Lecture 13

\section*{Zeros of Holomorphic Functions}

Let $f$ be holomorphic in a domain $D$, $a \in D$. Assume that all the derivatives of $f$ vanish at $a$, and $f(a) = 0$. Then from the Taylor series representation, it follows that $f(z) = 0$ for all $z$ in $B_r(a) \subset D$. In fact, we have,

\textbf{Theorem 1} Let $f$ be a holomorphic function in a domain $\Omega$. Suppose there is a point $a \in \Omega$ such that $f^{(k)}(a) = 0$ for all $k \geq 0$. Then $f \equiv 0$ on $\Omega$.

\textbf{Proof:} Given any point $b \in \Omega$ choose a continuous map

$$\gamma : [0,1] \rightarrow \Omega, \text{ such that } \gamma(0) = a, \gamma(1) = b.$$ 

Let

$$A = \{t \in [0,1] : f^{(k)}(\gamma(t)) = 0 \ \forall k \geq 0\}.$$ 

(Here, by $f^{(0)}$ we mean $f$ itself.) By the hypothesis, $0 \in A$. It is enough to prove that $1 \in A$. Let $s = \text{Sup} A$. Then $0 \leq s \leq 1$. Also there exists $t_n \in A$ such that $t_n \rightarrow s$. Since $f^{(k)}$ is continuous, and $\gamma$ is also continuous, it follows that

$$f^{(k)}(\gamma(s)) = \lim_{n \rightarrow \infty} f^{(k)}(\gamma(t_n)) = 0.$$ 

Since this is true for all $k \geq 0$, this implies that $s \in A$.

Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which $f$ is represented by its Taylor's series. But all coefficients of Taylor series are zero. It follows that $f(z) = 0$ for all $z \in B_r(\gamma(t_0))$. But then it also follows that $f^{(k)}(z) = 0$ for all $z \in B_r(\gamma(s))$ and for all $k \geq 0$. By continuity of $\gamma$, it follows that there exists $\epsilon > 0$ such that $\gamma(s - \epsilon, s + \epsilon) \subset B_r(\gamma(t_0))$. Therefore, $(s - \epsilon, s + \epsilon) \cap [0,1] \subset A$. 

Since $s$ is the supremum of $A$, this is possible only if $s = 1$. Therefore, $1 \in A$ as claimed.

**Remark 1** Observe that the connectivity of the domain plays a crucial role here. For if the function were to be considered on the union of two disjoint, non empty open sets, $A$ and $B$ say, we could simply take $f \equiv 0$ on $A$ and $\equiv 1$ on $B$. Then even though $f$ is a holomorphic function which is not identically zero, its zero set consisted of an open set $A$. Also, observe that, the above theorem implies that if a holomorphic function vanishes on a non empty open subset of a domain, then it vanishes on the entire region. This result will be strengthened soon.

In contract, see the example below for a typical example of a $C^\infty$-function of a real variable with the property $f^{(k)}(0) = 0$, $\forall k$, but $f \not\equiv 0$.

The above theorem thus leads us to the following definition exclusively for holomorphic functions:

**Definition 1** Let $f$ be a holomorphic function which is not identically zero in a region $\Omega$. Let $a \in \Omega$. Then by the above theorem there exists a non negative integer $k$ such that $f^{(j)}(a) = 0$ for all $j < k$ and $f^{(k)}(a) \neq 0$. We say that $a$ is a zero of order $k$ of $f$. Of course a zero of order zero is not a zero at all!

**Remark 2** In terms of the above terminology, we may say that for a holomorphic function $f$, which is not identically zero, there are no zeros of infinite order. Thus we have

$$f(z) = (z - a)^k f_k(z)$$
where $f_k(z)$ is some holomorphic function in $\Omega$, and $f_k(a) \neq 0$. By continuity of $f_k(z)$, this implies that in a neighborhood of $a$, $a$ is the only zero of $f(z)$. Let us make a formal definition, before summing up what we have seen just now.

**Definition 2** A subset $K$ of $\Omega$ is said to be *isolated in* $\Omega$, if $\forall z \in \Omega$, we can find a disc $\Delta$ around $z$, such that either $\Delta \cap K = \{z\}$ or $= \emptyset$. Observe that an isolated set is a closed subset of $\Omega$ and does not have any limit points in $\Omega$. Also observe that every subset of an isolated set is isolated. These are also called *discrete subsets*, because the induced topology on the subset will be discrete, i.e., every singleton subset is open.

We can sum up our observations in the following theorem.

**Theorem 2** Let $f$ be a holomorphic function not identically zero, in a region $\Omega$. Then the zero set of $f$

$$Z_f := \{z \in \Omega : f(z) = 0\}$$

is an isolated subset of $\Omega$.

As an immediate corollary we have:

**Theorem 3** **Identity Theorem:** Let $f$ and $g$ be holomorphic functions on a region $\Omega$. Suppose $K \subset \Omega$ is such that for every $z \in K$, $f(z) = g(z)$ and $K$ has a limit point in $\Omega$. Then $f \equiv g$ on $\Omega$.

**Proof:** For the function $f - g$, the set $K$ happens to be a subset of the set of all zeros. Since this set has a limit point, it follows that the set of all zeros of $f - g$ is not an isolated set. Hence, by the above theorem, $f - g \equiv 0$ on $\Omega$.  

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Remark 3 Typical instances of the above corollary are:
(i) Two holomorphic functions agreeing on a non empty subregion, will have to agree on the whole region.
(ii) Even if they agree on an arc which is not a single point then they agree on the whole region.

Thus we see that a holomorphic function is well determined once we know its values on a set which has a limit point. This does not necessarily mean that we can effectively compute its value everywhere. In contrast, by Cauchy’s integral formula, we could actually know the value of a holomorphic function inside a disc, the moment we know it on the boundary circle. However, both the results have their own importance and it is perhaps not wise to say that one result is superior to the other.

Example 1 Let $f, g : \Omega \rightarrow \mathbb{C}$ be holomorphic functions on a non empty domain $\Omega$. Suppose $fg \equiv 0$. Then $f \equiv 0$ or $g \equiv 0$.

Proof: Assume on the contrary that neither $f$ nor $g$ is identically zero. Let $A = \mathbb{Z}_f, B = \mathbb{Z}_g$ be the zero sets. Then $\Omega = A \cup B$. By the above theorem, $A, B$ are both isolated sets and therefore $\Omega$ is an isolated set. This is absurd, since $\Omega$ is a non empty open set.

Example 2 Consider

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

(1)

It can be easily seen that $f$ has continuous derivatives of all order. (The only point to be worried about is at the origin. Differentiate the function on the positive interval and take limit as $t \to 0^\pm$.)
All the derivatives at 0 vanish. Yet the function is not identically zero. So, that is a counter example for the above theorem, in case of $C^\infty$ functions. Now take $g(t) = f(-t)$. Then we see that $g$ is also smooth and $fg \equiv 0$. But neither $f$ nor $g$ is identically zero.

**Exercise:** Give a direct proof of the fact that the zero set of an analytic function is an isolated set using the power series definition.

§Singularities

Let $\Omega$ be a region in $\mathbb{C}$. If $f(z)$ is a function on a subset of $\Omega$ then the points at which $f$ is not defined or those points at which $f$ is not holomorphic are referred to as singularities. In section 1, we began by discussing the so called removable singularities. We shall now discuss other types of singularities. However we shall restrict ourselves to the study of isolated singularities only.

**Definition 3** A point $z \in \Omega$ is called an isolated singularity of $f$ if $f$ is defined and holomorphic in a neighborhood of $z$ except perhaps at $z$.

**Example 3**

(i) If $p(z)$ is a polynomial, then $1/p(z)$ has all its singularities isolated and these are nothing but the zeros of $p(z)$.

(ii) Since for any holomorphic function $f$, the zeros of $f$ are isolated, it follows that all the singularities of $1/f$ are isolated.

(iii) Natural examples of holomorphic functions which have non isolated singularities are branches of logarithmic function and inverse-trigonometric functions. For instance, $\text{Ln}(z)$ has singularities along the negative real axis.
Removable Singularities

**Definition 4** An isolated singularity $z_0$ of a holomorphic function is called a **removable singularity** if $f(z_0)$ can defined in such a way that $f$ becomes complex differentiable at $z_0$. That is there exists a holomorphic function $g: \Omega \to \mathbb{C}$ such that for all $z \in \Omega \setminus \{a\}$ we have $f(z) = g(z)$.

**Theorem 4** Let $\Omega$ be a domain, $a \in \Omega$ be any point. Suppose $f$ is holomorphic in $\Omega \setminus \{a\}$. A necessary and sufficient condition that there exists a holomorphic function $g: \Omega \to \mathbb{C}$ such that $g(z) = f(z), z \in \Omega \setminus \{a\}$.

**Proof:** If such a $g$ exists as stated in the definition, then

$$\lim_{z \to a} f(z) = \lim_{z \to a} g(z) = g(a)$$

exists.

Conversely, suppose the above limit exists. Then we can first of all define $f$ at $a$ if necessary and make it continuous at $a$. Now take a circular region $D$ around $a$ contained in $\Omega$ and apply C.I.F.:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw$$

for all $z \in D$ and not equal to $a$. But we know that the R.H.S. is a holomorphic function throughout $\Omega \setminus \partial D$. Hence, if we define

$$g(z) = \begin{cases} f(z) & z \in \Omega \setminus \{a\} \\ \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw & z \in D, \end{cases}$$

then $g$ is holomorphic throughout $\Omega$ and equals $f$ on $\Omega \setminus \{a\}$. ♠
**Example 4** Consider the function \( f(z) = \frac{\sin z}{z}, \ z \neq 0 \). Obviously \( z = 0 \) is an isolated singularity. We easily see that \( \lim_{z \to 0} f(z) \) exists. Hence, \( z = 0 \) is a removable singularity. Also we see that \( \lim_{z \to 0} f(z) = 1 \). So we can define \( f(0) = 1 \) and make \( f \) holomorphic at \( z = 0 \) also.

Another similar example is \( z \cot z \) for which \( z = 0 \) is a removable singularity.

**Remark 4** One easy way a removable singularity \( z_0 \) can arise is by taking a genuine holomorphic function \( f \) around this point and then brutally redefining the value of \( f \) to be something else only at \( z_0 \) or merely pretending as if \( f \) is not defined at \( z_0 \).