A Proof of Fundamental Theorem of Algebra Through Linear Algebra due to Derksen

Anant R. Shastri

February 13, 2011
We present a proof of the Fundamental Theorem of Algebra (FTA)

Every non-constant polynomial in one variable with coefficients in \( \mathbb{C} \) has a root in \( \mathbb{C} \).

The proof uses elementary linear algebra except that we begin with the intermediate value theorem in proving that every odd degree real polynomial has a real root.
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MA106-Linear Algebra 2011

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through a sequence of easily do-able exercises. The proof uses elementary linear algebra except that we begin with
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Based on an article in
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In what follows $\mathbb{K}$ will denote any field. However, we need worry about only two cases here: $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. 
We begin with a basic result in real analysis:

Intermediate Value Theorem immediately yields:

Every odd degree polynomial $p(t) \in \mathbb{R}[t]$ has a real root.

It should be noted that there is no purely algebraic proof of FTA which does not use IVT. Indeed, all proofs of FTA use IVT explicitly or implicitly. The simple reason is that IVT is equivalent to any other axiom that is used in the construction of real numbers.

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- Indeed, all proofs of FTA use IVT explicitly or implicitly. The simple reason is that IVT is equivalent to any other axiom that is used in the construction of real numbers.
- From now onwards we only use linear algebra.
Companion Matrix Let

\[ p(t) = t^n + a_1 t^{n-1} + \cdots + a_n \]

be a monic polynomial of degree \( n \). Its companion matrix \( C_p \) is defined to be the \( n \times n \) matrix

\[
C_p = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 \\
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Ex.1. Show that the characteristic polynomial of \( C_p \) is \( p \), i.e., \( \det(C_p - tl_n) = (-1)^n p(t) \).
Example 2. Show that every non constant polynomial $p(t) \in \mathbb{K}[t]$ of degree $n$ has a root in $\mathbb{K}$ iff every endomorphism $\mathbb{K}^n \to \mathbb{K}^n$ has an eigenvalue in $\mathbb{K}$.

Recall that by an endomorphism of a vector space $V$, we mean a linear transformation $V \to V$. Example 3. Show that every $\mathbb{R}$-linear map $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ has a real eigenvalue.
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Ex. 3 Show that every $\mathbb{R}$-linear map $f : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ has a real eigenvalue.
For positive integers $r$, let $S_1(\mathbb{K}, r)$ denote the following statement:

Every endomorphism $A : \mathbb{K}^n \to \mathbb{K}^n$ has an eigenvector for all $n$ not divisible by $2^r$.

Also let $S_2(\mathbb{K}, r)$ denote the statement

Any two commuting endomorphisms $\mathbb{K}^n \to \mathbb{K}^n$ have a common eigenvector for all $n$ not divisible by $2^r$. 
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Sketch of Proof

▶ (a) $S_1(K, r) \iff S_2(K, r), r \geq 1$. 

▶ Begin with IVT which is the same as $S_1(R, 1)$.

Putting $K = R$ in (a) we get $S_2(R, 1)$.

Now (b) gives $S_1(C, 1)$ and repeated application of (c) gives $S_1(C, k)$ for all $k \geq 1$.

▶ Given $n$, write $n = 2^k \ell$ where $\ell$ is odd. Then $S_1(C, k+1)$ implies that every polynomial of degree $n$ has a root. And that completes the proof of FTA.
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- (c) $S_1(\mathbb{C}, r) \implies S_1(\mathbb{C}, r + 1), r \geq 1$.

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So assume that the statement is true for smaller values of \( n \) and let \( A, B \) be two commuting \( n \times n \) matrices over \( \mathbb{K} \), and \( n \) is not divisible by \( 2^r \).
Let $\lambda$ be an eigenvalue of $A$ and put $V_1 = \mathcal{N}(A - \lambda I), V_2 = \mathcal{R}(A - \lambda I)$. Since $B$ commutes with $A$, it follows that $B$ commutes with $A - \lambda I$ also. Therefore $B$ restricts to endomorphisms $\beta : V_1 \to V_1$ and $\beta : V_2 \to V_2$.
(Refer: Tut. sheet 9.18.)
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By rank-nullity theorem,

$$\dim V_1 + \dim V_2 = n$$

and hence $\dim V_1$ or $\dim V_2$ is not divisible by $2^r$. 
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If $\dim V_1$ is not divisible by $2^r$ then $\beta : V_1 \to V_1$ has an eigenvector which is also the eigenvector for $A$.

If not, then $\dim V_2 < n$ and not divisible by $2^r$.

Now we consider $\alpha, \beta : V_2 \to V_2$ which are restrictions of $A$ and $B$ and hence are commuting endomorphisms.

By induction we are through there is a common eigenvector $v \in V_2 \subset V$ for $\alpha$ and $\beta$ which is then a common eigenvector for $A$ and $B$ as well.
Ex. 5 Show that the space $\text{HERM}_n(\mathbb{C})$ of all complex Hermitian $n \times n$ matrices is a $\mathbb{R}$ vector space of dimension $n^2$.
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Recall that a matrix \( A \) is hermitian if it is equal to its transpose conjugate \( A^* \).
and the next one is from 8.6.
Ex. 6 Given \( A \in M_n(\mathbb{C}) \), the mappings
\[
\alpha_A(B) = \frac{1}{2}(AB + BA^*); \quad \beta_A(B) = \frac{1}{2i}(AB - BA^*)
\]
define \( \mathbb{R} \)-linear maps \( \text{HERM}_n(\mathbb{C}) \rightarrow \text{HERM}_n(\mathbb{C}) \).
Show that \( \alpha_A, \beta_A \) commute with each other.
Answer:

$$4v \alpha_A \circ \beta_A(B)$$
$$= 2\alpha_A(AB - BA^*)$$
$$= [A(AB - BA^*) + (AB - BA^*)A^*]$$
$$= [A(AB + BA^*) - (AB + BA^*)A^*]$$
$$= 2v \beta_A(AB + BA^*)$$
$$= 4v \beta_A \circ \alpha_A(B).$$
Ex. 7. If $\alpha_A$ and $\beta_A$ have a common eigenvector then $A$ has an eigenvalue in $\mathbb{C}$. 

Answer: If $\alpha_A(B) = \lambda B$ and $\beta_A(B) = \mu B$ then consider $AB = (\alpha_A + i\beta_A)(B) = (\lambda + i\mu)B$. Since $B$ is an eigenvector, at least one of the column vectors of $B$, say $u \neq 0$. It follows that $A u = (\lambda + i\mu)u$ and hence $\lambda + i\mu$ is an eigenvalue for $A$. 

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Since $B$ is an eigenvector, at least one of the column vectors of $B$, say $u \neq 0$. 
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Since $B$ is an eigenvector, at least one of the column vectors of $B$, say $u \neq 0$.

It follows that $Au = (\lambda + \iota\mu)u$ and hence $\lambda + \iota\mu$ is an eigenvalue for $A$. 

Anant R. Shastri
Derksen’s Proof of FTA
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\[ S_2(\mathbb{R}, 1) \iff S_1(\mathbb{C}, 1). \] (Exercise 11.4)
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Solution: By Ex. 6, given \( A \in M_n(\mathbb{C}) \), \( \alpha_A, \beta_A : \text{Herm}_n(\mathbb{C}) \rightarrow \text{Herm}_n(\mathbb{C}) \) are two commuting endomorphisms of the real vector space \( \text{Herm}_n \) which is of dimension \( n^2 \), by ex. 5.
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If \( n \) is odd, so is \( n^2 \).
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- By Ex. 4, it follows that \( \alpha_A, \beta_A \) have a common eigenvector.
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By Ex. 7, \( A \) has an eigenvalue.
Ex. 9. Show that the space $\text{Sym}_n(\mathbb{K})$ of symmetric $n \times n$ matrices forms a subspace of dimension $n(n + 1)/2$ of $M_n(\mathbb{K})$. (Exercise 8.5)
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Ex. 10 Given $A \in M_n(\mathbb{K})$, show that

$$\phi_A : B \mapsto \frac{1}{2}(AB + BA^t); \quad \psi_A : B \mapsto ABA^t$$

define two commuting endomorphisms of $\text{Sym}_n(\mathbb{K})$. Show that if $B$ is a common eigenvector of $\phi_A, \psi_A$, then

$$(A^2 + aA + b I_n)B = 0$$

for some $a, b \in \mathbb{K}$; further if $\mathbb{K} = \mathbb{C}$, conclude that $A$ has an eigenvalue.
Answer: The first part is easy.
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To see the second part, suppose

$$\phi_A(B) = AB + BA^t = \lambda B$$

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Multiply first relation by \( A \) on the left and use the second to obtain

\[ A^2B + \mu B - \lambda AB = 0 \]

which is the same as

\[ (A^2 - \lambda A + \mu I_n)B = 0. \]
For the last part, first observe that since $B$ is an eigenvector there is at least one column vector $\mathbf{v}$ which is non zero. Therefore
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\[(A^2 + aA + bI_n)v = 0.\]

Now write $A^2 + aA + bI_n = (A - \lambda_1 I_n)(A - \lambda_2 I_n)$. If $(A - \lambda_2 I_n)v = 0$, then $\lambda_2$ is an eigenvalue of $A$ and we are through.
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Now write $A^2 + aA + bI_n = (A - \lambda_1 I_n)(A - \lambda_2 I_n)$. If $(A - \lambda_2 I_n)\mathbf{v} = 0$, then $\lambda_2$ is an eigenvalue of $A$ and we are through.

Otherwise put $\mathbf{u} = (A - \lambda_2 I_n)\mathbf{v} \neq 0$. Then $(A - \lambda_1 I_n)\mathbf{u} = 0$ and hence $\lambda_1$ is an eigenvalue of $A$. 

Ex. 11 Prove statement (c) \[ S_1(\mathbb{C}, r) \iff S_1(\mathbb{C}, r + 1). \] Hence conclude that \( S_1(\mathbb{C}, r) \) is true for all \( r \geq 1 \).
Ex. 11 Prove statement (c) 
\( S_1(\mathbb{C}, r) \implies S_1(\mathbb{C}, r + 1) \). Hence conclude that 
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Answer:
Let \( n = 2^k \ell \) where \( 1 \leq k \leq r \) and \( \ell \) is odd. Let 
\( A \in M_n(\mathbb{C}) \). Then \( \phi_A, \psi_A \) are two mutually 
commuting operators on \( \text{Sym}_n(\mathbb{C}) \) which is of 
dimension \( n(n + 1)/2 = 2^{k-1} \ell(2^k \ell + 1) \) which is 
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Answer:
Let $n = 2^k \ell$ where $1 \leq k \leq r$ and $\ell$ is odd. Let $A \in M_n(\mathbb{C})$. Then $\phi_A, \psi_A$ are two mutually commuting operators on $Sym_n(\mathbb{C})$ which is of dimension $n(n + 1)/2 = 2^{k-1} \ell(2^k \ell + 1)$ which is not divisible by $2^r$.

By Ex. 10, with $\mathbb{K} = \mathbb{C}$ along with the induction hypothesis, it follows that $A$ has an eigenvalue.
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By Ex. 8, this holds for odd $n$. This implies that $S_1(\mathbb{C}, 1)$. By Ex. 11 applied repeatedly, we get $S_1(\mathbb{C}, r)$ for all $r$.

Given any polynomial of degree $n$, write $n = 2^r \ell$, where $\ell$ is odd. Then $S_1(\mathbb{C}, r + 1)$ implies that every polynomial of degree $n$ has a root.
THANK YOU
FOR YOUR ATTENTION