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Product distance matrix of a tree with matrix weights

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ABSTRACT

Let T be a tree on n vertices and let the $n-1$ edges $e_1, e_2, \dots, e_{n-1}$ have weights that are $s \times s$ matrices $W_1, W_2, \dots, W_{n-1}$, respectively. For two vertices i, j, let the unique ordered path between i and j be $p_{i,j} = e_{r_1} e_{r_2} \dots e_{r_k}$. Define the distance between i and j as the $s \times s$ matrix $E_{i,j} = \prod_{p=1}^{k} W_{e_p}$. Consider the $ns \times ns$ matrix D whose (i,j)-th block is the matrix $E_{i,j}$. We give a formula for $\det(D)$ and for its inverse, when it exists. These generalize known results for the product distance matrix when the weights are real numbers.

1. Introduction

Let T be a tree with vertex set $[n] = \{1, 2, \dots, n\}$. Let $D = (d_{i,j})_{1 \leqslant i,j \leqslant n}$ be its distance matrix, i.e. $d_{i,j}$ is the distance between vertices i and j. A classical result of Graham and Pollak [7] is the following.

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Theorem 1. *Let T be a tree on n vertices and let D be its distance matrix. Then* $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$.

Thus, $\det(D)$ only depends on n and is independent of the structure of the tree T. Later, Graham and Lovász [4,6] gave a formula for the inverse of D. Motivated by this, Bapat and Sivasubramanian, building on the work of Bapat, Lat and Pati [3], considered the exponential distance matrix of a tree T. Let the tree T have n vertices and let $e_1, e_2, \ldots, e_{n-1}$ be an ordering of its edges. Let edge e_i have weight q_i, $i = 1, \ldots, n-1$, where $q_1, \ldots, q_{n-1}$ are commuting indeterminates, and define $\mathsf{E}_T = (e_{i,j})$, the exponential distance matrix of T as follows. For vertices i, j, let $p_{i,j}$ be the unique path between i and j. Define $e_{i,j} = \prod_{k \in p_{i,j}} q_k$. Note that $e_{i,j} = e_{j,i}$ as the q_k's commute with each other. By convention, for all i, we set $e_{i,i} = 1$. With this, Bapat and Sivasubramanian [4] showed the following.

Theorem 2. *Let T be a tree on n vertices with edges having weights $q_1, q_2, \ldots, q_{n-1}$ and let E be the exponential distance matrix E. Then,* $\det(\mathsf{E}) = \prod_{i=1}^{n-1}(1 - q_i^2)$.

In [4], a slightly more general setup was considered and the inverse of E was also determined.

In this work, we consider the product distance matrix of a tree with matrix weights. The motivation for considering matrix weights may be described as follows. When we consider product distance, it is natural to let the weights be noncommutative, since the edges on a path come with a natural order. The entries of the product distance matrix are then elements of an underlying ring. The formula for the inverse given in Theorem 4 holds in the case of noncommutative weights, even though we have chosen to formulate the result with the weights being matrices which provide a natural example of noncommutative weights. In the case of the formula for the determinant of the product distance matrix, given in Theorem 3, matrix weights are justified since there are difficulties in defining the determinant of a matrix with general noncommutative elements. It is apparent from our results that noncommutative weights do not present any obstacle in obtaining formulas for the determinant and the inverse of distance matrices.

An application of weighted graphs arises naturally in circuit theory, where the graph represents an electrical network, and the weights on the edges are the resistances. Thus the weights are nonnegative numbers. Ando and Bunce [1], motivated by the work of Duffin [5], consider the case where nonnegative weights are replaced by positive semidefinite matrices, and show that certain operator inequalities extend naturally to the more general setting.

In the context of classical distance, matrix weights have been considered by Bapat in [2] where an analogue of Theorem 1 is proved. It is natural to consider a similar setup in the case of product distance.

Thus, we have a tree T on n vertices and each edge e_i has an $s \times s$ matrix weight W_i, $i = 1, 2, \ldots, n-1$. The matrices $W_1, \ldots, W_{n-1}$ may be over an arbitrary field, or more

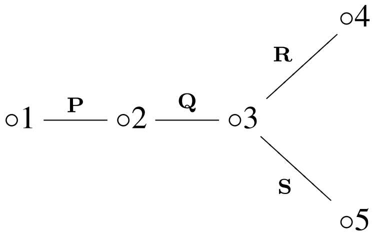

Fig. 1. A matrix-weighted tree T.

generally, over a commutative ring. Given two vertices i, j, let $p_{i,j}$ be the sequential path from i to j in T. This means that $p_{i,j}$ is a *sequence* of edges of T with the order of the edges taken into account. Formally, if $p_{i,j} = (e_{k_1}, e_{k_2}, \dots, e_{k_r})$, then $i \in e_{k_1}$, $j \in e_{k_r}$ and successive edges in $p_{i,j}$ have a common vertex. Define $D_{i,j} = W_{k_1} \cdots W_{k_r}$, and when $i = j$, define $D_{i,i}$ to be the $s \times s$ identity matrix for all $1 \leqslant i \leqslant n$. Denote the $ns \times ns$ matrix whose (i,j)-th block is $D_{i,j}$ by $\mathcal{D}_T$. When, the tree T is clear from the context, we abuse notation and denote $\mathcal{D}_T$ as $\mathcal{D}$.

We show that as in the commutative case, the determinant of the $ns \times ns$ matrix $\mathcal{D}$ is again independent of the structure of T (see Theorem 3) and explicitly find the inverse of $\mathcal{D}$ when it exists (see Theorem 4).

2. Determinant of $\mathcal{D}$

Consider the tree in Fig. 1. The product distance matrix of the tree in Fig. 1 is given by

$$\mathcal{D} = \begin{pmatrix} \mathrm{I} & \mathrm{P} & \mathrm{PQ} & \mathrm{PQR} & \mathrm{PQS} \\ \mathrm{P} & \mathrm{I} & \mathrm{Q} & \mathrm{QR} & \mathrm{QS} \\ \mathrm{QP} & \mathrm{Q} & \mathrm{I} & \mathrm{R} & \mathrm{S} \\ \mathrm{RQP} & \mathrm{RQ} & \mathrm{R} & \mathrm{I} & \mathrm{RS} \\ \mathrm{SQP} & \mathrm{SQ} & \mathrm{S} & \mathrm{SR} & \mathrm{I} \end{pmatrix}.$$

Recall that the matrix P is called unimodular if $\det(P) = \pm 1$. The transpose of the matrix P is denoted P^t.

Lemma 1. *Let T be a tree on the vertex set $[n]$ with edge e_i having weight W_i, $i = 1, \dots, n-1$ and let $\mathcal{D}$ be its distance matrix. Then there exists a unimodular matrix $\mathcal{P}$ such that*

$$\mathcal{P}^t \mathcal{D} \mathcal{P} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I - W_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I - W_{n-1}^2 \end{pmatrix}.$$

Proof. We induct on n, the number of vertices of T. When $n = 2$, $\mathcal{D} = \begin{pmatrix} I & W_1 \\ W_1 & I \end{pmatrix}$ and performing elementary operations in blocks, we get the result. Let T have n vertices with vertex n being a pendant vertex adjacent to $n-1$. Let $T' = T - \{n\}$ be the smaller tree obtained by deleting vertex n and its adjacent edge and let $\mathcal{D}'$ be the product distance matrix of T'. Let V be the "column" of $s \times s$ block-matrices indexed by the vertex $n-1$ and U the "row" of $s \times s$ block-matrices indexed by $n-1$. Writing $\mathcal{D}$ as blocks, we get $\mathcal{D} = \begin{pmatrix} \mathcal{D}' & VW_{n-1} \\ (W_{n-1}U)^t & I \end{pmatrix}$.

Consider the following matrix, where each I and 0 (a block of zeroes) is of order $s \times s$.

$$R = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & -W_{n-1} \\ 0 & 0 & \cdots & 0 & I \end{pmatrix}.$$

Clearly, R is unimodular and we have $R^t\mathcal{D}R = \begin{pmatrix} \mathcal{D}' & 0 \\ 0 & I-W_{n-1}^2 \end{pmatrix}$. By induction, there exists a unimodular matrix Q such that

$$Q^t\mathcal{D}'Q = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I-W_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I-W_{n-2}^2 \end{pmatrix}.$$

Define $S = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$ and $P = SR$. Then it is easy to see that

$$P^t\mathcal{D}P = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I-W_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I-W_{n-1}^2 \end{pmatrix}.$$

It is also clear that P is unimodular and the proof is complete. □

The matrix $\mathcal{P}$ in Lemma 1 admits a simple combinatorial description. Let T be a tree with vertex set $[n]$ and with matrices $W_1, \dots, W_{n-1}$ as edge weights. We fix a root, which we take to be n. Orient the edges of T so that each edge points towards the root. The matrix $\mathcal{P}$ has its rows and columns indexed by $[n]$. Let $i \in [n]$, $i \neq n$, be a vertex and let $e_k = \{i, j\}$ be the edge incident to i such that j is on the path from i to n. For $i > 1$, the i-th block of $\mathcal{P}$ has I at the position $n - i + 1$, $-W_k$ at the position j and zeros elsewhere. The first block of $\mathcal{P}$ has only I at position n and zeros elsewhere. This combinatorial description is easily proved using induction. For the tree in Fig. 1, we have

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & -P \\ 0 & -R & I & -Q & 0 \\ 0 & I & 0 & 0 & 0 \\ I & 0 & -S & 0 & 0 \end{pmatrix}.$$

It may be verified that $\mathcal{P}'\mathcal{D}\mathcal{P}$ is a diagonal matrix with I, $I - R^2$, $I - S^2$, $I - Q^2$, $I - P^2$ on the diagonal, as in Lemma 1. The following is a simple consequence of Lemma 1.

Theorem 3. *Let T be a tree with vertex set $[n]$. For $1 \leqslant i \leqslant n-1$, let each edge e_i be assigned an $s \times s$ matrix W_i and let $\mathcal{D}$ be its non-commutative product distance. Then, $\det(\mathcal{D}) = \prod_{i=1}^{n-1} \det(I - W_i^2)$.*

Proof. By Lemma 1 there exists a unimodular matrix P such that

$$P^t\mathcal{D}P = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I - W_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I - W_{n-1}^2 \end{pmatrix}.$$

The result follows by taking the determinant of both sides of the preceding equation in view of the fact that $\det(P) = \det(P^t) = \pm 1$. $\square$

If we set $s = 1$ in Theorem 3, then we obtain Theorem 2.

It might be instructive to compare the proof technique adopted in this paper to the existing proofs of similar results. In proving the determinant formula, the common technique is to identify a pendant vertex, eliminate the corresponding row and column by row and column operations, and then to use induction. The technique used here is similar but we have combined the row and column operations in a single identity proved in Lemma 1.

As an application of Theorem 3, let A be an $m \times m$ matrix, and consider the partitioned matrix

$$B = \begin{pmatrix} I & A & A^2 & \cdots & A^{m-1} \\ A & I & A & \cdots & A^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{m-2} & \cdots & A & I & A \\ A^{m-1} & \cdots & A^2 & A & I \end{pmatrix}.$$

The matrix B is the product distance matrix of the path on m vertices, with each edge weighted by the matrix A. It follows from Theorem 3 that

$$\det B = \bigl(\det\bigl(I - A^2\bigr)\bigr)^{m-1}.$$

3. Inverse of D_T

In this section, we give an explicit formula for the inverse of D_T when D_T is nonsingular. Thus assume that for all i, $1 \leqslant i \leqslant n-1$, the matrix $I - W_i^2$ is invertible. For an $s \times s$ matrix P, it is easy to see that when $I - P^2$ is invertible,

$$\begin{pmatrix} I & P \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-P^2)^{-1} & -P(I-P^2)^{-1} \\ -P(I-P^2)^{-1} & (I-P^2)^{-1} \end{pmatrix}. \tag{1}$$

For each edge $e_r = \{i, j\}$ of the tree T, consider the following $ns \times ns$ matrix M_{e_r} which we describe in terms of $s \times s$ blocks as follows. The (i,i)-th and (j,j)-th blocks of M_{e_r} are $(I - W_r^2)^{-1}$ and the (i,j)-th and (j,i)-th blocks are $-W_r(I - W_r^2)^{-1}$. For other indices (p,q), define all (p,q) blocks to be the $s \times s$ zero block. We have $n-1$ such matrices M_{e_r}, one for each $1 \leqslant r \leqslant n-1$. If A and B are matrices of order $m \times n$ and $p \times q$ respectively then recall that their Kronecker product $A \otimes B$ is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Let the degree sequence of the tree T be $d_1, d_2, \ldots, d_n$ and define the $n \times n$ diagonal matrix $\mathsf{Deg} = \mathsf{Diag}(d_1, d_2, \ldots, d_n)$. Define the $ns \times ns$ matrix $\Delta = I_s \otimes \mathsf{Deg}$. With these definitions, we are ready to prove the following.

Theorem 4. *Let $\mathcal{D}$ be the product distance matrix of the tree T on n vertices with edge weight matrices W_r for $1 \leqslant r \leqslant n-1$. Then*

$$\mathcal{D}^{-1} = I - \Delta + \sum_{r=1}^{n-1} M_{e_r}.$$

Proof. We induct on the number of vertices n of the tree T. The base case when $n = 2$ is easy and we assume that $n > 2$. Let vertex n be a pendant vertex adjacent to vertex $n-1$, and let $T' = T - \{n\}$ be the smaller tree obtained by deleting vertex n. Let $\mathcal{D}'$ be the $(n-1)s \times (n-1)s$ product distance matrix of T'. Let V be the columns of $\mathcal{D}'$ corresponding to vertex $n-1$. Thus, V is an $(n-1)s \times s$ matrix. Similarly, let U be the rows of $\mathcal{D}'$ corresponding to vertex $n-1$. Hence, U has dimension $s \times s(n-1)$.

We recall that the edge $\{n-1, n\}$ has weight W_{n-1}. Clearly,

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}' & VW_{n-1} \\ W_{n-1}U & I \end{pmatrix}.$$

Let $K = I - \Delta + \sum_{r=1}^{n-1} M_{e_r}$. Let $\mathsf{Deg}_{T'}$ be the $(n-1) \times (n-1)$ diagonal matrix with the degrees of T' on the diagonal, and let $\Delta_{T'} = I_s \otimes \mathsf{Deg}_{T'}$ be the counterpart of Δ for the tree T'. Similarly, for $1 \leqslant r < n-1$, if we define M'_{e_r} as the $(n-1)s \times (n-1)s$ matrix for T', then by induction assumption, we have

$$(\mathcal{D}')^{-1} = I - \Delta_{T'} + \sum_{r=1}^{n-2} M'_{e_r}. \tag{2}$$

Let E_{nn} be the $n \times n$ matrix with its (n,n)-th entry equal to 1 and all the other entries equal to zero. If we augment $\mathsf{Deg}_{T'}$ with the last row and the last column of zeroes, then note that the resulting matrix is $\mathsf{Deg} - E_{nn}$. These observations and (2) allow us to write K as

$$K = \left(\begin{array}{c|c} (\mathcal{D}')^{-1} & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{cccc|c} 0 & 0 & & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & & (I - W_{n-1}^2)^{-1} - I & -W_{n-1}(I - W_{n-1}^2)^{-1} \\ \hline 0 & \cdots & & -W_{n-1}(I - W_{n-1}^2)^{-1} & (I - W_{n-1}^2)^{-1} \end{array}\right) \tag{3}$$

We must show that $\mathcal{D}K = I$. With the above notation, we claim that

$$\left(\begin{array}{c|c} \mathcal{D}' & VW_{n-1} \\ \hline W_{n-1}U & I \end{array}\right) \left(\begin{array}{ccc|c} & (\mathcal{D}')^{-1} & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} & I & & 0 \\ \hline 0 & \cdots & W_{n-1} & 0 \end{array}\right). \tag{4}$$

In order to show (4) it is sufficient to show that the $s \times (n-1)s$ matrix $W_{n-1}U(\mathcal{D}')^{-1}$ has its last $s \times s$ block as W_{n-1} and all other $s \times s$ blocks as zero. To see this, note that U is the $s \times (n-1)s$ submatrix of $\mathcal{D}'$ corresponding to vertex $n-1$. Thus, $U(\mathcal{D}')^{-1} = (0, \dots, 0, I)$ where all blocks are of order $s \times s$. Clearly $W_{n-1}(U(\mathcal{D}')^{-1}) = W_{n-1}(0, \dots, 0, I) = (0, \dots, 0, W_{n-1})$, which shows (4). A simple block-wise matrix multiplication shows that

$$\left(\begin{array}{c|c} \mathcal{D}' & VW_{n-1} \\ \hline W_{n-1}U & I \end{array}\right) \left(\begin{array}{cccc|c} 0 & 0 & & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & & (I - W_{n-1}^2)^{-1} - I & -W_{n-1}(I - W_{n-1}^2)^{-1} \\ \hline 0 & \cdots & & -W_{n-1}(I - W_{n-1}^2)^{-1} & (I - W_{n-1}^2)^{-1} \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & -W_{n-1} & I \end{array}\right) \tag{5}$$

Adding (4) and (5) gives $\mathcal{D}K = I$ and hence $\mathcal{D}^{-1} = K$, completing the proof. $\square$

It may be remarked that the special case of Theorem 4 with scalar weights was proved in Bapat and Sivasubramanian [4]. Furthermore, if the weights are all equal to q, then we recover the formula for the inverse of the exponential distance matrix given in [3, Proposition 3.3].

The proof of Theorem 4 involves a constructive and explicit description of the inverse. This is in contrast to the proof of the inverse formula in [3, Theorem 2.1], where the main tool employed was the formula for the inverse of a partitioned matrix, involving the Schur complement.

We illustrate Theorem 4 by an example. For the tree in Fig. 1, let us label the edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$ and $\{3,5\}$ as e_1, e_2, e_3 and e_4, respectively. Then

$$M_{e_1} = \begin{pmatrix} (I-P^2)^{-1} & -P(I-P^2)^{-1} & 0 & 0 & 0 \\ -P(I-P^2)^{-1} & (I-P^2)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & (I-Q^2)^{-1} & -Q(I-Q^2)^{-1} & 0 & 0 \\ 0 & -Q(I-Q^2)^{-1} & (I-Q^2)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (I-R^2)^{-1} & -R(I-R^2)^{-1} & 0 \\ 0 & 0 & -R(I-R^2)^{-1} & (I-R^2)^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (I-S^2)^{-1} & 0 & -S(I-S^2)^{-1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S(I-S^2)^{-1} & 0 & (I-Q^2)^{-1} \end{pmatrix},$$

and $\Delta = I_s \otimes \mathsf{Diag}(1,2,3,1,1)$. Then according to Theorem 4,

$$\mathcal{D}^{-1} = I_{5s} - \Delta + M_{e_1} + M_{e_2} + M_{e_3} + M_{e_4}.$$

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