Derksen's Proof of FTA [1]

We present a proof of Fundamental Theorem of Algebra through a sequence of easily do-able exercises. The proof uses only elementary linear algebra and of course the intermediate value theorem.

1. Show that every odd degree polynomial $p(t) \in \mathbb{R}[t]$ has a real root. (This is where IVT is used. From now onwards we only use linear algebra.)

2. **Companion Matrix** Let $p(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be a monic polynominal of degree n. Its companian matrix C_p is defined to be the $n \times n$ matrix

$$C_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & & \cdots & \cdots & -a_1 \end{bmatrix}.$$

   Show that $\det(tI - C_p) = p(t)$.

3. Show that every non constant polynomial $p(t) \in \mathbb{K}[t]$ of degree n has a root in $\mathbb{K}$ iff every linear map $\mathbb{K}^n \to \mathbb{K}^n$ has an eigen value in $\mathbb{K}$.

4. Every $\mathbb{R}$-linear map $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ has real eigen value.

5. Show that the space $\text{HERM}_n(\mathbb{C})$ of all complex Hermitian $n \times n$ matrices is a $\mathbb{R}$ vector space of dimension n^2.

6. Given $A \in M_n(\mathbb{C})$, the mappings

$$\alpha_A(B) = \frac{1}{2}(AB + BA^*); \quad \beta_A(B) = \frac{1}{2\imath}(AB - BA^*)$$

   define $\mathbb{R}$-linear maps $\text{HERM}_n(\mathbb{C}) \to \text{HERM}_n(\mathbb{C})$. Show that α, β commute with each other.

[1] From Amer. Math. Monthly- 110,(2003), pp. 620-623. (Presented by Anant Shastri at ATML-2006, on 13th June 2006)

7. If α_A and β_A have a common eigen vector then A has an eigen value in $\mathbb{C}$.

8. Show that any two commuting linear maps $\alpha, \beta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ have a common eigen vector.(Use induction and subspaces kernel and image of $\alpha - \lambda I_n$ where λ is an eigen value of α.)

9. Every $\mathbb{C}$-linear map $\mathbb{C}^{2n+1} \to \mathbb{C}^{2n+1}$ has an eigen value.

10. Show that the space $\mathrm{SKEW}_n(\mathbb{K})$ of skew symmetric $n \times n$ matrices forms a subspace of dimension $n(n-1)/2$ of $M_n(\mathbb{K})$.

11. Given $A \in M_n(\mathbb{K})$, show that

$$\phi_A : B \mapsto \frac{1}{2}(AB + BA^t); \quad \psi_A : B \mapsto ABA^t$$

define endomorhisms of $\mathrm{SKEW}_n(\mathbb{K})$. Show that if B is a common eigen vector of ϕ_A, ψ_A then $(A^2 + aA + b)B = 0$ for some $a, b \in \mathbb{K}$. Further if $\mathbb{K} = \mathbb{C}$, conlude that A has an eigen value.

Let $E(\mathbb{K}, k, r)$ denote the following statememt: *Any mutually commuting endomorphisms* $A_1, \ldots, A_r : \mathbb{K}^n \to \mathbb{K}^n$ *have a common eigen vector for all* n *not divisible by* 2^k.

12. Prove that $E(\mathbb{K}, k, 1) \implies E(\mathbb{K}, k, 2)$.

13. Prove that $E(\mathbb{C}, k, 1) \implies E(\mathbb{C}, k+1, 1)$. Hence conclude $E(\mathbb{C}, k, 1)$ is true for all $k \geq 1$.

14. Conclude that every non constant polynomial over complex numbers has a root.