Lecture 9: Cauchy’s Integral Formula

Holomorphic implies Analytic

Zero set of Holomorphic functions
Cauchy’s Integral Formula

- Recall that we have proved the following:
Recall that we have proved the following:

Let \( f : \Omega \to \mathbb{C} \) be a holomorphic function and \( \gamma \) is a simple closed curve enclosing a region \( R \) in \( \Omega \) then every point \( w \in R \) we have

\[
f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} \, dz.
\]  

(1)
Cauchy’s Integral Formula for derivatives

We also have seen that differentiation under integral sign is valid here repeatedly, and gives us

**Theorem**

**Cauchy’s Integral formula for Derivatives:** Let $f$ be holomorphic in a domain $\Omega$. Then $f$ has derivatives of all order in $\Omega$. Moreover, if $C$ is a circle in $\Omega$ and $z$ is a point inside the circle $C$ then for all integers $n \geq 0$, we have,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z) \, dz}{(z - w)^{n+1}}.$$  

(2)
Remark

Behold! We have shown that a function which is once complex differentiable is differentiable any number of times. Certainly this is something that we never bargained for while launching the theory of complex differentiation. There is more to come. It is time for us to reap the harvest:
Let $f$ be a holomorphic function in a domain $\Omega$. 
Holomorphic implies Analyticity

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Holomorphy implies Analyticity

- Let $f$ be a holomorphic function in a domain $\Omega$.
- In (1), the curve $\gamma$ can be chosen to be the oriented boundary of a closed disc $D$ contained in $\Omega$.
- Then the formula is valid for all points $w$ in the interior of $D$. Fix one such $w$.
- Let $a$ denote the center of this disc.
Holomorphy implies Analyticity

Then we have

\[ \left| \frac{w - a}{z - a} \right| < 1, \ z \in \partial D. \]

and hence the geometric series

\[ \frac{1}{z - w} = \frac{1}{(z - a) - (w - a)} = \frac{1}{z - a} \left( \sum_{n=0}^{\infty} \left( \frac{w - a}{z - a} \right)^n \right) \]

is uniformly convergent on the boundary of \( D \).
Holomorphy implies Analyticity

Therefore, we can substitute (3) in (1) and interchange the order of integration and summation to obtain

\[
f(w) = \frac{1}{2\pi i} \sum_0^\infty \left( \int_{\partial D} \frac{f(z)}{(z - a)^{n+1}} \right) (w - a)^n
\]

\[i.e., \quad f(w) = \sum_0^\infty \frac{f^{(n)}(a)}{n!} (w - a)^n \quad (4)\]

which is valid for all points \(w\) inside \(D\).
Holomorphy implies Analyticity

Thus, we have found a power series representation for $f$ around each point of its domain.
Holomorphy implies Analyticity

Thus, we have found a power series representation for $f$ around each point of its domain.

Note that the only condition on the size of the disc to be chosen is that it should be contained in the domain $R$. Therefore we have following theorem.
Holomorphy implies Analyticity

Theorem

Every holomorphic function on a domain $R$ is analytic with a unique power series representation around each point $a \in R$; the power series so obtained has radius of convergence at least as big as $\sup\{r : B_r(a) \subset R\}$. 
Definition

The power series (4), obtained in the previous theorem is called the **Taylor’s series** for $f$ around the point $z = a$. When the point $a = 0$, it is also called the **Maclaurin’s series**.
Cauchy’s Estimates

Theorem

Cauchy’s Estimate: If $f$ is holomorphic in an open set containing the closure of the disc $B_r(z)$ and let $M_r = \text{Sup}\{|f(w)| : |w - z| = r\}$. Then for all $n \geq 1$ we have,

$$|f^{(n)}(z)| \leq \frac{n! M_r}{r^n}.$$  (5)
Cauchy’s Estimates

**Proof:** Take $C$ to be the circle of radius $r$ around $z$. 
Cauchy’s Estimates

**Proof:** Take $C$ to be the circle of radius $r$ around $z$. Then we have,

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(w) \, dw}{(w - z)^{n+1}} \right| \leq \frac{n! M_r}{2\pi r^{n+1}} \int_C |dw| = \frac{n! M_r}{r^n}.$$

This proves the theorem. ♠
Definition

A function that is holomorphic on the entire plane $\mathbb{C}$ is called an *entire function*. 
Liouville’s Theorem

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Theorem
**Liouville’s Theorem**: *A bounded entire function is a constant.*
Liouville’s Theorem

**Proof:** Putting $n = 1$ in the Cauchy’s estimate, we obtain that $|f'(z)| \leq M_r/r$. 
Liouville’s Theorem

Proof: Putting $n = 1$ in the Cauchy’s estimate, we obtain that $|f'(z)| \leq M_r/r$.
Since $f$ is bounded, let $M$ be such that $M_r \leq M$ for all $r$. Now take the limit as $r \to \infty$. Therefore $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f$ is a constant.
The Fundamental Theorem of Algebra: Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_i \in \mathbb{C}$, $a_n \neq 0$ be a polynomial function in one variable of degree $n \geq 1$ over the complex numbers. Then the equation $p(z) = 0$ has at least one solution in $\mathbb{C}$. 
Proof: Assume that $p(z)$ is never zero. It follows that $f(z) = 1/p(z)$ is differentiable everywhere, i.e., $f(z)$ is an entire function.
Proof of FTA

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Proof of FTA continued

Now to show that $f$ is bounded, we first show that

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- Observe that
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  \]

- $\lim_{|z| \to \infty} |p(z)| = \lim_{|z| \to \infty} |a_n||z^n| = \infty$. 

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The Fundamental Theorem of Algebra:

- Hence we can find large $r$ such that

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- On the other hand, by continuity of \( f \) there exists \( K \) such that
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  |f(z)| \leq k \quad \forall \quad |z| \leq r.
  \]
- If \( M = \max\{1, K\} \), then
  \[
  |f(z)| \leq M, \quad \forall \quad z \in \mathbb{C}.
  \]

This completes the proof of FTA.
Let $f$ be holomorphic in a domain $D$, $a \in D$. Assume that all the derivatives of $f$ vanish at $a$, and $f(a) = 0$. Then from the Taylor series representation, it follows that $f(z) = 0$ for all $z$ in $B_{r}(a) \subset D$. In fact, we have,
Zeros of Holomorphic Functions

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**Theorem**

Let $f$ be a holomorphic function in a domain $\Omega$. Suppose there is a point $a \in \Omega$ such that $f^{(k)}(a) = 0$ for all $k \geq 0$. Then $f \equiv 0$ on $\Omega$. 
Proof: Given any point \( b \in \Omega \) choose a continuous map

\[ \gamma : [0, 1] \rightarrow \Omega, \text{ such that } \gamma(0) = a, \gamma(1) = b. \]
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Let

$$A = \{ t \in [0, 1] : f^{(k)}(\gamma(t)) = 0 \ \forall k \geq 0 \}.$$ 

(Here, by $f^{(0)}$ we mean $f$ itself.)
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Let

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By the hypothesis, \( 0 \in A \). It is enough to prove that \( 1 \in A \).
Let $s = \text{Sup } A$. Then $0 \leq s \leq 1$. 
Zeros of Holomorphic Functions

- Let $s = \text{Sup } A$. Then $0 \leq s \leq 1$.
- Also there exists $t_n \in A$ such that $t_n \to s$. 
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- Let $s = \text{Sup } A$. Then $0 \leq s \leq 1$.
- Also there exists $t_n \in A$ such that $t_n \to s$.
- Since $f^{(k)}$ is continuous, and $\gamma$ is also continuous, it follows that

$$f^{(k)}(\gamma(s)) = \lim_{n \to \infty} f^{(k)}(\gamma(t_n)) = 0.$$
Let $s = \text{Sup } A$. Then $0 \leq s \leq 1$.

Also there exists $t_n \in A$ such that $t_n \to s$.

Since $f^{(k)}$ is continuous, and $\gamma$ is also continuous, it follows that

$$f^{(k)}(\gamma(s)) = \lim_{n \to \infty} f^{(k)}(\gamma(t_n)) = 0.$$ 

Since this is true for all $k \geq 0$, this implies that $s \in A$. 

Zeros of Holomorphic Functions
Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which $f$ is represented by its Taylor’s series.
Zeros of Holomorphic Functions

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Zeros of Holomorphic Functions

- Choose a disc $B_r(\gamma(s)) \subset \Omega$ on which $f$ is represented by its Taylor’s series.
- But all coefficients of Taylor series are zero, because $s \in A$.
- It follows that $f(z) = 0$ for all $z \in B_r(\gamma(s))$. But then it also follows that $f^{(k)}(z) = 0$ for all $z \in B_r(\gamma(s))$ and for all $k \geq 0$. 
By continuity of $\gamma$, it follows that there exists $\epsilon > 0$ such that $\gamma(s - \epsilon, s + \epsilon) \subset B_r(\gamma(s))$. Therefore, $(s - \epsilon, s + \epsilon) \cap [0, 1] \subset A$. Since $s$ is the supremum of $A$, this is possible only if $s = 1$. Therefore, $1 \in A$ as claimed.
The above theorem thus leads us to the following definition exclusively for holomorphic functions:

Let $f$ be a holomorphic function which is not identically zero in a region $\Omega$. Let $a \in \Omega$. Then by the above theorem there exists a non-negative integer $k$ such that $f^{(j)}(a) = 0$ for all $j < k$ and $f^{(k)}(a) \neq 0$. We say that $a$ is a zero of order $k$ of $f$. Of course a zero of order zero is not a zero at all!
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Zeros of Holomorphic Functions

- The above theorem thus leads us to the following definition exclusively for holomorphic functions:
- Let $f$ be a holomorphic function which is not identically zero in a region $\Omega$. Let $a \in \Omega$.
- Then by the above theorem there exists a non-negative integer $k$ such that $f^{(j)}(a) = 0$ for all $j < k$ and $f^{(k)}(a) \neq 0$. 
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- Thus we have

$$f(z) = (z - a)^k g(z)$$

where $g(z)$ is some holomorphic function in $\Omega$, and $g(a) \neq 0$. By continuity of $g(z)$, this implies that in a neighborhood of $a$, $a$ is the only zero of $f(z)$. 
Zeros of Holomorphic Functions

Let us make a formal definition, before summing up what we have seen just now.

**Definition**

A subset $K$ of $\Omega$ is said to be *isolated in* $\Omega$, if $\forall z \in \Omega$, we can find a disc $\Delta$ around $z$, such that either $\Delta \cap K = \{z\}$ or $= \emptyset$. 

Observe that an isolated set is a closed subset of $\Omega$ and does not have any limit points in $\Omega$. Also observe that every subset of an isolated set is isolated.
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Example

Consider the following subsets of $\mathbb{C}$.

(1) Any finite subset is isolated.
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(1) Any finite subset is isolated.

(2) The set of integers $\{m + in : m, n \in \mathbb{Z}\}$ is isolated.

(3) The set $\{1/m : m \in \mathbb{Z}^+\}$ is not isolated.
Zeros of Holomorphic Functions

We can sum up our observations in the following theorem.

Theorem

Let $f$ be a holomorphic function not identically zero, in a region $\Omega$. Then the zero set of $f$

$$\mathcal{Z}_f := \{z \in \Omega \mid f(z) = 0\}$$

is an isolated subset of $\Omega$. 
Zeros of Holomorphic Functions

We can sum up our observations in the following theorem.

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Let $f$ be a holomorphic function not identically zero, in a region $\Omega$. Then the zero set of $f$

$$Z_f := \{ z \in \Omega : f(z) = 0 \}$$

is an isolated subset of $\Omega$.

As an immediate corollary we have:
Identity Theorem

Theorem

Identity Theorem: Let $f$ and $g$ be holomorphic functions on a region $\Omega$. Suppose $K \subset \Omega$ is such that for every $z \in K$, $f(z) = g(z)$ and $K$ has a limit point in $\Omega$. Then $f \equiv g$ on $\Omega$. 
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Proof: For the function $f - g$, the set $K$ happens to be a subset of the set of all zeros.
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Proof: For the function $f - g$, the set $K$ happens to be a subset of the set of all zeros. Since this set has a limit point, it follows that the set of all zeros of $f - g$ is not an isolated set. Hence, by the above theorem, $f - g \equiv 0$ on $\Omega$. 
Zeros of Holomorphic Functions

- Typical instances of the above corollary are:
  
  (i) Two holomorphic functions agreeing on a non empty subregion, will have to agree on the whole region.
  
  (ii) Even if they agree on an arc which is not a single point then they agree on the whole region.
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- (ii) Even if they agree on an arc which is not a single point then they agree on the whole region.
Zeros of Holomorphic Functions

Thus we see that a holomorphic function is well determined once we know its values on a set which has a limit point. This does not necessarily mean that we can effectively compute its value everywhere. In contrast, by Cauchy’s integral formula, we could actually know the value of a holomorphic function inside a disc, the moment we know it on the boundary circle. However, both the results have their own importance and it is perhaps not wise to say that one result is superior to the other.
An Example

Consider

\[
    f(t) = \begin{cases} 
        e^{-1/t} & \text{if } t > 0 \\
        0 & \text{if } t \leq 0.
    \end{cases}
\]  (6)
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\end{cases} \quad (6) \]

It can be easily seen that \( f \) has continuous derivatives of all order.

(The only point to be worried about is at the origin. Differentiate the function on the positive interval and take limit as \( t \to 0^+ \). All the derivatives at 0 vanish. Yet the function is not identically zero.)
An Example

Consider

\[ f(t) = \begin{cases} \frac{e^{-1/t}}{t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \quad (6) \]

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