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Laplacian immanantal polynomials and the GTS poset on trees

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A R T I C L E I N F O

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Let T be a tree on n vertices with Laplacian L_T and let GTS_n be the generalised tree shift poset on the set of unlabelled trees on n vertices. Inequalities are known for coefficients of the characteristic polynomial of L_T as we go up the poset GTS_n. In this work, we generalise these inequalities to the q-Laplacian $\mathcal{L}_T^q$ of T and to the coefficients of all immanantal polynomials.

1. Introduction

Csikvári in [10] defined a poset on the set of unlabelled trees with n vertices that we denote in this paper as GTS_n (see Definition 5). Among other results, he showed that going up on GTS_n has the following effect: the coefficients of the characteristic polynomial of the Laplacian L_T of T decrease in absolute value. Let $\mathbb{R}^+$ denote the set

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of non-negative real numbers and $\mathbb{R}^+[q]$ denote the set of polynomials in one variable q with coefficients from $\mathbb{R}^+$. In this paper, we prove the following more general result about immanantal polynomials (see (3) for the definition of immanantal polynomial) of the q-Laplacian matrix of trees (see Definition 4).

Theorem 1. *Let T_1 and T_2 be trees with n vertices and let T_2 cover T_1 in GTS_n. Let $\mathcal{L}_{T_1}^q$ and $\mathcal{L}_{T_2}^q$ be the q-Laplacians of T_1 and T_2 respectively. For $\lambda \vdash n$, let*

$$f_\lambda^{\mathcal{L}_{T_1}^q}(x) = d_\lambda(xI - \mathcal{L}_{T_1}^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) x^{n-r} \text{ and}$$

$$f_\lambda^{\mathcal{L}_{T_2}^q}(x) = d_\lambda(xI - \mathcal{L}_{T_2}^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) x^{n-r}.$$

Then, for all $\lambda \vdash n$ and for all $0 \le r \le n$, we assert that $c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) - c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \in \mathbb{R}^+[q^2]$.

For a positive integer n, let $[n] = \{1, 2, \dots, n\}$. Let $\mathfrak{S}_n$ be the group of permutations of $[n]$. Let χ_λ be the irreducible character of the $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by the partition λ of n. We refer the reader to the book by Sagan [26] as a reference for results on representation theory that we use in this work. We denote partitions λ of n as $\lambda \vdash n$. This means we have $\lambda = \lambda_1, \lambda_2, \dots, \lambda_l$ where $\lambda_i \in \mathbb{Z}$ for all i with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0$ and with $\sum_{i=1}^{l} \lambda_i = n$. We also write partitions using the exponential notation, with multiplicities of parts written as exponents. Since characters of $\mathfrak{S}_n$ are integer valued, we think of χ_λ as a function $\chi_\lambda : \mathfrak{S}_n \to \mathbb{Z}$. Let $\lambda \vdash n$ and let $A = (a_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Define its immanant as $d_\lambda(A) = \sum_{\psi \in \mathfrak{S}_n} \chi_\lambda(\psi) \prod_{i=1}^{n} a_{i,\psi_i}$. It is well known that $d_{1^n}(A) = \det(A)$ and $d_n(A) = \mathsf{perm}(A)$ where $\mathsf{perm}(A)$ is the permanent of A.

For an $n \times n$ matrix A, define $f_\lambda^A(x) = d_\lambda(xI - A)$. The polynomial $f_\lambda^A(x)$ is called the immanantal polynomial of A corresponding to $\lambda \vdash n$. Thus, in this notation, $f_{1^n}^A(x)$ is the characteristic polynomial of A. Let T be a tree with n vertices with Laplacian matrix L_T and define

$$f_\lambda^{L_T}(x) = d_\lambda(xI - L_T) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L_T} x^{n-r} \tag{1}$$

where the $c_{\lambda,r}^{L_T}$'s are coefficients of the Laplacian immanantal polynomial of T in absolute value. Immanantal polynomials were studied by Merris [21] where the Laplacian immanantal polynomial corresponding to the partition $\lambda = 2, 1^{n-2}$ (also called the second immanantal polynomial) of a tree T was shown to have connections with the centroid of T. Botti and Merris [6] showed that almost all trees share a complete set of Laplacian immanantal polynomials. When $\lambda = 1^n$, Gutman and Pavlovic [16] conjectured the following inequality which was proved by Gutman and Zhou [17] and independently by Mohar [24].

Theorem 2 *(Gutman and Zhou, Mohar). Let T be any tree on n vertices and let S_n and P_n be the star and the path trees on n vertices respectively. Then, for $0 \leq r \leq n$, we have*

$$c_{1^n,r}^{L_{S_n}} \leq c_{1^n,r}^{L_T} \leq c_{1^n,r}^{L_{P_n}}.$$

Thus, in absolute value, any tree T has coefficients of its Laplacian characteristic polynomial sandwiched between the corresponding coefficients of the star and the path trees. Mohar actually proves stronger inequalities than this result, see Csikvári [11, Section 10] for information on Mohar's stronger results. Much earlier, Chan, Lam and Yeo in their preprint [9], proved the following.

Theorem 3 *(Chan, Lam and Yeo). Let T be any tree on n vertices with Laplacian L_T and let S_n and P_n be the star and the path trees on n vertices respectively. Then, for all $\lambda \vdash n$ and $0 \leq r \leq n$,*

$$c_{\lambda,r}^{L_{S_n}} \leq c_{\lambda,r}^{L_T} \leq c_{\lambda,r}^{L_{P_n}}. \tag{2}$$

In this work, we consider the q-Laplacian matrix $\mathcal{L}_T^q$ of a tree T on n vertices.

Definition 4. Let T be a tree on n vertices with adjacency matrix A and let D be the $n \times n$ diagonal matrix with degrees on the diagonal. Let I denote the $n \times n$ identity matrix. For a variable q, define $\mathcal{L}_T^q = I + q^2(D - I) - qA$ as the q-Laplacian of T.

$\mathcal{L}_T^q$ can be defined for arbitrary graphs G analogously and it is clear that when $q = 1$, $\mathcal{L}_G^q = L_G$. The matrix $\mathcal{L}_G^q$ has occurred previously in connection with the Ihara–Selberg zeta function of G (see Bass [5] and Foata and Zeilberger [13]). For trees, $\mathcal{L}_T^q$ has connections with the inverse of T's exponential distance matrix (see Bapat, Lal and Pati [2]). As done in (1), define

$$f_\lambda^{\mathcal{L}_T^q}(x) = d_\lambda(xI - \mathcal{L}_T^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_T^q}(q) x^{n-r}. \tag{3}$$

We consider the following counterpart of inequalities like (2) when each coefficient is a polynomial in the variable q: we want the difference $c_{\lambda,r}^{\mathcal{L}_T^q}(q) - c_{\lambda,r}^{\mathcal{L}_{S_n}^q}(q) \in \mathbb{R}^+[q]$. That is, the difference polynomial has non-negative coefficients. This is the standard way to get q-analogue of inequalities. Similarly, we want $c_{\lambda,r}^{\mathcal{L}_{P_n}^q}(q) - c_{\lambda,r}^{\mathcal{L}_T^q}(q) \in \mathbb{R}^+[q]$.

We mention a few lines about our proof of Theorem 1. In [11, Theorem 5.1], Csikvári gives a "General Lemma" from which he infers properties about polynomials associated to trees. In that lemma, the following crucial property is needed when dealing with characteristic polynomials of matrices. Let $M = A \oplus B$ be an $n \times n$ matrix that can be

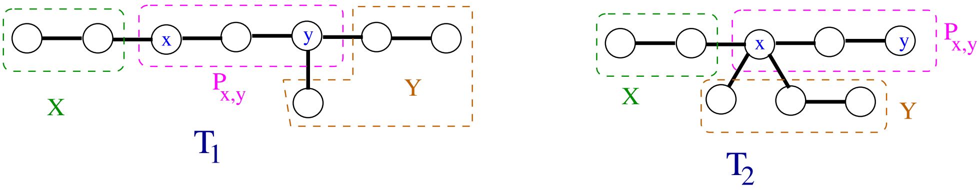

Fig. 1. Two trees with $T_2 \geq_{\mathsf{GTS}_n} T_1$ and T_2 covering T_1.

written as a direct sum of two square matrices. Then, clearly $\det(M) = \det(A)\det(B)$. This property is sadly not true for other immanants. That is, $d_\lambda(M) \neq d_\lambda(A)d_\lambda(B)$ (indeed, the definition of $d_\lambda(A)$ is not clear when $\lambda \vdash n$ and A is an $m \times m$ matrix with $m < n$). We thus combinatorialise the immanant as done by Chan, Lam and Yeo [9] and express the immanantal polynomial in terms of matchings and vertex orientations. Section 2 gives preliminaries on the GTS_n poset and Section 3 gives the necessary background on B-matchings, B-vertex orientations and their connection to coefficients of immanantal polynomials. We give our proof of Theorem 1 in Section 4 and draw several corollaries in Sections 5, 6 and 7 involving the q^2-analogue of vertex moments in a tree, q,t-Laplacian matrices which include the Hermitian Laplacian of T and T's exponential distance matrices.

2. The poset GTS_n

Though Csikvári in [10] defined the poset on unlabelled trees with n vertices, we will label the vertices of the trees according to some convention (see Remark 17). We recall the definition of this poset.

Definition 5. Let T_1 be a tree on n vertices and x, y be two vertices of T_1. Let $P_{x,y}$ be the unique path in T_1 between x and y. Assume that x and y are such that all the interior vertices (if they exist) on $P_{x,y}$ have degree 2. Let z be the neighbour of y on the path $P_{x,y}$. Consider the tree T_2 obtained by moving all neighbours of y except z to the vertex x. This is illustrated in Fig. 1. This move helps us to partially order the set of unlabelled trees on n vertices. We denote this poset on trees with n vertices as GTS_n. We say T_2 is above T_1 in GTS_n or that T_1 is below T_2 in GTS_n and denote it as $T_2 \geq_{\mathsf{GTS}_n} T_1$. The poset GTS_6 is illustrated in Fig. 2.

If $T_2 \geq_{\mathsf{GTS}_n} T_1$ and there is no tree T with $T \neq T_1, T_2$ such that $T_2 \geq_{\mathsf{GTS}_n} T \geq_{\mathsf{GTS}_n} T_1$, then we say T_2 covers T_1 (see Fig. 1). If either x or y is a leaf vertex in T_1, then it is easy to check that T_2 is isomorphic to T_1. If neither x nor y is a leaf in T_1, then T_2 is said to be obtained from T_1 by a proper generalised tree shift (PGTS henceforth). Clearly, if T_2 is obtained by a PGTS from T_1, then, the number of leaf vertices of T_2 is one more than the number of leaf vertices of T_1. Csikvári in [10] showed the following.

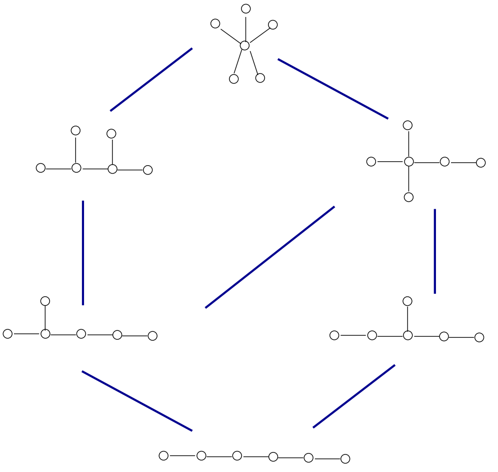

Fig. 2. The poset GTS_6 on trees with 6 vertices.

Lemma 6 *(Csikvári). Every tree T with n vertices other than the path, lies above some other tree T' on GTS_n. The star tree on n vertices is the maximal element and the path tree on n vertices is the minimal element of GTS_n.*

3. B-matchings and B-vertex orientations

As done in earlier work [25], we use matchings in T to index terms that arise in the computation of the immanant $d_\lambda(\mathcal{L}_T^q)$. A dual concept of vertex orientations was used to get a near positive expression for immanants of $\mathcal{L}_T^q$.

In this work, we need to find $f_\lambda^{\mathcal{L}_T^q}(x) = d_\lambda(xI - \mathcal{L}_T^q)$. As done by Chan, Lam and Yeo [9], we index terms that occur in the computation of $f_\lambda^{\mathcal{L}_T^q}(x)$ by partial matchings that we term as B-matchings. Let T have vertex set V and edge set E. Let $B \subseteq V$ with $|B| = r$ and let F_B be the forest induced by T on the set B. A B-matching of T is a subset $M \subseteq E(F_B)$ of edges of F_B such that each vertex $v \in B$ is incident to at most one edge in M. Alternatively, a B-matching is a matching in the graph induced by the vertices in B. If the number of edges in M equals j, then M is called a j-sized B-matching in T. Let $\mathcal{M}_j(B)$ denote the set of j-sized B-matchings in T. Note that we could have $B = [n]$ as well. For vertex v, we denote its degree $\deg_T(v)$ in T alternatively as d_v. For $M \in \mathcal{M}_j(B)$, define a polynomial weight $\mathsf{wt}_{B,M}(q) = q^{2j} \prod_{v \in B-M} [1 + q^2(d_v - 1)]$. Define

$$m_{B,j}(q) = \sum_{M \in \mathcal{M}_j(B)} \mathsf{wt}_{B,M}(q) \text{ and } \quad m_{r,j}(q) = \sum_{B \subset V, |B|=r} m_{B,j}(q).$$

Define $\chi_\lambda(j)$ to be the character $\chi_\lambda(\cdot)$ evaluated at such a permutation with cycle type $2^j, 1^{n-2j}$. The following lemma is straightforward from the definition of immanants.

Lemma 7. *Let T be a tree on vertex set $[n]$ with q-Laplacian $\mathcal{L}_T^q$. Let $\lambda \vdash n$ and let $0 \leq r \leq n$. Then, the coefficient $c_{\lambda,r}^{\mathcal{L}_T^q}(q)$ as defined in (3) equals*

$$c_{\lambda,r}^{\mathcal{L}_T^q}(q) = \sum_{j=0}^{\lfloor r/2 \rfloor} \chi_\lambda(j) m_{r,j}(q).$$

Proof. Let $B \subseteq [n]$ with $|B| = r$. Then, clearly $c_{\lambda,B}^{\mathcal{L}_T^q}(q) = d_\lambda \begin{bmatrix} \mathcal{L}_T^q[B|B] & 0 \\ 0 & I \end{bmatrix}$, where $\mathcal{L}_T^q[B|B]$ is the sub-matrix of $\mathcal{L}_T^q$ induced on the rows and columns with indices in the set B and I is the $(n-r) \times (n-r)$ identity matrix. Further, it is clear that $c_{\lambda,r}^{\mathcal{L}_T^q}(q) = \sum_{B \subseteq [n], |B|=r} c_{\lambda,B}^{\mathcal{L}_T^q}(q)$.

Note that there is no cycle in T, and hence in the forest F_B. Thus, each permutation $\psi \in \mathfrak{S}_n$ which in cycle notation has a cycle of length strictly greater than 2, will satisfy $\prod_{i=1}^n \ell_{i,\psi_i} = 0$. Therefore, only permutations $\psi \in \mathfrak{S}_n$ which fix the set $[n] - B$ and have cycle type $2^j, 1^{n-2j}$ contribute to $c_{\lambda,B}^{\mathcal{L}_T^q}(q)$. It is easy to see that such permutations can be identified with j-sized B-matchings in F_B and that this correspondence is reversible.

Recall $\mathcal{M}_j(B)$ is the set of j-sized B matchings in T. Clearly, the contribution to $c_{\lambda,B}^{\mathcal{L}_T^q}(q)$ from permutations which fix $[n] - B$ and have cycle-type $2^j, 1^{n-2j}$ is $\chi_\lambda(j) m_{B,j}(q)$. Thus, we see that

$$c_{\lambda,B}^{\mathcal{L}_T^q}(q) = \sum_{j=0}^{\lfloor r/2 \rfloor} \chi_\lambda(j) m_{B,j}(q). \tag{4}$$

Summing over various B's of size r completes the proof. □

3.1. B-vertex orientations

As done by Chan, Lam and Yeo [9], we next express coefficients of the immanantal polynomial as a sum of almost positive summands where the summands are indexed by partial vertex orientations that we term as B-vertex orientations.

Let T be a tree with vertex set $V = [n]$. For $B \subseteq [n]$, we orient each vertex $v \in B$ to one of its neighbours (which may or may not be in B). Such vertex orientations are termed as B-vertex orientations. Let O be a B-vertex orientation. Each $v \in B$ has d_v orientation choices. We depict the orientation O in pictures by drawing an arrow on the edge from v to its oriented neighbour and directing the arrow away from v. We do not distinguish between O and its picture from now on. In O, edges thus get arrows and there may be edges which have two arrows, one in each direction (see Figs. 4, 6 and 7 for examples). We call such edges as *bidirected arcs* and let $\mathsf{bidir}(O)$ denote the set of bidirected arcs in O. We extend this notation to vertices $v \in B$ and say $v \in \mathsf{bidir}(O)$ if $\{u, v\} \in \mathsf{bidir}(O)$ for some $u \in B$. We also say $v \in B$ is *free* in O if $v \in B - \mathsf{bidir}(O)$ and denote by $\mathsf{free}(O)$ the set of free vertices of O.

In T, let $\mathcal{O}_{B,i}^T$ be the set of B-orientations O, such that O has i bidirected arcs. We need to separate the case $B = V$ from the cases $B \neq V$. First, let $B \neq V$. For such a $B \subseteq V$, let $m = \min_{v \in [n]-B} v$ be the minimum numbered vertex outside B and let $O \in \mathcal{O}_{B,i}^T$. For each $v \in \mathsf{free}(O)$, as there is a unique path from v to m in T, we can tell if v is oriented *"towards"* m or if v is oriented *"away from"* m. Formally, for $O \in \mathcal{O}_{B,i}^T$, define a 0/1 function $\mathsf{away} : \mathsf{free}(O) \to \{0, 1\}$ by

$$\mathsf{away}(v) = \begin{cases} 1 & \text{if } v \text{ is oriented away from } m, \\ 0 & \text{if } v \text{ is oriented towards } m. \end{cases}$$

For each $O \in \mathcal{O}_{B,i}^T$ assign the following non-negative integer:

$$\mathsf{Aw}_B^T(O) = 2i + 2 \sum_{v \in \mathsf{free}(O)} \mathsf{away}(v).$$

Define the generating function of the statistic $\mathsf{Aw}_B^T(\cdot)$ in the variable q as follows:

$$a_{B,i}^T(q) = \sum_{O \in \mathcal{O}_{B,i}^T} q^{\mathsf{Aw}_B^T(O)}, \tag{5}$$

$$a_{r,i}^T(q) = \sum_{B \subset V, |B|=r} a_{B,i}^T(q) = \sum_{B \subseteq V, |B|=r} \sum_{O \in \mathcal{O}_{B,i}^T} q^{\mathsf{Aw}_B^T(O)}. \tag{6}$$

Example 8. Let T_2 be the tree given in Fig. 3 and let $B = \{2, 4, 6, 7, 8\}$ with $|B| = r = 5$. Below we give $a_{B,i}^{T_2}(q)$ for i from 0 to $\lfloor r/2 \rfloor$.

i	0	1	2
$a_{B,i}^{T_2}(q)$	$1 + 2q^2 + q^4$	$q^2(1 + 2q^2 + q^4)$	0

Remark 9. For any tree T and all r, j, it is easy to see from the definitions that both $m_{r,j}(q)$ and $a_{r,i}^T(q)$ are polynomials in q^2.

For the Laplacian L_T of a tree T, Chan and Lam had in [8, Theorem 2.2] proved an identity involving numerical counterparts of $m_{B,j}(q)$'s and $a_{B,i}^T(q)$'s for the special case when $B = [n]$. Later, Chan, Lam and Yeo in [9, Theorem 3.1] extended the same identity for all $B \subseteq [n]$. Earlier, we had in [25, Theorem 11] obtained a q-analogue of this identity when $B = [n]$. There, care had to be taken to define $a_{[n],0}^T(q) = 1 - q^2$. We give a q-analogue below in Lemma 10 when B can be an arbitrary subset. In [25], since $B = [n]$, there was no vertex outside B and hence the minimum vertex m could not be defined. Thus, the lexicographically minimum edge of the matching M was used in place of m there. It is easy to see that we could have used the lexicographically minimum edge of M when $B \neq [n]$ as well. Since the proof is identical to that of [25, Theorem 11], we omit it and merely state the result. From now onwards, we are free from this restriction $B \neq [n]$.

Lemma 10. *Let T be a tree with vertex set $[n]$ and B be an r-subset of $[n]$. Then,*

$$m_{B,j}(q) = \sum_{i=j}^{\lfloor r/2 \rfloor} \binom{i}{j} a^T_{B,i}(q). \text{ Moreover, } m_{r,j}(q) = \sum_{i=j}^{\lfloor r/2 \rfloor} \binom{i}{j} a^T_{r,i}(q).$$

Chan and Lam in [7] showed the following non-negativity result on characters summed with binomial coefficients as weights. Let $n \geq 2$ and let $\lambda \vdash n$. Recall $\chi_\lambda(j)$ is the character χ_λ evaluated at a permutation with cycle type $2^j, 1^{n-2j}$.

Lemma 11 *(Chan and Lam). Let $\lambda \vdash n$ and let $\chi_\lambda(j)$ be as defined above. Let $0 \leq i \leq \lfloor n/2 \rfloor$. Then $\sum_{j=0}^{i} \chi_\lambda(j)\binom{i}{j} = \alpha_{\lambda,i} 2^i$, where $\alpha_{\lambda,i} \geq 0$. Further, if $\lambda = k, 1^{n-k}$, then $\alpha_{\lambda,i} = \binom{n-i-1}{k-i-1}$.*

Combining Lemmas 10 and 11 with Lemma 7 gives us the following Corollary whose proof we omit. This gives an interpretation of the coefficient $c^{\mathcal{L}^q_T}_{\lambda,r}(q)$ in the immanantal polynomial as functions of the $a^T_{r,i}(q)$'s. Since all the $a^T_{r,i}(q)$'s except $a^T_{[n],0}(q)$ have positive coefficients, this is an almost positive expression.

Corollary 12. *For $0 \leq r \leq n$, the coefficient of the immanantal polynomial of $\mathcal{L}^q_T$ in absolute value is given by*

$$c^{\mathcal{L}^q_T}_{\lambda,r}(q) = \sum_{i=0}^{\lfloor r/2 \rfloor} \alpha_{\lambda,i} 2^i a^T_{r,i}(q), \quad \text{where } \alpha_{\lambda,i} \geq 0, \ \forall\ \lambda \vdash n, i.$$

Combining (4), Lemmas 11 and 10 gives us another corollary when the partition is $\lambda = 1^n$, which we again merely state.

Corollary 13. *When $\lambda = 1^n$, we have $\alpha_{\lambda,i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$. Further, let $B \subseteq [n]$ with $|B| = r$. Then,*

$$\det(\mathcal{L}^q_T[B|B]) = a^T_{B,0}(q). \text{ Moreover, } c^{\mathcal{L}^q_T}_{1^n,r}(q) = a^T_{r,0}(q).$$

Remark 14. Let tree T have vertex set $[n]$ and let $B \subseteq [n]$ with $|B| = n-1$. Then, for all $q \in \mathbb{R}$, $a^T_{B,0}(q) = 1$. This implies that $a^T_{n-1,0}(q) = n$.

Let $B \subseteq [n]$ with $|B| = r$. Let $\mathcal{L}^q_T[B|B]$ denote the $r \times r$ submatrix of $\mathcal{L}^q_T$ induced on the rows and columns indexed by B. From Corollary 13, we get $\det(\mathcal{L}^q_T[B|B]) \geq 0$ when $B \neq [n]$. When $B = [n]$, Bapat, Lal and Pati [2] have shown that $\det(\mathcal{L}^q_T) = 1 - q^2$. As remarked in Section 1, when $q \in \mathbb{R}$ with $|q| \leq 1$, the matrix $\mathcal{L}^q_T$ is positive semidefinite.

Remark 15. By Sturm's Theorem (see [14]), the number of negative eigenvalues of $\mathcal{L}^q_T$ equals the number of sign changes among the leading principal minors. When $|q| > 1$,

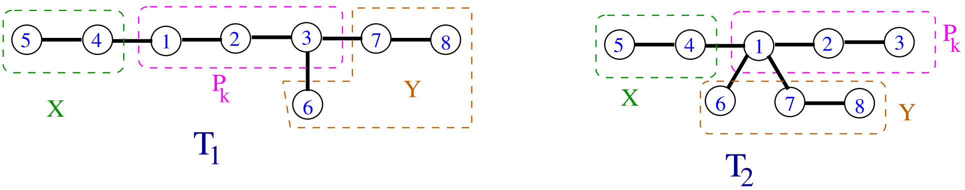

Fig. 3. Two labelled trees with $T_2 \geq_{\mathsf{GTS}_n} T_1$ and T_2 covering T_1.

the number of sign changes equals 1 by Corollary 13. This gives a short proof of a result of Bapat, Lal and Pati [2, Proposition 3.7] that the signature of $\mathcal{L}_T^q$ is $(n-1,1,0)$ when $|q|>1$, where signature of a Hermitian matrix A is the vector (p,n,z) with p,n being the number of positive, negative eigenvalues of A respectively and z being the nullity of A.

Remark 16. By (5), for any T, all $a_{B,i}^T(q) \in \mathbb{R}^+[q]$ when $B \neq [n]$. In [25, Corollary 13], it was shown that $a_{[n],i}^T(q) \in \mathbb{R}^+[q]$ when $i>0$. By definition, $a_{[n],0}^T(q) = 1-q^2$ has negative coefficients. In [25, Theorem 2.4], it was shown that $c_{\lambda,n}^{\mathcal{L}_T^q} \in \mathbb{R}^+[q]$ for all $\lambda \vdash n$ except $\lambda = 1^n$.

By these and Corollary 12, it is easy to see that barring $c_{1^n,n}^{\mathcal{L}_T^q}(q)$, which equals $1-q^2$, $c_{\lambda,r}^{\mathcal{L}_T^q}(q) \in \mathbb{R}^+[q]$ for all $\lambda \vdash n$ and for all $0 \leq r \leq n$. Thus all statements in this work can be made about $c_{\lambda,r}^{\mathcal{L}_T^q}(q)$ or alternatively about the *absolute value* of the coefficient of x^{n-r} in $f_\lambda^{\mathcal{L}_T^q}(x)$ (which equals $(-1)^r c_{\lambda,r}^{\mathcal{L}_T^q}(q)$).

4. Proof of Theorem 1

We begin with a few preliminaries towards proving Theorem 1. Let T_1 and T_2 be trees on n vertices with $T_2 \geq_{\mathsf{GTS}_n} T_1$. We assume that both T_1 and T_2 have vertex set $V=[n]$.

Remark 17. Since immanants are invariant under a relabelling of vertices (see Littlewood's book [19] or Merris [22]), without loss of generality, we label the vertices of T_1 as follows: first label the vertices on the path P_k as $1,2,\ldots,k$ in order with 1 being the closest vertex to X and k being the closest vertex to Y. Then, label vertices in X with labels $k+1, k+2, \ldots, k+|X|$ in increasing order of distance from vertex 1 (say in a breadth-first manner starting from vertex 1) and lastly, label vertices of Y from $n-|Y|+1$ to n again in increasing order of distance from vertex 1. See Fig. 3 for an example.

Recall our notation $a_{B,i}^{T_1}(q)$ and $a_{B,i}^{T_2}(q)$ for the trees T_1 and T_2 respectively. Also recall $\mathcal{O}_{B,i}^{T_1}$ denotes the set of B-orientations in T_1 with i bidirected-arcs and let $\mathcal{O}_{r,i}^{T_1} = \cup_{B\subseteq V, |B|=r} \mathcal{O}_{B,i}^{T_1}$. Recall that $\mathcal{O}_{r,i}^{T_2}$ is defined analogously. It would have been nice if for all $B \subseteq V$ with $|B| = r$ and for all $0 \leq i \leq \lfloor r/2 \rfloor$, we could prove that $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$. Unfortunately, this is not true as the example below illustrates.

Example 18. Let T_2 and T_1 be the trees given in Fig. 3. Let $B = \{1, 4, 6, 7, 8\}$ and let $i = 2$. It can be checked that $a_{B,i}^{T_2}(q) = 2q^4 + q^6$ and that $a_{B,i}^{T_1}(q) = q^4$.

Nonetheless, by combining all sets B of size r, we will for all r, i construct an injective map $\gamma : \mathcal{O}_{r,i}^{T_2} \to \mathcal{O}_{r,i}^{T_1}$ that preserves the "away" statistic. For each r, note that there are $\binom{n}{r}$ sets B that contribute to $\mathcal{O}_{r,i}^{T_2}$ and $\mathcal{O}_{r,i}^{T_1}$. We partition the r-sized subsets B into three disjoint families and apply three separate lemmas. Recall that vertices 1 and k are the endpoints of the path P_k used in the definition of the poset GTS_n. The first family consists of those sets B with both $1, k \notin B$.

Lemma 19. *Let $B \subseteq [n]$, $|B| = r$ be such that both $1, k \notin B$. Then, there is an injective map $\phi : \mathcal{O}_{B,i}^{T_2} \to \mathcal{O}_{B,i}^{T_1}$ such that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(\phi(O))$. Thus, for all $0 \leq i \leq \lfloor r/2 \rfloor$, we have $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$.*

Proof. Let $O \in \mathcal{O}_{B,i}^{T_2}$. Clearly, $1 = \min_{u \in [n]-B} u$ and for O, define $O' = \phi(O)$ as follows. In O', for each vertex $v \in B$, assign the same orientation as in O. Clearly, $O' \in \mathcal{O}_{B,i}^{T_1}$ and it is clear that ϕ is an injective map from $\mathcal{O}_{B,i}^{T_2}$ to $\mathcal{O}_{B,i}^{T_1}$. Further, it is easy to see that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(\phi(O))$, hence proving that $a_{B,i}^{T_1}(q) - a_{B,i}^{T_2}(q) \in \mathbb{R}^+[q^2]$, completing the proof. $\square$

We next consider those B with $|\{1, k\} \cap B| = 1$. We use the notation B for r-sized subsets with $1 \in B$, $k \notin B$ and B' for r-sized subsets with $k \in B'$, $1 \notin B'$. The next lemma below considers such subsets B' and those B-orientations O with the orientation of vertex 1 in $X \cup P_k$, that is $O(1) \in X \cup P_k$. Note that for such B-orientations O, $\min_{v \in [n]-B} v \in P_k$.

Lemma 20. *Let $O \in \mathcal{O}_{B,i}^{T_2}$, where $1 \in B, k \notin B$ and let $O(1)$ denote the oriented neighbour of vertex 1 in O. If $O(1) \in X \cup P_k$, then there exists an injective map $\mu : \mathcal{O}_{B,i}^{T_2} \to \mathcal{O}_{B,i}^{T_1}$ such that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(\mu(O))$. Similarly, let $B' \subseteq V$ be such that $1 \notin B', k \in B'$. Then, there is an injective map $\nu : \mathcal{O}_{B',i}^{T_2} \to \mathcal{O}_{B',i}^{T_1}$ such that for $P \in \mathcal{O}_{B',i}^{T_2}$, $\mathsf{Aw}_{B'}^{T_2}(P) = \mathsf{Aw}_{B'}^{T_1}(\nu(P))$.*

Proof. The proof for both cases are similar. Let $O \in \mathcal{O}_{B,i}^{T_2}$ and let $O(1) \in X \cup P_k$. In this case, the same injection of Lemma 19 works. That is, we form O' by assigning all vertices of B the same orientation as in O. Clearly, $O' \in \mathcal{O}_{B,i}^{T_1}$ and $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(O')$.

Similarly, let $P \in \mathcal{O}_{B',i}^{T_2}$. Form $P' \in \mathcal{O}_{B',i}^{T_1}$ by assigning all vertices of B' the same orientation as in P. Clearly, $\mathsf{Aw}_{B'}^{T_2}(P) = \mathsf{Aw}_{B'}^{T_1}(P')$. Note that in both P and P', the orientation of k equals $k - 1$ as k is a leaf vertex in T_2. The proof is complete. $\square$

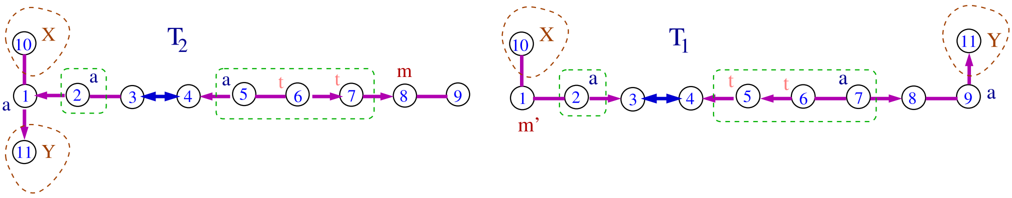

Fig. 4. Illustrating the injection when $O(1) \in Y$, $m \in P_k$.

We continue to use the notation B for an r-sized subset of V with $1 \in B$. We now handle B-orientations $O \in \mathcal{O}^{T_2}_{B,i}$ with $O(1) \in Y$.

Lemma 21. *Let B be an r-sized subset of $[n]$ with $1 \in B$, $k \notin B$. Define $B' = (B - \{1\}) \cup \{k\}$. Let $O \in \mathcal{O}^{T_2}_{B,i}$ with $O(1) \in Y$. There is an injective map $\delta : \mathcal{O}^{T_2}_{B,i} \to \mathcal{O}^{T_1}_{B',i}$ such that $\mathsf{Aw}^{T_2}_B(O) = \mathsf{Aw}^{T_1}_{B'}(\delta(O))$. Further, if $N = \delta(O)$, then we have $N(k) = O(1)$.*

Proof. The proof is identical to the proof of [25, Lemma 7]. We hence only sketch our proof. In T_1, define $m' = \min_{v \in [n]-B'} v$ and recall that $m = \min_{v \in [n]-B} v$ in T_2. Since $1 \notin B'$, note that in T_1, we have $m' = 1$. Thus, we reverse the orientation of some vertices in T_2 on the subpath from $(1, m)$ of P_k. To decide the vertices whose orientations are to be reversed, we break the $(1, m)$ path into segments separated by bidirected arcs. In each segment, if the ℓ-th closest vertex to m in T_2 was oriented "towards m", then in T_1, orient the ℓ-th closest vertex to 1 "towards 1". Likewise, if the ℓ-th closest vertex to m in T_2 was oriented "away from m", then in T_1, orient the ℓ-th closest vertex to 1 "away from 1".

See Fig. 4 for an example, where the letter "t" is used to denote a vertex whose orientation is towards m and "a" is used to denote a vertex whose orientation is away from m. This convention of "t" and "a" will be used in later figures as well. For the example in the Fig. 4, note that $k = 9$. If δ is the map described above, then it is clear that $\mathsf{Aw}^{T_2}_B(O) = \mathsf{Aw}^{T_1}_{B'}(\delta(O))$ and that $(\delta(O))(k) = O(1)$. The proof is complete. $\square$

Corollary 22. *Let $B \subseteq V$ with $1 \in B$, $k \notin B$ and define $B' = (B - \{1\}) \cup \{k\}$. For all i, there is an injection $\omega : \mathcal{O}^{T_2}_{B,i} \cup \mathcal{O}^{T_2}_{B',i} \to \mathcal{O}^{T_1}_{B,i} \cup \mathcal{O}^{T_1}_{B',i}$. Thus, $a^{T_1}_{B,i}(q) + a^{T_1}_{B',i}(q) - a^{T_2}_{B,i}(q) - a^{T_2}_{B',i}(q) \in \mathbb{R}^+[q^2]$.*

Proof. If $O \in \mathcal{O}^{T_2}_{B,i}$ is such that $O(1) \in X \cup P_k$, use Lemma 20. On the other hand, if $O(1) \in Y$, then we use Lemma 21. Let $O' = \omega(O)$. Note that in this case, vertex k is oriented with $O'(k) \in Y$.

Similarly, if $O \in \mathcal{O}^{T_2}_{B',i}$, then, we use Lemma 20. Note that in this case if $O' = \omega(O)$, then $O'(k) = k - 1 \in P_k$. Thus, the case mentioned in the earlier paragraph and this case are disjoint and hence ω is an injection. $\square$

Our last family consists of subsets B with both $1, k \in B$. Define another subset $B' \subseteq [n]$ using B as follows: Let $B_{xy} = B \cap (X \cup Y)$ and let $B_p = B \cap P_k$. The set B' will be used when $m \in P_k$. In this case, $m = \min_{v \in P_k, v \notin B} v$ is the minimum vertex outside B in P_k. Define $l = \max_{v \in P_k, v \notin B} v$ to be the maximum numbered vertex in P_k not in B. Define $m' = k+1-l$ and $l' = k+1-m$. Form B_p^t by taking the union of the three sets $A' = \{1, \ldots, m'-1\}$, $C' = \{l'+1, \ldots, k\}$ and $\{m'-m+x : x \in B \cap \{m+1, \ldots, l-1\}\}$. See Fig. 9 for an example. Define $B' = B_{xy} \cup B_p^t$. Clearly, both $1, k \in B'$ and $(B')' = B$.

Lemma 23. *Let $B \subseteq [n]$ be such that both $1, k \in B$ and let B' be as defined above. For all i, there is an injective map $\theta : \mathcal{O}_{B,i}^{T_2} \cup \mathcal{O}_{B',i}^{T_2} \to \mathcal{O}_{B,i}^{T_1} \cup \mathcal{O}_{B',i}^{T_1}$ that preserves the away statistic. Thus, $a_{B,i}^{T_1}(q) + a_{B',i}^{T_1}(q) - a_{B,i}^{T_2}(q) - a_{B',i}^{T_2}(q) \in \mathbb{R}^+[q^2]$.*

Proof. We denote the orientation of vertex 1 in O as $O(1)$. Given B, recall $m = \min_{v \notin B} v$ is the minimum vertex outside B and that we have labelled vertices on the path P_k first, vertices in X next and vertices of Y last. There are nine cases based on m and $O(1)$. Only one of the nine cases will involve B getting changed to B'. For now, let $O \in \mathcal{O}_{B,i}^{T_2}$. Define a map $\theta : \mathcal{O}_{B,i}^{T_2} \to \mathcal{O}_{B,i}^{T_1}$ as follows. Let $O \in \mathcal{O}_{B,i}^{T_2}$. We construct a unique $O' \in \mathcal{O}_{B,i}^{T_1}$ by using the algorithms tabulated below. Though it seems that there are a large number of cases, the underlying moves are very similar.

For vertices u, v, a, b, we explain an operation that we denote as $\mathsf{reverse\_on\_path}(u, v; a, b)$ that will be needed when $m \in Y$. We will always have $d_{u,v} = d_{a,b}$ in T_1 where $d_{u,v}$ is the distance between vertices u and v in T_1. Further, all vertices w on the u, v path $P_{u,v}$ in T_1 will be in B and hence be oriented. $\mathsf{reverse\_on\_path}(u, v; a, b)$ will change orientations of all vertices on $P_{u,v}$. We will use this operation in all the three cases when $m \in Y$. Due to our labelling convention and the fact that $m \in Y$, all vertices of $P_k \cup X$ will be contained in B. In T_2, vertex m has vertex 1 as its closest vertex among the vertices in P_k, whereas in T_1, vertex m has vertex k as its closest vertex among those in P_k. Denote vertices on $P_{u,v}$ as $u = u_1, u_2, \ldots, u_s = v$ and the vertices on the (a, b) path as $a = a_1, a_2, \ldots, a_s = b$. In O, if vertex a_i is oriented "towards m", then orient vertex u_{s+1-i} "towards m" and likewise if vertex a_i is oriented "away from m", then orient vertex u_{s+1-i} "away from m". We give the map θ using several algorithms below.

	$m \in P_k$	$m \in X$	$m \in Y$
$O(1) = 2 \in P_k$	Use algorithm 1	Use algorithm 1	Use algorithm 2
$O(1) = x \in X$	Use algorithm 1	Use algorithm 1	Use algorithm 4
$O(1) = y \in Y$	Use algorithm 5	Use algorithm 3	Use algorithm 2

Algorithm 1: This is a trivial copying algorithm. Define $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T_1}$ as follows. In O', retain the same orientation for all vertices $v \in B$. It is clear that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(O')$.

Algorithm 2: Since $m \in Y$, by our labelling convention, this means all the vertices of P_k and X are in B. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T_1}$ by first copying the orientation

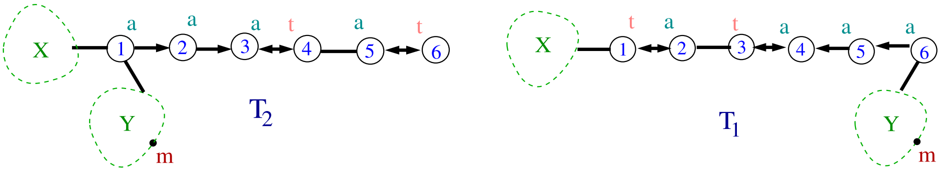

Fig. 5. Illustrating Algorithm 2 with $O(1) \in P_k$, $m \in Y$.

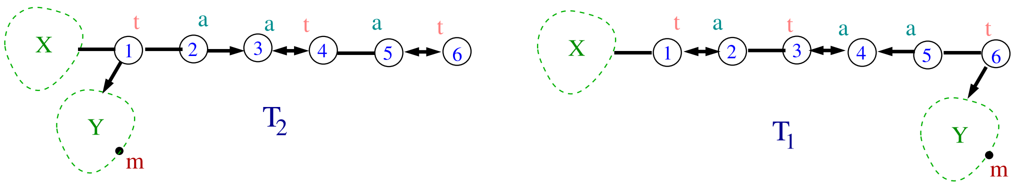

Fig. 6. Illustrating Algorithm 2 with both $O(1)$, $m \in Y$.

O for each vertex. Then perform $\mathsf{reverse\_on\_path}(1, k; 1, k)$. This is illustrated in Fig. 5 when $O(1) = 2$ and $m \in Y$ and in Fig. 6 when both $O(1)$, $m \in Y$. It is clear that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(O')$.

Algorithm 3: We have $m \in X$ and $O(1) \in Y$. Recall that we have labelled the vertices of X in increasing order of distance from vertex 1. We claim that there exists a unique edge $e = \{x, y\}$ on the path from 1 to m satisfying the following two conditions:

1. There is no arrow on e. That is, either both $x, y \in B$ with $O(x) \neq y$ and $O(y) \neq x$ or $x \in B$ and $y = m$.
2. Among such edges, x is the closest vertex to 1 distance-wise (that is, e is the unique closest edge to 1).

That there exists such an edge e satisfying condition (1) above is easy to see. Condition (2) is just a labelling of vertices of such an edge. Further, we label the vertices on the path from 1 to x in increasing order of distance from vertex 1 as $1, x_1, x_2, \ldots, x_l = x$. (See Fig. 7 for an example.)

It is easy to see that $O(x_1) = 1$ and $O(x_i) = x_{i-1}$ for $2 \leq i \leq l$ and recall that $O(1) \in Y$. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T_1}$ as follows. Vertices of B not on the path from x_l to k in T_1 get the same orientation as in O. We orient the last $l+1$ vertices in T_1 on the x_l to k path $P_{x_l,k}$ away from m, and then orient the first $k-1$ vertices on $P_{x_l,k}$ as they were on P_k. See Fig. 7 for an example. As $k = 6$ and $l = 3$, the last $l+1$ vertices on the $(x_3, 6)$ path means that the last 4 vertices are oriented away from m. The orientation of the remaining vertices is inherited from T_2. It is clear that $|\mathsf{bidir}(O)| = |\mathsf{bidir}(O')|$ and that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(O')$.

Algorithm 4: We have $O(1) \in X$ and $m \in Y$. As done in Algorithm 3, find the closest edge $e = \{x, y\}$ to vertex 1 with e having no arrow on the 1 to m path. As before, label

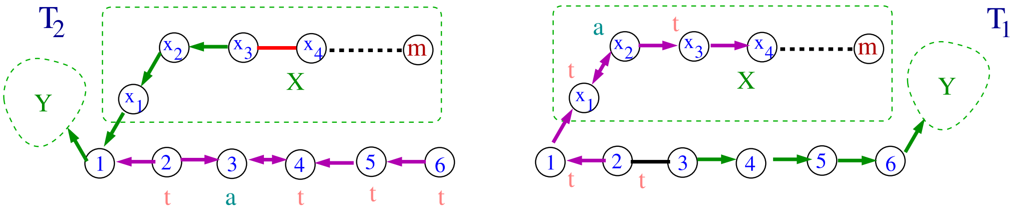

Fig. 7. Illustrating Algorithm 3.

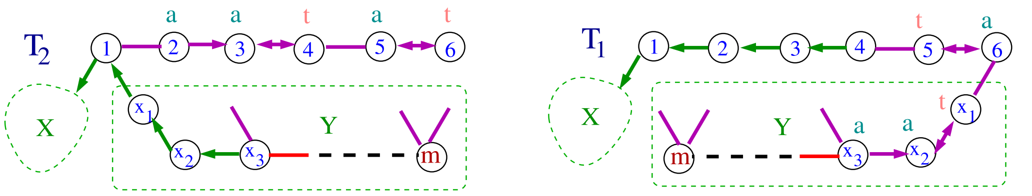

Fig. 8. Illustrating Algorithm 4.

e as $\{x, y\}$ with x being closer to 1 than y, and label the vertices on the path from 1 to x as $1, x_1, x_2, \ldots, x_l = x$ (see Fig. 8).

It is easy to see that $O(x_1) = 1$ and $O(x_i) = x_{i-1}$ for $2 \leq i \leq l$. Note that there is a continuous string of $l+1$ vertices that are oriented away from m. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B,i}^{T_1}$ as follows. Vertices of B not on the path from x_l to k in T_1 get the same orientation as in O. The closest $l+1$ vertices of B on the path from 1 to x_l in T_1 get oriented away from m. Denote the path comprising the last $k-1$ vertices on the $(1, x_l)$-path as P_ℓ. Let α, β be the first and last vertices of P_ℓ. Perform $\mathsf{reverse\_on\_path}(\alpha, \beta; 2, k)$. See Fig. 8 for an example. It is clear that $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_B^{T_1}(O')$.

Algorithm 5: We have $O(1) = y \in Y$ and $m \in P_k$. Recall $B' = B_{xy} \cup B_p^t$. Recall $l = \max_{v \in P_k, v \notin B} v$. Note that the minimum vertex $m' \notin B'$ will be $m' = k + 1 - l$. Form $O' = \theta(O)$ with $O' \in \mathcal{O}_{B',i}$ as follows. Note that in T_2, there is a continuous sequence A of $m-1$ oriented vertices from 1 to $m-1$ and another continuous sequence C of $k-l$ oriented vertices from $l+1$ to k in the path P_k (see Fig. 9 for an example). Similarly, in T_1, there is a continuous sequence A' of $m'-1$ oriented vertices from 1 to $m'-1$ and another continuous sequence C' of $k-l'$ oriented vertices from $l'+1$ to k in the path P_k.

It is easy to see that $|A| = |C'|$ and $|C| = |A'|$. If vertex $s \in A$ is oriented away from (or towards) m in O, then in O' orient vertex $k+1-s$ away from (or towards respectively) m'. Likewise, if vertex $s \in C$ is oriented away from (or towards) m in O, then in O' orient vertex $k+1-s$ away from (or towards respectively) m'.

Lastly, in O' copy the orientation of vertices in B that lie between m and l in T_2 as they were to the vertices in B' between m' and l' in T_1. Formally, if vertex $s \in P_k$ with $m < s < l$ is oriented away from (or towards) m in O, then in O' orient vertex $(m'-m)+s$ away from (or towards respectively) m'.

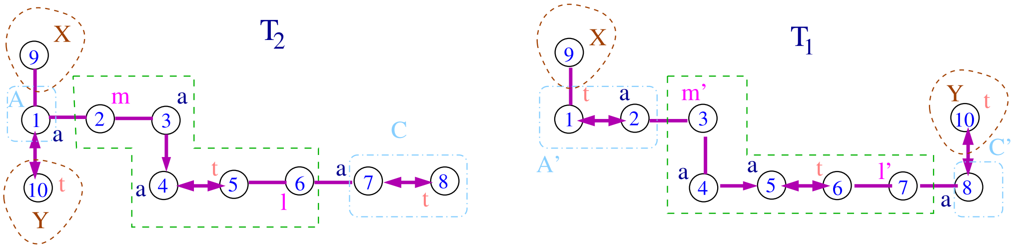

Fig. 9. Illustrating Algorithm 5 with $B = \{1, 3, 4, 5, 7, 8, 10\}$ and $B' = \{1, 2, 4, 5, 6, 8, 10\}$.

For vertices $s \in B_p$, see Fig. 9 for an example. Clearly, $|\mathsf{bidir}(O)| = |\mathsf{bidir}(O')|$ and $\mathsf{Aw}_B^{T_2}(O) = \mathsf{Aw}_{B'}^{T_1}(O')$. This completes Algorithm 5.

When $B = [n]$, note that all vertices are oriented and hence there exists at least one bidirected edge. In this case, we have $\mathsf{Aw}_B(O) = \mathsf{away}(O, e)$, where as defined in [25], $\mathsf{away}(O, e)$ is found with respect to the lexicographic minimum bidirected edge $e \in O$. If the lexicographic edge is $e = \{u, v\}$, we let $m = \min(u, v)$ be the smaller numbered vertex among u, v. We find the statistic $\mathsf{away}(O, e)$ with respect to m. It is simple to note that among the nine cases, the following will not occur when $B = [n]$ due to our labelling convention:

(1) $m \in X$ and $O(1) \in P_k$,
(2) $m \in Y$ and $O(1) \in P_k$ and
(3) $m \in Y$ and $O(1) \in X$.

In the remaining cases, we follow the same algorithms. It is easy to see that the pair $(m, O(1))$ is different in all the nine cases. We do not change B in eight cases, except in Algorithm 5. Thus, we get an injection in these eight cases. When Algorithm 5 is run, we get an injection from $\mathcal{O}_{B,i}^{T_2}$ to $\mathcal{O}_{B',i}^{T_1}$ and similarly we get an injection from $\mathcal{O}_{B',i}^{T_2}$ to $\mathcal{O}_{B,i}^{T_1}$. Thus, we get an injection from $\mathcal{O}_{B',i}^{T_2} \cup \mathcal{O}_{B,i}^{T_2}$ to $\mathcal{O}_{B,i}^{T_1} \cup \mathcal{O}_{B',i}^{T_1}$, completing the proof. $\square$

With these Lemmas in place, we can now prove Theorem 1.

Proof of Theorem 1. We group the set of r-sized subsets B into three categories: those without $1, k$, those with either 1 or k and those with both $1, k$. By Lemmas 19, 23 and Corollary 22 it is clear that there is an injective map from $\mathcal{O}_{r,i}^{T_2}$ to $\mathcal{O}_{r,i}^{T_1}$ for all r and i. By Corollary 12, $c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) - c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) \in \mathbb{R}^+[q^2]$ for all λ, r. $\square$

Corollary 24. *Setting $q = 1$ in $\mathcal{L}_T^q$, we infer that for all r, the coefficient of x^{n-r} in the immanantal polynomial of the Laplacian L_T of T decreases in absolute value as we go up GTS_n. Using Lemma 6, we thus get a more refined and hence stronger result than Theorem 3.*

Corollary 25. *Let T_1, T_2 be trees on n vertices with respective q-Laplacians $\mathcal{L}^q_{T_1}, \mathcal{L}^q_{T_2}$. Let $T_2 \geq_{\mathsf{GTS}_n} T_1$ and let $d_\lambda(\mathcal{L}^q_{T_i})$ denote the immanant of $\mathcal{L}^q_{T_i}$ for $1 \leq i \leq 2$ corresponding to the partition $\lambda \vdash n$. By comparing the constant term of the immanantal polynomial, for all $\lambda \vdash n$, we infer $d_\lambda(\mathcal{L}^q_{T_2}) \leq d_\lambda(\mathcal{L}^q_{T_1})$. This refines the inequalities in Theorem 3.*

5. q^2-analogue of vertex moments in a tree

Merris in [21] gave an alternate definition of the centroid of a tree T through its vertex moments. He then showed that the sum of vertex moments appears as a coefficient of the immanantal polynomial of L_T corresponding to the partition $\lambda = 2, 1^{n-2}$. In this section, we define a q^2-analogue of vertex moments and through it, the centroid of a tree. We then show that the sum of q^2-analogue of the vertex moments of all vertices appears as a coefficient in the second immanantal polynomial of $\mathcal{L}^q_T$. Thus, by Theorem 1, the sum of the q^2-analogue of vertex moments decreases as we go up on GTS_n. We further show that as we go up on GTS_n, the value of the minimum q^2-analogue of the vertex moments also decreases.

The following definition of vertex moments is from Merris [21]. Let T be a tree with vertex set $[n]$. For a vertex $i \in [n]$, define $\mathsf{Moment}^T(i) = \sum_{j\in[n]} d_j d_{i,j}$ where d_j is the degree of vertex j in T and $d_{i,j}$ is the distance between vertices i and j in T. Define the q^2-analogue of the distance $d_{i,j}$ between vertices i and j to be $[d_{i,j}]_{q^2} = 1 + q^2 + (q^2)^2 + \cdots + (q^2)^{d_{i,j}-1}$ and define for all $i \in [n]$, $[d_{i,i}]_{q^2} = 0$. We define the q^2-analogue of the moment of vertex i of T as

$$\mathsf{Moment}^T_{q^2}(i) = \sum_{j\in[n]} [1 + q^2(d_j - 1)][d_{i,j}]_{q^2}. \tag{7}$$

Fix $q \in \mathbb{R}$, $q \neq 0$. Vertex i is called the centroid of T if $\mathsf{Moment}^T_{q^2}(i) = \min_{j\in[n]} \mathsf{Moment}^T_{q^2}(j)$. We clearly recover Merris' definition of moments when we plug in $q = 1$ in (7). Merris showed that his definition of centroid coincides with the usual definition of the centroid of a tree T. In [4], Bapat and Sivasubramanian while studying the third immanant of $\mathcal{L}^q_T$ proved a lemma that we need. The following lemma is obtained by setting $s = q^2$ in [4, Lemma 3].

Lemma 26 *(Bapat and Sivasubramanian). Let T be a tree with vertex set $V = [n]$ and let $i \in [n]$. Then,*

$$\sum_{j\in[n]} q^2(d_j - 1)[d_{i,j}]_{q^2} = \sum_{j\in[n]} [d_{i,j}]_{q^2} - (n - 1). \tag{8}$$

The following alternate expression for $\mathsf{Moment}^T_{q^2}(i)$ is easy to derive using Lemma 26 and the definition (7). As the proof is a simple manipulation, we omit it.

Lemma 27. *Let T be a tree with vertex set $[n]$ and let $i \in [n]$. Then,*

$$\mathsf{Moment}^T_{q^2}(i) = (n-1) + 2q^2 \sum_{j\in[n]} (d_j - 1)[d_{i,j}]_{q^2}. \tag{9}$$

The following lemma gives an algebraic interpretation for the q^2-analogue of vertex moments in T.

Lemma 28. *Let T be a tree with vertex set $[n]$. Let $i \in [n]$ be a vertex and let $B = [n]-\{i\}$. Then,*

$$\mathsf{Moment}^T_{q^2}(i) = (n-1)a^T_{B,0}(q) + 2a^T_{B,1}(q). \tag{10}$$

Proof. Clearly for $B = [n] - \{i\}$, we have a unique B-orientation $O \in \mathcal{O}_{B,0}$ with $\mathsf{Aw}^T_B(O) = 0$. This is the orientation where every vertex $j \in [n] - i$ gets oriented towards i. Thus $a^T_{B,0}(q) = 1$.

We will show that $a^T_{B,1}(q) = q^2 \sum_{j\in[n]}(d_j - 1)[d_{i,j}]_{q^2}$ and appeal to (9). By (8), equivalently, we need to show that

$$a^T_{B,1}(q) = \sum_{j\in[n]} [d_{i,j}]_{q^2} - (n-1) = q^2 \sum_{j\in[n],j\neq i} [d_{i,j} - 1]_{q^2}.$$

Root the tree T at the vertex i and recall $B = [n] - \{i\}$. Thus $m = i$. Let O be a B-orientation with one bidirected arc $e = \{u, v\}$ where we label the edge e such that $d_{i,v} = d_{i,u} + 1$. That is, u occurs on the path from i to v in T. Since $n - 1$ vertices are oriented and one edge is bidirected, there must be one edge without any arrows (when seen pictorially). It is easy to see that all edges $f \in T$ not on the path $P_{i,u}$ from i to u must be oriented towards i. Moreover, it is clear that the edge f without arrows must be on the path $P_{i,u}$. Thus, our choice lies in orienting vertices in $P_{i,u}$ such that one edge does not get any arrows. Let $f = \{x, y\}$ with x being on the path from i to y in T (x could be i or y could be u). Thus, there are $d_{i,u} - 1$ choices for the edge f. In O, clearly, all vertices from y till u on the path $P_{i,u}$ must be oriented away from i. Hence the contribution of all such orientations will be $q^2 + q^4 + \cdots + q^{2d_{i,u}-2}$. Thus vertex u contributes $q^2[d_{i,u} - 1]_{q^2}$ to $a_{B,1}(q)$. Summing over all vertices u completes the proof. □

Theorem 29. *Let T be a tree with vertex set $[n]$ and q-Laplacian $\mathcal{L}^q_T$. Let $\lambda = 2, 1^{n-2} \vdash n$. Then,*

$$c^{\mathcal{L}^q_T}_{\lambda,n-1}(q) = \sum_{i=1}^{n} \mathsf{Moment}^T_{q^2}(i).$$

Proof. Summing (10) over all B with cardinality $n - 1$, we get

$$\sum_{i=1}^{n} \mathsf{Moment}^T_{q^2}(i) = (n-1)a^T_{n-1,0}(q) + 2a^T_{n-1,1}(q) = c^{\mathcal{L}^q_T}_{\lambda,n-1}(q)$$

where the last equality follows from Corollary 12 and Lemma 11 with $k = 2$. The proof is complete. □

On setting $q = 1$ in Theorem 29, we recover Merris' result [21, Theorem 6]. From Theorem 1 and Theorem 29, we get the following.

Theorem 30. *Let T_1 and T_2 be trees with n vertices and let T_2 cover T_1 in GTS_n. Then,*

$$\sum_{i=1}^{n} \mathsf{Moment}_{q^2}^{T_2}(i) \leq \sum_{i=1}^{n} \mathsf{Moment}_{q^2}^{T_1}(i).$$

Theorem 30 implies that the sum of the vertex moments decreases as we go up on the poset GTS_n. We next show that the minimum value of the q^2-analogue of vertex moments also decreases as we go up on GTS_n.

Lemma 31. *Let T_1 and T_2 be two trees with vertex set $[n]$ such that T_2 covers T_1 in GTS_n. Then, for all $q \in \mathbb{R}$, we have $\min_{i\in[n]} \mathsf{Moment}_{q^2}^{T_2}(i) \leq \min_{j\in[n]} \mathsf{Moment}_{q^2}^{T_1}(j)$.*

Proof. Let $l \in [n]$ be the vertex in T_1 with $\mathsf{Moment}_{q^2}^{T_1}(l) = \min_{i\in[n]} \mathsf{Moment}_{q^2}^{T_1}(i)$. Let $l \in X \cup Y \cup P_{\lfloor k/2 \rfloor}$ (see Fig. 1 for X, Y and $P_k = P_{x,y}$). Here $P_{\lfloor k/2 \rfloor}$ is the path P_k restricted to the vertices $1, 2, \ldots, \lfloor k/2 \rfloor$. Then, using the fact that the distance $d_{x,y}^{T_1} \geq d_{x,y}^{T_2}$ for all pairs $(x, y) \in X \times Y$, we have

$$\mathsf{Moment}_{q^2}^{T_1}(l) \geq \mathsf{Moment}_{q^2}^{T_2}(l) \geq \min_{i\in[n]} \mathsf{Moment}_{q^2}^{T_2}(i).$$

If $l \geq \lfloor k/2 \rfloor$ then $\mathsf{Moment}_{q^2}^{T_1}(l) \geq \mathsf{Moment}_{q^2}^{T_2}(k + 1 - l) \geq \min_{i\in[n]} \mathsf{Moment}_{q^2}^{T_2}(i)$. Thus we can find a vertex i in T_2 such that $\mathsf{Moment}_{q^2}^{T_1}(l) \geq \mathsf{Moment}_{q^2}^{T_2}(i)$, completing the proof. □

Corollary 32. *Let T_1, T_2 be two trees on n vertices with $T_2 \geq_{\mathsf{GTS}_n} T_1$. Then, for all $q \in \mathbb{R}$, the minimum q^2-analogue of the vertex moments of T_2 is less than the minimum q^2-analogue of the vertex moments of T_1.*

An identical statement about the maximum q^2-analogue of vertex moments is not true as shown in the following example.

Example 33. Let T_1, T_2 be trees on the vertex set [8] given in Fig. 10. Vertices 2 and 3 are both centroid vertices in T_1, while in T_2, the centroid is vertex 1. The q^2-analogue of their vertex moments are as follows: $\mathsf{Moment}_{q^2}^{T_1}(2) = \mathsf{Moment}_{q^2}^{T_1}(3) = 9 + 2q^2(7 + 3q^2)$ and $\mathsf{Moment}_{q^2}^{T_2}(1) = 9 + 2q^2(2 + q^2)$. The q^2-analogue of vertex moments of leaf vertices of T_1 and T_2 are as follows.

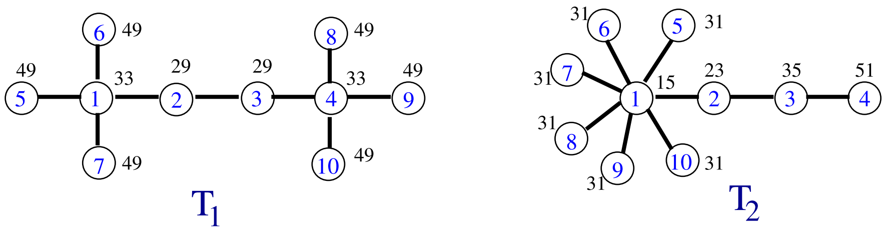

Fig. 10. q^2-moments of vertices of T_1 and T_2 when $q = 1$.

$$\mathsf{Moment}_{q^2}^{T_1}(i) = 9 + 2q^2(8 + 5q^2 + 4q^4 + 3q^6) \ \text{ for } \ i = 5, 6, 7, 8, 9, 10.$$
$$\mathsf{Moment}_{q^2}^{T_2}(i) = 9 + 2q^2(8 + 2q^2 + q^4) \ \text{ for } \ i = 5, 6, 7, 8, 9, 10.$$
$$\mathsf{Moment}_{q^2}^{T_2}(4) = 9 + 2q^2(8 + 7q^2 + 6q^4).$$

When $q = 1$, the moments of vertices of T_1 and T_2 are given in Fig. 10 alongside the vertices. Clearly, when $q = 1$, $\max_{j\in[8]} \mathsf{Moment}_{q^2}^{T_2}(j) = 51 \not\leq 49 = \max_{j\in[8]} \mathsf{Moment}_{q^2}^{T_1}(j)$.

Associated to a tree are different notions of "median" and "generalised centers", see the book [18]. It would be nice to see the behaviour of these parameters as one goes up GTS_n.

6. q, t-Laplacian $\mathcal{L}_{q,t}$ and Hermitian Laplacian of a tree T

All our results work for the bivariate Laplacian matrix $\mathcal{L}_{q,t}$ of a tree T on n vertices defined as follows. Let T be a tree with edge set E. Replace each edge $e = \{u, v\}$ by two bidirected arcs, (u, v) and (v, u). Assign one of the arcs, say (u, v) a variable weight q and its reverse arc, a variable weight t and let $A_{n\times n} = (a_{i,j})_{1\leq i,j\leq n}$ be the matrix with $a_{u,v} = q$ and $a_{v,u} = t$. Assign $a_{u,v} = 0$ if $\{u, v\} \notin E$. Let $D_{n\times n} = (d_{i,j})$ be the diagonal matrix with entries $d_{i,i} = 1 + qt(\deg(i) - 1)$. Define $\mathcal{L}_{q,t} = D - A$. Note that when $q = t$, $\mathcal{L}_{q,t} = \mathcal{L}_T^q$ and that when $q = t = 1$, $\mathcal{L}_{q,t} = L_T$ where L_T is the usual combinatorial Laplacian matrix of T.

It is easy to see that our proof relies on the fact that the difference in the coefficients of the immanantal polynomial is a non-negative combination of the $a_{r,i}^T(q)$'s which are polynomials in q^2 and that $q^2 \geq 0$ for all $q \in \mathbb{R}$. When $B = [n]$, bivariate versions of $m_{n,j}(q, t)$ and $a_{n,i}^T(q, t)$ were defined in [25]. Define bivariate versions $m_{r,j}(q, t)$ and $a_{r,i}^T(q, t)$ as done in Section 3 but replace all occurrences of q^2 with qt.

With this definition, it is simple to see that all results go through for $\mathcal{L}_{q,t}$, the q, t-Laplacian of T whenever $q, t \in \mathbb{R}$ and $qt \geq 0$ or $q, t \in \mathbb{C}$ and $qt \geq 0$. One special case of $\mathcal{L}_{q,t}$ is obtained when we set $q = \imath$ and $t = -\imath$ where $\imath = \sqrt{-1}$. In this case, the weighted adjacency matrix becomes the Hermitian adjacency matrix of T with edges oriented in the direction of the arc labelled q. The Hermitian adjacency matrix is a matrix defined and studied by Bapat, Pati and Kalita [1] and later independently by Liu and Li [20] and by Guo and Mohar [15]. With these complex numbers as weights,

$\mathcal{L}_{q,t}$ reduces to what is defined as the Hermitian Laplacian of T by Yu and Qu [27]. We get the following corollary of Theorem 1.

Corollary 34. *Let T_1, T_2 be trees on n vertices with $T_2 \geq_{\mathsf{GTS}_n} T_1$. Then, in absolute value, the coefficients of the immanantal polynomials of the Hermitian Laplacian of T_1 are larger than the corresponding coefficient of the immanantal polynomials of the Hermitian Laplacian of T_2.*

Let T be a tree on n vertices with Laplacian L_T and q,t-Laplacian $\mathcal{L}_{q,t}$. When $q = z \in \mathbb{C}$ with $z \neq 0$, and $t = 1/q$ then it is simple to see that the matrix $\mathcal{L}_{q,t}$ need not be Hermitian. In this case, for all $i \geq 0$, we have $a^T_{r,i}(q)_{q=1} = a^T_{r,i}(z, 1/z)$. This implies that for all $\lambda \vdash n$ and for $0 \leq r \leq n$, $c^{\mathcal{L}_{q,t}}_{\lambda,r} = c^{L_T}_{\lambda,r}$. Thus, we obtain the following simple corollary.

Corollary 35. *Let T be a tree on n vertices with Laplacian L_T and q,t-Laplacian $\mathcal{L}_{q,t}$. Then, for all $z \in \mathbb{C}$ with $z \neq 0$ and for all $\lambda \vdash n$*

$$f_\lambda^{\mathcal{L}_{z,1/z}}(x) = f_\lambda^{L_T}(x).$$

7. Exponential distance matrices of a tree

In [2], Bapat, Lal and Pati introduced the exponential distance matrix ED_T of a tree T. In this section, we prove that when $q \neq \pm 1$, the coefficients of the characteristic polynomial of ED_T, in absolute value decrease when we go up GTS_n. We show a similar relation on immanants of ED_T indexed by partitions with two columns. We recall the definition of ED_T from [2]. Let T be a tree with n vertices. Then, its exponential distance matrix $\mathsf{ED}_T = (e_{i,j})_{1 \leq i,j \leq n}$ is defined as follows: the entry $e_{i,j} = 1$ if $i = j$ and $e_{i,j} = q^{d_{i,j}}$ if $i \neq j$, where $d_{i,j}$ is the distance between vertex i and vertex j in T. For $\lambda \vdash n$, define

$$f_\lambda^{\mathsf{ED}_T}(x) = d_\lambda(xI - \mathsf{ED}_T) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathsf{ED}_T}(q) x^{n-r}. \tag{11}$$

We need the following lemma of Bapat, Lal and Pati [2].

Lemma 36 *(Bapat, Lal and Pati). Let T be a tree with n vertices. Let $\mathcal{L}_T^q$ and ED_T be the q-Laplacian and exponential distance matrix of T respectively. Then,* $\det(\mathsf{ED}_T) = (1-q^2)^{n-1}$ *and if $q \neq \pm 1$, then*

$$\mathsf{ED}_T^{-1} = \frac{1}{1-q^2} \mathcal{L}_T^q.$$

Using Jacobi's Theorem on minors of the inverse of a matrix (see DeAlba's article [12, Section 4.2]), we get the following easy corollary, whose proof we omit.

Corollary 37. *Let T be a tree with n vertices. Let $\mathcal{L}_T^q$ and ED_T be the q-Laplacian and exponential distance matrix of T respectively. Let $q \neq \pm 1$. Then, for $0 \leq r \leq n$*

$$c_{1^n,r}^{\mathsf{ED}_T}(q) = (1-q^2)^{r-1} c_{1^n,n-r}^{\mathcal{L}_T^q}(q),$$

where $c_{1^n,n-r}^{\mathcal{L}_T^q}(q)$ is the coefficient of $(-1)^{n-r}x^r$ in $f_{1^n}^{\mathcal{L}_T^q}(x)$.

The following corollary is an easy consequence of Theorem 1 and Corollary 37, we omit its proof.

Corollary 38. *Let T_1 and T_2 be two trees with n vertices such that $T_2 \geq_{\mathsf{GTS}_n} T_1$. Then, for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and for $0 \leq r \leq n$,*

$$\left| c_{1^n,r}^{\mathsf{ED}_{T_2}}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_{T_1}}(q) \right|.$$

In particular, for an arbitrary tree T with n vertices,

$$\left| c_{1^n,r}^{\mathsf{ED}_{S_n}}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_T}(q) \right| \leq \left| c_{1^n,r}^{\mathsf{ED}_{P_n}}(q) \right|.$$

We give some results for the immanant $d_\lambda(\mathsf{ED}_T)$, when $\lambda \vdash n$ is a two column partition. That is $\lambda = 2^k, 1^{n-2k}$ with $0 \leq k \leq \lfloor n/2 \rfloor$. When λ is a two column partition of n, Merris and Watkins in [23] proved the following lemma for invertible matrices.

Lemma 39 *(Merris, Watkins). Let A be an invertible $n \times n$ matrix. Then $\lambda \vdash n$ is a two column partition if and only if*

$$d_\lambda(A)\det(A^{-1}) = d_\lambda(A^{-1})\det(A).$$

Lemma 40. *Let T be a tree with n vertices with q-Laplacian and exponential distance matrices $\mathcal{L}_T^q$ and ED_T respectively. Then for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and $\lambda = 2^k, 1^{n-2k}$ for $0 \leq k \leq \lfloor n/2 \rfloor$*

$$d_\lambda(\mathsf{ED}_T) = d_\lambda(\mathcal{L}_T^q)(1-q^2)^{n-2}.$$

Proof. For all $q \in \mathbb{R}$ with $q \neq \pm 1$, ED_T is invertible. By Lemma 39, we have

$$d_\lambda(\mathsf{ED}_T)\det\left(\frac{1}{1-q^2}\mathcal{L}_T^q\right) = d_\lambda\left(\frac{1}{1-q^2}\mathcal{L}_T^q\right)\det(\mathsf{ED}_T).$$

$$\text{Thus, } d_\lambda(\mathsf{ED}_T)\det\left(\mathcal{L}_T^q\right) = d_\lambda\left(\mathcal{L}_T^q\right)\det(\mathsf{ED}_T).$$

Hence, $d_\lambda(\mathsf{ED}_T) = d_\lambda(\mathcal{L}_T^q)(1-q^2)^{n-2}$, completing the proof. □

Combining Lemma 40 and Theorem 1 gives us another corollary whose straightforward proof we again omit.

Corollary 41. *Let T_1 and T_2 be two trees on n vertices with $T_2 \geq_{\mathsf{GTS}_n} T_1$. Then, for all $q \in \mathbb{R}$ with $q \neq \pm 1$ and for all $\lambda = 2^k, 1^{n-2k}$, we have*

$$|d_\lambda(\mathsf{ED}_{T_2})| \leq |d_\lambda(\mathsf{ED}_{T_1})|.$$

7.1. q,t-exponential distance matrix

We consider the bivariate exponential distance matrix in this subsection. Orient the tree T as done above. Thus each directed arc e of $E(T)$ has a unique reverse arc e_{rev} and we assign a variable weight $w(e) = q$ and $w(e_{rev}) = t$ or vice versa. If the path $P_{i,j}$ from vertex i to vertex j has the sequence of edges $P_{i,j} = (e_1, e_2, \ldots, e_p)$, assign it weight $w_{i,j} = \prod_{e_k \in P_{i,j}} w(e_k)$. Define $w_{i,i} = 1$ for $i = 1, 2, \ldots, n$. Define the bivariate exponential distance matrix $\mathsf{ED}_T^{q,t} = (w_{i,j})_{1 \leq i,j \leq n}$. Clearly, when $q = t$, we have $\mathsf{ED}_T^{q,t} = \mathsf{ED}_T$. Bapat and Sivasubramanian in [3] showed the following bivariate counterpart of Lemma 36.

Lemma 42 *(Bapat, Sivasubramanian). Let T be a tree with n vertices and let $\mathcal{L}_T^{q,t}$ and $\mathsf{ED}_T^{q,t}$ be its q,t-Laplacian and q,t exponential distance matrix respectively. Then, $\det(\mathsf{ED}_T^{q,t}) = (1-qt)^{n-1}$ and if $qt \neq 1$, then*

$$(\mathsf{ED}_T^{q,t})^{-1} = \frac{1}{1-qt}\mathcal{L}_T^{q,t}.$$

It is easy to see that all results about ED_T go through for the bivariate q,t-exponential distance matrix $\mathsf{ED}_T^{q,t}$ when $q,t \in \mathbb{R}$ with $qt \neq 1$ or when $q,t \in \mathbb{C}$ with $qt \neq 1$. In particular, Corollary 41 goes through for the bivariate exponential distance matrix.

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