Lectures 22 and 23
Harmonic Functions

As Cauchy -Riemann equations tell us, the real and the imaginary parts of a complex analytic function have some special properties. Apart from being inter-related they have the special property of possessing partial derivatives of all order. Similarly, the maximum modulus theorem tells us about certain distinct features of the modulus function of a complex analytic function. Such properties of real valued functions can be studied on their own and such a study can either be carried out using the knowledge of complex functions or independently. Also, an independent enquiry can lead to better understanding of the theory of complex functions themselves. The class of harmonic functions and the wider class of subharmonic functions substantiate this view with many such instances. From the application point of view, few ideas surpass the notion of harmonic functions.

Definition 1 A real valued function $u = u(x, y)$ defined on a domain $D$ in $\mathbb{C}$, is called harmonic with respect to the variables $x, y$, if it possesses continuous second order partial derivatives and satisfies the Laplace’s equation:

\[ \nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]  

(1)

Polar coordinate form of Laplace:

\[ x = r \cos \theta, \ y = r \sin \theta, \]

the Laplace’s equation takes the form
\[ r \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \theta^2} = 0. \] (2)

Verify this. Observe that this form of Laplace’s equation is not applicable at the origin, since the polar coordinate transformation is singular there. However, the polar coordinate form has many advantages of its own and is loved by physicists and engineers.

As an application, from (2), it follows easily that \( \ln r = \ln |z| \) is well defined and harmonic throughout \( \mathbb{C} \setminus \{0\} \). In fact as an easy exercise, prove that any function of \( r \) alone, i.e., independent of \( \theta \), is harmonic iff it is of the form \( a \ln r + b \), where \( a, b \) are constants.

**Remark 1**

(i) Harmonic functions arise in the study of gravitational fields, electrostatic fields, steady-state heat conduction, incompressible fluid flows etc. etc.

(ii) Mathematically, they are very close to holomorphic functions. Given a holomorphic function \( f = u + iv \), as a straightforward consequence of Cauchy-Riemann equations and the property of an analytic function possessing continuous derivatives of all order, it follows that \( u \) and \( v \) are both harmonic. In this case we call \( v \) the harmonic conjugate of \( u \). Observe that, by considering the function \( if(x, y) \), it then follows that \( u \) is the harmonic conjugate of \(-v\). We shall see the converse of this as a theorem.

(iii) There is nothing very special about considering only real valued functions for the definition of harmonic functions. We could even allow complex valued functions in the above definition. Then it follows that a complex function is harmonic iff its real and imaginary parts are harmonic. Thus, it suffices to treat only the real
valued functions, in the study of harmonic functions.

iv) From the linearity of the differential operator, $\nabla^2$, it follows that the set of all harmonic maps on a domain forms a vector space. In particular all linear functions $ax + by$ are harmonic. However, it is not true that product of two harmonic functions is harmonic. For example, $xy$ is harmonic but $x^2y^2$ is not.

iv) Harmonicity is quite a delicate property. If $\phi$ is a smooth real valued function of a real variable and $u$ is harmonic, then, in general, $\phi \circ u$ need not be harmonic. Indeed, $\phi \circ u$ is harmonic for all harmonic $u$ iff $\phi$ is linear (exercise). Likewise, if $f : D_1 \rightarrow D_2$ is a smooth complex valued function of two real variables then $u \circ f$ need not be harmonic. However, under conformal mapping we have some positive result which we shall see below.

**Theorem 1** Let $u(x, y)$ be a harmonic function on a simply connected domain. Then $u$ is the real part of an analytic function in $D$, i.e., $u$ has a harmonic conjugate $v$ throughout $D$.

**Proof:** We first consider the function $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. From Cauchy-Riemann equations it follows that $g$ is holomorphic. (Thus $\frac{\partial u}{\partial x}$ is also harmonic and has an harmonic conjugate $-\frac{\partial u}{\partial y}$.) If $v$ were a conjugate of $u$, then the differential $dv$ is given by,

$$dv := \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy =: *du$$

Thus we must try to show that $*du$ is an exact differential. Now the differential $g(z)dz$ can be written as,

$$g(z)dz = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(dx + idy) = du + i*du$$
and by Cauchy’s theorem we have,

\[ \int_\gamma g(z)\,dz = 0 \]

for all cycles \( \gamma \) in \( D \), since \( D \) is simply connected. On the other hand, since \( du \) is the exact differential, its integral on any cycle is zero (see primitive existence theorem). Hence it follows that

\[ \int_\gamma *du = 0 \] (3)

for all cycles \( \gamma \) in \( D \). This in turn implies that \( *du \) is an exact differential, i.e., \( *du = dv \) for some \( v \) defined throughout \( D \).

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**Remark 2** The statement of the theorem for convex domains can be proved just by integration. Observe that the simply connectivity assumption on the domain is important. A typical example is the \( \ln r \) which is harmonic throughout \( \mathbb{C} \setminus \{0\} \) but there is no holomorphic function on \( \mathbb{C} \setminus \{0\} \) whose real part is equal to \( \ln r \). For any function which may fit this bill has to be a branch of \( \ln z \) none of which is available all over \( \mathbb{C} \setminus \{0\} \).
Theorem 2 Suppose \( w = f(z) \) is a one-to-one, conformal mapping of a domain \( D_1 \) in the \( xy \)-plane onto a domain \( D_2 \) \( uv \)-plane. Let \( C_1 \) be a smooth curve in \( D_1 \) and \( C_2 = f(C_1) \). Let \( \phi(u,v) \) be a real valued function with continuous partial derivatives of second order on \( D_2 \) and let \( \psi \) be the composite function \( \phi \circ f \) on \( D_1 \). Then

(i) \( \psi \) is harmonic if and only if \( \phi \) is harmonic (ii) \( \phi \) is constant on \( C_2 \) if and only if \( \psi \) is constant on \( C_1 \) (or in other words, \( f \) takes the level curves of \( \psi \) to those of \( \phi \).

(iii) The normal derivative of \( \phi \) along \( C_2 \) vanishes iff the normal derivative of \( \psi \) along \( C_1 \) vanishes.

Proof: (i) Take \( z_1 \in D_1 \) and let \( z_2 = f(z_1) \). We can then choose open discs \( B_j \) around these points contained in \( D_1 \) and \( D_2 \) respectively and such that \( f(B_1) \subset B_2 \). Assuming \( \phi \) is harmonic, we can choose a conjugate \( \hat{\phi} \) to it in \( B_2 \) so that \( g = \phi + i\hat{\phi} \) is holomorphic. But then \( g \circ f \) is holomorphic in \( B_1 \). Its real part is nothing but \( \phi \circ f = \psi \). Therefore, \( \psi \) is harmonic at \( z_1 \). Since \( z_1 \) is arbitrary, \( \psi \) is harmonic in \( D_1 \). The converse follows by taking \( f^{-1} : D_2 \to D_1 \) and interchanging the role of \( \phi \) and \( \psi \).

(ii) This is just set theory.

(iii) Let \( z_1 \) be a point on \( C_1 \), \( T, N \) be tangent and normal vectors to \( C_1 \) at \( z_1 \). Then the derivative \( Df \) of \( f \) at \( z_1 \) maps \( T \) onto a tangent vector to \( C_2 \) at \( z_2 \). By conformality of \( f \) it also maps \( N \) onto a nonzero normal vector, \( \lambda n \), where \( n \) is the unit normal vector to \( C_2, \lambda \neq 0 \). to \( C_2 \) at \( z_2 \). We also have, by chain rule \( (\nabla \phi) \circ Df = \nabla \psi \). Now the normal derivative of \( \psi \) along \( C_1 \) vanishes iff \( \nabla \psi \cdot N = 0 \) iff \( [(\nabla \phi) \circ Df] \cdot N = 0 \) iff
grad\phi \cdot Df(N) = 0 \text{ iff } grad\phi \cdot n = 0 \text{ iff the normal derivative of } \phi \text{ along } C_2 \text{ vanishes.} \phantom{♠}

**Example 1** It is not true, in general that the normal derivative of \( \psi \) along \( C_2 \) is a constant if normal derivative of \( \phi \) along \( C_1 \) is a constant. An example take \( D_1 = \{(x, y) : x > 0, \} \), \( f(z) = z_2 \) and \( C_1 = \{(x, 0) : x_0 \} \) and \( \phi(u, v) = v \). Then \( C_2 \) is the positive \( u \)-axis, the normal derivative of \( u \) along \( C_2 \) is \( \frac{\partial v}{\partial u} = 1 \). We have \( \psi(x, y) = 2xy \) and the normal derivative of along \( C_1 \) is \( \frac{\partial \psi}{\partial y} = 2x \) which is not a constant along \( C_1 \).

**Mean Value Theorem and Maximum Principle**

**Theorem 3 Mean Value Property** : Let \( u \) be harmonic in a domain \( D \) and \( B_r(z_0) \subset D \). Then

\[
u(z_0) = \frac{1}{2\pi} \int_{|z-z_0|=r} u(z) \, d(arg(z - z_0)) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta. (4)
\]

**Proof:** Let \( v \) be a harmonic conjugate of \( u \) on \( B_r(z_0) \) so that \( f = u + iv \) is holomorphic. By Cauchy’s integral formula we have

\[
f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0} \, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta.
\]

Upon comparing real parts we get (4). \phantom{♠}

**Theorem 4** Let \( u \) be a harmonic function on a domain \( D \). If \( u \) is constant on a non empty open subset, then it is a constant on the whole of \( D \).

**Proof:** First assume that \( D \) is simply connected. Let \( f \) be a holomorphic function \( f = u + w \). If \( U \) is a non empty open connected set on which \( u \) is a constant, by CR equations it follows
that \( f \) is also a constant on \( U \). By Identity Theorem, this implies that \( f \) is a constant on \( D \) and hence \( u \) is also so.

In the general case, fix \( z_0 \in U \). For any fixed \( z \in D \) we can find an arc joining \( z_0 \) to \( z \) in \( D \). A small nbd \( D' \) of this arc is then simply connected. Applying the previous case, \( u \) will be a constant on \( D' \). But then \( u(z) = u(z_0) \). Since \( z \in D \) is arbitrary, this means \( u \) is a constant.

\[ \text{♠} \]

**Lecture 23**

**Theorem 5** Let \( u \) be a non constant harmonic function in a domain \( D \). Then \( u \) does not attain its maximum or minimum in \( D \).

**Proof:** Let us suppose, on the contrary that \( z_0 \in D \) is such that \( u(z_0) \geq u(z) \) for all \( z \in D \). We can choose \( R > 0 \) such that the closure of the disc \( B_R(z_0) \) is contained in \( D \). By the previous theorem, it suffices to prove that \( u \) is a constant on \( B_R(z_0) \). If possible, suppose this is not the case. Then for some \( z_1 \) with \(|z_1 - z_0| = r < R\) we have \( u(z_1) < u(z_0) \). By continuity, we can choose a subarc \( A \) of length \( L \) on circle \(|z - z_0| = r\) on which \( u(z) < u(z_0) - \epsilon \) for a suitable \( \epsilon > 0 \).

By (4), and ML-inequality, we have

\[
\begin{align*}
u(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + e^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \left( \int_A u(z_0 + e^{i\theta}) d\theta + \int_B u(z_0 + e^{i\theta}) d\theta \right) \\
&\leq \frac{1}{2\pi} \left[ (u(z_0) - \epsilon)L + u(z_0)(2\pi - L) \right] < u(z_0)
\end{align*}
\]

which is a contradiction. The proof for minimum is similar. Alternatively, we can consider \(-u\) and take the maximum.  

\[ \text{♠} \]
Poisson Integral Formula:

The maximum principle tells us the boundary values of a harmonic function determine it completely inside the domain. Poisson’s integral formula which is a direct consequence of Cauchy integral formula gives an exact expression of the harmonic function inside a disc in terms of its values on the boundary of the circle.

We can now generalize the Mean Value Theorem to obtain expression for \( u(z) \) for any \( z \) inside the disc.

**Boundary Value Problems:**

**Theorem 6** Let \( D \) be a bounded domain with boundary \( \partial D \). Let \( u_1, u_2 \) be harmonic functions on an open set containing \( D \cup \partial D \). If \( u_1 = u_2 \) on \( \partial D \) then \( u_1 = u_2 \) on \( D \).

**Proof:** Consider the function \( h = u_1 - u_2 \) which is also harmonic. By continuity \( h \) must attain its maximum on the closure \( D \cup \partial D \) of \( D \). By Maximum principle, this cannot happen inside \( D \). Therefore, the maximum is attained on the boundary \( \partial D \). But on \( \partial D \), we have \( h \equiv 0 \). This means \( h(z) \leq 0 \) for all \( z \in D \). Thus \( u_1(z) \leq u_2(z), z \in D \). By symmetry, \( u_2(z) \leq u_1(z), z \in D \) and we are done.

\( \spadesuit \)

**Remark 3** Thus in principle, the value of a harmonic function is entirely determined once we know its value on the boundary of a bounded domain. Whether we can actually determine this value is another matter. The problem is known as Dirichlet’s problem. Its solutions heavily depend upon the ‘shape’ of the domain and have tremendous importance both theoretic as well as practical.
One of the important tools here is to use conformal transformations to ‘simplify’ the shape of the domain. However, for the simplest domain viz., a disc, this problem has a nice solution and this goes under the name **Poisson Integral formula**.

Suppose, we have a continuous function $u$ on the closed disc $\bar{B}_R(z_0)$ which is harmonic in the interior $B_R(z_0)$. Consider the fractional linear transformations

$$T(w) = \frac{R^2(w + z_0)}{R^2 + \bar{z}_0 w}; \quad S(z) = \frac{R^2(z - z_0)}{R^2 - \bar{z}_0 z}.$$ 

Check that $T, S$ are inverses of each other, $T$ maps the disc $|z| < R$ onto itself and maps 0 to $z_0$. Now the function $u \circ T$ is again harmonic on the disc $|z| \leq R$ and by (4), we have

$$u(z_0) = u \circ T(0) = \frac{1}{2\pi} \int_{|w|=R} (u \circ T)(w)d(\text{arg} w).$$

Now

$$d(\text{arg} w) = -\frac{dw}{w} = -\frac{d(S(z))}{S(z)}dz = \frac{R^2(R^2 - \bar{z}_0 z) + R^2(z - z_0)\bar{z}_0}{R^2 - \bar{z}_0 z} \frac{R^2 - \bar{z}_0 z}{R^2(z - z_0)} \frac{rz}{R^2 - \bar{z}_0 z} d\theta$$

$$= \frac{R^2 - r^2}{|z - z_0|^2} d\theta$$

the last equality being obtained by using the fact $z \bar{z} = R^2$ on the circle of integration $|z| = R$. Thus we obtain:

**Theorem 7 Poisson Integral Formula:** Let $u(r, \theta)$ be a harmonic function on an open set containing the closed disc $|z| \leq R$. Then

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \alpha) \frac{R^2 - r^2}{R^2 - 2Rr(\cos(\theta - \alpha) + r^2)} d\alpha \quad (5)$$

for all $0 \leq r < R, 0 \leq \theta \leq 2\pi$. 9
We shall now discuss some examples, all from electrostatic potential. We quote from your book: ‘We could have equally well phrased everything in terms of (time independent) heat flows; then instead of voltages, we would have temperatures, the equipotential lines would have become isotherms....What we see again here is the unifying power of Mathematics.’

Example 2 Potential Between Parallel Plates along which it is a constant. The basic assumption is that the harmonic function is well determined once we know it on the boundary of an infinite strip such as:

\[ \{x + iy : a \leq x \leq b\} \]

We can simply make a guess work: Try some functions which are \(y\)-independent. This just means that we are looking for the solution of

\[ u_{xx} = 0 \]

given \(u(a)\) and \(u(b)\). Therefore, \(u(x) = \alpha x + \beta\) where

\[ \alpha = \frac{u(b) - u(a)}{b - a}; \quad \beta = \frac{bu(a) - au(b)}{b - a}. \]

Remark 4 The assumption that we made in the above example does not hold in general. However, if we put some extra condition such as \(u\) is a bounded function, then this will be true and requires a proof.

Example 3 Find the Potential Between co-axial Cylinders on each of which the potential is a constant. This problem is similar to the above one except that now, we can use polar
coordinates and try for \( u \) which is independent of \( \theta \). The Laplace equation becomes
\[
r^2 u_{rr} + ru_r = 0
\]
which after simplification and separating the variables, becomes
\[
\frac{u_{rr}}{u_r} = \frac{1}{r}; \quad u_r = \frac{c}{r}; \quad u(r) = a \ln r + b
\]
where \( a, b \) are determined by the values \( u(r_1), u(r_2) \).

**Example 4 Potential between two non co-axial Cylinders** Assume now that we have to determine a harmonic function on the domain bounded by two circles, one interior to the other, given the value of \( u \) on the two boundary components to be constants. Since this problem has been solved for the case when the circles are concentric, we shall try to find a conformal mapping of the given domain with the region between two concentric circles. Without loss of generality, by scaling and translating if necessary we may assume that the outer circle is the unit circle.

We recall that any FLT which maps the unit disc onto itself is of the form
\[
T(z) = \frac{cz - a}{1 - \bar{a}z}
\]
with \( |c| = 1 \) and \( |a| < 1 \). Thus we have two freedoms in the choice of \( c \) and \( a \) so that \( T \) will map the inner circle onto some circle with center 0. By performing a rotation, we can bring the center of the inner circle \( C \) to be at \( b \) where \( 0 < b < 1 \). This already uses up the freedom in the choice of \( c \). Therefore, we may now assume \( c = 1 \) and try to fix the value of \( a \) in such a way that the inner circle is mapped onto some circle with center 0.
Note that mapping the center of the inner circle to 0 does not help! viz., taking $a = b$ we no doubt get $T(C)$ to be a circle with its diameter as the real axis. This is not enough to conclude that its center is 0.

By choosing $a$ to be a real number we see that $T$ maps the real axis onto itself. If $\alpha, \beta$ are the two points at which $C$ meets the $x$-axis, then $T(\alpha)$ and $T(\beta)$ will be the points at which the real axis will meet $T(C)$. By conformality, they meet perpendicularly. Hence, the mid point of $[T(\alpha), T(\beta)]$ is the centre of $T(C)$ which we want to be the origin.

Therefore, we set up the equation

$$T(\alpha) = -T(\beta)$$

Upon simplification this becomes

$$a^2(\alpha + \beta) - 2a(1 + \alpha \beta) + (\alpha + \beta) = 0$$

which is a quadratic equation for $a$. This admits precisely one real solution $a$ with $|a| < 1$ since $(1 + \alpha \beta)^2 > (\alpha + \beta)^2$ for $0 < \alpha < 1$ and $-1 < \beta < 1$. An example with $\alpha = 4/5, \beta = 0$ is in your text book (Kreyszig: article 16.2, example 1).

**Example 5** Steady state temperature $T(x, y)$ in a thin semi-infinite plate $y \geq 0$, whose faces have been insulated and whose edge $y = 0$ is kept at temperature 0 except in the segment $-1 < x < 1$ where the temperature is 1. This translates into the following boundary value problem: Find a harmonic function $T(x, y)$ in upper half plane such that

$$T(x, 0) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| < 1. \end{cases}$$
There are several (slightly) different ways to approach this problem, none of which is too obvious.

(A) One function that we are familiar with which assumes two constant values on the two portions of the real axis is $\text{arg} z$. In order to handle three segments, we should try to find a conformal mapping which will bring the two segments $|x| > 1$ together, say onto the positive real axis and map the segment $|x| < 1$ onto the negative real axis. Such a conformal map must map 1 to 0 and −1 to $\infty$. Lo! we know one such FLT:

$$\phi(z) = \frac{z - 1}{z + 1}.$$ 

Therefore, $\text{arg}(\phi(z))/\pi$ is the solution that we are looking for.

(B) Geometrically, the points $|x| < 1$ on the real axis are characterized by the property that the angle between the two segments $[-1, x]$ and $[1, x]$ is equal to $\pi$ whereas the points $|x| > 1$ are characterized by the property that the angle between these segments is 0. For points $z$ in the upper-half plane the angle between the segments $[-1, z]$ and $[1, z]$ is the imaginary part of a holomorphic function viz., $\text{Ln} \left( \frac{z - 1}{z + 1} \right)$ and hence is a harmonic function, which leads to the same solution after dividing by $\pi$ and simplifying:

$$T(x, y) = \frac{1}{\pi} \tan^{-1} \left( \frac{2y}{x^2 + y^2 - 1} \right)$$

is the answer.