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Root numbers of Asai L -functions.

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The main result of this paper is the computation of the epsilon factor $\varepsilon(\frac{1}{2}, \pi, r, \psi)$, where π is a square integrable representation of $\mathrm{GL}(n)$ over a p -adic field E and r is the Asai L -function. Recall that, if E/F is a quadratic extension, π is a representation of the L -group

$${}^L[R_{E/F}\mathrm{GL}(n)] = \mathrm{GL}_n(\mathbf{C}) \times \mathrm{GL}_n(\mathbf{C}) \rtimes \mathrm{Gal}(E/F),$$

which is in fact equivalent to the adjoint action of the L -group on the Lie algebra of the unipotent radical of a certain parabolic subgroup P of the unitary group $\mathrm{U}(n, n)$. It is then clear, from the Langlands-Shahidi philosophy, that the Asai L -function occurs in the constant term of certain Eisenstein series on $\mathrm{U}(n, n)$.

Theorem 1.2 of the paper states that, if E/F is a quadratic extension of number fields and π is irreducible, cuspidal, and conjugate self-dual, then $\varepsilon(\frac{1}{2}, \pi, r) = 1$.

The main idea of the proof is to make use of an ingenious technique developed by E. M. Lapid and S. J. Rallis [Ann. of Math. (2) **157** (2003), no. 3, 891–917; [MR1983784 \(2004d:11039\)](#)]: let $I(\pi, s)$ be the normalized induced representation $\mathrm{Ind}_P^G(\pi \otimes |\cdot|^s)$, and let $M(s, \pi)$ be the usual intertwining operator, associated to the longest element in the Weyl group. Shahidi has explained how to define a normalized intertwining operator $R(s, \pi)$ which is holomorphic and nonvanishing for $\mathrm{Re}(s) \geq 1/2$. The crucial step is then to prove that the Hermitian form on $I(s, \pi)$, defined by $(R(s, \pi)\varphi, \varphi')$, is semi-positive definite (here (\cdot, \cdot) is a naturally defined sesquilinear form). From this, it follows quite easily that $\varepsilon(\frac{1}{2}, \pi, r) = 1$. The technical heart of the proof is actually contained in [op. cit.] and is only sketched here.

Theorem 1.1 provides a formula for the local epsilon factor $\varepsilon(\frac{1}{2}, \pi, r, \psi)$, where E/F is a quadratic extension of p -adic fields, and π is irreducible, square integrable and conjugate self-dual.

The proof is somewhat technical and exploits, among others, results of C. J. Bushnell and G. M. Henniart [Bull. London Math. Soc. **31** (1999), no. 5, 534–542; [MR1703873 \(2000f:22020\)](#)], and of Frölich and Queyrut.

In the final section of the paper, the author makes the conjecture that $\varepsilon(\frac{1}{2}, \pi_1 \times \pi_2, \psi) = 1$, for irreducible representations π_i of $\mathrm{GL}_{n_i}(E)$, distinguished with respect to $\mathrm{GL}_{n_i}(F)$. He then provides some evidence for this conjecture. When $n_1 = 2, n_2 = 1$, the conjecture is a theorem [J. L. Hakim, Duke Math. J. **62** (1991), no. 1, 1–22; [MR1104321 \(92c:22037\)](#)].

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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