Advances in Applied Mathematics 47 (2011) 783–794

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Signed excedance enumeration via determinants

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ARTICLE INFO

Article history:
Received 6 June 2010
Accepted 28 February 2011
Available online 31 March 2011

MSC:
05A15
05A30
05E99

Keywords:
Determinant
Signed-excedance
Enumeration

ABSTRACT

We show a simple connection between determinants and signed-excedance enumeration of permutations. This gives us an alternate proof of a result of Mantaci about enumerating signed excedances in permutations. The connection also gives an alternate proof of a result of Mantaci and Rakotondrajao about enumerating signed excedances over derangements.

Motivated by this connection, we define several excedance-like statistics on permutations and show interesting values for their signed enumerator. In some cases, we also obtain the signed excedance-like statistic enumerator with respect to positive integral weights.

1. Introduction

Enumerating permutations by statistics is a well-studied area. The most famous statistics on a permutation are *descent, inversion, major-index and excedance* and their enumerators are classical results, see [4,10]. The course notes of Foata and Han [6] gives a self-contained introduction to this area.

For a positive integer n, define $[n] = \{1, 2, \dots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations on $[n]$. For a $\pi = (\pi_1, \pi_2, \dots, \pi_n) \in \mathfrak{S}_n$, define its number of excedances as $\mathsf{exc}(\pi) = |\{i \in [n]\colon\ \pi_i > i\}|$, its number of non-excedances as $\mathsf{nexc}(\pi) = |\{i \in [n]\colon\ \pi_i < i\}|$ and its number of fixed-points as $\mathsf{fix}(\pi) = |\{i \in [n]\colon\ \pi_i = i\}|$. For a $\pi \in \mathfrak{S}_n$, let its number of *weak non-excedances* be $\mathsf{wknexc}(\pi) = \mathsf{nexc}(\pi) + \mathsf{fix}(\pi)$ and its number of *weak-excedances* be $\mathsf{wkexc}(\pi) = \mathsf{exc}(\pi) + \mathsf{fix}(\pi)$.

Define the number of descents of $\pi \in \mathfrak{S}_n$ as $\mathsf{des}(\pi) = |\{i \in [n-1]\colon\ \pi_i > \pi_{i+1}\}|$ and its number of ascents as $\mathsf{asc}(\pi) = |\{i \in [n-1]\colon\ \pi_i < \pi_{i+1}\}|$. For $\pi \in \mathfrak{S}_n$, define its number of inversions as $\mathsf{inv}(\pi) = |\{1 \leqslant i < j \leqslant n\colon\ \pi_i > \pi_j\}|$.

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.02.003

It is well known (see [4]) that excedances and descents are equidistributed when summed over the elements of $\mathfrak{S}_n$. Thus, if $\mathsf{Des}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{des}(\pi)+1}$ and $\mathsf{Exc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{exc}(\pi)+1}$, then $\mathsf{Des}_n(q) = \mathsf{Exc}_n(q) = \mathsf{Eul}_n(q)$ where $\mathsf{Eul}_n(q)$ is the Eulerian polynomial of degree n defined by

$$\sum_{n \geqslant 0} \frac{\mathsf{Eul}_n(q)}{(1-q)^{n+1}} \frac{t^n}{n!} = \sum_{s \geqslant 0} q^s \exp\bigl(t(s+1)\bigr) = \frac{e^t}{1-qe^t}.$$

The coefficient of x^k in $\mathsf{Eul}_n(q)$ is called an Eulerian number and denoted $A_{n,k}$. There are several other equivalent definitions, see [6, Definition 10.2]. We give this definition of $\mathsf{Eul}_n(q)$ as our Theorem 5 is presented using similar generating function language.

Several signed-statistic enumeration results over permutations are known. Here, for a statistic $s : \mathfrak{S}_n \mapsto \mathbb{N} \cup \{0\}$, we count $\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{s(\pi)}$. Sometimes, the sum is not over all of $\mathfrak{S}_n$, but over some subset. Loday [9] defined $\mathsf{SgnDes}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{des}(\pi)}$ as the signed descent enumerator and conjectured a recurrence relation which Foata and Desarmenien [5] proved. Wachs [15] gave a sign reversing involution on $\mathfrak{S}_n$, thereby giving an alternate, bijective proof of Foata and Desarmenien's result. Tanimoto [14] has shown divisibility for some coefficients of the signed-descent enumerator by prime numbers. Recently, Barnabei, Bonetti and Silimbani [2] have enumerated signed ascents over involutions using properties of the RSK correspondence.

Enumerating signed-excedance-like statistics over both the set of permutations and over derangements is the content of this work. Mantaci [12] showed the following remarkable result.

Theorem 1. *For $n \geqslant 1$, let $\mathsf{SgnExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$ be the signed-excedance enumerator. Then $\mathsf{SgnExc}_n(q) = (1-q)^{n-1}$.*

In a later paper, Mantaci [11] gave a bijective proof of this result. Mantaci and Rakotondrajao in [13] determined the signed-excedance enumerator for derangements. If $\mathfrak{D}_n$ is the set of derangements on $[n]$, and if the number of signed derangements is defined as $\mathsf{SgnDer}_n = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)}$, then it is easy to note (see for example Remark 5) that the $\mathsf{SgnDer}_n = (-1)^{n-1}(n-1)$. We recall that for non-negative integers, i, its q-analogue is defined as $[i]_q = 1 + q + q^2 + \cdots + q^{i-1}$, where q is an indeterminate and $[0]_q = 0$. Mantaci and Rakotondrajao [13] showed the following.

Theorem 2. *For $n \geqslant 2$, define $\mathsf{DSgnExc}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$ as the signed excedance enumerator over derangements. Then, $\mathsf{DSgnExc}_n(q) = (-1)^{n-1} q \cdot [n-1]_q$.*

An equivalent, though differently phrased statement, was proved by Ksavrelof and Zeng [8]. For a permutation $\pi \in \mathfrak{S}_n$, let $\mathsf{cyc}(\pi)$ be its number of cycles. It is easy to show that for even n, $(-1)^{\mathsf{cyc}(\pi)} = (-1)^{\mathsf{inv}(\pi)}$ and that for odd n, $(-1)^{\mathsf{cyc}(\pi)} = -(-1)^{\mathsf{inv}(\pi)}$, both statements being for all $\pi \in \mathfrak{S}_n$. They considered the polynomial $p_n(x, y, \beta) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathsf{exc}(\pi)} y^{\mathsf{fix}(\pi)} \beta^{\mathsf{cyc}(\pi)}$ and by exhibiting a sign reversing involution, showed the following.

Theorem 3. *For $n \geqslant 1$, let $p_n(x, y, \beta)$ be the polynomial defined above. Then $p_n(x, 1, -1) = -(x-1)^{n-1}$ and $p_n(x, 0, -1) = -x - x^2 - \cdots - x^{n-1}$.*

In this work, we give a framework for enumerating signed-excedance by determinants of appropriate matrices. This framework allows us to derive in a unified manner, alternate proofs of the above results of Mantaci [12] and Mantaci and Rakotondrajao [13], apart from presenting new results on signed excedance-like enumeration.

This paper is organized as follows. In Section 2, we show a simple connection between determinants and signed-excedance enumeration. Our framework of evaluating determinants to enumerate signed excedance statistics is easily seen in this section. In Section 2.1, we show several new results, by modifying the definitions of quantities that are sign-enumerated. In Section 2.2, we define new excedance-like statistics which we sign-enumerate in Sections 3 and 4. The motivation for sign-enumerating such new excedance-like statistics is the neat results that we obtain. In Section 3, we sign-enumerate excedance-like statistics over permutations. In Section 4, we first give our alternate proof of the result of Mantaci and Rakotondrajao [13] and then sign-enumerate some excedance-like statistics over derangements.

Finally, Bagno and Garber [1] have obtained hyperoctahedral and wreath product analogues of Theorem 3. Several results in this work have similar analogues which we will treat in a future paper.

2. Determinants and signed excedance enumeration

We will deal with $n \times n$ matrices, and for any such matrix denote its i-th column as Col_i for $1 \leqslant i \leqslant n$. Fix an integer $n > 0$ and consider $\mathfrak{S}_n$. We recall $\mathsf{SgnExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$ is the signed-excedance enumerator. We begin with a determinant based proof of Theorem 1.

Proof of Theorem 1. Consider the following $n \times n$ matrices

$$M_n = \begin{pmatrix} 1 & q & q & \cdots & q \\ 1 & 1 & q & \cdots & q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad L_n = \begin{pmatrix} t & q & \cdots & q \\ t & t & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & t \end{pmatrix}$$

i.e. if $M_n = (m_{i,j})_{1 \leqslant i,j \leqslant n}$, then $m_{i,j} = q$ if $i < j$ and $m_{i,j} = 1$ otherwise. Similarly, if $L_n = (\ell_{i,j})_{1 \leqslant i,j \leqslant n}$, then $\ell_{i,j} = q$ if $i < j$, and $\ell_{i,j} = t$ otherwise.

We claim that $\mathsf{SgnExc}_n(q) = \det(M_n)$. To see this, we only need to note that $\det(M_n) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} \prod_{i=1}^n m_{i,\pi_i}$. For a $\pi \in \mathfrak{S}_n$, let $T_\pi = \prod_{i=1}^n m_{i,\pi_i}$ be the term occuring in the determinant expansion corresponding to π. Since $m_{i,j} = q$ if $i < j$ and $m_{i,j} = 1$ otherwise, we get $T_\pi = q^{\mathsf{exc}(\pi)}$. Hence, $\det(M_n) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$. It is straightforward to verify that $\det(M_n) = (1-q)^{n-1}$. The proof is complete. $\square$

Remark 1. Just as $\det(M_n) = (1-q)^{n-1}$, it is simple to note that $\mathsf{perm}(M_n) = \mathsf{Eul}_n(q) = \sum_{k=0}^n A(n,k) q^k$, where $\mathsf{perm}(M)$ is the permanent of any matrix M. A bivariate generalisation of this permanent result is also easy to see. Consider the matrix L_n given above. Then $\mathsf{perm}(L_n) = t(\sum_{k=0}^{n-1} A(n,k) q^k t^{n-k})$.

2.1. Some similar results

Some simple modifications of the matrix M_n above give rise to interesting signed-excedance enumerators. For $n \geqslant 1$, let $\mathsf{SgnSkExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{exc}(\pi)}$ be the *signed skew excedance enumerator* and let $\mathsf{SgnSkExc}_0(q) = 1$. Also define the *signed weak-skew excedance enumerator* as $\mathsf{SgnWkSkExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{wknexc}(\pi)} q^{\mathsf{exc}(\pi)}$. Consider the following two $n \times n$ matrices.

$$T_n = \begin{pmatrix} -1 & q & \cdots & q \\ -1 & -1 & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 \end{pmatrix} \quad \text{and} \quad U_n = \begin{pmatrix} 1 & q & q & \cdots & q \\ -1 & 1 & q & \cdots & q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

i.e. if $T_n = (t_{i,j})_{1 \leqslant i,j \leqslant n}$, then $t_{i,j} = q$ if $i < j$ and $t_{i,j} = -1$ otherwise. Similarly, if $U_n = (u_{i,j})_{1 \leqslant i,j \leqslant n}$, then $u_{i,j} = q$ if $i < j$, $u_{i,j} = 1$ if $i = j$ and $u_{i,j} = -1$ otherwise.

Theorem 4. *For $n \geqslant 1$, $\mathsf{SgnWkSkExc}_n(q) = (-1)^n (1+q)^{n-1}$.*

Consider the matrix T_n. Arguing as above, it is clear that $\det(T_n) = \mathsf{SgnWkSkExc}_n(q)$. It is again simple to see that $\det(T_n) = (-1)^n (1+q)^{n-1}$, completing the proof.

Remark 2. $\mathrm{perm}(T_n) = (-1)^n \mathrm{Eul}_n(q)$, where $(-1)^n \mathrm{Eul}_n(q)$ is the n-th Eulerian polynomial with alternate coefficients having opposite sign, with the largest degree coefficient being -1. This follows by setting $t = -1$ in the bivariate generalization mentioned in Remark 1.

Theorem 5. $\sum_{n \geqslant 0} \mathrm{SgnSkExc}_n(q) t^n = \frac{1}{1-t} + \sum_{n \geqslant 1} (\frac{(-t)^{n+1}}{(1-2t)(1-t)^{n+1}}) q^n$.

Proof. Consider the matrix U_n. Clearly, $\mathrm{SgnSkExc}_n(q) = \det(U_n)$. Denote $\det(U_n)$ as $d_n(q)$. A list of $d_i(q)$ for $1 \leqslant i \leqslant 5$ is given below

$$\begin{aligned} d_1(q) &= 1, \\ d_2(q) &= 1 + q, \\ d_3(q) &= 1 + 4q - q^2, \\ d_4(q) &= 1 + 11q - 5q^2 + q^3, \\ d_5(q) &= 1 + 26q - 16q^2 + 6q^3 - q^4. \end{aligned}$$

By performing the elementary column transformation $\mathrm{Col}_1 := \mathrm{Col}_1 - \mathrm{Col}_2$ and evaluating the determinant after this transformation, we get the following recurrence for $d_n(q)$, where $d_1(q) = 1$, $d_2(q) = 1 + q$ are easily seen

$$d_n(q) = (3 - q) d_{n-1}(q) + 2(q - 1) d_{n-2}(q).$$

Hence if $d_n(q) = \sum_{k=0}^{n-1} a_{n,k} q^k$, it is easy to see that $a_{n,0} = 1$ for all $n \geqslant 0$ and that $a_{n,1} = (2^n - n - 1)$ (i.e. it is the Eulerian number $A_{n,2}$). The above recurrence gives

$$a_{n,k} = 3a_{n-1,k} - a_{n-1,k-1} + 2a_{n-2,k-1} - 2a_{n-2,k}, \quad \text{for } n \geqslant 3.$$

From this, for $k \geqslant 2$ by inducting on n for a fixed k, we get the following (equivalent) recurrences

$$a_{n,k} = -\left(\sum_{j=k+1}^{n-1} a_{j,k-1} \right) \quad \text{OR} \quad a_{n,k} = a_{n-1,k} - a_{n-1,k-1}. \tag{1}$$

For $k = 1$ (i.e. the coefficient of q), when we sum over various n's, we clearly get the generating function $\frac{t^2}{(1-2t)(1-t)^2}$. The generating function for higher powers of q follows from recurrence (1), completing the proof. $\square$

Consider the $n \times n$ matrices

$$E_n = \begin{pmatrix} q & q & \cdots & q \\ 1 & q & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & q \end{pmatrix}; \qquad F_n = \begin{pmatrix} -q & q & \cdots & q \\ -1 & -q & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -q \end{pmatrix} \quad \text{and} \quad G_n = \begin{pmatrix} q & q & \cdots & q \\ -1 & q & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & q \end{pmatrix}$$

i.e. if $E_n = (e_{i,j})_{1 \leqslant i,j \leqslant n}$, then $e_{i,j} = q$ if $i \leqslant j$ and $e_{i,j} = 1$ otherwise. Similarly, if $F_n = (f_{i,j})_{1 \leqslant i,j \leqslant n}$, then $f_{i,j} = q$ if $i < j$, $f_{i,j} = -q$ if $i = j$ and $f_{i,j} = -1$ otherwise and if $G_n = (g_{i,j})_{1 \leqslant i,j \leqslant n}$, then $g_{i,j} = q$ if $i \leqslant j$, and $g_{i,j} = -1$ otherwise.

For $n \geqslant 1$, let $\mathsf{SgnWkExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{wkexc}(\pi)}$ as the *signed weak-excedance* enumerator. Define $\mathsf{SgnSkWkExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{wkexc}(\pi)}$ as the *signed skew weak-excedance* enumerator and let $\mathsf{SgnWkSkWkExc}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{wknexc}(\pi)} q^{\mathsf{wkexc}(\pi)}$ be the *signed weak-skew weak-excedance* enumerator.

Theorem 6. *For $n \geqslant 1$, $\mathsf{SgnWkExc}_n(q) = q(q-1)^{n-1}$. For $n \geqslant 1$, $\mathsf{SgnSkWkExc}_n(q) = q(q+1)^{n-1}$.*

Proof. From the definitions, it is easy to see that $\mathsf{SgnWkExc}_n(q) = \det(E_n)$. Further, it is straightforward to see (by applying elementary column operations) that $\det(E_n) = q(q-1)^{n-1}$. Similarly, it is easy to see that $\mathsf{SgnSkWkExc}_n(q) = \det(G_n)$ and that $\det(G_n) = q(q+1)^{n-1}$, completing the proof. $\square$

Remark 3. It is easy to see from Remark 1 that $\mathsf{perm}(E_n) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{wkexc}(\pi)} = q \cdot \mathsf{Eul}_n(q)$. It is also easy to see that $\mathsf{perm}(G_n) = q \cdot (-1)^n \mathsf{Eul}_n(q)$.

For the next result, we need the following sequence $r_n(q)$ of polynomials for $n \geqslant 1$. For $n \geqslant 1$ and $0 \leqslant k \leqslant n-2$, let $c_{n,k} = \sum_{j=0}^{k} \binom{n}{j}$ be the sum of the first k binomial coefficients. Define $r_n(q) = \sum_{k=0}^{n-2} (-1)^k c_{n,k} q^{k+1} + (-1)^n q^n$. The polynomials $r_n(q)$ for $1 \leqslant n \leqslant 5$ are given below. We note that these polynomials have coefficients similar to that of the polynomials $d_n(q)$ given in the proof of Theorem 5

$$\begin{aligned}
r_1(q) &= -q, \\
r_2(q) &= q + q^2, \\
r_3(q) &= q - 4q^2 - q^3, \\
r_4(q) &= q - 5q^2 + 11q^3 + q^4, \\
r_5(q) &= q - 6q^2 + 16q^3 - 26q^4 - q^5.
\end{aligned}$$

Lemma 1. *For $n > 2$, $r_n(q) = (1-3q) \cdot r_{n-1}(q) - 2q(q-1) \cdot r_{n-2}(q)$.*

Proof. We equivalently show that $c_{n,k} = c_{n-1,k} + 3c_{n-1,k-1} - 2c_{n-2,k-2} - 2c_{n-2,k-1}$. This easily follows from the definition that $c_{n,k} = \sum_{j=0}^{k} \binom{n}{j}$. The following alternate recurrence is also easy to see: $c_{n,k} = c_{n-1,k} - c_{n-1,k-1}$ for all $k \neq n-1$ and $c_{n,n-1} = (-1)^n (2^n - n - 1)$ for $n \geqslant 2$. The proof is complete. $\square$

Theorem 7. *For $n \geqslant 1$, $\mathsf{SgnWkSkWkExc}_n(q) = r_n(q)$.*

Proof. It is clear that $\mathsf{SgnWkSkWkExc}_n(q) = \det(F_n)$. Thus, we only need to show that $\det(F_n) = r_n(q)$ for all $n \geqslant 1$. Since the result is true for $n = 1, 2$ we assume $n > 2$. Let f_n denote $\det(F_n)$. Applying the elementary column transformation $\mathsf{Col}_n := \mathsf{Col}_n - \mathsf{Col}_{n-1}$, and evaluating the determinant after this, we get the following recurrence for $n \geqslant 3$

$$f_n = (1-3q) \cdot f_{n-1} - 2q(q-1) \cdot f_{n-2}.$$

The proof is complete by combining with Lemma 1. $\square$

2.2. Excedance like statistics

For $\pi \in \mathfrak{S}_n$, define its *excedance-set* as $\mathsf{ExcSet}(\pi) = \{i \in [n]: \pi_i > i\}$ and its *weak-excedance set* as $\mathsf{WkExcSet}(\pi) = \{i \in [n]: \pi_i \geqslant i\}$.

For a $\pi \in \mathfrak{S}_n$, define its *excedance-sum* as $\mathsf{ExcSum}(\pi) = \sum_{i \in \mathsf{ExcSet}(\pi)} i$ and its *excedance-length* as $\mathsf{ExcLen}(\pi) = \sum_{i \in \mathsf{ExcSet}(\pi)} (\pi_i - i)$. The motivation for these definitions come from similar determinant expressions for their signed enumerators and the attractive results obtained.

3. Enumeration over permutations

We recall the following notation used in q-analogue enumeration (see the lecture notes [6]). Let q be a variable and for a non-negative integer n, define

$$(q;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{i=1}^{n}(1-q^i) & \text{if } n > 0. \end{cases} \tag{2}$$

3.1. Signed excedance like statistic enumeration

Consider the following two $n \times n$ matrices

$$P_n = \begin{pmatrix} 1 & q & q & \cdots & q \\ 1 & 1 & q^2 & \cdots & q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q^{n-1} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} 1 & q & q^2 & \cdots & q^{n-1} \\ 1 & 1 & q & \cdots & q^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

i.e. if $P_n = (p_{i,j})_{1 \leqslant i,j \leqslant n}$, then $p_{i,j} = q^i$ if $i < j$ and $p_{i,j} = 1$ otherwise. Similarly, if $S_n = (s_{i,j})_{1 \leqslant i,j \leqslant n}$, then $s_{i,j} = q^{j-i}$ if $i < j$ and $s_{i,j} = 1$ otherwise.

Theorem 8. *For $n \geqslant 1$, define $\mathsf{SgnExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{ExcSum}(\pi)}$ to be the signed excedance-sum enumerator. Then $\mathsf{SgnExcSum}_n(q) = (1-q)^{n-1} \prod_{i=1}^{n} [i]_q = (q;q)_{n-1}$.*

Proof. Clearly, $\mathsf{SgnExcSum}_n(q) = \det(P_n)$. It is easy to check that $\det(P_n) = (q;q)_{n-1}$, completing the proof. $\square$

Theorem 9. *For $n \geqslant 1$, let $\mathsf{SgnExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{ExcLen}(\pi)}$ be the signed excedance-length enumerator. Then $\mathsf{SgnExcLen}_n(q) = (1-q)^{n-1}$.*

Proof. It is clear that $\mathsf{SgnExcLen}_n(q) = \det(S_n)$. It is again easy to see that $\det(S_n) = (1-q)^{n-1}$, completing the proof. $\square$

$$\mathsf{WtdP}_n = \begin{pmatrix} 1 & q^{w_1} & q^{w_1} & \cdots & q^{w_1} \\ 1 & 1 & q^{w_2} & \cdots & q^{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q^{w_{n-1}} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{WtdS}_n = \begin{pmatrix} 1 & q^{w_1} & q^{w_2} & \cdots & q^{w_{n-1}} \\ 1 & 1 & q^{w_1} & \cdots & q^{w_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q^{w_1} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Next, consider the two $n \times n$ matrices given above, depending on $n-1$ positive integral "weights" $w_1, w_2, \ldots, w_{n-1}$. i.e. if $\mathsf{WtdP}_n = (p_{i,j})_{1 \leqslant i,j \leqslant n}$, then $p_{i,j} = q^{w_i}$ if $i < j$ and $p_{i,j} = 1$ otherwise. Similarly, if $\mathsf{WtdS}_n = (s_{i,j})_{1 \leqslant i,j \leqslant n}$, then $s_{i,j} = q^{w_{j-i}}$ if $i < j$ and $s_{i,j} = 1$ otherwise.

Consider a *weighted* signed-excedance sum statistic, where we assign for an excedance $i \in \mathsf{ExcSet}(\pi)$, a positive integral weight w_i. Thus we have $n-1$ weights $w_1, \ldots, w_{n-1}$ where w_i is the weight when the i-th position is an excedance. With respect to these weights, define a weighted signed-excedance sum for a permutation as $\mathsf{WtdExcSum}(\pi) = \sum_{i \in \mathsf{ExcSet}(\pi)} w_i$ and its enumerator as $\mathsf{SgnWtdExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{WtdExcSum}(\pi)}$. It is easy to note that when $w_i = i$ for $1 \leqslant i < n$, then $\mathsf{WtdExcSum}(\pi) = \mathsf{ExcSum}(\pi)$ and when $w_i = 1$ for all i, then $\mathsf{WtdExcSum}(\pi) = \mathsf{exc}(\pi)$. As the following is easy to prove on similar lines, we omit its proof.

Theorem 10. *For positive integers $n \geqslant 2$ and positive integral weights $w_1, w_2, \ldots, w_{n-1}$, we have* $\mathsf{SgnWtdExcSum}_n(q) = \det(\mathsf{WtdP}_n) = \prod_{i=1}^{n-1}(1 - q^{w_i})$. *Thus the weighted signed-excedance sum enumerator only depends on the set $\{w_1, w_2, \ldots, w_{n-1}\}$, i.e. it does not depend on the sequence $(w_1, w_2, \ldots, w_n)$ and thus permuting the weights w_i does not change the signed enumerator.*

Similarly, consider a *weighted* signed-excedance length statistic, where we assign for an excedance $i \in \mathsf{ExcSet}(\pi)$, a positive integral weight $w_{\pi_i - i}$. Thus we have $n-1$ weights $w_1, \ldots, w_{n-1}$ where w_i is the weight when index j is an excedance and the difference $\pi_j - j = i$. With respect to the weights w_i, define a weighed signed-excedance for a permutation as $\mathsf{WtdExcLen}(\pi) = \sum_{i \in \mathsf{ExcSet}(\pi)} w_i$ and its enumerator as $\mathsf{SgnWtdExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{WtdExcLen}(\pi)}$. It is easy to note that when $w_i = i$ for $1 \leqslant i < n$, then $\mathsf{WtdExcLen}(\pi) = \mathsf{ExcLen}(\pi)$ and when $w_i = 1$ for all i, then $\mathsf{WtdExcSum}(\pi) = \mathsf{exc}(\pi)$. The following is immediate.

Theorem 11. *For positive integers $n \geqslant 2$ and positive integral weights $w_1, w_2, \ldots, w_{n-1}$, we have* $\mathsf{SgnWtdExcLen}_n(q) = \det(\mathsf{WtdS}_n) = (1 - q^{w_1})^{n-1}$. *Thus the weighted signed-excedance length enumerator only depends on w_1.*

3.2. Signed weak-excedance like enumeration

We recall the definition of WkExcSet from Section 2.2. For a permutation $\pi \in \mathfrak{S}_n$, define its *weak-excedance sum* denoted $\mathsf{WkExcSum}(\pi) = \sum_{i \in \mathsf{WkExcSet}(\pi)} i$.

Consider the $n \times n$ matrix

$$B_n = \begin{pmatrix} q & q & q & \cdots & q \\ 1 & q^2 & q^2 & \cdots & q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q^n \end{pmatrix}; \qquad \mathsf{WB}_n = \begin{pmatrix} q^{w_1} & q^{w_1} & q^{w_1} & \cdots & q^{w_1} \\ 1 & q^{w_2} & q^{w_2} & \cdots & q^{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & q^{w_n} \end{pmatrix}$$

i.e. if $B_n = (b_{i,j})_{1 \leqslant i,j \leqslant n}$, then $b_{i,j} = q^i$ if $i \leqslant j$ and $b_{i,j} = 1$ otherwise and similarly given n positive integers $w_1, w_2, \ldots, w_n$, if $\mathsf{WB}_n = (b_{i,j})_{1 \leqslant i,j \leqslant n}$, then $b_{i,j} = q^{w_i}$ if $i \leqslant j$ and $b_{i,j} = 1$ otherwise. For n positive integral weights $W = (w_1, w_2, \ldots, w_n)$ and a $\pi \in \mathfrak{S}_n$, define its weighted weak-excedance sum as $\mathsf{WtdWkExcSum} = \sum_{i \in \mathsf{WkExcSet}(\pi)} w_i$. For $n \geqslant 1$, define $\mathsf{SgnWtdWkExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{WtdWkExcSum}(\pi)}$.

Theorem 12. *For $n \geqslant 1$, let the signed weak-excedance sum enumerator as* $\mathsf{SgnWkExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{WkExcSum}(\pi)}$. *Then,* $\mathsf{SgnWkExcSum}_n(q) = \frac{q}{q-1}(-1)^n (q;q)_n$. *If we have weights $W = (w_1, w_2, \ldots, w_n)$, then* $\mathsf{SgnWtdWkExcSum}_n(q) = q^{w_1} \prod_{i=2}^n (q^{w_i} - 1)$.

Proof. It is easy to see that $\mathsf{SgnWkExcSum}_n(q) = \det(B_n)$ and that $\det(B_n) = q \prod_{i=2}^n (q^i - 1)$. It is also easy to see from Eq. (2) that this expression is identical to $\frac{q}{q-1}(-1)^n (q;q)_n$, completing the proof. The proof for the weighted case is identical, except that we change i to w_i and is hence omitted. □

3.3. Signed skew weak-excedance like statistic enumeration

Consider the $n \times n$ matrices

$$C_n = \begin{pmatrix} q & q & q & \cdots & q \\ -1 & q^2 & q^2 & \cdots & q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^n \end{pmatrix} \quad \text{and} \quad D_n = \begin{pmatrix} q & q^2 & q^3 & \cdots & q^n \\ -1 & q & q^2 & \cdots & q^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q \end{pmatrix}$$

i.e. if $C_n = (c_{i,j})_{1 \leqslant i,j \leqslant n}$, then $c_{i,j} = q^i$ if $i \leqslant j$ and $c_{i,j} = -1$ otherwise. Similarly, if $D_n = (d_{i,j})_{1 \leqslant i,j \leqslant n}$, then $d_{i,j} = q^{j-i+1}$ if $i \leqslant j$ and $d_{i,j} = -1$ otherwise. For a $\pi \in \mathfrak{S}_n$, define its *weak-excedance length* $\mathsf{WkExcLen}(\pi) = \sum_{i \in \mathsf{WkExcSet}} (\pi_i - i + 1)$. Similarly define its *weak-excedance sum* as $\mathsf{WkExcSum}(\pi) = \sum_{i \in \mathsf{WkExcSet}} i$. For $n \geqslant 1$, define the *signed skew weak-excedance sum* enumerator as $\mathsf{SgnSkWkExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{WkExcSum}(\pi)}$. More generally, when there are n positive integral weights $w_1, w_2, \ldots, w_n$, define $\mathsf{WkWtdExcSum}(\pi) = \sum_{i \in \mathsf{WkExcSet}} w_i$ and the weighted analogue $\mathsf{SgnSkWkWtdExcSum} = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{WkWtdExcSum}(\pi)}$.

Likewise, let $\mathsf{SgnSkWkExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{WkExcLen}(\pi)}$ be the *signed skew weak-excedance length* enumerator.

Theorem 13. *For $n \geqslant 1$, $\mathsf{SgnSkWkExcSum}_n(q) = q \prod_{i=2}^n (1 + q^i)$. Further, given weights $W = (w_1, w_2, \ldots, w_n)$, $\mathsf{SgnSkWkWtdExcSum} = q^{w_1} \prod_{i=2}^n (1 + q^{w_i})$.*

Proof. It is easy to see that $\mathsf{SgnSkWkExcSum}_n(q) = \det(C_n)$. It is straightforward to check that $\det(C_n) = q \prod_{i=2}^n (1 + q^i)$, completing one proof. The proof for the weighted version is similar, just that we replace i by w_i. This completes the proof. □

Theorem 14. $\mathsf{SgnSkWkExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{WkExcLen}(\pi)} = 2^{n-1} q^n$.

Proof. From the definition of D_n, it is easy to see that $\mathsf{SgnSkWkExcLen}_n(q) = \det(D_n)$. It is again straightforward to see that $\det(D_n) = 2^{n-1} q^n$, completing the proof. □

3.4. Signed skew excedance like statistic enumeration

Let R_n be the following $n \times n$ matrix.

$$R_n = \begin{pmatrix} 1 & q & q^2 & \cdots & q^{n-1} \\ -1 & 1 & q & \cdots & q^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

i.e. if $R_n = (r_{i,j})_{1 \leqslant i,j \leqslant n}$, then $r_{i,j} = q^{j-i}$ if $i < j$, $r_{i,j} = 1$ if $i = j$ and $r_{i,j} = -1$ otherwise.

Theorem 15. *For $n \geqslant 1$, if $\mathsf{SgnSkExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{ExcLen}(\pi)}$ is the signed skew excedance-length i enumerator, then $\mathsf{SgnSkExcLen}_n(q) = (1 + q)^{n-1}$.*

Proof. It is clear that $\mathsf{SgnSkExcLen}_n(q) = \det(R_n)$. It is again easy to see that $\det(R_n) = (1 + q)^{n-1}$, completing the proof. □

3.5. Signed weak-skew excedance like statistic enumeration

The following two are immediate from the results of Section 3.1.

Theorem 16. *For $n \geqslant 1$, let $\mathsf{SgnWkSkExcSum}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{wknexc}(\pi)} q^{\mathsf{ExcSum}(\pi)}$ be the signed weak-skew excedance-sum enumerator. Then, $\mathsf{SgnWkSkExcSum}_n(q) = (-1)^n \prod_{i=1}^{n-1} (1 + q^i)$.*

Proof. Follows from the simple observation that $\mathsf{SgnWkSkExcSum}_n(-q) = (-1)^n \det(P_n)$. □

Theorem 17. *For $n \geqslant 1$, let $\mathsf{SgnWkSkExcLen}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{wknexc}(\pi)} q^{\mathsf{ExcLen}(\pi)}$ be the signed weak-skew excedance-length enumerator. Then $\mathsf{SgnWkSkExcLen}_n(q) = (-1)^n (1+q)^{n-1}$.*

Proof. Follows immediately by noting that $\mathsf{SgnWkSkExcLen}_n(-q) = (-1)^n \det(S_n)$. □

4. Enumeration over derangements

Let $\mathfrak{D}_n \subseteq \mathfrak{S}_n$ be the set of derangements on $[n]$. We will consider several matrices considered in Sections 2 and 3 but with their diagonal elements made zero. If α_n is a matrix from Section 2 or Section 3, we will denote the same matrix with all diagonal elements made zero as D_{α_n}. Thus, for example, we have the two $n \times n$ matrices

$$D_{M_n} = \begin{pmatrix} 0 & q & q & \cdots & q \\ 1 & 0 & q & \cdots & q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad D_{L_n} = \begin{pmatrix} 0 & q & q & \cdots & q \\ t & 0 & q & \cdots & q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \cdots & 0 \end{pmatrix}.$$

We begin with an alternate proof of Mantaci and Rakotondrajao's result. Let $\mathsf{DSgnExc}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$ be the signed excedance enumerator.

Proof of Theorem 2. Consider the matrix D_{M_n}. We claim that $\mathsf{DSgnExc}_n(q) = \det(D_{M_n})$. Denote the (i,j)-th entry of D_{M_n} for $1 \leqslant i, j \leqslant n$ as $d_{i,j}$. To see that $\det(D_{M_n}) = \mathsf{DSgnExc}_n(q)$, we note as before that $\det(D_{M_n}) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} \prod_{i=1}^n d_{i,\pi_i}$. As in the proof of Theorem 1, it is clear that $\det(D_{M_n}) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{exc}(\pi)}$. It is simple to show that $\det(D_{M_n}) = (-1)^{n-1} q \cdot [n-1]_q$, completing the proof. □

Just as $\det(D_{M_n}) = (-1)^{n-1} q [n-1]_q$, we note that $\mathsf{perm}(D_{M_n}) = P_n(q)$, where $P_n(q) = \sum_{k=0}^n a_{n,k} q^k$ is the polynomial occurring in Mantaci and Rakotondrajao's work [13, Table 1] and $\mathsf{perm}(D_{M_n})$ is the permanent of D_{M_n}. This polynomial $P_n(q)$ will appear again in Corollary 1.

Remark 4. More generally, if we consider the matrix D_{L_n} given above, it is easy to see that $\mathsf{perm}(D_{L_n})$ is a bivariate generalisation of the polynomial $P_n(q)$ in the sense that $\mathsf{perm}(D_{L_n}) = P_n(q,t)$ where $P_n(q,t) = \sum_{k=0}^n a_{n,k} q^k t^{n-k}$.

Remark 5. If $D_n = |\mathfrak{D}_n|$ is the number of derangements in $\mathfrak{S}_n$, then it is well known (see for example [3]) that $D_n = (n-1)(D_{n-1} + D_{n-2})$ where $D_0 = 1, D_1 = 0$. If $\mathsf{SgnDer}_n = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)}$ is the signed derangement number, then since $\mathsf{SgnDer}_n = \det(J - I)_{n \times n}$ where J and I are the $n \times n$ all-ones and identity matrix, we get an analogous relation $\mathsf{SgnDer}_n = (n-1)(-\mathsf{SgnDer}_{n-1} - \mathsf{SgnDer}_{n-2})$ where $\mathsf{SgnDer}_0 = 1, \mathsf{SgnDer}_1 = 0$. Theorem 2 yields the following q-analogue of this recurrence:

$$\mathsf{DSgnExc}_n(q) = [n-1]_q \big[-\mathsf{DSgnExc}_{n-1}(q) - q \times \mathsf{DSgnExc}_{n-2}(q) \big].$$

Theorem 18. *Define $\mathsf{DSgnSkExc}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{exc}(\pi)}$. Then, $\mathsf{DSgnSkExc}_n(q) = \sum_{i=1}^{n-1} (-1)^i q^{i-1}$.*

Proof. It is clear that $\mathsf{DSgnSkExc}_n(q) = \det(D_{F_n})$. To evaluate $\det(D_{F_n})$, we perform the elementary column operation $\mathsf{Col}_n := \mathsf{Col}_n - \mathsf{Col}_{n-1}$ and then evaluate the determinant of the modified matrix. Denoting $\det(D_{F_n})$ as $d_n(q)$, it is simple to see that we get the following recurrence $d_n(q) = (1-q) \cdot$

$d_{n-1}(q) + q \cdot d_{n-2}(q)$. It is clear that for $n = 2, 3$ that we get $d_2(q) = q$ and $d_3(q) = q - q^2$. From the above recurrence, we get $d_n(q) = \sum_{i=1}^{n-1} (-1)^i q^{i-1}$, completing the proof. $\square$

Corollary 1. *If $s_n(q) = \mathsf{perm}(D_{F_n})$ is the permanent of D_{F_n}, then $s_n(q) = \sum_{k=0}^{n} a_{n,k} q^k (-1)^{n-k}$, where the $a_{n,k}$'s are coefficients that appeared in Mantaci and Rakotondrajao's work, mentioned in Remark 4.*

4.1. Signed excedance-like statistic enumeration

Theorem 19. *Let $\mathsf{DSgnExcSum}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{ExcSum}(\pi)}$ be the signed index-excedance enumerator summed over $\mathfrak{D}_n$. Then $\mathsf{DSgnExcSum}_n(q) = (-1)^{n-1} (\sum_{i=2}^{n} q^{\binom{i}{2}})$.*

Proof. It is clear that $\mathsf{DSgnExcSum}_n(q) = \det(D_{P_n})$. Let $p_n(q) = \det(D_{P_n})$. After performing the elementary column operation $\mathsf{Col}_n := \mathsf{Col}_n - \mathsf{Col}_{n-1}$, we get the following recurrence: $p_n(q) = -(1 + q^{n-1}) p_{n-1}(q) - q^{n-1} \cdot p_{n-2}(q)$. It is easy to see that $p_2(q) = -q$ and $p_3(q) = q + q^3$. With these initial values and the above recurrence, it is easy to see that $p_n(q) = (-1)^{n-1} (\sum_{i=2}^{n} q^{\binom{i}{2}})$, completing the proof. $\square$

More generally, consider positive integral weights $w_1, w_2, \ldots, w_{n-1}$ and suppose an excedance at position i contributes w_i to $\mathsf{WtdExcSum}(\pi)$ as done in Section 3.1. With respect to the underlying weights $W = (w_1, w_2, \ldots, w_{n-1})$, define $\mathsf{DSgnWtdExcSum}_n(q) = \sum_{\pi \in \mathfrak{D}_n} q^{\mathsf{WtdExcSum}(\pi)}$. We have the following common generalisation of Theorems 2 and 19.

Theorem 20. *For positive integral weights $W = (w_1, w_2, \ldots, w_{n-1})$ and for $i \leqslant i < n$, define $y_i = \sum_{j=1}^{i} w_j$. For all $n \geqslant 2$, with respect to the weights W, we have $\mathsf{DSgnWtdExcSum}_n(q) = (-1)^{n-1} (\sum_{i=1}^{n-1} q^{y_i})$.*

Proof. We induct on n, the dimension of the matrix, with the base case $n = 2$ being clear. Let $n > 2$ and given a weight sequence $W = (w_1, w_2, \ldots, w_{n-1})$, denote the $n \times n$ matrix D_{WtdP_n} with respect to weights W as $D^W_{\mathsf{WtdP}_n}$. For a weight sequence W, denote by W_1, W_2 and $W_{1,2}$ the sequences obtained from W by deleting w_1, w_2 and both w_1, w_2 respectively. For a sequence W with r weights, denote $f_W = \det(\mathsf{WtdP}^W_{r+1})$. In D_{WtdP_n}, performing the column operation $\mathsf{Col}_1 := \mathsf{Col}_1 - \mathsf{Col}_2$ and then evaluating the determinant, we get the following recurrence

$$f_W = q^{w_1} f_{W_1} - q^{w_1} f_{W_2} - f_{W_{1,2}}. \tag{3}$$

Since each of the vectors $W_1, W_2, W_{1,2}$ have lesser components, their values are known by induction. Combining them completes the proof. $\square$

As a preliminary for the next result, we recall the definition of Chebysheff polynomials $U_n(x)$ for $n \geqslant 0$ of the second kind (see [7, Appendix C]). $U_n(x)$ is a sequence of polynomials defined by $U_0(x) = 1$; $U_1(x) = 2x$; and

$$U_{n+1}(x) = 2x \cdot U_n(x) - U_{n-1}(x). \tag{4}$$

Alternatively, they are also defined by the generating function $\sum_{n=0}^{\infty} U_n(x) t^n = 1/(1 - 2xt + t^2)$.

Theorem 21. *Let $\mathsf{DSgnExcLen}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)} q^{\mathsf{ExcLen}(\pi)}$ be the signed excedance-length enumerator summed over $\mathfrak{D}_n$. Then $\mathsf{DSgnExcLen}_{n+2}(q) = (-1)^{n+1} q^{\frac{n+2}{2}} U_n(\sqrt{q})$.*

Proof. It is easy to see that $\mathsf{DSgnExcLen}_n(q) = \det(D_{S_n})$. Denote by $m_n(q)$, $\det(D_{S_n})$. It is easy to check that $m_1(q) = 0$, $m_2(q) = -q$. Hence, the theorem follows for $n = 1, 2$. Further, for $n \geqslant 0$, we derive the following recurrence

$$m_{n+2}(q) = -2q \cdot m_{n+1}(q) - q \cdot m_n(q). \tag{5}$$

To see this recurrence, perform the elementary column operation $\mathsf{Col}_n := \mathsf{Col}_n - q \cdot \mathsf{Col}_{n-1}$ and then evaluate $m_n(q)$. From recurrence (4), it is easy to see that the polynomials $(-1)^{n+1} q^{\frac{n+2}{2}} U_n(\sqrt{q})$ also satisfy recurrence (5) and have the same initial terms. Since both polynomials $m_n(q)$ and $(-1)^{n+1} q^{\frac{n+2}{2}} U_n(\sqrt{q})$ have the same initial values as well, they are identical for all $n \geqslant 1$, completing the proof. $\square$

4.2. Signed skew excedance-like statistic enumeration

Consider the following $n \times n$ matrices.

$$D_{H_n} = \begin{pmatrix} 0 & q & q & \cdots & q \\ -1 & 0 & q^2 & \cdots & q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^{n-1} \\ -1 & -1 & -1 & \cdots & 0 \end{pmatrix}; \qquad D_{\mathsf{WtdH}_n} = \begin{pmatrix} 0 & q^{w_1} & q^{w_1} & \cdots & q^{w_1} \\ -1 & 0 & q^{w_2} & \cdots & q^{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^{w_{n-1}} \\ -1 & -1 & -1 & \cdots & 0 \end{pmatrix};$$

i.e. if $D_{H_n} = (h_{i,j})_{1 \leqslant i,j \leqslant n}$, then $h_{i,j} = q^i$ if $i < j$, $h_{i,j} = 0$ if $i = j$ and $h_{i,j} = -1$ otherwise. Similarly, if $D_{\mathsf{WtdH}_n} = (h_{i,j})_{1 \leqslant i,j \leqslant n}$, then $h_{i,j} = q^{w_i}$ if $i < j$, $h_{i,j} = 0$ if $i = j$ and $h_{i,j} = -1$ otherwise. We recall that for a given $w_1, w_2, \ldots, w_{n-1}$, $y_k = \sum_{j=1}^{k} w_j$.

Theorem 22. *With respect to weights $W = (w_1, w_2, \ldots, w_{n-1})$, let the weighted skew signed-excedance sum enumerator be* $\mathsf{DSgnSkWtdExcSum}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{WtdExcSum}(\pi)}$. *Then for $n \geqslant 2$,* $\mathsf{DSgnSkWtdExcSum}_n(q) = \sum_{i=1}^{n-1} (-1)^{i-1} q^{y_i}$.

Proof. This proof is very similar to the proof of Theorem 20. In particular, we get the same recurrence given in Eq. (3), but with different initial values. We omit the routine details. $\square$

We single out the following corollary of Theorem 22 obtained by setting $w_i = i$. With this weights W, for all $\pi \in \mathfrak{S}_n$, $\mathsf{WtdExcSum}(\pi) = \mathsf{ExcSum}(\pi)$.

Corollary 2. *Let* $\mathsf{DSgnSkExcSum}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{ExcSum}(\pi)}$ *be the skew signed excedance sum polynomial summed over derangements. Then for $n \geqslant 2$,* $\mathsf{DSgnSkExcSum}_n(q) = \sum_{i=2}^{n} (-1)^i q^{\binom{i}{2}}$.

Below, we give an analog of Theorem 15 for derangements. Hence, we consider the matrix D_{R_n}. Let $\mathsf{DSgnSkExcLen}_n(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathsf{inv}(\pi)+\mathsf{nexc}(\pi)} q^{\mathsf{ExcLen}(\pi)}$ be the skew signed excedance length enumerator summed over derangements.

Theorem 23. *For $n \geqslant 1$,*

$$\mathsf{DSgnSkExcLen}_n(q) = \begin{cases} q^{n/2} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k+1. \end{cases}$$

Proof. It is clear that $\mathsf{DSgnSkExcLen}_n(q) = \det(D_{R_n})$. Let $r_n(q) = \det(D_{R_n})$. As in the proof of Theorem 18, performing the elementary column operation $\mathsf{Col}_n := \mathsf{Col}_n - q \cdot \mathsf{Col}_{n-1}$ and then evaluating the determinant of the resulting matrix gives us the following recurrence: $r_n(q) = q \cdot r_{n-2}(q)$. It is easy to see that $r_2(q) = q$ and $r_3(q) = 0$. Using the above recurrence with these initial values yields us $r_{2n}(q) = q^n$ and $r_{2n+1}(q) = 0$, completing the proof. $\square$

Acknowledgments

Some theorems in this work were in their conjecture form, tested using the computer package “Sage”. We thank the authors for generously releasing Sage as an open-source package. The *Online Encyclopedia of Integer Sequences* was consulted for the results in Theorems 5, 7 and 21. We are grateful to Professor N.J.A. Sloane for putting his searchable encyclopedia on the web.

We acknowledge support from project grant P07 IR052, given by IIT Bombay.

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