham et al. [11] without proof.

Lemma 10. *Let A be an $n \times n$ matrix. Subtract the first row of A from every other row, then the first column from every other column, and delete the first row and column in the resulting matrix. If B is the matrix thus obtained, then $\operatorname{cofsum} A = \det B$.*

We now prove the following special case of Theorem 8.

Theorem 11. *Let T be a tree with vertex set $[n]$. Let L be the Laplacian matrix and D the distance matrix of T. Let $S \subseteq [n]$ with $|S| = k$. Then*

$$\operatorname{cofsum} D(S, S) = (-2)^{n-k-1} \det L[S, S]. \tag{27}$$

Proof. Let $S \subseteq [n]$ with $|S| = k$ and let $S = \{n-k+1, \ldots, n\}$, without loss of generality. We first claim that it is possible to relabel the vertices in S^c as $w_1, \ldots, w_{n-k}$, such that for any $1 \leqslant i < j < \ell \leqslant n-k$, w_i is not on the unique path from w_j to w_ℓ. This claim is easily proved by induction and we omit the proof. From now on we assume that the vertices $w_1, \ldots, w_{n-k}$ are ordered as stated. When the vertices are ordered in this fashion we also make the following observation. For any $i < j < \ell$, the vertex on the path from w_j to w_ℓ that is closest to w_i, is the same for all $\ell > j$.

Perform the following operations on $D(S, S)$. Subtract the first row from every other row, then the first column from every other column, and delete the first row and column in the resulting matrix. Let M be the resulting matrix. Then by Lemma 10, $\operatorname{cofsum} D(S, S) = \det M$. It will be convenient to index the rows and columns of M as $2, \ldots, n-k$. For $i = 2, \ldots, n-k$, let α_i be the vertex closest to 1 on the path from i to j, for any $i < j \leqslant n-k$. Note that α_i is well-defined in view of the preceding observation (see Fig. 1 for an illustration). The (i, j)-element of M is $d(i, j) - d(1, i) - d(1, j)$, which is easily seen to be $-2d(1, \alpha_1)$. Thus M has the form $M = -2\tilde{M}$, where

$$\tilde{M} = \begin{bmatrix} d(1,2) & d(1,\alpha_1) & d(1,\alpha_1) & \cdots & d(1,\alpha_1) \\ d(1,\alpha_1) & d(1,3) & d(1,\alpha_2) & \cdots & d(1,\alpha_2) \\ \vdots & & \vdots & & \vdots \\ d(1,\alpha_1) & d(1,\alpha_2) & d(1,\alpha_3) & \cdots & d(1,\alpha_{n-k}) \end{bmatrix}.$$

We will prove the result by induction on the cardinality of S^c. The base cases $|S^c| = 1, 2$ are easy. We thus assume $|S^c| \geq 3$. By induction assumption and by Theorem 9, $\det M(1, 1)$ is the number of ways to break the tree T into $n-k-1$ components, with vertices $w_1, w_3, \ldots, w_{n-k}$ into separate components. We denote the number of such possibilities (i.e., the number of such forests with $n-k-1$ components) as $\operatorname{sep}(1, 3, 4 \ldots, n-k)$. Similarly, we denote the number of forests of T with $n-k-2$ components, each of which contains a vertex from $\{w_3, w_4, \ldots, w_{n-k}\}$ as $\operatorname{sep}(3, 4, \ldots, n-k)$.

Note that the rows and the columns of $\tilde{M}$ are indexed by $2, \ldots, n-k$. Perform the following operations on $\tilde{M}$. Subtract row 3 from row i for $4 \leqslant i \leqslant n-k$, and then subtract column 3 from column i for $4 \leqslant i \leqslant n-k$ and let $\overline{M}$ be the resulting matrix. It is easy to see by expanding $\det \overline{M}$ by the first row that

$$\det \overline{M} = d(1, 2)\operatorname{sep}(1, 3, 4, \ldots, n-k) - d(1, \alpha_1)^2 \operatorname{sep}(3, 4, \ldots, n-k). \tag{28}$$

It is clear (see Fig. 1) that the recurrence (28) is satisfied by $\operatorname{sep}(1, 2, \ldots, n-k)$. To see this, we note that $d(1, 2)\operatorname{sep}(1, 3, 4, \ldots, n-k)$ is the number of ways to break T into forests with $n-k-1$ components, each component containing one of the vertices $w_1, w_3, w_4, \ldots, w_{n-k}$, and

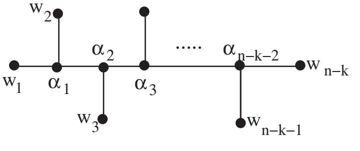

Fig. 1. The subtree induced on S.

to choose an edge f on the w_1w_2-path. Thinking of f as a choice to break the w_1w_2-path, we see that this way of counting gives us choices of breaking T into $n-k$ components, with each component containing one vertex from S^c. Clearly, the choices of f which lie on the $1\alpha_1$-path, and the choices of $\text{sep}(1, 3, 4, \ldots, n-k)$ where the separating edge on the w_1w_3-path is also from the $1\alpha_1$ path, will not separate the vertices w_2, w_3, and thus we need to remove these choices. It is simple to see that such choices, which we might call as w_2w_3-nonseparators, are precisely $d(1, \alpha_1)^2\text{sep}(3, 4, \ldots, n-k)$ in number. Thus $\det \overline{M} = \det \tilde{M} = \text{sep}(1, 2, \ldots, n-k)$, completing the proof. $\square$

5. Graphs with each block as K_r

In this section we illustrate another application of Theorem 4, thereby obtaining a generalization of Theorem 8. Let G be a connected graph with vertex set $[n]$. Recall that a block of a graph is defined as a maximal 2-connected subgraph. We assume that each block of G is K_r, the complete graph on r vertices. Let G have p blocks. Then it is easy to see that $n = pr - p + 1$. Let d_i be the degree of the vertex i, and let $\tau_i = r - d_i$, $i = 1, \ldots, n$. Let τ be the $n \times 1$ vector with components $\tau_1, \tau_2, \ldots, \tau_n$. With this notation we have the following result of Sivasubramanian [20].

Theorem 12. *Let G be a connected graph with vertex set $[n]$. Let D be the distance matrix and L the Laplacian of G. Then the following assertions hold.*

(i) $\det D = (-1)^{n-1}(n-1)r^{\frac{n-r}{r-1}}$.
(ii) $D\tau = (n-1)\mathbf{1}$.
(iii)

$$D^{-1} = -\frac{1}{r}L + \frac{1}{r(n-1)}\tau\tau'. \tag{29}$$

Note that if we set $r = 2$ in Theorem 12, then we recover the Graham–Lovaśz formula for the inverse of the distance matrix of a tree. We now have the following result which is an immediate consequence of Theorems 4 and 12.

Theorem 13. *Let G be a connected graph with vertex set $[n]$. Let D be the distance matrix and L the Laplacian of G. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then*

$$\text{cofsum}\, D(K, S) = (-1)^{\alpha(S)+\alpha(K)}(-r)^{\frac{n-1}{r-1}-|S|}\det(L[S, K]). \tag{30}$$

We remark that Theorem 8 is a special case of Theorem 13, obtained when $r = 2$.

6. q-Analogues for trees

In this section we consider only trees. We begin with some preliminaries. For a positive integer i, let $i_q = 1 + q + q^2 + \cdots + q^{i-1}$, called the q-analogue of i, denote the polynomial in the variable q. The q-distance matrix D_q of any tree is obtained from the distance matrix D by replacing each entry i by i_q, where $0_q = 0$.

Let A be the adjacency matrix and Δ the diagonal matrix of vertex degrees. Define the q-analogue of the Laplacian as $\mathcal{L}_q = q^2\Delta - qA - (q^2-1)I$, where q is an indeterminate. The matrix $\mathcal{L}_q$ has been studied in the literature [3,5,6,16,18,21]. Note that setting $q = 1$ in both D_q, $\mathcal{L}_q$ gives the matrices D, L respectively.

Let T be a tree with vertex set $[n]$ and let $D, L, D_q, \mathcal{L}_q$ be its appropriate matrices. Let $S, K \subseteq [n]$ with $|S| = |K|$. Define

$$\text{qcofsum}\, D_q(S, K) = \text{cofsum}\, D_q(S, K) - (1-q)\det D(S, K).$$

Again, note that setting $q = 1$ gives $\mathrm{qcofsum}\, D_q(S, K) = \mathrm{cofsum}\, D_q(S, K)$ for all $S, K \subseteq [n]$. For a tree T, with distances between vertices i and j given by $d(i, j)$, define the exponential distance matrix $\mathrm{ED}_T = ((e_{i,j}))$ as $e_{i,j} = 1$ if $i = j$ and $e_{i,j} = q^{d(i,j)}$ where q is an indeterminate, and where $q^0 = 1$. We abuse notation and refer to the matrix as ED instead of ED_T when the tree T is clear from the context. The following result of Bapat et al. [5] will be used.

Theorem 14. *With the notation above, for any tree T, $\mathcal{L}_q^{-1} = \frac{1}{1-q^2}\mathrm{ED}$ and $\det(\mathcal{L}_q) = 1 - q^2$.*

The following result is yet another generalization of Theorem 8, which is obtained by setting $q = 1$. It is not possible to derive this result from Theorem 4 since $\mathcal{L}_q$ does not have zero row and column sums. We give a proof using the Jacobi and Sherman–Morrison formulae.

Theorem 15. *Let T be a tree with vertex set $[n]$. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then,*

$$(-1)^{\alpha(S)+\alpha(K)}(-1-q)^{n-|S|-1} \det \mathcal{L}_q[S, K] = \mathrm{qcofsum}\, D_q(K, S). \tag{31}$$

Proof. By Theorems 1 and 14 we see that

$$\det \mathcal{L}_q[S, K] = (-1)^{\alpha(S)+\alpha(K)} \frac{\det \mathrm{ED}(K, S)}{(1-q^2)^{n-|S|}}(1-q^2) \tag{32}$$

$$= (-1)^{\alpha(S)+\alpha(K)} \frac{\det \mathrm{ED}(K, S)}{(1-q^2)^{n-|S|-1}}. \tag{33}$$

Note that $\mathrm{ED}(K, S) = J-(1-q)D_q(K, S)$, where J is the all ones matrix of dimension $(n-|S|)\times(n-|S|)$ and hence by Theorem 2,

$$\begin{aligned}
&\det \mathrm{ED}(K, S)\\
&= (-1)^{n-|S|} \det((1-q)D_q(K, S) - J)\\
&= (-1)^{n-|S|} \det((1-q)D_q(K, S)) \left[1 - \frac{\mathrm{cofsum}\,((1-q)D_q(K, S))}{\det((1-q)D_q(K, S))}\right]\\
&= (-1)^{n-|S|} \left[\det((1-q)D_q(K, S)) - \mathrm{cofsum}\,((1-q)D_q(K, S))\right]\\
&= (-1)^{n-|S|} \left[(1-q)^{n-|S|} \det(D_q(K, S)) - (1-q)^{n-|S|-1}\mathrm{cofsum}\,(D_q(K, S))\right]\\
&= (-1)^{n-|S|}(1-q)^{n-|S|-1} \left[(1-q) \det(D_q(K, S)) - \mathrm{cofsum}\,(D_q(K, S))\right]\\
&= (-1)^{n-|S|-1}(1-q)^{n-|S|-1} \left[\mathrm{qcofsum}\, D_q(K, S)\right].
\end{aligned} \tag{34}$$

Using (32) and (34) we get

$$\det(\mathcal{L}_q[S, K]) = (-1)^{\alpha(S)+\alpha(K)} \frac{(-(1-q))^{n-|S|-1}\mathrm{qcofsum}\, D_q(K, S)}{(1-q^2)^{n-|S|-1}}$$

$$= (-1)^{\alpha(S)+\alpha(K)} \frac{(-1)^{n-|S|-1}}{(1+q)^{n-|S|-1}}\mathrm{qcofsum}\, D_q(K, S). \tag{35}$$

The proof is complete in view of (35). $\square$

We obtain the following interesting q^2-analogue of (25) as a consequence of Theorem 15. For a positive integer i, we recall $i_q = 1 + q + \cdots + q^{i-1}$, the q-analogue of i, and denote $(i)_{q^2} = 1 + q^2 + q^4 + \cdots + q^{2i-2}$ as the q^2-analogue of i.

Corollary 16. *Let $\mathcal{L}_q$ be the q-analogue of the Laplacian of a tree T with vertex set $[n]$. Let $u, v \in [n]$, $u \neq v$ and $d = d(u, v)$. Then, $\det(\mathcal{L}_q(\{u, v\}, \{u, v\})) = (d)_{q^2}$. i.e., $\det(\mathcal{L}_q(\{u, v\}, \{u, v\}))$ equals the q^2-analogue of $d(u, v)$.*

Proof. Let $S = K = [n] \setminus \{u, v\}$ and let $d_q = (d(u, v))_q$ be the q-analogue of $d(u, v)$. Since $|S| = n-2$, by Theorem 15,

$$(1+q)\det \mathcal{L}_q[S, S] = -\mathsf{qcofsum}\begin{pmatrix} 0 & d_q \\ d_q & 0 \end{pmatrix}. \tag{36}$$

Using the definition of qcofsum we see that

$$\begin{aligned} \mathsf{qcofsum}\begin{pmatrix} 0 & d_q \\ d_q & 0 \end{pmatrix} &= -2d_q - (1-q)\left(-d_q^2\right) \\ &= d_q(-2 + (1-q)d_q) \\ &= d_q\left(-2 + (1-q)\frac{1-q^d}{1-q}\right) \\ &= -d_q(1+q^d). \end{aligned} \tag{37}$$

It follows from (36) and (37) that

$$\det \mathcal{L}_q[S, S] = \frac{(1+q^d)d_q}{1+q}. \tag{38}$$

Since $d_q = \frac{1-q^d}{1-q}$, we get from (38) that

$$\det \mathcal{L}_q[S, S] = \frac{(1+q^d)(1-q^{d)})}{(1+q)(1-q)} = (d)_{q^2},$$

and the proof is complete. □

Acknowledgements

Several theorems in this work were in conjectural phase tested using the package "Sage". We thank the authors of Sage for releasing their software on an open source platform.

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