Research Summary

H. Ananthnarayan

∞

My research interests lie in Commutative Algebra, the study of commutative rings and modules over them. The problems I have worked on include the study of Gorenstein rings via Gorenstein colength (sections 2, 3 and 4) and connected sums (section 7). Other projects include work on 3-standardness of the maximal ideal in a local ring (section 8) and the study of minimal reductions and multiplicities (sections 9 and 10) and a simplification of the proof of a theorem of A. Braun (section 11). I am currently involved in studying various aspects of fiber products of rings and establishing connections with connectedness theorems and exploring links to the theory of Gorenstein liaison (section 12).

1. Gorenstein Rings: A Brief Introduction

An *Artinian* local ring R is the quotient of a regular local ring $(T, \mathfrak{m}_T, \mathsf{k})$ (e.g., the ring of formal power series in d variables, $\mathsf{k}[|X_1, \ldots, X_d|]$ over a field k where $\mathfrak{m}_T = (X_1, \ldots, X_d)$) by an $\mathfrak{m}_T$-primary ideal I (e.g., an $\mathfrak{m}_T$-primary ideal in $\mathsf{k}[|X_1, \ldots, X_d|]$ is one which, for each i, contains $X_i^{n_i}$ for some n_i). We say that I is a defining ideal of R in T. In particular, R is said to be a *complete intersection* when I is generated by d elements. The generating set of such an ideal is said to be a *system of parameters*.

One can study Artinian local rings by looking at their duals or *canonical modules*. The canonical module ω_R of an Artinian local ring $(R, \mathfrak{m}, \mathsf{k})$ is isomorphic to the *injective hull* of the residue field k over R. When the canonical module ω_R is cyclic, we say that R is a *Gorenstein Artin* or a *zero-dimensional Gorenstein* local ring. It is a well known that $(R, \mathfrak{m}_R, \mathsf{k})$ is a Gorenstein Artin local ring if and only if the k-vector space $\operatorname{ann}_R(\mathfrak{m}_R)$ is 1-dimensional. In such a case, one can see that $R \simeq \omega_R$. Complete intersections are examples of Gorenstein rings.

Many properties of Gorenstein rings in higher dimensions can be studied by going modulo a *maximal regular sequence* and reducing it to the study of Gorenstein Artin rings.[1]

2. Gorenstein Colength

A problem I have worked on involves approximating Artinian local rings by Gorenstein Artin local rings using the notion of *Gorenstein colength*. Part of my work deals with studying bounds on this number and the other involves constructing Gorenstein Artin local rings. Techniques used include studying inverse systems, maps from the injective hull of the residue field to the ring and self-dual ideals, the Hoskin-Deligne formula, the strong Lefschetz property and fibre products and connected sums of Gorenstein Artin local rings.

Let T be a commutative Noetherian ring and I an ideal in T such that $R := T/I$ is *Cohen-Macaulay*.[2] A problem of interest to many mathematicians is to find Gorenstein rings S such that R is a quotient of S. We are interested not only in finding such a Gorenstein ring, but also find one as "close" to R as possible. More specifically, the problem we are interested in is: Given an Artinian local ring $(R, \mathfrak{m}, \mathsf{k})$, how close can we get to R using Gorenstein Artin local rings mapping onto R?

For an Artinian local ring R, there is a notion of length, denoted $\lambda(R)$, which is finite. (If R is an Artinian quotient of $\mathsf{k}[|X_1, \ldots, X_d|]$, then $\lambda(R) = \dim_{\mathsf{k}}(R)$.) Thus we can make the idea

[1] A sequence $x_1, \ldots, x_r$ in $\mathfrak{m}_R$ is regular if x_i is a non-zerodivisor on R_{i-1}, where $R_0 = R$ and $R_i = R_{i-1}/(x_i)$.
[2] A Noetherian local ring $(R, \mathfrak{m}_R)$ is Cohen-Macaulay if R modulo a maximal regular sequence is Artinian.

of "being close" precise using length. We define the Gorenstein colength of R, denoted $\mathrm{gcl}(R)$ to be

$$\mathrm{gcl}(R) = \min\{\lambda(S) - \lambda(R) : S \text{ is a Gorenstein Artin local ring mapping onto } R\}.$$

The number $\mathrm{gcl}(R)$ gives a numerical value to how close one can get to an Artinian ring R by a Gorenstein Artin local ring.

The main questions one would like to answer are the following: Given an Artinian local ring $(R, \mathfrak{m}, \mathsf{k})$, (a) how does one intrinsically compute $\mathrm{gcl}(R)$ and (b) how does one construct a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = \mathrm{gcl}(R)$?

The answer to part (b) is not an easy one, since the structure of Gorenstein Artin local rings is quite mysterious. The process of classifying Gorenstein Artin local rings is far from complete. For example, in a recent paper ([8]), J. Elias and G. Valla classify the structure of Gorenstein Artin local rings with Hilbert function (1,2,2,2,1,1,1). In another paper, Elias and M. E. Rossi([7]) show that Gorenstein Artin local rings with Hilbert function (1,h,n,1) all arise in similar fashion. It is interesting to note that these are examples of *connected sums* (see section 7).

When both R and S are graded, we can use tools like the Hilbert function to give some lower bounds on $\mathrm{gcl}(R)$. However, in the non-graded case, the problem is much more complex. One way of studying Gorenstein Artin local rings is to look at a dual polynomial ring, an approach used by A. Iarrobino in the construction of compressed Gorenstein Artin algebras (see section 5).

The main tool we use is the fact that canonical module ω_R of R, is an ideal in any Gorenstein Artin ring S mapping onto R. In fact, the image of the canonical module $\mathfrak{a} \subseteq R$ under the surjective map $S \twoheadrightarrow R$ is a *self-dual* ideal, i.e., $\mathfrak{a} \simeq \mathrm{Hom}_R(\mathfrak{a}, \omega_R)$. In particular, we get the following bounds on $\mathrm{gcl}(R)$ in [1]:

$$\boxed{\lambda(R/\omega_R^*(\omega_R)) \leq \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is self-dual}\} \leq \mathrm{gcl}(R) \leq \lambda(R),}$$

where $\omega^*(\omega) = \langle f(\omega) : f \in \mathrm{Hom}_R(\omega_R, R)\rangle$.

3. Gorenstein Colength and Self-dual Ideals

One of the first results relating to Gorenstein colength can be found in Teter's paper([27]). Teter gives a criterion for writing a given Artinian local ring as the quotient of a Gorenstein Artin local ring by its socle (which is the annihilator of its maximal ideal). Since the length of the socle of a Gorenstein Artin ring is 1, this theorem tells us when $\mathrm{gcl}(R) \leq 1$.

Teter needs an awkward technical condition in his theorem which Huneke and Vraciu remove in [14]. However, they need to assume 2 is invertible and that $\mathrm{ann}_R(\mathfrak{m}) \subseteq \mathfrak{m}^2$. Using connected sums, we can show in [3] that the condition on $\mathrm{ann}_R(\mathfrak{m})$ is not necessary (see Theorem 8).

We extend Teter's theorem and the Huneke-Vraciu theorem, and as a consequence, can characterize rings R with $\mathrm{gcl}(R) \leq 2$. We prove the following in [1]:

Theorem 1 *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring and I an $\mathfrak{m}_T$-primary ideal. Let $R = T/I$, $^-$ denote going modulo I and $(\_)^\vee = \mathrm{Hom}_R(\_, \omega)$, where ω is the canonical module of R. Assume that $I \subseteq \mathfrak{m}_T^6$ and 2 is invertible in R. Then the following are equivalent:*
1) $\mathrm{gcl}(R) \leq 2$.
2) *There exists an ideal $\bar{\mathfrak{a}} \subseteq R$, $\lambda(R/\bar{\mathfrak{a}}) \leq 2$ such that $\bar{\mathfrak{a}}$ is self-dual.*

Thus, we see that with some mild hypothesis, $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is self-dual}\} = \mathrm{gcl}(R)$ when $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is self-dual}\} \leq 2$ or when $\mathrm{gcl}(R) \leq 3$.

A natural question at this juncture is the following:

Question 2 Given any Artinian local ring $(R, \mathfrak{m}, \mathsf{k})$, is $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is self-dual}\} = \text{gcl}(R)$?

This question has a positive answer in some cases which can be seen as a corollary to Theorem 5. A stronger question one can ask is:

Question 3 Given a self-dual ideal $\mathfrak{a}$ in R, is $\text{gcl}(R) \leq \lambda(R/\mathfrak{a})$? More specifically, is there a Gorenstein Artin local ring S such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$?

We answer the above question in [1] when $R/\mathfrak{a}$ is an *algebra retract* of R.[3]

4. Computing Gorenstein Colength

One can easily see that $\text{ann}_R(\mathfrak{m}) \subseteq R$ is a self-dual ideal. So a particular case of the Question 3 is:

Question 4 Is $\text{gcl}(R) \leq \lambda(R/\text{ann}_R(\mathfrak{m}))$?

We give a positive answer to Question 4 in [2] in two cases.
(a) Let $R = T/\mathfrak{b}$ where T is a 2-dimensional regular local ring with infinite residue field and $\mathfrak{b}$ is primary to the maximal ideal of T. We use a formula of Hoskin and Deligne (e.g., See [18]) to show that Question 4 has a positive answer in this case.
(b) In general, given an Artinian local ring $(R, \mathfrak{m})$, it is not easy to (i) determine what $\text{gcl}(R)$ is and (ii) find a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = \text{gcl}(R)$. We have been able to do both when $R = T/\mathfrak{m}_T^n$, where T is a polynomial ring over a field of characteristic zero and $\mathfrak{m}_T$ is its unique homogeneous maximal ideal.

This construction is a key ingredient in giving a positive answer to Question 4 when $R = T/\mathfrak{b}$ where T is a power series ring over a field of characteristic zero and $\mathfrak{b}$ an ideal that is the power of a system of parameters. We deduce this as a corollary to the following:

Theorem 5 *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field k of characteristic zero. Let $f_1, \ldots, f_d$ be a system of parameters in T and $R = T/(f_1, \ldots, f_d)^n$. Then $\text{gcl}(R) = \lambda(T/(f_1, \ldots, f_d)^{n-1}) \leq \lambda(R/\text{ann}_R(\mathfrak{m}))$.*

Thus, in particular, we see that Question 2 has a positive answer for $R \simeq T/\mathfrak{m}_T^n$.

5. Compressed Gorenstein Algebras

Let S be a graded Gorenstein Artin quotient of $T = \mathsf{k}[X_1, \ldots, X_d]$ such that $S_j \simeq \mathsf{k}$ and $S_i = 0$ for $i > j$, where S_i represents the i-th graded piece of S. We say that S has *socle degree* j. The ring S is said to be a *compressed Gorenstein algebra* if $\dim_{\mathsf{k}}(S_i) = H_S(i)$ is the maximum possible for each i given the number of variables d and the socle degree j. Compressed algebras have been extensively studied by Iarrobino([15]), Fröberg and Laksov([9]). We use a different approach to construct a compressed Gorenstein Artin algebra in [2]. Using a result of Reid, Roberts and Roitman([24]) which guarantees the existence of a linear form in $T/(X_1^n, \ldots, X_d^n)$ satisfying the *strong Lefschetz property*, we prove the following

[3]We say $R/\mathfrak{a}$ is an algebra retract of R if there is a subring R' of R such that $\pi \circ i$ is an isomorphism, where $i : R' \longrightarrow R$ is the inclusion and $\pi : R \longrightarrow R/\mathfrak{a}$ is the natural projection.

Theorem 6 *Let $T = \mathsf{k}[X_1, \dots, X_d]$ be a polynomial ring over k, where $\mathrm{char}(\mathsf{k}) = 0$. The ring $S = T/((X_1^n, \dots, X_d^n) :_T (X_1 + \cdots + X_d)^{(n-1)(d-2)-i})$ is a compressed Gorenstein Artin algebra with socle degree $2n - 2 + i$, where $i = 0, 1$.*[4]

6. The Glicci Problem

An ideal $I \subseteq T$ is said to be in the *linkage class of a complete intersection (licci)* if there is a sequence of ideals $J_n \subseteq I_n$, $I_0 = I$, satisfying (1) T/J_n is a complete intersection for every n, (2) $I_n = J_{n-1} :_T I_{n-1}$ and $I_{n-1} = J_{n-1} :_T I_n$ and (3) I_n is a complete intersection for some n. We say that I is *linked* to I_n via complete intersections in n steps.

We say that an ideal $I \subseteq T$ is said to be in the *Gorenstein linkage class of a complete intersection (glicci)* if we replace condition (1) above by the condition: T/J_n is Gorenstein for every n.

There are ideals which are glicci but not licci. A result of Huneke and Ulrich in [13] shows that $\mathfrak{m}^n \subseteq \mathsf{k}[X_1, \dots, X_d]$ is not licci for $d \geq 3$, $n \geq 2$.

The Glicci problem: Given any homogeneous ideal $I \subseteq T := \mathsf{k}[X_1, \dots, X_d]$, such that $R := T/I$ is Cohen-Macaulay, is I glicci?

Remark 7

1. A possible approach is the following: Choose $J_n \subseteq I_n$ to be the "closest" Gorenstein ideal. The question is: Does this ensure that I_n is a complete intersection for some n? We show in [2] that this strategy does work for $I = (f_1, \dots, f_d)^n \subseteq T = \mathsf{k}[X_1, \dots, X_d]$ when k is a field of characteristic zero, where $f_1, \dots, f_d$ is a regular system of parameters.
2. Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring. Let $\mathfrak{d} = (f_1, \dots, f_d)$, where $f_1, \dots, f_d$ form a system of parameters. In [19], Kleppe, Migliore, Miro-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. The technique used in [2] can be used to show that $\mathfrak{d}^n$ can be linked directly via the Gorenstein ideal $(f_1^n, \dots, f_d^n) :_T (f_1 + \cdots + f_d)^{(d-2)(n-1)}$ to $\mathfrak{d}^{n-1}$, and hence to $\mathfrak{d}$, a complete intersection, in $n - 1$ steps.
3. In a private conversation, Migliore asked if this technique will show that $\mathfrak{d}^n$ is self-linked. We see that $\mathfrak{d}^n$ is linked to itself via the Gorenstein ideal $(f_1^n, \dots, f_d^n) :_T (f_1 + \cdots + f_d)^{(d-2)(n-1)-1}$.

7. Connected Sums

In joint work with Avramov and Moore, we introduce, study, and apply a new construction of local Gorenstein rings in [3]. The starting point is the classical *fiber product* $R \times_T S$ of a pair of surjective homomorphisms $\varepsilon_R \colon R \to T \leftarrow S : \varepsilon_S$ of local rings.[5] It is well known that this ring is local. If R, S, and T are Cohen-Macaulay of dimension d, then so is $R \times_T S$, but this ring is Gorenstein only in trivial cases.

Our main construction involves, in addition to the ring homomorphisms ε_R and ε_S, a T-module V and homomorphisms $\iota_R \colon V \to R$ of R-modules and $\iota_S \colon V \to S$ of S-modules, for the structures induced through ε_R and ε_S, respectively. When these maps satisfy $\varepsilon_R \iota_R = \varepsilon_S \iota_S$, we define a *connected sum* ring by the formula

$$R \#_T S = (R \times_T S)/\{(\iota_R(v), \iota_S(v)) \mid v \in V\}\,.$$

In case R, S, and T have dimension d (for some $d \geq 0$), R and S are Gorenstein, T is Cohen-Macaulay, and V is a canonical module for T, one can choose ι_R and ι_S to be isomorphisms

[4] If I and J are ideals in T, $(I :_T J)$ represents the ideal $\{f \in T : fJ \subseteq I\}$.

[5] $R \times_T S = \{(r, s) \in R \times S : \varepsilon_R(r) = \varepsilon_S(s)\}$

onto $(0 : \mathrm{Ker}(\varepsilon_R))$ and $(0 : \mathrm{Ker}(\varepsilon_S))$, respectively. We prove that if $\varepsilon_R\iota_R = \varepsilon_S\iota_S$ holds, then $R\#_T S$ is Gorenstein of dimension d.

As an application, we obtain new estimates for the Gorenstein colength of Artinian rings which are fiber products. We use them in the proof of Theorem 8 to remove a restrictive hypothesis from a result of the Huneke-Vraciu Theorem mentioned in section 3. We prove the following:

Theorem 8 *Let $(R, \mathfrak{m}_R, \mathsf{k})$ be an Artinian local ring with $1/2 \in R$. Then $\mathrm{gcl}(R) \leq 1$ if and only if either R is Gorenstein or $\mathfrak{m}_R$ is self-dual.*

Some questions for further research are:

Question 9
When is a given Gorenstein ring indecomposable as a connected sum?

We give a partial answer in [3], but the complete characterization is unknown.

8. Three-standardness of the Maximal Ideal

Let $(T, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field k and I be an $\mathfrak{m}$-primary ideal. An ideal $J \subseteq I$ is called a *minimal reduction* of I if it is minimal (under inclusion) with respect to the property that $I^n = JI^{n-1}$ for some n. We define the *multiplicity* of I, $e(I) = \lambda(T/J)$. We say that I is *n-standard w.r.t.* J if $I^k \cap J = I^{k-1}J$, $k = 1, \ldots, n$ and define I to be *n-standard* if it is n-standard w.r.t. to all its minimal reductions.

It is well known that when $(T, \mathfrak{m}, \mathsf{k})$ is Cohen-Macaulay, $\mathfrak{m}^2 \cap J = \mathfrak{m}J$ for any minimal reduction J of $\mathfrak{m}$. In an attempt to see when $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3)$ is independent of J, part of my work with Huneke in [4] was focussed on understanding when $\mathfrak{m}$ is 3-standard, i.e., studying when the condition $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ is true. Using results about the existence of big Cohen-Macaulay algebras in characteristic p (e.g., see [10]), the following result can be proved.

Theorem 10 *Let $(T, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring such that $\mathrm{char}(\mathsf{k}) = p > 0$. If the associated graded ring of the maximal ideal, $\mathfrak{G} = \oplus_{i\geq 0}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ is a domain, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ for every minimal reduction J of $\mathfrak{m}$.*

The conditions on the associated graded ring can be weakened by assuming that $\mathfrak{G}$ is *reduced* and *connected in codimension 1.*

One can then use the *method of reduction to prime characteristic* and prove the above theorem when $\mathrm{char}(\mathsf{k}) = 0$, under the assumption that $\mathfrak{G}$ is an *absolute domain.*[6]

9. Invariance of a Length Associated to a Reduction

In [23], T. Puthenpurakal proved that in a Cohen-Macaulay local ring $(T, \mathfrak{m}, \mathsf{k})$ if J is a minimal reduction of $\mathfrak{m}$, then the length $\lambda(\mathfrak{m}^3/\mathfrak{m}^2 J)$ is independent of J. In [4], in joint work with C. Huneke, we prove a similar formula for any $\mathfrak{m}$-primary ideal I that is n-standard. We prove:

Theorem 11 *Let $(T, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with an infinite residue field k. If I is an $\mathfrak{m}$-primary ideal and J is a minimal reduction of I such that I is n-standard with respect to J, then*

$$\lambda(I^{k+1}/JI^k) = e(I) + \sum_{i=0}^{k} (-1)^{i+1} \binom{d-1}{i} \lambda(I^{k-i}/I^{k-i+1}). \quad \textit{for } 0 \leq k \leq n \quad (\sharp)$$

[6]We say a k-algebra $\mathfrak{G}$ is an absolute domain if $\mathfrak{G} \otimes_{\mathsf{k}} \overline{\mathsf{k}}$ is a domain, where $\overline{\mathsf{k}}$ is an algebraic closure of k.

Remark 12

a) If $\mathfrak{G} = \sum_{j=0}^{\infty} I^j/I^{j+1}$ is Cohen-Macaulay, then by the work of Valabrega and Valla([28]), I is n-standard for all n. Hence the above formula holds for all k.

b) If I is n-standard w.r.t. J and $\lambda(I^{n+1}/I^nJ) \leq 1$, then by combining works of Marley([21]) and Rossi([25]), one can see that the above formula holds for all k.

c) If I is an integrally closed $\mathfrak{m}$-primary ideal, then I is 2-standard (e.g., see [12] and [17]). Hence $\lambda(I^3/JI^2)$ is independent of J for integrally closed ideals I.

d) If either $\mathfrak{G}$ is an absolute domain or $\text{char}(\mathsf{k}) = p > 0$ and $\mathfrak{G}$ is reduced and connected in codimension 1, then we show in [4] that $\mathfrak{m}$ is 3-standard. Hence $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3)$ does not depend on J.

Some questions for further research are:

Question 13

a) When is an integrally closed $\mathfrak{m}$-primary ideal 3-standard?

b) When is $\mathfrak{m}$ 4-standard?

Rossi([25]) shows that if I is n-standard w.r.t. J and $\lambda(I^{n+1}/JI^n) \leq 1$, then $\text{depth}(\mathfrak{G}) \geq \dim(T) - 1$. Furthermore, if I is n-standard w.r.t. J and $\lambda(I^{n+1}/I^nJ) = 0$, then, by [28], we see that $\mathfrak{G}$ is Cohen-Macaulay, i.e., $\text{depth}(\mathfrak{G}) = \dim(T)$, which leads us to the following question:

Question 14 *Is there any relation between* $\text{depth}(\mathfrak{G})$ *and* $\lambda(I^{n+1}/JI^n)$*, where I is n-standard w.r.t. J?*

10. Lech's Inequality

Let $(T, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field k, I be an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I. It is clear that $\lambda(T/I) \leq \lambda(T/J) = e(I)$. In [20], C. Lech shows that when T is a regular local ring, $e(I) \leq (d!)\lambda(T/I)$.

Asymptotically, this is the best possible bound since $\lim_{n \leftarrow \infty} \frac{(d!)\lambda(T/\mathfrak{m}^n)}{e(\mathfrak{m}^n)} = 1$. However, in general, this is a very weak bound. The following question was raised by C. Huneke.

Question 15 (Huneke) *Let T be a regular local ring with infinite residue field and I an ideal primary to the maximal ideal of T. Is* $e(I) + \binom{d}{2} e(I)^{(d-1)/d} \leq (d!)\lambda(T/I)$*?*

In general, the answer to Question 15 is not known. In unpublished work, J. Validashti and I have shown that the inequality holds in some cases.

a) Using the formula of Hoskin and Deligne mentioned earlier, we show that the above inequality holds when $d = 2$.

b) The inequality is easily seen to be true in any dimension for powers of the maximal ideal. When T contains a coefficient field k, if $\mathfrak{m} = (x_1, \ldots, x_d)$ and $f_1, \ldots, f_d$ is a system of parameters, using the fact that $\mathsf{k}[|x_1, \ldots, x_d|]$ is flat over $\mathsf{k}[|f_1, \ldots, f_d|]$, we can show that the inequality also holds for all the powers of $(f_1, \ldots, f_d)$.

c) The algebra package Macaulay 2 was also used to verify the inequality in Question 15 for a large number of monomial ideals in a polynomial ring over a field.

11. On a Question of Auslander (after A. Braun)

In joint work with J. Striuli, we simplify the proof of a theorem of A. Braun([5]) which gives a positive answer to the following question of M. Auslander:

Question 16 (Auslander) *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, M a finitely generated reflexive R-module of finite projective dimension. If $\operatorname{Hom}_R(M, M)$ is free, then is M free?*

Braun looks at $\operatorname{Hom}_R(M, M)$ as a non-commutative ring and uses techniques from non-commutative algebra such as Brauer groups and Azumaya algebras to answer Auslander's question. His proof of the theorem is based on studying central simple algebras inside the matrix ring $M_{\beta_i}(k)$, where β_i is the ith betti number of the module M.

In unpublished work, Striuli and I simplify the proof by reducing the problem to that of an indecomposable module over a complete local ring. Using liftings of homomorphisms in free resolutions, we study division rings contained in $M_{\beta_i}(k)$. We prove and use the fact that $\operatorname{Hom}(M, M)$ has a unique maximal ideal $\mathfrak{m}\operatorname{Hom}(M, M)$ and that $\operatorname{Hom}(M, M)/\mathfrak{m}\operatorname{Hom}(M, M)$ is a division ring. The idea of lifting idempotents reduces the proof to the case where M has rank 1.

12. Current Work: Fiber Products

Consider the fiber product $R \times_T S$ of a pair of surjective homomorphisms $\varepsilon_R \colon R \to T \leftarrow S : \varepsilon_S$ of local rings. When $\varepsilon_R = \varepsilon_S$, D'Anna([6]) and Shapiro([26]) proposed and partly proved a criterion for $R \times_T R$ to be Gorenstein. In [3], we complete and strengthen their results by proving: $R \times_T R$ is Gorenstein if and only if R is Cohen-Macaulay and $\operatorname{Ker}(\varepsilon_R)$ is a canonical module for R.

In current work with E. Celikbas and Z. Yang, we are studying various homological and connectedness properties of $P = R \times_T S$. In particular, we have shown that:

Theorem 17 *Let P be a Noetherian ring, K a ideal in P. Then $\operatorname{Spec}(P) \backslash V(K)$ is disconnected if and only if $P \simeq R \times_T S$, where $\sqrt{K} = \sqrt{\operatorname{ann}_P(T)}$.*[7]

In particular this establishes a connection of fiber products with the notion of being *connected in codimension i* and with *local cohomology*.[8]

Some questions for further research are:
a) By a theorem of Hochster and Huneke([11]), it is known that if $\dim(P) = d$, then the top local cohomology module $H^d_{\mathfrak{m}}(P)$ is indecomposable if and only if P is connected in codimension 1, i.e., whenever $P \simeq R \times_T S$, then $\operatorname{ht}(\operatorname{ann}_P(T)) \leq 1$, which leads us to the following question:

Question 18 *Is there a connection between some of the local cohomology modules $H^j_{\mathfrak{m}}(P)$ being indecomposable and P being connected in codimension i?*

b) T. Ogoma([22]) gives a characterization for a fiber product ring to be Gorenstein, but needs to assume that $\ker(\varepsilon_R)$ (and hence $\ker(\varepsilon_S)$) contains a non-zerodivisor.

We have shown that if R and S are Cohen-Macaulay local rings of dimension d, then $R \times_T S$ is Cohen-Macaulay if and only if $\operatorname{depth}(T) \geq d-1$. But in general, we do not know the complete answer to the following questions:

Question 19
i) When is the fiber product Cohen-Macaulay?
ii) When is the fiber product Gorenstein?

[7] We define $V(K) = \{\mathfrak{p} \in \operatorname{Spec}(P) : K \subseteq \mathfrak{p}\}$ and $\sqrt{K} = \{x \in P : x^n \in K \text{ for some } n \geq 0\}$.
[8] We say that P is connected in codimension i if $\operatorname{Spec}(P) \setminus V(K)$ being disconnected forces $\operatorname{ht}(K) \leq i$.

c) One can also see that if $(R, \mathfrak{m}_R, \mathsf{k})$ and $(S, \mathfrak{m}_S, \mathsf{k})$ are graded Gorenstein Artin local rings with different socle degrees, then the associated graded ring $\mathrm{gr}_{\mathfrak{m}_Q}(Q)$ is a fiber product, where $Q = R\#_{\mathsf{k}}S$. A natural question that arises, which we are currently investigating is: Is the converse true? i.e.,

Question 20 *Suppose $(Q, \mathfrak{m})$ is graded Gorenstein Artin local ring such that its associated graded ring $\mathrm{gr}_{\mathfrak{m}}(Q)$ is a non-trivial fiber product. Can Q be written as a non-trivial connected sum?*

d) Another related project I am interested in is to extend some of the work of Iarrobino and H. Srinivasan in [16] to the non-graded case.

References

[1] H. Ananthnarayan, *The Gorenstein Colength of an Artinian Local Ring,* Journal of Algebra, **320** (2008), 3438-3446.

[2] H. Ananthnarayan, *Computing Gorenstein Colength,* Journal of Commutative Algebra, **1**, no. 3, (2009), 343-359. .

[3] H. Ananthnarayan, L. L. Avramov, W. F. Moore, *Connected Sums of Gorenstein Rings,* Crelle J. (to appear)

[4] H. Ananthnarayan, C. Huneke, *Three-standardness of the maximal ideal*, Journal of Pure and Applied Algebra, **215** vol. 11 (2011), 2674-2683.

[5] A. Braun, *On a question of M. Auslander*, J. Algebra **276** (2004), 674-684.

[6] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306** (2006), 507-519.

[7] J. Elias, M. E. Rossi, *Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system*, Trans. Amer. Math. Soc. (to appear).

[8] J. Elias and G. Valla, *Structure theorems for certain Gorenstein ideals*, Michigan Math. J. **57** (2008), 269-292..

[9] R. Fröberg and D. Laksov, *Compressed Algebras,* Conference on Complete Intersections in Acireale, Lecture Notes in Mathematics, **No. 1092** (1984), 121-151, Springer Verlag.

[10] M. Hochster, C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras,* Annals Math., **135** (1992), 53-89.

[11] M. Hochster, C. Huneke, *Indecomposable canonical modules and connectedness*, Contemporary Mathematics, **159** (1994) 197208.

[12] C. Huneke, *Hilbert Functions and Symbolic Powers*, Michigan Math. J., **34** (1987) 293-318.

[13] C. Huneke, B. Ulrich, *Liaison of Monomial Ideals,* Bull. London Math. Soc., **39** (2007), 384-392.

[14] C. Huneke, A. Vraciu, *Rings which are almost Gorenstein,* Pacific J. Math., **225** (2006) no. 1, 85 - 102.

[15] A. Iarrobino, *Compressed Algebras: Artin algebras having given socle degrees and maximal length,* Trans. Amer. Math. Soc., **285** (1984), 337-378.

[16] A. Iarrobino, H. Srinivasan, *Artinian Gorenstein algebras of embedding dimension four: Components of $\mathbb{P}Gor(H)$ for H=(1,4,7,..., 1)*, Journal of Pure and Applied Algebra, **201** (2005), 62-96.

[17] S. Itoh, *Integral Closures of Ideals Generated by Regular Sequences*, J. Algebra, **117** (1988), 390-401.

[18] B. Johnston, J. K. Verma, *On the length formula of Hoskin and Deligne and associated graded rings of two-dimensional regular local rings,* Math. Proc. Cambridge Phil. Soc., **111** (1992), 423 - 432.

[19] J. Kleppe, J. Migliore, R.M. Miro-Roig, U. Nagel, C. Peterson, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness,* Mem. Amer. Math. Soc., **154** (2001), no. 732, viii+116 pp.

[20] C. Lech, *Note on multiplicities of ideals,* Ark. Mat., **4** (1960), 6386.

[21] T. Marley, *Hilbert Functions of Ideals in Cohen-Macaulay Rings*, Ph.D. Thesis, Purdue University (1989).

[22] T. Ogoma, *Fibre Products of Noetherian Rings*, Adv. Stud. Pure Math, **11** (1987), 173-182.

[23] T. Puthenpurakal, *Invariance of a Length Associated to a Reduction,* Communications in Algebra, **33** (2005), 2039-2042.

[24] L. Ried, L. Roberts, M. Roitman, *On complete intersections and their Hilbert functions,* Canadian Mathematical Bulletin, **34** (1991), 525-535.

[25] M. E. Rossi, *Primary ideals with good associated graded ring ,* Journal of Pure and Applied Algebra, **145** (2000), 75-90.

[26] J. Shapiro, *On a construction of Gorenstein rings proposed by M. D'Anna*, J. Algebra **323** (2010), 1155-1158.

[27] W. Teter, *Rings which are a factor of a Gorenstein ring by its socle,* Invent. Math., **23** (1974), 153-162.

[28] P. Valabrega, G. Valla, *Form Rings and Regular Sequences*, Nagoya Math. J., **72** (1978), 93-101.