1. Fourier transform: Basic properties

In this section we show that there exists a remarkable unitary operator \( F \) on the Hilbert space \( L^2(\mathbb{R}) \) which we call the Fourier transform and study some of the basic properties of that operator.

1.1. Unitary operators: some examples. We begin with some definitions. Given two Hilbert spaces \( H_1 \) and \( H_2 \) consider a bounded linear operator \( T : H_1 \rightarrow H_2 \). We define its adjoint, denoted by \( T^* \) the unique operator from \( H_2 \) into \( H_1 \) determined by the condition

\[
(Tu,v)_2 = (u,T^*v)_1, \quad u \in H_1, v \in H_2
\]

where \((\cdot,\cdot)_j\) stand for the inner product in \( H_j \). Note that \( T^* \) is bounded.

We say that \( T \) is unitary if \( TT^* = I_2, T^*T = I_1 \) where \( I_j \) is the identity operator on \( H_j \). If \( T \) is unitary then we have \((u,v)_1 = (Tu,Tv)_2\) for all \( u, v \in H_1 \). In particular \( \|Tu\|_2 = \|u\|_1, u \in H_1 \).

We give some examples of unitary operators. Let \( H_1 = L^2(S^1) \) and \( H_2 = L^2(\mathbb{Z}) \). Take \( T \) to be the operator \( Tf(k) = \hat{f}(k) \) where

\[
\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}dt.
\]

Then it can be checked that \( T^* \) is given by

\[
T^*\varphi(t) = \sum_{-\infty}^{\infty} \varphi(k)e^{-2\pi ikt}.
\]

The Plancherel theorem for the Fourier series shows that \( T \) is unitary. Another simple example is provided by the translation \( \tau_a f(x) = f(x-a) \) defined from \( L^2(\mathbb{R}) \) into itself. We give some more examples below.
Let us take the nonabelian group $\mathbb{H}^1$ which is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + xy').$$

Then it is clear that $\mathbb{H}^1$ is nonabelian and the Lebesgue measure $dxdydt$ is both left and right invariant Haar measure on $\mathbb{H}^1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}^1)$. Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then it is easy to check that $\Gamma$ is a subgroup of $\mathbb{H}^1$ so that we can form the quotient $M = \Gamma/\mathbb{H}^1$ consisting of all right cosets of $\Gamma$. Functions on $M$ are naturally identified with left $\Gamma-$invariant functions on $\mathbb{H}^1$. As the Lebesgue measure $dxdydt$ is left $\Gamma-$ invariant we can form $L^2(M)$ using the Lebesgue measure restricted to $M$. As a set we can identify $M$ with $[0, 1)^3$ and we just think of $L^2(M)$ as $L^2([0, 1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = e^{2\pi ik}f(x, y, t).$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k \in \mathcal{H}_k.$$

We are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right.(why?) It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of $\Gamma$.

Our next example of a unitary operator is the following. Consider the map $J : \mathbb{H}^1 \rightarrow \mathbb{H}^1$ given by $J(x, y, t) = (y, -x, t - xy)$. Then $J$ is an automorphism of the group $\mathbb{H}^1$ which satisfies (i)$J^4 = I$, (ii) $J(\Gamma) = \Gamma$(i.e. $J$ leaves $\Gamma$ invariant) and (iii) $J$ restricted to the center of $\mathbb{H}^1$ is just the identity; i.e. $J(0, 0, t) = (0, 0, t)$. Using this automorphism we define an operator, denoted by the same symbol, on $\mathcal{H}_1$ by

$$Jf(x, y, t) = f(J(x, y, t)) = f(y, -x, t - xy).$$
It is clear that $J^* f(x, y, t) = f(-y, x, t - xy)$ so that $J$ is unitary. We also observe that $J^2 f(x, y, t) = f(-x, -y, t)$.

We now define another very important unitary operator which takes $L^2(\mathbb{R})$ onto $\mathcal{H}_1$. This operator used by Weil and Brezin is called the Weil-Brezin transform and is defined as follows. For $f \in L^2(\mathbb{R})$,

$$V f(x, y, t) = e^{2\pi it} \sum_{n=-\infty}^{\infty} f(x + n)e^{2\pi iny}.$$ 

As $f \in L^2(\mathbb{R})$ we know that $f(x + n)$ is finite for almost every $x \in \mathbb{R}$. The above series converges in $L^2([0,1])$ as a function of $y$ and we have

$$\int_0^1 |V f(x, y, t)|^2 dy = \sum_{n=-\infty}^{\infty} \int_0^1 |f(x + n)|^2.$$ 

Thus it follows that $V f \in \mathcal{H}_1$ and

$$\int_{[0,1]^3} |V f(x, y, t)|^2 dx dy dt = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

**Proposition 1.1.** $V$ is a unitary operator from $L^2(\mathbb{R})$ onto $\mathcal{H}_1$.

To prove this proposition we need to calculate $V^*$. It is clear that $V$ is one to one but is also onto. To see this, given $F \in \mathcal{H}_1$ consider $f$ defined as follows. For $x \in [m, m + 1)$ define

$$f(x) = \int_0^1 F(x - m, y, 0)e^{-2\pi imy} dy.$$ 

Then it is clear that $f \in L^2(\mathbb{R})$ and

$$V f(x, y, t) = e^{2\pi it} \sum_{m=-\infty}^{\infty} \left( \int_0^1 F(x, u, 0)e^{-2\pi imu} du \right) e^{2\pi imy} = F(x, y, t).$$

Moreover, if $f, g \in L^2(\mathbb{R})$ then

$$(f, g) = \sum_{m=-\infty}^{\infty} \int_0^1 f(x + m)\overline{g(x + m)} dx.$$ 

The sum is nothing but

$$\int_0^1 V f(x, y, t)\overline{V g(x, y, t)} dy.$$
and hence we have \((f, g) = (Vf, Vg)\). This shows that \(V^* = V^{-1}\) and hence \(V\) is unitary.

1.2. Fourier transform: Plancherel and inversion theorems.

**Definition 1.2.** The unitary operator \(V^*JV\) from \(L^2(\mathbb{R})\) onto itself is called the Fourier transform and is denoted by \(\mathcal{F}\).

We record some important properties of the Fourier transform in the following theorem.

**Theorem 1.3.** The Fourier transform \(\mathcal{F}\) satisfies: (i) \(\mathcal{F}^4 f = f\), for every \(f \in L^2(\mathbb{R})\) (ii) \(\mathcal{F}^2 f(x) = f(-x)\) for almost every \(x \in \mathbb{R}\) and (iii) \(\|\mathcal{F}f\|_2 = \|f\|_2\).

We only need to check (ii) as (i) follows immediately since \(J^4 = I\). As \(J^2 f(x, y, t) = f(-x, -y, t)\) we have

\[
\mathcal{F}^2 f(x) = \int_0^1 Vf(-x - m, -y, t)e^{-2\pi i m y} dy
\]
whenever \(x \in [-m, m + 1]\). If we recall the definition of \(Vf\) the above is simply \(f(-x)\).

The property (iii), namely \(\|\mathcal{F}f\|_2 = \|f\|_2\) is called the Plancherel theorem for the Fourier transform.

Before proceeding further let us calculate the Fourier transforms of some well known functions. As our first example let us take the Gaussian \(\varphi(x) = e^{-\pi x^2}\).

**Proposition 1.4.** The Fourier transform of \(\varphi\) is itself: \(\mathcal{F}\varphi = \varphi\).

**Proof.** By definition, when \(x \in [0, 1)\),

\[
\mathcal{F}\varphi(x + m) = \int_0^1 \sum_{n=-\infty}^{\infty} \varphi(y + n)e^{-2\pi i(y+n)(x+m)} dy
\]

which can be rewritten as

\[
e^{-\pi(x+m)^2} \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(y+n+i(x+m))^2} dy.
\]
We claim that the integral is a constant. To see this, note that
\[ G(w) = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(y+n+iv)^2} dy = \int_{-\infty}^{\infty} e^{-\pi(y+iw)^2} dy \]
is an entire function of \( w = u + iv \) and \( G(iv) = G(v) \). Hence \( G(x+m) \) is a constant and we get \( \mathcal{F}\varphi = c\varphi \). But \( G(0) = 1 \) and so \( c = 1 \) proving the proposition.

The above proposition shows that the Gaussian \( \varphi \) is an eigenfunction of the Fourier transform. We will say more about the spectral decomposition of \( \mathcal{F} \) in the next section.

We introduced the Fourier transform as a unitary operator on \( L^2(\mathbb{R}) \). Now we extend the definition to \( L^1(\mathbb{R}) \) and prove a useful inversion formula.

**Theorem 1.5.** For \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) the Fourier transform is given by
\[ \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi} dx. \]
If we further assume that \( \mathcal{F}f \in L^1(\mathbb{R}) \) then for almost every \( x \) we have
\[ f(x) = \int_{\mathbb{R}} \mathcal{F}f(\xi)e^{2\pi ix\xi} d\xi. \]

**Proof.** If \( \xi = x + m, x \in [0,1) \) it follows from the definition that
\[ \mathcal{F}f(\xi) = \int_0^1 \sum_{n=-\infty}^{\infty} f(y + n)e^{-2\pi i(y+n)(x+m)} dy. \]
As \( f \) is integrable we can interchange the order of summation and integration to arrive at the formula
\[ \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi} dx. \]
Under the assumption that \( \mathcal{F}f \) is also integrable the inversion formula \( \mathcal{F}^2 f(x) = f(-x) \) leads to
\[ f(x) = \int_{\mathbb{R}} \mathcal{F}f(\xi)e^{2\pi ix\xi} d\xi. \]
This completes the proof of the theorem. \( \square \)
It is customary to denote the Fourier transform $\mathcal{F}f$ of integrable functions by $\hat{f}$. Thus

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$ 

It is clear from this that the Fourier transform can be defined on all of $L^1(\mathbb{R})$. Note that $\hat{f}$ for $f \in L^1(\mathbb{R})$ is a bounded function and

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)| dx = \|f\|_1.$$ 

It can be easily checked, by an application of the Lebesgue dominated convergence theorem, that $\hat{f}$ is in fact continuous. But something more is true. The following result is known as the Riemann-Lebesgue lemma in the literature.

**Theorem 1.6.** For all $f \in L^1(\mathbb{R})$, $\hat{f}$ vanishes at infinity; i.e., $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

**Proof.** As $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ it is enough to prove the result for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that $\hat{f}(x + m), x \in [0, 1)$ is the $m$–th Fourier coefficient of the integrable periodic function

$$F(y) = \sum_{n=-\infty}^{\infty} f(y+n)e^{-2\pi i (y+n)}$$

and hence it is enough to show that the Fourier coefficients of an integrable function vanish at infinity. It is clearly true of trigonometric polynomials and as they are dense in $L^1([0, 1))$ the same true for all integrable functions.

Another immediate consequence of our definition of the Fourier transform is the so called Poisson summation formula. If the integrable function $f$ satisfies the estimate $|f(y)| \leq C(1 + y^2)^{-1}$ then the series defining $Vf(x, y, t)$ converges uniformly. The same is true of $V\hat{f}$ if $\hat{f}$ also satisfies such an estimate. For such functions we have the following result.
Theorem 1.7. Assume that $f$ is measurable and satisfies $|f(y)| \leq C(1 + y^2)^{-1}$ and $|\hat{f}(\xi)| \leq C(1 + \xi^2)^{-1}$. Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. Since $\hat{f} = V^* J V f$ we have $J V f(x, y, t) = V \hat{f}(x, y, t)$. As both series defining $J V f$ and $V \hat{f}$ converge uniformly we can evaluate them at $(0, 0, 0)$ which gives the desired result.

When we take $f(x) = t^{-\frac{1}{2}} \varphi(t^{-\frac{1}{2}} x)$ for $t > 0$, it follows that $\hat{f}(\xi) = \varphi(t^{\frac{1}{2}} \xi)$ and hence Poisson summation formula gives the interesting identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = t^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi t^{-1} n^2}.$$

We can obtain several identities of this kind by considering eigenfunctions of the Fourier transform.

1.3. Spectral decomposition of $\mathcal{F}$. The spectrum of the Fourier transform $\mathcal{F}$ is contained in the unit circle as $\mathcal{F}$ is unitary. Moreover, as $\mathcal{F}^4 = I$ any $\lambda$ in its spectrum $\sigma(\mathcal{F})$ satisfies $\lambda^4 = 1$. Hence, $\sigma(\mathcal{F}) = \{1, -1, i, -i\}$. In this subsection we describe explicitly the orthogonal projections associated to each point of the spectrum.

We have identified at least one eigenfunction, namely the Gaussian. Let us search for eigenfunctions of the form $f(x) = p(x) e^{-\pi x^2}$ where $p$ is a real valued polynomial. The reason is the following: the Fourier transform of such a function is given by,

$$\mathcal{F} f(x + m) = e^{-\pi(x+m)^2} \int_{0}^{1} \sum_{n=-\infty}^{\infty} p(y + n) e^{-\pi(y+n+i(x+m))^2} dy$$

for $x \in [0, 1)$. As before we are led to consider the function

$$G(w) = \int_{-\infty}^{\infty} p(y) e^{-\pi(y+\omega)^2} dy$$

which is entire. For $v \in \mathbb{R}$,

$$G(iv) = \int_{-\infty}^{\infty} p(y + v) e^{-\pi y^2} dy$$
is a polynomial and hence $\mathcal{F} f(x) = q(x)e^{-\pi x^2}$ for another polynomial $q$. So it is reasonable to expect eigenfunctions among this class of functions. Let us record this in the following.

**Proposition 1.8.** Let $p$ be a polynomial with real coefficients. Then $f(x) = p(x)e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue $\lambda$ if and only if

$$\int_{-\infty}^{\infty} p(x - iy)e^{-\pi x^2} dx = \lambda p(y).$$

From the above equation we can infer several things. Calculating the the derivatives at the origin we have

$$(-i)^k \int_{-\infty}^{\infty} p^{(k)}(x)e^{-\pi x^2} dx = \lambda p^{(k)}(0).$$

Since $p$ is real valued and $\lambda = (-i)^n$ for $n = 0, 1, 2, 3$ the degree $m$ of $p$ should satisfy the condition $(-i)^{m+n}$ is real. This means $m$ should be odd (even) whenever $n$ is odd (resp. even). Moreover, since $\mathcal{F}^2 f(x) = f(-x)$ we infer that the polynomials $p$ corresponding to real (imaginary) eigenvalues are even (resp. odd) functions. If we assume that each $p$ is monic then we also get that $p$ should be of degree $4k + n$ if $\lambda = (-i)^n, n = 0, 1, 2, 3$. With these preparations we can easily show the existence of eigenfunctions of the Fourier transform.

**Theorem 1.9.** There exist monic polynomials $p_k$ of degree $k \in \mathbb{N}$ such that $p_k(x)e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue $(-i)^k$.

**Proof.** We consider only the case of $\lambda = 1$. The other cases can be treated similarly. In this case we have to find polynomials $p$ of degree $4k$ such that

$$\int_{-\infty}^{\infty} p(x - iy)e^{-\pi x^2} dx = p(y).$$

This leads to the equations

$$\sum_{j=0}^{4k} \frac{1}{j!} C_j (-iy)^j = \sum_{j=0}^{4k} \frac{1}{j!} p^{(j)}(0)y^j$$

where

$$C_j = \int_{-\infty}^{\infty} p^{(j)}(x)e^{-\pi x^2} dx.$$
As \( p \) is a function of \( x^2 \) it follows that \( C_j = 0 \) unless \( j \) is even. Thus we are led to the equations

\[
p^{(2j)}(0) = (-1)^j \int_{-\infty}^{\infty} p^{(2j)}(x)e^{-\pi x^2} dx.
\]

These equations can be solved recursively starting with \( p^{(4k)}(0) = (4k)! \). The details are left to the reader.

The polynomials whose existence is guaranteed by the above theorem are called the Hermite polynomials and denoted by \( H_k(x) \). We define the Hermite functions \( \varphi_k(x) = c_k H_k(x)e^{-\pi x^2} \) with suitably chosen \( c_k \) so as to make \( \| \varphi_k \|_2 = 1 \). The importance of the Hermite functions lie in the following theorem.

**Theorem 1.10.** The Hermite functions \( \varphi_k, k \in \mathbb{N} \) form an orthonormal basis for \( L^2(\mathbb{R}) \).

**Proof.** Here we only prove that they form an orthonormal system. The completeness will be proved later. Since \( (f, g) = (\hat{f}, \hat{g}) \) for all \( f, g \in L^2(\mathbb{R}) \) it follows that

\[
(\varphi_{4k+n}, \varphi_{4j+m}) = (-i)^{n-m}(\varphi_{4k+n}, \varphi_{4j+m})
\]

for any \( n, m \in \{0, 1, 2, 3\} \). Thus \( (\varphi_{4k+n}, \varphi_{4j+m}) = 0 \) whenever \( n \neq m \). This argument does not prove the orthogonality within the same eigenspace.

Consider the operator \( H = -\frac{d^2}{dx^2} + 4\pi^2 x^2 \). By integration by parts we can easily verify that \( \mathcal{F}(Hf) = H(\mathcal{F}f) \) for all functions of the form \( f(x) = p(x)e^{-\pi x^2} \) with \( p \) polynomial. The above shows that \( H\varphi_k \) is an eigenfunction of \( \mathcal{F} \) with the same eigenvalue and hence there are constants \( \lambda_k \) such that \( H\varphi_k = \lambda_k \varphi_k \). If we can show that \( \lambda_k \neq \lambda_j \) for \( k \neq j \) then we are done. To check this the equation \( Hf = \lambda_k f \) for \( f(x) = p(x)e^{-\pi x^2} \) reduces to

\[
-p^{(2)}(x) + 4\pi xp^{(1)}(x) + 2\pi p(x) = \lambda_k p(x).
\]

By direct calculation one can show that this equation has a degree \( k \) polynomial solution only for certain discrete values of \( \lambda_k \) and the \( \lambda_k \) are all distinct. The details are left to the reader. \( \square \)
We are now ready to state the explicit spectral decomposition of \( F \). For \( j = 0, 1, 2, 3 \) define \( L^2_j(\mathbb{R}) \) be the subspace of \( L^2(\mathbb{R}) \) for which \( \{ \varphi_{4k+j} : k \in \mathbb{N} \} \) is an orthonormal basis. Let \( P_j \) stand for the orthogonal projection of \( L^2(\mathbb{R}) \) onto \( L^2_j(\mathbb{R}) \).

**Theorem 1.11.** We have \( L^2(\mathbb{R}) = \bigoplus_{j=0}^{3} L^2_j(\mathbb{R}) \). Every \( f \in L^2(\mathbb{R}) \) can be uniquely written as \( f = \sum_{j=0}^{3} P_j f \). The projections are explicitly given by the Hermite expansion

\[
P_j f = \sum_{k=0}^{\infty} (f, \varphi_{4k+j}) \varphi_{4k+j}.
\]

We conclude this section with the following generalisation of the Poisson summation formula.

**Theorem 1.12.** Let \( f = \varphi_{4k+j} \) be any of the Hermite functions. Then we have

\[
\sum_{n=-\infty}^{\infty} f(y+n)e^{-2\pi i x(y+n)} = (-i)^j \sum_{n=-\infty}^{\infty} f(x+n)e^{2\pi i y n}.
\]

**Proof.** As \( F = V^*J V \) the equation \( Ff = (-i)^j f \) translates into \( J V f(x, y, t) = (-i)^j V f(x, y, t) \). This proves the theorem. \( \square \)

2. **Invariant subspaces of \( L^2(\mathbb{R}^n) \)**

In this section we define the Fourier transform for functions on \( \mathbb{R}^n \) and study some subspaces of \( L^2(\mathbb{R}^n) \) that are left invariant by the Fourier transform.

2.1. **Fourier transform on \( \mathbb{R}^n \).** In the previous section we defined the Fourier transform on functions defined on \( \mathbb{R} \). We can easily extend the definition to functions on \( \mathbb{R}^n \). Instead of \( \mathbb{H}^1 \) we consider the \((2n+1)\) dimensional group \( \mathbb{H}^n \) which is \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) with the group law

\[
(x, y, t)(x', y', t') = (x + x', y + y', t + t' + x \cdot y').
\]

Let \( \Gamma = \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z} \). Then \( \Gamma \) is a subgroup of \( \mathbb{H}^n \) so that we can form the quotient \( M = \Gamma/\mathbb{H}^n \) consisting of all right cosets of \( \Gamma \). Functions on \( M \) are naturally identified with left \( \Gamma \)-invariant functions on \( \mathbb{H}^n \). As
the Lebesgue measure $dx dy dt$ is left $\Gamma$-invariant we can form $L^2(M)$ using the Lebesgue measure restricted to $M$. As a set we can identify $M$ with $[0, 1)^{2n+1}$ and we just think of $L^2(M)$ as $L^2([0, 1)^{2n+1})$.

As before Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $H_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = e^{2\pi i ks} f(x, y, t).$$

Then $H_k$ is orthogonal to $H_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=\infty}^{\infty} f_k, \ f_k \in H_k.$$

We are mainly interested in $H_1$ which is a Hilbert space in its own right.

Define $J$ as in the one dimensional case and let $V : L^2(\mathbb{R}^n) \rightarrow H_1$ be defined by

$$V f(x, y, t) = e^{2\pi i t} \sum_{m \in \mathbb{Z}^n} f(x + m) e^{2\pi i m \cdot y}.$$

Then $V$ is unitary and we simply define $F = V^* J V$. It is then clear that $F$ is a unitary operator on $L^2(\mathbb{R}^n)$. All the results proved for functions on $\mathbb{R}$ remain true now. For example, for $f \in L^1(\mathbb{R}^n)$ we have

$$F f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}.$$

The eigenfunctions of $F$ are obtained by taking tensor products of the one dimensional Hermite functions.

For each $\alpha \in \mathbb{N}^n$ we define

$$\Phi_\alpha(x) = \varphi_{\alpha_1}(x_1) \varphi_{\alpha_2}(x_2) \cdots \varphi_{\alpha_1}(x_n).$$

Then it follows that $F \Phi_\alpha = (-i)^{|\alpha|} \Phi_\alpha$ where $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Moreover, $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Thus every $f \in L^2(\mathbb{R}^n)$ has an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} (f, \Phi_\alpha) \Phi_\alpha.$$
the series being convergent in the $L^2$ sense. Defining $P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha) \Phi_\alpha$ we can write the above in the compact form $f = \sum_{k=0}^\infty P_k f$. Note that $\mathcal{F}(P_k f) = (-i)^k P_k f$.

2.2. The Schwartz space. Our first example of an invariant subspace of $L^2(\mathbb{R}^n)$ is provided by the the space of Schwartz class functions. First we make

**Definition 2.1.** A subspace $W$ of $L^2(\mathbb{R}^n)$ is said to be invariant under $\mathcal{F}$ if $\mathcal{F} f \in W$ whenever $f \in L^2(\mathbb{R}^n)$.

As $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ it follows that $f \in L^2(\mathbb{R}^n)$ if and only if $\sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_\alpha)|^2 < \infty$. Since $\mathcal{F} f, \Phi_\alpha = (-i)^{\|\alpha\|} (f, \Phi_\alpha)$ any subspace of $L^2(\mathbb{R}^n)$ defined in terms of the behaviour of $|\mathcal{F} f, \Phi_\alpha|$ will be invariant under the Fourier transform. We can define a whole family of invariant subspaces. Indeed, for each $s > 0$ define $W^s_H(\mathbb{R}^n)$ to be the subspace of $L^2(\mathbb{R}^n)$ consisting of functions $f$ for which

$$
\|f\|_{2,s}^2 = \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^s |(f, \Phi_\alpha)|^2 < \infty.
$$

We let $\mathcal{S}(\mathbb{R}^n) = \cap_{s>0} W^s_H(\mathbb{R}^n)$. This is called the Schwartz space, members of which are called Schwartz functions.

**Theorem 2.2.** The Schwartz space has the following properties. (i) $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^2(\mathbb{R}^n)$; (ii) $\mathcal{S}(\mathbb{R}^n)$ is invariant under $\mathcal{F}$ and (iii) $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is one to one and onto.

The density follows from the fact that finite linear combinations of Hermite functions form a subspace of $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$. The invariance follows from that of each of $W^s_H(\mathbb{R}^n)$. As $\mathcal{F}(\mathcal{F}^* f) = f$ the surjectivity is proved.

The Schwartz space can be made into a locally convex topological vector space such that $\mathcal{S}(\mathbb{R}^n)$ is continuously embedded in $W^s_H(\mathbb{R}^n)$ for every $s > 0$. The dual of $\mathcal{S}(\mathbb{R}^n)$ denoted by $\mathcal{S}'(\mathbb{R}^n)$ is called the space of tempered distributions. It can be shown that $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only
if
\[ \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{-s}|(f, \Phi_\alpha)|^2 < \infty \]
for some \( s > 0 \). The Fourier transform has a natural extension to \( \mathcal{S}'(\mathbb{R}^n) \) given by \((\mathcal{F}u, f) = (u, \mathcal{F}f)\) where the brackets here stand for the action of a tempered distribution on a Schwartz function.

2.3. **Weighted Bergman spaces.** From the definition it follows that \( f \in \mathcal{S}(\mathbb{R}^n) \) if and only if for every \( s > 0 \), there are constants \( C_s \) such that
\[ |(f, \Phi_\alpha)| \leq C_s(2|\alpha| + n)^{-s}, \ \alpha \in \mathbb{N}^n. \]
It is natural to consider conditions of the form
\[ |(f, \Phi_\alpha)| \leq Ce^{-2s|\alpha| + n}^s, \ \alpha \in \mathbb{N}^n \]
for some \( s > 0 \). This will lead to another family of invariant subspaces which can be identified with certain Hilbert spaces of entire functions.

For each \( t > 0 \) let us consider the weight function
\[ U_t(x, y) = 2^n (\sinh(4t))^{-\frac{n}{2}}e^{2\pi(\tanh(2t)|x|^2 - \coth(2t)|y|^2)}. \]
We define \( \mathcal{H}_t(\mathbb{R}^n) \) to be the subspace of \( L^2(\mathbb{R}^n) \) consisting all \( f \) which extend to \( \mathbb{C}^n \) as entire functions and satisfy
\[ \|f\|_{\mathcal{H}_t}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 U_t(x, y) dxdy < \infty. \]
These are examples of weighted Bergman spaces. We call them Hermite-Bergman spaces for reasons which will become clear soon.

**Theorem 2.3.** For each \( t > 0 \) the space \( \mathcal{H}_t(\mathbb{R}^n) \) is invariant under the Fourier transform.

Since \( \Phi_\alpha(x) = H_\alpha(x)e^{-\pi|x|^2} \) where \( H_\alpha \) is a polynomial we can extend \( \Phi_\alpha \) to \( \mathbb{C}^n \) as an entire function simply by setting \( \Phi_\alpha(z) = H_\alpha(z)e^{-\pi z^2} \) where \( z^2 = \sum_{j=1}^n z_j^2, z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n \). The reader can verify by direct calculation that \( \Phi_\alpha \in \mathcal{H}_t(\mathbb{R}^n) \) for any \( t > 0 \). Moreover, it can be shown that the functions \( \Phi_\alpha(x) = e^{-(2|\alpha| + n)^s}\Phi_\alpha(x) \) form an orthonormal system in \( \mathcal{H}_t(\mathbb{R}^n) \). More is true.
Theorem 2.4. The family \( \{ \Phi_\alpha : \alpha \in \mathbb{N}^n \} \) is an orthonormal basis for \( \mathcal{H}_t(\mathbb{R}^n) \).

We will not attempt a proof of this theorem but only indicate a major step involved in the proof. Before that let us see how this theorem can be used to prove the invariance of \( \mathcal{H}_t(\mathbb{R}^n) \). Any \( f \in \mathcal{H}_t(\mathbb{R}^n) \) has an expansion

\[
f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \Phi_\alpha
\]

where the sequence \( c_\alpha \) is square summable. It is now obvious that

\[
\mathcal{F} f = \sum_{\alpha \in \mathbb{N}^n} (-i)^{\|\alpha\|} c_\alpha \Phi_\alpha
\]

also belongs to \( \mathcal{H}_t(\mathbb{R}^n) \).

Let \( \mathcal{B}_t(\mathbb{R}^n) \) be the Bergman space consisting of all \( f \in L^2(\mathbb{R}^n) \) which extend to \( \mathbb{C}^n \) as entire functions that are square integrable with respect to the weight function

\[
p_{t/2}(y) = e^{-\frac{t}{4} |y|^2}.
\]

That is, \( f \in \mathcal{B}_t(\mathbb{R}^n) \) if and only if \( f(x + iy) \) is entire and

\[
\|f\|_{\mathcal{B}_t(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy < \infty.
\]

We have the following characterisation of these Bergman spaces.

Theorem 2.5. A function \( f \in \mathcal{B}_t(\mathbb{R}^n) \) if and only if \( f = g * p_t \) for some \( g \in L^2(\mathbb{R}^n) \). Moreover,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy = c \int_{\mathbb{R}^n} |g(x)|^2 dx.
\]

Proof. When \( f = g * p_t \) so that \( \mathcal{F} f(\xi) = \mathcal{F} g(\xi) e^{-2\pi t |\xi|^2} \) the inversion formula shows that the prescription

\[
f(x + iy) = \int_{\mathbb{R}^n} \mathcal{F} g(\xi) e^{-2\pi t |\xi|^2} e^{2\pi i (x + iy) \cdot \xi} d\xi
\]

gives an entire extension of \( f \). Moreover, by Plancherel theorem

\[
\int_{\mathbb{R}^n} |f(x + iy)|^2 dx = \int_{\mathbb{R}^n} |\mathcal{F} g(\xi)|^2 e^{-4\pi t |\xi|^2} e^{-4\pi y \cdot \xi} d\xi.
\]
Integrating both sides against $p_{t/2}(y)$ and simplifying we get
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dxdy = c \int_{\mathbb{R}^n} |g(x)|^2 dx. \]
As all the steps are reversible we get the theorem. □

2.4. **Spherical harmonics.** In this subsection we look for some more eigenfunctions of the Fourier transform which have some symmetry. As in the one dimensional case we consider functions of the form $f(x) = p(x)e^{-\pi|x|^2}$. This will be an eigenfunction of $\mathcal{F}$ if and only if $p$ satisfies
\[ \int_{\mathbb{R}^n} p(x - iy)e^{-\pi|x|^2} dx = \lambda p(y). \]
If this is true for all $y \in \mathbb{R}^n$ then we should also have
\[ \int_{\mathbb{R}^n} p(x + y)e^{-\pi|x|^2} dx = \lambda p(iy). \]
Integrating in polar coordinates the integral on the left is
\[ \int_0^\infty |S^{n-1}| \left( \int_{S^{n-1}} p(y + r\omega)d\sigma(\omega) \right) e^{-\pi r^2} r^{n-1} dr \]
where $d\sigma$ is the normalised surface measure on the unit sphere $S^{n-1}$. If $p$ is homogeneous of degree $m$ then $p(iy) = i^m p(y)$ and hence for such polynomials the equation
\[ \int_{\mathbb{R}^n} p(x + y)e^{-\pi|x|^2} dx = \lambda i^m p(y) \]
will be satisfied for $\lambda = (-i)^m$ if $p$ has the mean value property
\[ \int_{S^{n-1}} p(y + r\omega)d\sigma(\omega) = p(y). \]
Such functions are precisely the harmonic functions satisfying $\Delta u = 0$. Thus we have proved

**Theorem 2.6.** Let $f(x) = p(x)e^{-\pi|x|^2}$ where $p$ is homogeneous of degree $m$ and harmonic. Then $\mathcal{F} f = (-i)^m f$.

Let $\mathcal{P}_m$ stand for the finite dimensional space of homogeneous harmonic polynomials of degree $m$. The above theorem says that the finite dimensional subspace of $L^2(\mathbb{R}^n)$ consisting of functions of the form
$p(x)e^{-\pi|x|^2}, p \in \mathcal{P}_m$ is invariant under the Fourier transform. We claim that the following extension is true.

**Theorem 2.7.** Let $f \in L^2(\mathbb{R}^n)$ be of the form $f(x) = p(x)g(|x|)$ where $p \in \mathcal{P}_m$. Then $\mathcal{F}f(\xi) = p(\xi)G(|\xi|)$. Thus the subspace of functions of the form $f(x) = p(x)g(|x|), p \in \mathcal{P}_m$ is invariant under the Fourier transform.

**Proof.** When $f(x) = p(x)g(|x|), p \in \mathcal{P}_m$ is from $L^2$ the function $g$ satisfies

$$\int_0^\infty |g(r)|^2 r^{n+2m-1}dr < \infty.$$ 

We let $\mathcal{D}_{n+2m}$ to stand for the space of all such functions with the obvious norm. We claim that the subspace $W$ consisting of finite linear combinations of $e^{-\pi t|x|^2}$ as $t$ runs through positive reals is dense in $\mathcal{D}_{n+2m}$. To see this suppose $g \in \mathcal{D}_{n+2m}$ satisfies

$$\int_0^\infty e^{-\pi tr^2}g(r)r^{n+2m-1}dr = 0$$

for all $t > 0$. Differentiating the integral $k$ times at $t = 1$ we get

$$\int_0^\infty e^{-\pi tr^2}r^{2k}g(r)r^{n+2m-1}dr = 0$$

for all $k \in \mathbb{N}$. Thus the function $g(r)r^{n+2m-1}e^{-\frac{1}{2}\pi r^2}$ is orthogonal to all functions of the form $P(r^2)e^{-\frac{1}{2}\pi r^2}$ where $P$ runs through all polynomials. As this is a dense class in $L^2((0, \infty), dr)$ we get $g = 0$.

In view of this density, it is enough to prove that $W$ is invariant under Fourier transform. But this is easy to check. For $t > 0$ we let $f_t(x) = t^n f(tx)$ so that $\mathcal{F}(f_t)(\xi) = \mathcal{F}f(t^{-1}\xi)$. If $f(x) = p(x)e^{-\pi t^2|x|^2}$ take $g(x) = p(x)e^{-\pi|x|^2}$ and consider

$$\mathcal{F}(f)(\xi) = t^{-n-m}\mathcal{F}(g_t)(\xi) = t^{-n-m}\mathcal{F}(g)(t^{-1}\xi).$$

Since $\mathcal{F}g(\xi) = (-i)^mg(\xi)$ we get

$$\mathcal{F}(f)(\xi) = t^{-n-2m}(-i)^mp(x)e^{-\pi t^{-2}|x|^2}.$$ 

This proves that $W$ is invariant and hence the theorem follows. $\square$
The above theorem gives rise to an operator $T^n_m$ on the space $D_{n+2m}$ defined as follows. If $g \in D_{n+2m}$ then for $p \in P_m$ the function $p(x)g(|x|) \in L^2(\mathbb{R}^n)$ whose Fourier transform is of the form $p(x)G(|x|)$. As $F$ is unitary it follows that $G \in D_{n+2m}$. We define $T^n_m g = G$. Note that $k T^n_m g k = k g k$ where $k \cdot k$ is the norm in $D_{n+2m}$. We can think of $g(|x|)$ as a radial function on $\mathbb{R}^{n+2m}$ whose $n+2m$ dimensional Fourier transform will be a radial function, say $G_0(|x|)$. We define another operator $T^{n+2m}_{n+2m}$ on $D_{n+2m}$ by letting $T^{n+2m}_{n+2m} g = G_0$. It is also clear that $k T^{n+2m}_{n+2m} g k = k g k$. If we denote the Fourier transform on $\mathbb{R}^n$ by $F_n$ then $T^{n+2m}_{n+2m} = F_{n+2m}$. Calculations done in the proof of the above theorem shows that $T^n_m g = (-i)^m T^{n+2m}_{n+2m} g$ whenever $g \in W$. The density of this subspace gives

**Theorem 2.8.** Let $f \in L^2(\mathbb{R}^n)$ be of the form $f(x) = p(x)g(|x|), p \in P_m$. Then $F_n(f) = (-i)^m pF_{n+2m}g$.

The above result is known as the Hecke-Bochner formula for the Fourier transform. We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. Let $J_\alpha$ stand for the Bessel function of type $\alpha > -1$.

**Theorem 2.9.** If $f(x) = g(|x|)$ is radial and integrable then

$$F_n(f)(\xi) = c_n \int_0^\infty g(r) \frac{J_{\frac{n}{2}-1}(2\pi r|\xi|)}{(2\pi r|\xi|)^{\frac{n}{2}+1}} r^{n-1} dr.$$ 

**Proof.** As $f$ is radial

$$F_n(f)(\xi) = |S^{n-1}| \int_0^\infty g(r) \left( \int_{S^{n-1}} e^{-2\pi i r \omega \cdot \xi} d\sigma(\omega) \right) r^{n-1} dr.$$ 

The inner integral is clearly a radial function as the measure $d\sigma$ is rotation invariant. It can be shown that the inner integral is a constant multiple of $\frac{J_{\frac{n}{2}-1}(2\pi r|\xi|)}{(2\pi r|\xi|)^{\frac{n}{2}+1}}$. This completes the proof. 

3. **ULTRAVARIANT SUBSPACES OF $L^2(\mathbb{R}^n)$**

In the previous section we studied several subspaces of $L^2(\mathbb{R}^n)$ that are invariant under $F$. But not all subspaces are invariant. For example,
$L^1 \cap L^2(\mathbb{R}^n)$ is not invariant under $\mathcal{F}$. In this section we are interested in subspaces which are extremely sensitive to the action of the Fourier transform. We make this precise in the following definition.

**Definition 3.1.** *We say that a subspace $W$ of $L^2(\mathbb{R}^n)$ is ultravariant if the conditions $f \in W, \mathcal{F}f \in W$ imply $f = 0$.***

A priori it is not clear if there is any ultravariant subspace of $L^2(\mathbb{R}^n)$ but in this section we show that there are many such subspaces.

### 3.1. Paley-Wiener spaces.

Our first example of an ultravariant subspace is the Paley-Wiener space defined as follows. For each $a > 0$ let $PW_a(\mathbb{R}^n)$ stand for the subspace of $L^2(\mathbb{R}^n)$ consisting of functions having entire extensions to $\mathbb{C}^n$ and satisfying

$$\int_{\mathbb{R}^n} |f(x + iy)|^2 dx \leq Ce^{4\pi a|y|}$$

for all $y \in \mathbb{R}^n$. We define $PW(\mathbb{R}^n) = \cup_{a>0} PW_a(\mathbb{R}^n)$ and call it the Paley-Wiener space. The space $PW(\mathbb{R}^n)$ is not empty since any $f \in L^2(\mathbb{R}^n)$ whose Fourier transform is compactly supported belongs to the Paley-Wiener space. To see this, suppose $\mathcal{F}f$ vanishes for $|\xi| > a$ and consider the inversion formula

$$f(x) = \int_{|\xi| \leq a} \mathcal{F}f(\xi)e^{2\pi i x \cdot \xi} d\xi.$$

It is clear then that the prescription

$$f(x + iy) = \int_{|\xi| \leq a} \mathcal{F}f(\xi)e^{2\pi i (x + iy) \cdot \xi} d\xi$$

defines an entire function and by Plancherel we also have

$$\int_{\mathbb{R}^n} |f(x + iy)|^2 dx \leq Ce^{4\pi a|y|}.$$

We show below that the converse is also true.

**Theorem 3.2.** *An $L^2$ function $f$ belongs to $PW_a(\mathbb{R}^n)$ if and only if $\mathcal{F}f$ is supported in $\{\xi : |\xi| \leq a\}$.***
Proof. It is enough to show that $Ff$ is compactly supported in $\{\xi : |\xi| \leq a\}$ whenever $f \in PW_a(\mathbb{R}^n)$ since the converse has been already proved. First we claim that $PW(\mathbb{R}^n) \subset \cap_{t>0} \mathcal{B}_t(\mathbb{R}^n)$. To see this let $f \in PW_a(\mathbb{R}^n)$ and consider

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy \leq C t^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{4\pi t |y|^2} dy.$$ 

A simple calculation shows that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy \leq C (a^2 t)^n e^{4\pi a^2 t}.$$ 

This proves our claim. In view of Theorem 2.5 we get $g_t \in L^2(\mathbb{R}^n)$ such that $f = g_t \ast p_t$ and

$$\int_{\mathbb{R}^n} |g_t(x)|^2 dx = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy.$$ 

For each $\delta > 0$ consider

$$\int_{|\xi| > a + \delta} |\hat{f}(\xi)|^2 d\xi \leq e^{-4\pi t(a + \delta)^2} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 e^{4\pi t |\xi|^2} d\xi.$$ 

As $f = g_t \ast p_t$ the last integral is equal to

$$\int_{\mathbb{R}^n} |g_t(x)|^2 dx = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x + iy)|^2 p_{t/2}(y) dx dy.$$ 

This along with our earlier estimate gives

$$\int_{|\xi| > a + \delta} |\hat{f}(\xi)|^2 d\xi \leq C (a^2 t)^n e^{-4\pi t(a + \delta)^2} e^{4\pi t a^2}.$$ 

Letting $t$ go to infinity we conclude that $\hat{f}$ vanishes for $|\xi| > a + \delta$. Since $\delta$ is arbitrary we see that $\hat{f}$ is supported in $|\xi| \leq a$. \qed

**Theorem 3.3.** $PW(\mathbb{R}^n)$ is ultravariant.

The theorem follows immediately from the Paley-Wiener theorem. If $f \in PW(\mathbb{R}^n)$ then $\hat{f}$ is compactly supported and hence cannot have an entire extension unless $f = 0$.

The Paley-Wiener space is strictly contained in $\cap_{t>0} \mathcal{B}_t(\mathbb{R}^n)$. It turns out that the bigger space $\cap_{t>0} \mathcal{B}_t(\mathbb{R}^n)$ is also ultravariant. Even more
surprising is the following result. Note that the heat kernel \( p_s \in \mathcal{B}_t(\mathbb{R}^n) \) only for \( t < s \) since

\[
\mathcal{F}p_s(x) = e^{-2\pi s|x|^2} = (2s)^{-\frac{n}{2}} p_s^{1/4}(x).
\]

It is also clear that \( \mathcal{F}p_s \in \mathcal{B}_t(\mathbb{R}^n) \) only for \( t < \frac{1}{4s} \). Therefore, if \( 0 < t < 1/2 \) then for any \( s \) satisfying \( t < s < 1/4t \) the function \( p_s \) and \( \mathcal{F}p_s \) both belong to \( \mathcal{B}_t(\mathbb{R}^n) \). This means that for such \( t \) the space \( \mathcal{B}_t(\mathbb{R}^n) \) is not ultravariant. But the behaviour is different for other values of \( t \).

**Theorem 3.4.** The Bergman space \( \mathcal{B}_t(\mathbb{R}^n) \) is ultravariant for all \( t \geq 1/2 \).

This is a very special case of a theorem of Cowling and Price which is viewed as an uncertainty principle for the Fourier transform. There is a more general theorem due to Bonami et al from which Cowling-Price theorem can be deduced. We just state the result without proof.

**Theorem 3.5.** The only function \( f \in L^2(\mathbb{R}^n) \) for which

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)||\mathcal{F}f(y)|e^{2\pi|x-y|}dxdy < \infty
\]

is the trivial function \( f = 0 \).

In the one dimensional case this result is due to Beurling. Let us use this to prove the ultravariance of \( \mathcal{B}_t(\mathbb{R}^n) \) for \( t > 1/2 \). If both \( f \) and \( \mathcal{F}f \) are in \( \mathcal{B}_t(\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}} |g(x)|^2 e^4\pi|x|^2 dx < \infty
\]

for \( g = f \) as well as for \( g = \mathcal{F}f \). It is then easy to check that the hypothesis of Beurling’s theorem is satisfied.

**3.2. Theta transform and Hardy’s theorem.** In this section we return to the Hilbert space \( L^2(M) \) introduced in section 1.1. We introduce and study a transform called the theta transform. As applications we show that the Hermite functions form an orthonormal basis for \( L^2(\mathbb{R}) \) and prove a theorem of Hardy.
Let \( \varphi_{\iota x}(x) = e^{\pi \iota x^2} \) which belongs to \( L^2(\mathbb{R}) \) even for complex \( \tau \) provided \( \Im(\tau) > 0 \). Let \( \psi_{\iota x}(x) = \frac{1}{2\pi \iota x} \frac{\partial}{\partial x} \varphi_{\iota x}(x) = x \varphi_{\iota x}(x) \). Recall that for \( f \in L^2(\mathbb{R}) \) the Weil-Brezin transform is given by

\[
V f(x, y, t) = e^{2\pi it} \sum_{n=-\infty}^{\infty} f(x + n)e^{2\pi iny}.
\]

**Definition 3.6.** The theta transform is defined on \( L^2(M) \) by

\[
\Theta(F, \tau) = (V \varphi_{\iota x}, F) = \int_{M} V \varphi_{\iota x}(g) \overline{F}(g) dg.
\]

We also define \( \Theta^*(F, \tau) = (V \psi_{\iota x}, F) \).

Since \( V \) is a unitary operator we get the formulas

\[
\Theta(V \tilde{f}, \tau) = \int_{-\infty}^{\infty} f(x)e^{\pi \iota x^2} dx
\]

and

\[
\Theta^*(V \tilde{f}, \tau) = \int_{-\infty}^{\infty} x f(x)e^{\pi \iota x^2} dx.
\]

Note that \( \Theta(F, \tau) \) and \( \Theta^*(F, \tau) \) are functions defined on the upper half-plane \( \mathbb{R}^2_+ = \{ \tau \in \mathbb{C} : \Im(\tau) > 0 \} \).

**Theorem 3.7.** For \( F \in L^2(M) \) both \( \Theta(F, \tau) \) and \( \Theta^*(F, \tau) \) are holomorphic in the upper half-plane.

**Proof.** It is clear that \( \Theta(F, \tau) \) is holomorphic in the upper half-plane when \( F \in C^{\infty}(M) \). If \( F_n \in C^{\infty}(M) \) converges to \( F \in L^2(M) \) then \( \Theta(F_n, \tau) \) converges to \( \Theta(F, \tau) \) uniformly over compact subsets of the upper half-plane. This follows from the fact that \( V(\varphi_{\iota x}) \) is bounded when \( \tau \) is restricted to compact subsets. This shows that \( \Theta(F, \tau) \) is holomorphic. The proof for \( \Theta^*(F, \tau) \) is the same. \( \square \)

We can decompose \( \mathcal{H}_1 \) as \( \mathcal{H}_1^o \oplus \mathcal{H}_1^e \) where \( \mathcal{H}_1^o \) (resp. \( \mathcal{H}_1^e \)) is the image under \( V \) of all odd (even) functions. Note that \( V \varphi_{\iota x} \in \mathcal{H}_1^o \) and \( V \psi_{\iota x} \in \mathcal{H}_1^e \). Moreover, \( \Theta(F, \tau) = 0 \) if \( F \in \mathcal{H}_1^o \) and \( \Theta^*(F, \tau) = 0 \) if \( F \in \mathcal{H}_1^e \). We can now prove the following uniqueness theorem for the theta transform.
Theorem 3.8. For $F \in \mathcal{H}_1$, $F = 0$ if and only if $\Theta(F, \tau) = \Theta^*(F, \tau) = 0$ for all $\tau \in \mathbb{R}^2_+$. Consequently, the set of all functions $\{V\varphi_{i\tau}, V\psi_{i\tau}, \tau \in \mathbb{R}^2_+\}$ is dense in $\mathcal{H}_1$.

Proof. If $F = G + H$ where $G \in \mathcal{H}^e_1$ and $H \in \mathcal{H}^o_1$ then $\Theta(F, \tau) = \Theta(G, \tau)$. Therefore, $\Theta(F, \tau) = 0$ for all $\tau$ gives, by taking $\tau = t + i$,
\[
\int_0^{\infty} g(x)e^{-\pi x^2} e^{\pi itx} dx = 0
\]
where $G = V(\tilde{g})$. By making a change of variables we get
\[
\int_0^{\infty} g(s^\frac{1}{2})s^{-\frac{1}{2}}e^{-\pi s} e^{ist} ds = 0.
\]
As the integrand belongs to $L^1(\mathbb{R})$ by the uniqueness theorem for the Fourier transform we get $g = 0$. Similarly, the other condition $\Theta^*(F, \tau) = 0$ gives $h = 0$. Hence the theorem. \qed

Corollary 3.9. The Hermite functions $\{\varphi_k : k \in \mathbb{N}\}$ form an orthonormal basis for $L^2(\mathbb{R})$.

Proof. It is enough to show that the set of all functions $\{t^ne^{-\pi t^2} : n \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$. Suppose $f$ is orthogonal to all these functions. Let $F = V(f)$ and consider $\theta(\tau) = \Theta(F, \tau)$. Evaluating the derivatives of $\theta$ at $\tau = i$ we get
\[
\theta^{(n)}(i) = \int_{-\infty}^{\infty} f(t)t^{2n}e^{-\pi t^2} dt = 0.
\]
As $\theta$ is holomorphic we get $\Theta(F, \tau) = 0$. As before, if $F = G + H, G \in \mathcal{H}^e_1, H \in \mathcal{H}^o_1$ we have $\Theta(G, \tau) = \Theta(F, \tau) = 0$ and $\Theta^*(G, \tau) = 0$. Hence $G = 0$. This means that $f$ is odd. Working with $\Theta^*V(f)$ we can also conclude that $f$ is even. Hence $f = 0$ proving the corollary. \qed

We now use properties of the theta transform to prove a result on Fourier transform pairs due to Hardy. This result will be used to construct some more examples of invariant and ultravariant subspaces.

Theorem 3.10. Suppose $f \in L^2(\mathbb{R})$ satisfies the growth conditions
\[
|f(x)| \leq Ce^{-\pi tx^2}, \quad |Ff(y)| \leq Ce^{-\frac{\pi}{4}y^2}
\]
for some $t > 0$. Then $f(x) = Ce^{-\pi tx^2}$. 


Proof. By dilating by \( t \) we can assume that \( t = 1 \). Recalling the definitions of \( \varphi_{i\tau} \) and \( \psi_{i\tau} \) we can easily calculate that

\[
\Theta(V\varphi_{-1}, \tau) = (1 - i\tau)^{-\frac{1}{2}}, \quad \Theta^*(V\psi_{-1}, \tau) = (1 - i\tau)^{-\frac{3}{2}}.
\]

Given \( f \) as in the theorem we can write it as \( f = g + h \), where \( g(x) = \frac{1}{2}(f(x) + f(-x)) \) and \( h(x) = \frac{1}{2}(f(x) - f(-x)) \). Observe that both \( g \) and \( h \) satisfy the same growth conditions as \( f \). We therefore, prove the theorem for even and odd functions separately.

If \( f \) is even, consider the decomposition \( f = g + h \) where \( g = \frac{1}{2}(f + \mathcal{F}f) \) and \( h = \frac{1}{2}(f - \mathcal{F}f) \). Then \( \mathcal{F}g = g, \mathcal{F}h = -h \) and both satisfy the conditions of the theorem. If \( f \) is odd the decomposition \( g = \frac{1}{2}(f + i\mathcal{F}f) \) and \( h = \frac{1}{2}(f - i\mathcal{F}f) \) gives \( f = g + h \) with \( \mathcal{F}g = -ig \) and \( \mathcal{F}h = ih \). This shows that we can assume without loss of generality \( f \) is an eigenfunction of \( \mathcal{F} \). We start with the even case, \( \mathcal{F}f = cf \) where \( c = 1 \) or \(-1 \).

We consider the function \( \theta(\tau) = \Theta(V\bar{f}, \tau) \) which is given by the integral

\[
\alpha(\tau) = \int_{-\infty}^{\infty} f(x)e^{\pi i\tau x^2} \, dx.
\]

The growth condition on \( f \) shows that \( \alpha(\tau) \) is holomorphic in \( \Im(\tau) > -1 \). Since

\[
\Theta(V\bar{f}, \tau) = (V\varphi_{i\tau}, V\bar{f}) = (\varphi_{i\tau}, \bar{f})
\]

using the result \((f, g) = (\mathcal{F}f, \mathcal{F}g)\) we get

\[
\Theta(V\bar{f}, \tau) = (-i\tau)^{-\frac{1}{2}}(\varphi_{-1}, \bar{f}) = c(-i\tau)^{-\frac{1}{2}}\alpha(-\frac{1}{\tau}).
\]

In the above calculation we have used the facts that \( \mathcal{F}\varphi_{i\tau} = (-i\tau)^{-\frac{1}{2}}\varphi_{-1} \) and \( \mathcal{F}\bar{f} = c\bar{f} \). Therefore, \( \alpha \) satisfies \( \alpha(\tau) = c(-i\tau)^{-\frac{1}{2}}\alpha(-\frac{1}{\tau}) \) provided both \( \Im(\tau) > -1 \) and \( \Im(-\frac{1}{\tau}) > -1 \).

Define a new function \( \beta(\tau) = (1 - i\tau)^{\frac{1}{2}}\alpha(\tau) \). If we can show that \( \beta(\tau) \) is a constant which means that \( \alpha(\tau) = C\Theta(V\varphi_{-1}, \tau) \) then by the uniqueness theorem for the theta transform we get \( f = \varphi_{-1} \). This will take care of the even case.
An easy calculation shows that the function $\beta$ satisfies $\beta(\tau) = c\beta(\frac{1}{\tau})$ whenever both $\Im(\tau) > 1$ and $\Im(\frac{1}{\tau}) > 1$. If $\tau = a + ib$, $b < -1$, then $-b = |b| < b^2 + a^2$ so that $\Im(\frac{1}{\tau}) = \frac{b}{(a^2 + b^2)} > -1$. Therefore, $\beta$ can be extended to the entire complex plane except possibly $\tau = -i$. With $\tau = a + ib$ we have the estimate $|\alpha(\tau)| \leq C(1 + b)^{-\frac{1}{2}}, b > -1$ which follows from the integral defining $\alpha$ and the hypothesis on $f$. When $a^2 + b^2 \leq 1$ writing $b = -1 + \delta, \delta > 0$ we have $a^2 \leq 1 - b^2 = \delta(2 - \delta)$. This gives the estimate

$$|(1 - i\tau)\beta(\tau)| \leq C(1 + b)(1 + \frac{|a|}{1+b})^\frac{3}{2}$$

$$\leq C\delta(1 + 2\delta^{-\frac{1}{2}})^\frac{3}{2} \leq C\delta^\frac{3}{2}(2 + \delta^{-\frac{1}{2}})^\frac{3}{2}.$$ 

This together with the property $\beta(\tau) = c\beta(\frac{1}{\tau})$ shows that $(1 - i\tau)\beta(\tau)$ tends to zero as $\tau$ goes to $-i$. Hence $\beta$ is entire. It is also bounded (since $\beta(\tau) = c\beta(\frac{1}{\tau})$ ) and hence reduces to a constant.

When $f$ is an odd eigenfunction of the Fourier transform we work with $(1 - i\tau)^{\frac{3}{2}}\Theta^*(Vf, \tau)$ and show that $f(x) = Cxe^{-\pi x^2}$. But now the constant $C$ has to be zero if the growth condition on $f$ is satisfied. Thus the odd componet of $f$ is zero proving the theorem.

The above theorem can be generalised and extended to functions on $\mathbb{R}^n$. The following result is known as the Hardy’s uncertainty principle for the Fourier transform.

**Theorem 3.11.** Let $f \in L^2(\mathbb{R}^n)$ satisfy the estimates

$$|f(x)| \leq Cp(x)e^{-\pi t|x|^2}, \ |\mathcal{F}f(y)| \leq Cq(y)e^{-\pi s|y|^2}$$

for some $s, t > 0$ and $p, q$ polynomials. Then $f = 0$ whenever $st > 1$ and $f(x) = Cp(x)e^{-\pi t|x|^2}$ when $st = 1$.

**Proof.** First we consider the case $st > 1$ in one dimension. We can choose $\epsilon, \delta > 0$ such that $(s - \delta)(t - \epsilon) = 1$. Then we have the estimates

$$|f(x)| \leq C_{\epsilon}e^{-\pi(t-\epsilon)x^2}, \ |\mathcal{F}f(y)| \leq C_\delta e^{-\pi(s-\delta)y^2}.$$
By the previous theorem we conclude that \( f(x) = Ce^{-\pi(t-\epsilon)x^2} \). But this cannot satisfy the hypothesis unless \( C = 0 \) proving that \( f = 0 \).

To prove the theorem in \( n \) dimensions, we fix a vector \( \omega \in S^{n-1} \) and consider the function defined on \( \mathbb{R} \) by

\[
f_\omega(v) = \int_{\mathbb{R}^{n-1}} f(u \oplus v\omega) \, du.
\]

Then it is easy to see that

\[
\int_{\mathbb{R}} f_\omega(v)e^{-2\pi i vr} \, dv = \hat{f}(r\omega).
\]

Thus the function \( f_\omega \) and its Fourier transform satisfy the hypothesis with \( st > 1 \) and hence \( \hat{f}(r\omega) = 0 \). As this is true for all \( \omega \) we conclude that \( f = 0 \).

To prove the equality case \( st = 1 \) with polynomial factors, again we need to consider the one dimensional case. The proof of Theorem can be modified to take care of this case. The details are left to the reader. \( \square \)

Hardy’s theorem gives us one more example of ultravariant space. For each \( t > 0 \) consider the Hardy class \( H_t(\mathbb{R}^n) \) consisting of functions satisfying \( |f(x)| \leq Cp(x)e^{-\pi t|x|^2} \) for some polynomial. Then we have

**Theorem 3.12.** The Hardy class \( H_t(\mathbb{R}^n) \) is ultravariant for all \( t > 1 \).

Since the Hermite functions \( \Phi_\alpha \in H_t(\mathbb{R}^n) \) for all \( t \leq 1 \) it follows that \( H_t(\mathbb{R}^n) \) is not ultravariant for \( t \leq 1 \). We do not know if it is invariant or not.
Notes and references

There are several excellent books on Fourier transform and related topics. We recommend Dym and McKea [4], Stein and Weiss [5] and Sugiura [6]. We have followed Auslander and Meyer [1] and Howe [2] in introducing the Fourier transform via Heisenberg group. For more about this group we refer to Folland [3] and Thangavelu [7]. For the weighted Bergman spaces the reader may consult the CIMPA lecture notes [9]. The proof of Hardy’s theorem presented here is due to Tolimieri [10]. For other proofs and extensions of Hardy’s theorem we refer to Thangavelu [8].

References