1 Geometry of Complex Numbers

The picture below illustrates how to add two complex numbers geometrically.

![Diagram of adding complex numbers](image1)

Fig. 1

The ‘**parallelogram law**’ (B5) now becomes:

The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

**Remark 1.1 Geometric Multiplication of complex numbers:**

![Diagram of geometric multiplication](image2)

Fig. 2
In the picture above, the two triangles are similar. It tells you how to multiply two complex numbers.

Similarly, to get the inverse of \( z_2 \neq 0 \), we first re-scale the vector \( \overrightarrow{OP}_2 \) to have length \( r \) and then change its direction to have an amplitude \( -\theta \). Also, it follows that the amplitude of \( z_1z_2^{-1} \) is the angle that \( \overrightarrow{OP}_1 \) makes with \( \overrightarrow{OP}_2 \).

It follows that given \( 0 \neq \lambda \in \mathbb{C} \), the assignment \( z \mapsto \lambda z \) defines a linear map \( \mathbb{R}^2 \to \mathbb{R}^2 \), which is a composite of a rotation (through an angle \( \theta = \text{Arg} \lambda \)) and a dilation or a scaling (by a factor \( r = |\lambda| \)). Such linear maps \( \mathbb{R}^2 \to \mathbb{R}^2 \) are called similarities. For the converse part you will have to wait.

**The dot product** Think of two complex numbers as vectors in \( \mathbb{R}^2 \). Then their dot product is given by

\[
(x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + y_1y_2
\]

which we can rewrite in terms of multiplication of complex numbers as:

\[
z_1 \cdot z_2 = \Re(z_1 \bar{z}_2) = \Re(\bar{z}_1 z_2).
\]

Similarly, verify that the cross product can be rewritten is the form

\[
z_1 \times z_2 = \Im(\bar{z}_1 z_2) k
\]

where \( k \) denotes the unit vector in the positive \( z \)-direction perpendicular to the \( xy \)-plane.

**Equation of a line:** Let \( ax + by + c = 0 \) represent a line in cartesian coordinates. WLOG we may assume that \( a^2 + b^2 = 1 \). Put \( w = a + ib; z = x + iy \). Then \( ax + by = \text{Re}(w\bar{z}) = 1/2(w\bar{z} + \bar{w}z) \). Thus, we see that the general equation of a line in the plane can be given by complex numbers as follows:

\[
w\bar{z} + \bar{w}z = t, t \in \mathbb{R}.
\]

**Equation of a circle:**

\[
(z - w)(\bar{z} - \bar{w}) = r^2, r \in \mathbb{R}.
\]

**Definition 1.1** By a rigid motion or an isometry of the plane, we mean a mapping \( f : \mathbb{C} \to \mathbb{C} \) which preserves distances, i.e., \( |f(z) - f(w)| = |z - w| \) for all \( z, w \in \mathbb{C} \).

**Remark 1.2** Identity, translation, rotation about a point and reflexion in a line are obvious examples of rigid motions. The first three preserve the sense of orientation of the plane and the third one changes it. The first one keeps all points fixed. The second one has no fixed point but it preserves orientation. The third one fixes exactly one point and also preserves the orientation. The fourth one fixes exactly a line and (hence) changes the orientation. Are there other rigid motions such as those which fix no points and change the orientation? We shall investigate this right now.

It is clear that composite of two rigid motions is again a rigid motion. It is also clear that a rigid motion is a one-one mapping but it is not easy to see that it also onto. This will follow as an easy consequence of what we are going to do.
Reflection in a Line Let $$w\bar{z} + \bar{w}z = t$$ represent a line. If $$z^*$$ denotes the image of $$z$$ under the reflection in $$L$$, then $$z^* - z$$ is parallel to $$w$$ and is bisected by $$L$$. Therefore, we obtain,

$$z^* - z = sw, s \in \mathbb{R}; \quad w(z^* + z) + \bar{w}(z^* + z) = 2t.$$  

Upon simplification this yields

$$z^* = \frac{t - w\bar{z}}{\bar{w}} = wt - w^2\bar{z}.$$  

**Definition 1.2** By a glide-reflection we mean a RM which is a reflection in a line followed by a translation by a non zero vector in the direction of $$L$$.

It is easy to see that a glide-reflection does not have any fixed point and does not preserve the orientation. The converse follows from what we see below.

**Theorem 1.1** Let $$f : \mathbb{C} \to \mathbb{C}$$ be a rigid motion. Then there exist unique $$a, b \in \mathbb{C}$$ with $$|a| = 1$$ such that

$$f(z) = az + b, \forall z \in \mathbb{C} \quad OR \quad f(z) = a\bar{z} + b, \forall z \in \mathbb{C}.$$  

**Proof:** Put $$b = -f(0)$$ and define $$g(z) = f(z) + b$$. Then $$g$$ is also a RM. and $$g(0) = 0$$. Now $$|g(1)| = 1$$. So, put $$a = g(1)^{-1}$$ and define $$h(z) = ag(z)$$. Then $$h$$ is RM and $$h(0) = 0, h(1) = 1$$, and $$h(i) = \pm i$$.

**Case 1:** Assume $$h(i) = i$$. Now consider any $$z = x + iy \in \mathbb{C}$$ and put $$h(z) = u + iv$$. Then it follows that

$$u^2 + v^2 = x^2 + y^2;$$

$$(u - 1)^2 + v^2 = (x - 1)^2 = y^2;$$

$$u^2 + (v - 1)^2 = x^2 + (y - 1)^2.$$  

Solving these, yields, $$u = x, v = y$$. Thus $$h(z) = z, \forall z \in \mathbb{C}$$. This is the same as saying $$f(z) = az + b, \forall z \in \mathbb{C}$$.

**Case 2:** Assume that $$h(i) = -i$$. Put $$\bar{h}(z) = h(z)$$. Then $$\bar{h}$$ is a RM and $$\bar{h}(0) = 0, \bar{h}(1) = 1, \bar{h}(i) = i$$. So, we are in case 1. ♠.

**Theorem 1.2** Let $$f : \mathbb{C} \to \mathbb{C}$$ be a rigid motion.

(i) Suppose $$f$$ fixes two distinct points. Then all points on the line passing through these two points are also fixed by $$f$$.

(ii) Suppose $$f$$ fixes three non collinear points. Then $$f = Id$$.

(iii) Suppose $$f$$ fixes an entire line $$L$$. Then it is either $$Id$$ or the reflexion in that line.

(iv) Suppose $$f$$ fixes exactly one point. Then it is a rotation around that point.

(v) Suppose $$f$$ fixes no points. Then either $$f$$ is a translation or a glide reflexion.

**Proof:** (i) If $$z_1, z_2$$ are such that $$az_i + b = z_i, i = 1, 2$$ then for any real numbers $$t_1, t_2$$ such that $$t_1 + t_2 = 1$$, we have

$$a(t_1z_1 + t_2z_2) + b = t_1z_1 + t_2z_2.$$
(ii) Argue as in (i).

(iii) We may assume that 0, 1 are fixed by \( f \), so that the entire real line is fixed. Now if \( f \) fixes \( i \) also, then by (ii), \( f = Id \). If \( f(i) = -i \) then we consider the RM \( \bar{f} \) which must be \( Id \).

(iv) Choose the fixed point \( f \) as the origin. Then \( f(0) = 0 \). This means \( b = 0 \). Also, observe that if \( f(z) = e^{i\theta}z \) then \( f(re^{i\theta/2}) = re^{i\theta/2} \), shows that \( f \) fixes a whole line. Therefore \( f(z) = e^{i\theta}z \) for some \( \theta \).

(v) First consider the case \( f(z) = az + b \). If \( a \neq 1 \) then we can solve for \( f(z) = z \). Therefore \( a = 1 \) and \( f \) is a translation.

Now consider the case \( f(z) = a\bar{z} + b \). Choose \( w \) such that \( w^2 = -a \). We know that for any \( t \in \mathbb{R} \), the map \( z \mapsto -w^2z + tw \) represents the reflection in a line perpendicular to \( w \). So we write

\[
  f(z) = a\bar{z} + b = -w^2\bar{z} + tw + (b - tw)
\]

so that \( f \) is now a reflection followed by a translation. It remains to choose the real number \( t \) appropriately, so that the translation is along the line of reflection. This is the same as saying that \( b - tw = sw \) for some \( s \in \mathbb{R} \). Therefore, we have to choose \( t, s \in \mathbb{R} \) such that \( b = tw + sw \). This is always possible and is nothing but resolving the vector \( b \) in the direction of \( w \) and perpendicular to it. The real numbers \( t, s \) so obtained are unique. Observe that \( s \neq 0 \) for otherwise, \( f \) will fix a whole line. Thus \( f \) is a glide reflection.

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**Story Time**

This story is due to George Gamow, the well-known physicist and an ingenious story-teller. We quote from his book:

**ONE TWO THREE · · · INFINITY** (pp. 44-45).

There was a young and adventurous man who found among his great-grandfather’s papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:

“Sail to · · · North latitude and · · · West longitude where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine. There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn left by a right angle and see that thou takest the same number of steps, and put another spike into the ground. Dig half-way between the spikes; the treasure is there.”