Week 5 Lectures 13-15

Lecture 13

Definition 29 Let Y be a subset X. A subset $A \subset Y$ is open in Y if there exists an open set U in X such that $A = U \cap Y$. It is not difficult to show that the collection of all open subsets of Y defined in the above fashion forms a topology on Y. With this topology, we say Y is a subspace of X.

Remark 26 Indeed if (X, d) is a metric space, then d can be restricted to $Y \times Y$ to get a metric on Y. This then defines a topology on Y which is the same as the subspace topology that we have defined above.

Definition 30 By a cover of $A \subset X$ we mean a family $\{U_j\}$ of sets in X such that $A \subset \cup_j U_j$. It is called an open cover if every member U_j is open. A subcover we mean a cover $\{V_i\}$ which are members of the cover $\{U_j\}$. A subset K of X is compact, if every open cover of K admits a finite subcover.

Theorem 36 *Let Y be a subspace of X. Then $K \subset Y$ is compact (as a subset of Y iff $K \subset X$ is compact.*

Proof: Let K be compact in X and let $\{U_j\}$ any cover of K by open subset of Y. Then there exist open sets V_j in X such that $U_j = V_j \cap Y$.

But then $\{V_j\}$ is an open cover of K in X. Therefore there are finitely many say $V_{j_1}, \ldots, V_{j_k}\}$ such that $K \subset \cup_{i=1}^k V_{j_i}$. But then $K \subset \cup_{i=1}^k U_{j_i}$. We leave the proof of the converse to you. ♠

Theorem 37 *Every closed subset of a compact set is compact.*

Proof: Easy.

Corollary 4 *If F is closed and K is compact then $F \cap K$ is compact.*

Theorem 38 *Every compact subset of a metric space is closed and bounded.*

Proof: Let K be a compact subset of (X, d). We shall prove $X \setminus K$ is open. Fix a point $p \in X \setminus K$. For each $x \in K$ consider $\delta_x = \frac{1}{2}d(p, x)$. Then $\{B_{\delta_x}(x)\}_{x \in K}$ forms an open cover for K. Since K is compact, there exist $x_1, \ldots x_k$ such that $K \subset \cup_{i=1}^k B_{\delta_{x_i}}(x_i)$. It follows easily that $V = \cap_{i=1}^k B_{\delta_{x_i}}(p)$ is an open set contains p and $V \subset X \setminus K$.

To show that K is bounded, fix any point $x \in X$ and consider the family $\{B_\delta(x) : \delta > 0\}$ of open sets which actually cover the whole of X and hence K. A finite cover then gives a single δ such that $K \subset B_\delta(x)$. ♠

Theorem 39 *Let $\{K_j\}$ be a collection of compact subsets of a topological space X such that intersection of any finitely many members is non empty, then $\cap_j K_j \neq \emptyset$.*

Proof: Put $U_j = X \setminus K_j$. Then we know that each U_j is open. Now if $\cap_j K_j = \emptyset$, then it follows that $X = \cup_j U_j$. In particular $\{U_j\}$ is an open cover for K_1 which is compact. Therefore, there are finitely may $j_1, \ldots, j_k$ such that

$$K_1 \subset U_{j_1} \cup \cdots \cup U_{j_k}.$$

This means $K_1 \cap K_{j_1} \cap \cdots \cap K_{j_k} = \emptyset$ a contradiction. ♠

Corollary 5 *Let* $\cdots \supset K_n \supset K_{n+1} \supset \cdots$ *be a sequence of non empty compact sets in a topological space. Then* $\cap_n K_n \neq \emptyset$.

Theorem 40 *If* A *is an infinite subset of a compact subset* K *of a topological space, then* A *has limit point in* K.

Proof: If not then every point of $x \in K$ has a nbd U_x such that $U_x \cap A \subset \{x\}$. If $\{U_{x_1}, \ldots, U_{x_k}\}$ is a finite subcover of K this will imply $A \subset \cup_i U_{x_i}$. Therefore, $A \subset \cup_i (U_{x_i} \cap A) \subset \{x_1, \ldots, x_k\}$ which contradicts infiniteness of A. ♠

Lecture 14

We shall now examine compactness property inside $\mathbb{R}^n$.

Lemma 2 Let $I_n = [a_n, b_n]$ is a decreasing nested sequence of nonempty closed intervals, i.e.,

$$I_1 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

then $\cap_n I_n \neq \emptyset$.

Proof: Put $x = \sup a_n$. Claim $x \in I_n$ for all n. ♠

Definition 31 By a box in $\mathbb{R}^n$ we mean a product of n intervals. If each of these intervals is closed then we say the box is a closed box. If each of these intervals is open then we say the box is open.

Lemma 3 If B_n is a decreasing sequence of closed boxes in $\mathbb{R}^k$, then $\cap_n B_n \neq \emptyset$.

Theorem 41 *Every closed box* B *in* $\mathbb{R}^k$ *is compact.*

Proof: Let $\mathcal{U} = \{U_\alpha\}$ be a family of open subsets of X which covers B. Suppose there is no finite subfamily of $\mathcal{U}$ which covers B. Put $B_1 = B$. Cut B into 2^n boxes of equal size by bisecting each interval in the product. Then it follows that at least one of these boxes cannot be covered by any finite subfamily of $\mathcal{U}$. Choose one such and call it B_2. Note that diameter of B_2 is equal to $\delta/2$, where δ is the diameter of B. Repeat this process to obtain a sequence of closed boxes

$$B_1 \supset B_2 \supset \cdots$$

such that
(a) no finite subfamily of $\mathcal{U}$ covers B_n and
(b) diameter of $B_n = \delta/2^{n-1}$.
By lemma 3, there exists $x \in \cap_n B_n$. Since $\mathcal{U}$ covers B, there is some member U_α such that $x \in U_\alpha$. Since U_α is open, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset U_{\alpha.}$ Choose n large enough such that $\delta/2^{n-1} < \epsilon/$. Then $y \in B_n$ implies that $d(x, y) < \delta/2^{n-1} < \epsilon$. Therefore B_n is completely contained in U_α, a single member of $\mathcal{U}$. This is a contradiction.
♠

Theorem 42 *(Heine-Borel) A subset K of $\mathbb{R}^k$ is compact iff it is closed and bounded.*

Proof: We have to prove that if K is closed and bounded subset of $\mathbb{R}^k$, then it is compact. Since it is bounded, it is contained in a closed cell. Since it is a closed subset of a closed cell which is compact, it is compact. ♠

Theorem 43 *A subset K of $\mathbb{R}^k$ is compact iff every infinite subset of K has a limit point in K.*

Proof: Again, we have only to prove if part. We shall prove that K is closed and bounded.

If K is not bounded, then for each n we have $x_n \in K$ such that $|x_n| > n$. The subset $E = \{x_n\}$ has no limits points in $\mathbb{R}^k$ and hence none whatsoever in K. This is a contraction.

Now suppose K is not closed. This means there is a limit point x of K which is not in K. We now construct an infinite sequence $\{x_n\}$ in K which converges to x and hence no limit point inside K. Having found x_n, put $\delta_n = |x - x_n|/2$ and consider the open ball $B_{\delta_n}(x)$ which must a have point of K not equal to x; call this point x_{n+1}. ♠

Lecture 15

Theorem 44 *(Weierstrass) Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$.

Proof: The closure of this set is compact.

Theorem 45 *(Bolzano-Weierstrass) Let* A *be a bounded subset of* $\mathbb{R}^k$. *Then every infinite sequence in* A *has a subsequence which is convergent.*

Proof: Look at the image set and consider the two cases according to whether it is finite or infinite.

Theorem 46 *Let* $f : X \to Y$ *be a function: TFAE:*
(1) f *is continuous.*
(2) $f^{-1}(U)$ *is open in* X *for every open set* U *in* Y.
(3) $f^{-1}(F)$ *is closed in* X *for every closed set* F *in* Y.

Theorem 47 *Let* $f : X \to Y$ *be a continuous function of topological spaces. If* K *is a compact subset of* X, *then* $f(K)$ *is a compact subset of* Y.

Theorem 48 *Every continuous real valued function on a compact set attains its minimum and maximum.*

Proof: The image is closed and bounded and hence has maximum and minimum.

Theorem 49 (**Lebesgue Covering Lemma**) *Let $\{U_j\}$ be an open covering for a compact metric space. Then there exists a number $\delta > 0$ such that any ball of radius δ and center in K is contained in some member of $\{U_j\}$.*

Proof: By compactness of K we may assume that the cover is finite. Put $F_j = X \setminus U_j$ so that each F_j is a closed set. Now consider the function $f_j : X \to \mathbb{R}$ given by $f_j(x) = d(x, F_j)$. Check that it is continuous. Next put $f = max\{f_1, f_2, \dots, f_n\}$. Show that f is also continuous. Check that $f(x) > 0$ for $x \in K$. Now let $\delta = inf\{f(x) : x \in K\}$. Then by the previous theorem δ is actually the minimum and hence is positive. Now let $x \in K$, and consider $B_\delta(x)$. If it is not contained in any of $U_1, \dots, U_k$, that would mean that the ball contains points from each of F_j which means that the distance of x from each F_j is strictly less that δ. That means that the maximum of these distances viz. $f(x) < \delta$ which is absurd. ♠

Definition 32 Let $f : X \to Y$ be a function from one metric space to another metric space. We say f is uniformly continuous, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_X(x_1, x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.

Theorem 50 (Uniform Continuity) *Every continuous real valued function on a compact space is uniformly continuous.*

Proof: Given ϵ by continuity, for each $x \in K$ there exists $\delta_x > 0$ such that $d_Y(f(x), f(y)) < \epsilon/2$ for all $y \in B_\delta(x)$. Since K compact by LCL, there exists a $\delta > 0$ such that any ball of radius δ is contained in some member of $\{B_{\delta_x}(x)\}$. Now let $a, b \in K$ be such that $d(a, b) < \delta$. Choose $x \in K$ such that $a, b \in B_{\delta_x}(x)$. Then it follows that $d_Y(f(a), f(x)) < \epsilon/2, d_Y(f(b), f(x)) < \epsilon/2$ and therefore $d_Y(f(a), f(b)) < \epsilon$.

Example 12 $f : [0, \infty) \to [0, \infty)$ defined by x^2 is not uniformly continuous.