Lecture Notes in Real Analysis 2010

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Week 9: Lectures 25-27

Lecture 25

Example 1 A continuous function which is nowhere differentiable: Put

$$\phi(x) = |x|, \qquad -1 \leq x \leq 1$$

and extend this function all over $\mathbb{R}$ by periodicity:

$$\phi(x+2) = \phi(x).$$

This function is continuous on $\mathbb{R}$ and not differentiable at any integer value of x.

Let $\phi_n(x) = \phi(4^n x)$. Then each ϕ_n has similar properties to ϕ but the period has decreased and the number of points at which it is not differentiable has increased viz., at all those rational numbers q such that $4^n q \in \mathbb{Z}$. We now take

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi_n(x).$$

Observe that $0 \leq \phi_n(x) \leq 1$ for all n and hence the above series is uniformly convergent and hence defines a continuous function on $\mathbb{R}$. It is also clear that this function is not differentiable at any dyadic rational number. But there is a bonus: it is not differentiable anywhere:
Let $x \in \mathbb{R}$. For each integer m consider $4^m x$. Then one of the intervals

$(4^m x, 4^m x + 1/2), (4^m x - 1/2, 4^m x)$ will not contain any integer. Choose one such and accordingly define $\delta_m = \pm\frac{1}{2(4^m)}$ so that there is no integer between $4^m x$ and $4^m(x + \delta_m)$.

Now if $n > m$ then $4^n \delta_m$ is an even integer and hence $\phi_n(x+\delta_m) - \phi_n(x) = 0$. Also for $0 \le n \le m$, we have $|\phi_n(x + \delta_m) - \phi_n(x)| = |4^n \delta_m|$. (To see this use the property $|\phi(x) - \phi(y)| = |x - y|$ if the interval (x, y) contains no integer.) Therefore

$$\begin{aligned}\left|\frac{f(x + \delta_m) - f(x)}{\delta_m}\right| &= \left|\frac{1}{\delta_m}\sum_0^m \left(\frac{3}{4}\right)^n (\phi(4^n(x + \delta_m)) - \phi(4^n x))\right| \\ &\ge 3^m - \sum_0^{m-1} 3^n = 3^m - \frac{3^m - 1}{2} = \frac{3^m + 1}{2}.\end{aligned}$$

Therefore upon taking the limit as $m \to \infty$, we see that $f'(x)$ does not exist.

Such functions are collectively called *Weierstrass' functions*, because, Weierstrass was the first one to show the existence of such functions. Indeed, this discovery was a cultural shock to the mathematics community of that time and there were plenty of mathematicians who would not want to allow such weird things as functions. For, this is some thing the graph of which cannot be drawn on a paper.

There are many ways to get such functions. As a simple exercise show that the following function due to McCarthy [1] is also one such:

Another example Define

$$\psi(x) = \begin{cases} 1 + x, & \text{if} \quad -2 \le x \le 0; \\ 1 - x, & \text{if} \quad 0 \le x \le 2; \\ g(x - 4n), & \text{if} \quad -2 \le x - 4n \le 2, \text{ for some integer } n \ne 0. \end{cases}$$

Put $\psi_k(x) = g(2^{2^k} x)$ and $g(x) = \sum_0^\infty g_k(x)/2^k$. Show that g is a Weierstrass' function. [Hint: Consider the sequence $\{x + 2^{-2^k}\}$ or $\{x - 2^{-2^k}\}$ depending upon whether mod 2 you have $0 \le x \le 1$ or $1 \le x \le 2$.

[1]This appeared in Amer. Math. Monthly Vol. LX No. 10 Dec. 1953.

Uniform metric

Let X be any set and $B(X)$ be the set of all real (or complex) valued functions on X which are bounded. Then for each $f \in B(X)$,

$$\|f\| = sup\{|f(x)| \ : \ x \in X\} < \infty$$

and is called the *sup norm* of f. One easily checks that
(a) $f \equiv 0$ iff $\|f\| = 0$.
(b) $|\alpha f\| = |\alpha|\|f|$,$\alpha \in \mathbb{R}(\mathbb{C})$.
(c) $\|f + g\| \leq \|f\| + \|g\|$.

Therefore if we define $d(f, g) = \|f - g\|$, then d becomes a metric on $B(X)$ which is called the *uniform metric* on $B(X)$. Note that if X is a compact metric space then any continuous real valued function on X is bounded. In particular, $\mathcal{C}[a, b] \subset B[a, b]$.

Theorem 1 *A sequence $\{f_n\}$ in $B(X)$ is convergent with respect to the uniform metric iff it is uniformly convergent on X as a sequence of functions.*

Theorem 2 *$B(X)$ is a complete metric space.*

Remark 1 It follows from Weierstrass's theorems, that if K is compact subset of $\mathbb{R}^n$, then the space $\mathcal{C}(K)$ of continuous functions is a closed subset of $B(X)$.

Lecture 26

Let us now see a constructive proof of Weierstrass' approximation theorem.

Theorem 3 Weierstrass' Approximation Theoem *The set of all polynomial functions on $[a, b]$ is dense in $C[a, b]$.*

Proof: Given a continuous function $f : [a, b] \to \mathbb{R}$ and $\epsilon > 0$ we must find a polynomial P such that

$$|f(x) - P(x)| < \epsilon, \quad a \leq x \leq b.$$

Step 1 Enough to prove this for the case $[a, b] = [0, 1]$.
Put $g(t) = f(a + [b - a]t), \quad 0 \leq t \leq 1,$
get a polynomial Q such that

$$|g(y) - Q(y)| < \epsilon, \qquad \text{for } 0 \leq t \leq 1$$

and put $P(x) = Q\left(\frac{x-a}{b-a}\right)$.
Step 2 Bernstein's Polynomials. For $n \geq 1$, and $0 \leq x \leq 1$, define

$$B_n(x) := B_n^f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f(k/n).$$

We have
(I) If $f(x) \equiv 1$ then $B_n^f(x) = 1$.
(II) If $f(x) = x$ then $B_n^f(x) = x$.
(III) If $f(x) = x^2$ then $B_n^f(x) = x^2(1 - \frac{1}{n}) + \frac{x}{n}$.
(IV) $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1 - x)^{n-k} = \frac{x(1-x)}{n}$.
[**Proof:** I is obvious. For II and III consider the binomial expansion

$$(x + y)^n = \sum_{0}^{n} \binom{n}{k} x^k y^{n-k}$$

Differentiate this wrt x and multiply by x/n to obtain

$$x(x + y)^{n-1} = \sum_{0}^{n} \frac{k}{n} \binom{n}{k} x^k y^{n-k}.$$

If you put $y = 1 - x$ now you get II.
Differentiate this again with respect to x multiply by x/n and substitute $y = 1 - x$ to obtain III.
Finally (IV) is verified by expanding out and using I,II,III.]
Step 3 We shall now prove

Lemma 1 Given any continuous function $f : [a, b] \to \mathbb{R}$, the sequence B_n^f of Bernstein polynomials converges uniformly to f on $[0, 1]$.

Given $\epsilon > 0$ choose $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon/2, \text{ for } |x - y| < \delta, \quad x, y \in [0, 1].$$

Now for any $x \in [0, 1]$ by (I) above we have

$$\begin{aligned} & f(x) - B_n(x) \\ = \;& f(x)\sum_0^n \binom{n}{k} x^k(1-x)^{n-k} + \sum_{k=0}^n f(k/n)] \binom{n}{k} x^k(1-x)^{n-k} \\ = \;& \sum_{k=0}^n [f(x) - f(k/n)] \binom{n}{k} x^k(1-x)^{n-k} \\ = \;& \textstyle\sum_{k\in A} + \sum_{k\in B} \end{aligned}$$

where $A = \{k \;:\; |f(x) - f(k/n)| < \frac{\epsilon}{2}\}$ and $B = \{1, 2, \ldots, n\} \setminus B$. Note that A and B depend on x. In any case, we have

$$\left| \sum_{k\in A} [f(x) - f(k/n)] \binom{n}{k} x^k(1-x)^{n-k} \right| < \frac{\epsilon}{2} \sum_0^n \binom{n}{k} x^k(1-x)^{n-k} = \frac{\epsilon}{2}.$$

It is the second sum on the right that needs more careful handling. For $k \in B$ we have $|f(x) - f(k/n)| \geq \epsilon/2$ and therefore, $|x - k/n| \geq \delta$. Therefore

$$\begin{aligned} & \left| \sum_{k\in B} [f(x) - f(k/n)] \binom{n}{k} x^k(1-x)^{n-k} \right| \\ \leq \;& 2\|f\| \sum_{k\in B} \binom{n}{k} x^k(1-x)^{n-k} \frac{(\frac{k}{n} - x)^2}{\delta^2} \\ \leq \;& \frac{2\|f\|}{\delta^2} \frac{x(1-x)}{n} \text{ by (IV)} \\ \leq \;& \tfrac{2\|f\|}{n\delta^2}. \end{aligned}$$

Luckily this result is independent of x. All that we have to do now is to choose N such that $2\frac{\|f\|}{N\delta^2} < \frac{\epsilon}{2}$ i.e., $N > \frac{4\|f\|}{\delta^2\epsilon}$.

$$\left|\sum_{k\in A}[f(x)-f(k/n)]\binom{n}{k}x^k(1-x)^{n-k}\right| < \frac{\epsilon}{2}\sum_{0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = \frac{\epsilon}{2}.$$

♠

Remark 2 The above lemma actually implies, in probability theory the so called **Week law of large numbers.**

Exercise 1 (a) Write down B_1^f, B_2^f, B_3^f explicitly for $f(x) = x^2$, and $f(x) = x^3$.
(b) Learn about Bezier curves used in computer graphics, which are closely related to Bernstein polynomials.

Alternative proof of Weierstrass's theorem:

As before, we may assume that $[a, b] = [0, 1]$. We may further assume that $f(0) = f(1) = 0$, by considering the function $g(x) = f(x) - f(0) - x[f(1) - f(0)]$. Morever we can now extend f all over $\mathbb{R}$ by defining it to be 0 outside $[0, 1]$ so that f is uniformly continuous on $\mathbb{R}$.

Lemma 2 For any continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $supp f \subset [0, 1]$. Define the polynomial functions

$$P_n(f)(x) = \int_0^1 f(s)Q_n(s-x)ds \tag{1}$$

where

$$Q_n(f)(x) = c_n(1-x^2)^n$$

where the constant c_n is chosen so that

$$\int_{-1}^1 Q_n(x)dx = 1, \quad n \geq 1.$$

Then $\{P_n(f)\}$ is a sequence of polynomials converging uniformly to the function f on $\mathbb{R}$.

Proof: For each fixed $x \in \mathbb{R}$, the integrand in (1) is continuous function and hence is Riemann integrable in $[0,1]$. Also, since the integrand is a polynomial in x with coefficients which are continuous functions of s upon taking the definite integral w.r.t. s, we obtain $P_n(f)$ as a polynomial functions in x.

We begin with some estimate of the size of the constants c_n.

Claim: $c_n < \sqrt{n}$:

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^n dx &= 2\int_0^1 (1-x^2)^n dx \\ &\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \\ &\geq 2\int_0^{1/\sqrt{n}} (1-nx^2) dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.\end{aligned}$$

Now if $1 > \delta > 0$, then for $\delta \leq |x| \leq 1$, we have

$$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n.$$

Since $\sqrt{n}(1-\delta^2)^n \to 0$ as $n \to \infty$, $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Next we shall rewrite P_n : Putting $s = x + t$, we get

$$P_n(f)(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt.$$

Since $f = 0$ outside $[0,1]$ we see that for $x \in [0,1]$

$$P_n(f)(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt.$$

Given $\epsilon > 0$ choose $1 > \delta > 0$ so that
$|x-y| < \delta$ implies that $|f(x) - f(y)| < \epsilon/2$.
Let $M = sup\{|f(x)| \ : \ x \in \mathbb{R}\}$.

Then for any $x \in [0,1]$

$$\begin{aligned} |P_n(x) - f(x)| &= \left|\int_{-1}^{1} [f(x+t) - f(x)] Q_n(t) dt\right| \\ &\leq \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t) dt \\ &\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^{1} Q_n(t) dt \\ &\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

for sufficiently large n. ♠

Lecture 27

Remark 3 Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, it is not true that we can find a sequence of polynomials approximating f all over $\mathbb{R}$. For instance, in the above discsussions, the polynomials P_n wouild obviously diverge to $\pm\infty$ as $x \to \infty$ whereas the function f is identically 0 outside $[0,1]$.

Remark 4 The space $B(K)$ is not only a vector space but is also an algebra, i.e., if $f, g \in B(X)$ $fg \in B(X)$. We have earlier remarked that if K is a compact subset of $\mathbb{R}^n$ then $\mathcal{C}(K)$ is a closed subset of $B(K)$. Indeed we can also verify that $C(K)$ is a subalgebra. More generally we have,

Theorem 4 *If A is a subalgebra of $B(X)$ then $\bar{A}$ is a subalgebra of $B(X)$.*

Definition 1 Let A be a family of functions on a set X. We say A separates points in X if given any two distinct points $x_1, x_2 \in X$ there exists at least one $f \in A$ such that $f(x_1) \neq f(x_2)$. Likewise, we say A vanishes at no point of X if for each $x \in X$ there is at least one $f \in A$ such that $f(x) \neq 0$.

Example 2 A typical example of A satisfying the above properites is the family of polynomial functions where X is any subset of $\mathbb{R}^n$. On the other hand if we take the family of even polynomials on $[-1,1]$ it does not separates points and the family of odd polynomials does vanishes at $x = 0$.

Theorem 5 *Let A be an algebra of (real or complex valued) functions on a set X which separates points of X and which does not vanish at any point of X. Given $x_1 \neq x_2$ and constants c_1, c_2 there exists $f \in A$ such that $f(x_j) = c_j, j = 1, 2$.*

Proof: First find functions g, h, k such that

$$g(x_1) \neq g(x_2), h(x_1) \neq 0, \quad k(x_2) \neq 0.$$

Put

$$f(x) = c_1 \frac{(g(x) - g(x_2))h(x)}{(g(x_1) - g(x_2))h(x_1)} + c_2 \frac{(g(x) - g(x_1))k(x)}{(g(x_2) - g(x_1))k(x_2)}.$$

♠

Remark 5 Are you reminded of Lagrange's interpolation formula? Notice the role of the functions h, k in the above formula. Why does the simpler formula

$$f(x) = c_1 \frac{(g(x) - g(x_2)}{(g(x_1) - g(x_2))} + c_2 \frac{(g(x) - g(x_1))}{(g(x_2) - g(x_1))}$$

doe not work? Simply because we do not know whether the constant function 1 is in A or not.

Theorem 6 Stone-Weierstrass Theorem *Let A be an algebra of continuous real valued functions on a compact metric space X which separates points of X and vanishes at no point of X. Then $\bar{A} = C(X)$.*

Proof: Note that $A \subset C(X) \subset B(X)$ implies $\bar{A} \subset C(X)$ because the latter is closed in $B(X)$. So we have to show $C(X) \subset \bar{A}$.
Step 1: If $f \in \bar{A}$ then $|f| \in \bar{A}$.
Let $a = \sup \{|f(x)| \; : \; x \in X\}$. Now find polynomials $P_n(t)$ such that $|P_n(t) -$

$|t|| < \frac{1}{n}$ for $-a \leq t \leq a$ (exists by Weierstrass's theorem.) We can also assume that $P_n(0) = 0$ by considering $Q_n(t) = P_n(t) - P_n(0)$. Consider $g_n(x) = P_n(f(x)) = c_1 f(x) + c_2 f^2(x) + \cdots c_k f^k(x) \in A$. On the other hand for all $x \in X$, we have

$$|g_n(x) - |f(x)|| = |P_n(f(x)) - |(f(x)|| < \frac{1}{n}$$

This implies $g_n \to |f|$ and we are through.

Step 2 If $f, g \in \bar{A}$, then $max\{f, g\}, min\{f, g\} \in \bar{A}$.

This follows since

$$max\{f, g\} = \frac{f + g + |f - g|}{2}; \quad min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

By repeated application of this it follows that maximum (or minimum) of finitely many functions in $\bar{A}$ is again in $\bar{A}$.

Step 3 Let $f : X \to \mathbb{R}$ be a continuous function and $x \in X$. Given $\epsilon > 0$ there exists $g_x \in \bar{A}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(x) - \epsilon, \ t \in X. \tag{2}$$

Using the property of separation of points and nonvanishing, it follows that for every $t \in X$, we have a function $h_t \in A$ such that $h_t(x) = f(x), h_t(t) = f(t)$. By continuity of $h_t - f$, there is a nbd V_t of t in X such that $h_t(y) > f(y) - \epsilon$ for $y \in V_t$. Since X is compact, we get

$$X \subset V_{t_1} \cup V_{t_2} \cup \cdots \cup V_{t_k}.$$

Put

$$g_x = max\,\{h_{t_1}, \ldots, h_{t_k}\}.$$

Then $g_x(x) = f(x)$ and if $y \in X$ is such that $y \in V_{t_i}$, we have

$$g_x(y) \geq h_{t_i}(y) > f(y) - \epsilon, y \in V_{t_i}.$$

By Step 2, $g_x \in \bar{A}$.

Step 4 Given a continuous function $f : X \to \mathbb{R}$ and $\epsilon > 0$ there exists $g \in \bar{A}$ such that $|f(t) - g(t)| < \epsilon, t \in X$.

For each $x \in X$, let $g_x \in \bar{A}$ be a function as in Step 3. By continuity of $g_x - f$ there is a nbd U_x of x such that $g_x(y) < f(y) + \epsilon$ for all $y \in X$. Cover X with finitely many $U_{x_1}, \ldots, U_{x_m}$ and take $g = min\,\{g_{x_1}, \ldots, g_{x_m}\}$. By step 2, $g \in \bar{A}$. Since each g_{x_i} has the property (2), it follows that $g(y) > f(y) - \epsilon, y \in X$. On the other hand, if $y \in U_{x_i}$ then $g(y) \leq g_{x_i}(y) < f(y) + \epsilon$. Therefore for all $y \in X$, we have $f(y) - \epsilon < g(y) < f(y) + \epsilon$. ♠

Remark 6 The theorem does not hold for algebras of complex valued functions without the additional hypothesis that A is self-adjoint, i.e., it is closed under conjugation, i.e., if $f = u + \imath v \in A$ then $\bar{f} = u - \imath v \in A$. This can be illustrated by the following example.

Let $X = \mathbb{S}^1$, the unit circle and A be the algebra of all polynomial functions with complex coefficients. The A separates points and the polynomial $z \in A$ does not vanish on A. The function $f(z) = \frac{1}{z}$ is continuous on X. However, it does not belong to $\bar{A}$. For, we have $\int_{\mathbb{S}^1} P(z)dz = 0$ for all polynomials whereas $\int_{\mathbb{S}^1} \frac{dz}{z} = 2\pi\imath$. If there were a sequence of polynomials uniformly converging to $1/z$ then the integral should have been zero according to theoreom **??**.

The situation can be saved if we make one more assumption.

Theorem 7 *Let X be any compact metric space and A be a self adjoint algebra over $\mathbb{C}$, of complex valued continuous functions on X. Assume that A separates points of X and does not vanish anywhere on X. Then $\bar{A}$ contains all continuous complex valued functions on X.*

Proof: (Note that A has the additional property: $f \in A \Longrightarrow \imath f, \bar{f} \in A$ as compared with an algebra over $\mathbb{R}$ being an algebra over complex numbers, which is implict when we talk about self-adjoint algebras.)

Let $A_{\mathbb{R}}$ denote the subspace of all members of A which take real values only. Then A is a subalgebra which also has these two additional properties: For first of all observe that if $f \in A$ then $\Re(f) = (f + \bar{f})/2 \in A$ and $\Im(f) = (f - \bar{f})/2\imath \in A$. Therefore $\Re(f), \Im(f) \in A_{\mathbb{R}}$. Now given $x_1 \neq x_2 \in X$ let $f \in A$ be such that $f(x_1) \neq f(x_2)$. Then $\Re(f(x_1)) \neq Re(f(x_2))$ or $\Im(f(x_1)) \neq \Im(f(x_2))$ and accordingly, we get some $g \in A_{\mathbb{R}}$ with $g(x_1) \neq g(x_2)$. Similarly, if $f \in A$ is such that $f(x) \neq 0$ then one of $\Re(f)(x) \neq 0, \Im(f(x)) \neq 0$ is true and so we are done.

Now given any continuous function $f : X \to \mathbb{C}$ we can apply the real Stone-Weierstrass theorem to conclude that $\Re(f) \in \bar{A}$ and $\Im(f) \in \bar{A}$. Therefore $f \in \bar{A}$. ♠

Week 10 lectures 28-30

Lecture 28. Fourier Series

Some important Exercises on Integration:

Exercise 2 Throughout, let α be a fixed increasing function on $[a,b]$.

1. **Famous Inequalities** Let $p > 1$ be a positive real number. $\frac{1}{p} + \frac{1}{q} = 1$.
(a) Show that $\phi(x) = \frac{1}{p}x - x^{1/p}$, attains its minimum at $x = 1$. Put $\phi(1) = \frac{1}{p} - 1 = \frac{1}{q}$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Note that both $p, q > 1$. They are called 'dual pair' of numbers, i.e., q is the dual of p and p is the dual of q. Observe that if $p = 2$ then $q = 2$, i.e., 2 is dula to itself.
(b) If $u, v \geq 0$ then
$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$
Show that equality holds iff $u^p = v^q$.
(c) Let $f, g \in \mathcal{R}(\alpha)$ and $f, g \geq 0$ such that
$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha.$$
Then show that $\int_a^b fg d\alpha \leq 1$.
(d) Let f, g be any complex valued functions in $\mathcal{R}(\alpha)$. Then prove that
Holder's Inequality:
$$\left|\int_a^b fg d\alpha\right| \leq \left(\int_a^b |f|^p d\alpha\right)^{1/p} \left(\int_a^b |g|^q d\alpha\right)^{1/q}.$$

(e) **Schwarz's Inequality** With f, g as in (d), show that

$$\left|\int_a^b fg d\alpha\right| \leq \left(\int_a^b |f|^2 d\alpha\right)^{1/2} \left(\int_a^b |g|^2 d\alpha\right)^{1/2}.$$

(f) For any $u \in \mathcal{R}(\alpha)$ define and $p > 0$

$$\|u\|_p := \left[\int_a^b |u|^p d\alpha\right]^{1/p}.$$

For any $f, g, h \in \mathcal{R}(\alpha)$ prove **Minkowski Iniquality:**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

(g) Show that $d_p(f, g) = \|f - g\|_p$ satisfies **triangle inequality**.

Solution:

(a) $\phi'(x) = 0$ iff $x = 1$ and $\phi''(1) > 0$. The conclusion follows.

(b) Put $x = u^p/v^q$ in (a).

(c) $f, g \in \mathcal{R}(\alpha)$ implies $|f|^p, |g|^q \in \mathcal{R}(\alpha)$. (Why? Remember how we proved $f^2 \in \mathcal{R}(\alpha)$?) Now by (b) $f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}$. Upon taking integration and use the fact $\frac{1}{p} + \frac{1}{q} = 1$ we are done.

(d) Apply (c) to appropriate multiples of f, g.

(e) Put $p = q = 2$.

(f) Notice that $\frac{1}{p} + \frac{1}{q} = 1, p, q > 0$ implies $p, q \geq 1$. Put $k = \int_a^b (|f| + |g|)^p d\alpha$. Then

$$\begin{aligned} k &= \int_a^b (|f| + |g|)(|f| + |g|)^{p-1} d\alpha \\ &= \int_a^b (|f|(|f| + |g|)^{p-1} d\alpha + \int_a^b |g|(|f| + |g|)^{p-1} d\alpha \\ &\leq \left(\int_a^b |f|^p d\alpha\right)^{1/p} \left(\int_a^b (|f| + |g|)^{(p-1)q} d\alpha\right)^{1/q} \\ &\quad + \left(\int_a^b |g|^p d\alpha\right)^{1/p} \left(\int_a^b (|f| + |g|)^{(p-1)q} d\alpha\right)^{1/q} \\ &= \left[\left(\int_a^b |f|^p d\alpha\right)^{1/p} + \left(\int_a^b |g|^p d\alpha\right)^{1/p}\right] k^{1/q} \end{aligned}$$

because $(p-1)q = p$. The result follows.

(g) Easy.

2. Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Given $\epsilon > 0$ show that there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $\|f - g\|_2 < \epsilon$.

   Sol: Given ϵ positive it is enough construct a continuous function g and a partition $P = \{a = a_0 < a_1 \cdots < a_n = b\}$ such that

$$\sum_i |f(t_i) - g(t_i)|^2 \Delta\alpha_i < \epsilon.$$

   Let $M > 0$ be such that $|f| < M$. Choose P so that

$$\sum_i |f(t_i) - f(s_i)| \Delta\alpha_i < \epsilon^2/8M, \quad \text{for all } t_i, s_i \in [a_{i-1}, a_i].$$

   It follows that

$$\sum_i |f(t_i) - f(s_i)|^2 \Delta\alpha_i < \epsilon^2/4, \quad \text{for all } t_i, s_i \in [a_{i-1}, a_i].$$

   Put $\Delta x_i = a_i - a_{i-1}$ and

$$g(t) = \frac{a_i - t}{\Delta x_i} f(a_{i-1}) + \frac{t - a_{i-1}}{\Delta x_i} f(a_i), \quad a_{i-1} \le t \le a_i.$$

   Then clearly g is continuous. For $a_{i-1} \le t_i \le a_i$ we have,

$$f(t_i) - g(t_i) = \frac{a_i - t_i}{\Delta x_i}(f(t_i) - f(a_{i-1})) + \frac{t_i - a_{i-1}}{\Delta x_i}(f(t_i) - f(a_i))$$

   Therefore

$$|f(t_i) - g(t_i)| \le |(f(t_i) - f(a_{i-1})| + |(f(t_i) - f(a_i)| \le 2|f(t_i) - f(s_i)|$$

   where $s_i = a_i$ or a_{i-1}. Therefore

$$\sum_i |f(t_i) - g(t_i)|^2 \Delta\alpha_i \le 4 \sum_i |f(t_i) - f(s_i)|^2 \Delta\alpha \le 8M \sum_i |f(t_i) - f(s_i)| \Delta\alpha_i < \epsilon^2.$$

Definition 2 A function $f : \mathbb{R} \to \mathbb{R}, (\mathbb{C})$ is called periodic with period $\lambda > 0$ if $f(x + \lambda) = f(x)$ for all $x \in \mathbb{R}$.

As an immediate corollary of Theorem 7, (Stone-Weierstrass Theorem), we have

Theorem 8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Then there exists a sequence*

$$S_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx), \quad a_0, a_n, b_n \in \mathbb{R}, \tag{3}$$

which converges uniformly to f on the whole of $\mathbb{R}$.

Proof: Functions of the above form S_N are called trigonometric polynomials. Notice that each summand that occurs on the RHS of the formula for S_N has the property

$$g(x+2\pi) = g(x), \quad x \in \mathbb{R}.$$

Such functions are called periodic with period 2π. The important thing to note about them is that their behavior on $\mathbb{R}$ is completely known by their behaviour on any interval of length ($\geq$) 2π.

If we allow complex coefficients a_0, a_n, b_n in (3) then using the identities

$$\cos x = \frac{e^{\imath x} + e^{-\imath x}}{2}; \quad \sin x = \frac{e^{\imath x} - e^{-\imath x}}{2\imath},$$

it follows that we can rewrite (3) in the form

$$s_N(x) = \sum_{-N}^{N} c_n e^{inx}, c_n \in \mathbb{C}. \tag{4}$$

Let A denote the collection of all such functions s_N. Check that A is a self-adjoint algebra of continuous functions on the whole of $\mathbb{R}$ (but we shall consider these functions on the closed interval $[-\pi, \pi]$). Also check that this algebra separates points of $[-\pi, \pi]$ and does not vanish anywhere (since it contains constant functions). Therefore its closure contains the space $\mathcal{C}[-\pi, \pi]$.

Now given any continuous periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 2π restrict $f : [-\pi, \pi] \to \mathbb{R}$. Now by what we have concluded above, we get a sequence $\{s_N(x)\} \in A$ (with coefficients $a_0, a_n, b_n \in \mathbb{C}$) which uniformly converges to f. Upon rewriting it in terms of $\cos nx$ and $\sin nx$ and taking the real part the theorem follows. ♠

The above theorem prods us into studying many related concepts which lead us to the so called Theory of Fourier series. We shall only give a few basics of this vast theory here depending only on the mathematics that we have developed so far. Full justification to this topic cannot be done without the support of Lebesgue theory.

Lemma 3 Let n be an integer. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\imath nx}dx = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Definition 3 By a trigonometric series we mean a sum of the form

$$\sum_{-\infty}^{\infty} c_n e^{\imath nx} \tag{6}$$

whose N^{th}-partial sum s_N is given by (4). Given a Riemann integrable function f on $[-\pi, \pi]$, and an integer n, we define its n^{th} Fourier coefficient by the formula

$$c_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\imath nx}dx. \tag{7}$$

The Fourier series (also called trigonometric series) associated to f is defined to be $\sum_{-\infty}^{\infty} c_n(f)e^{\imath nx}$. We express this often by

$$f \sim \sum_{-\infty}^{\infty} c_n(f)e^{\imath nx}. \tag{8}$$

Remark 7 We observe that if S_N is a trigonometric polynomial as in (4), then $c_n(S_N) = c_n$, for $|n| \leq N$ and $c_n(S_N) = 0, |n| > N$. Thus the Fourier series of S_N reduces to a trigonometric polynomial. One of the fundamental problem in the theory is when can we write $=$ in place of $\sim$ in (8)? Of course there are many subquestions related to this as well viz., what should be the meaning of '$=$' here. For instance, it is clear that at all cost we should insist that RHS converges. If the convergence is uniform then it follows that the function represented is periodic and moreover continuous. The first property is desirable whereas the second one is NOT. The applications that we have in mind involve, quite often, functions which have discontinuities.

For instance if the series (6) converges to some function $\hat{f}$, then we would like that the so called Euler's formula

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(x)e^{\imath nx}dx$$

to be true. If we grant uniform convergence, then term-by-term integration is valid and so using (5) one easily checks that this property is true. (The situation is similar to the case of an analytic function whose n^{th} derivative at 0 determines the coefficeint of x^n in the power series expansion.) For trigonometric series or for more general Fourier series, we are looking for similar properties under more general conditions than uniform convergence.

Definition 4 Let $\{\phi_j\}$ be a family of complex valued integrable functions on $[a, b]$ with the property:

$$\int_a^b \phi_j(x)\overline{\phi_k(x)}dx = 0, j \neq k. \tag{9}$$

Then we say $\{\phi_j\}$ is an orthogonal family of functions. In addition if

$$\int_a^b |\phi_j(x)|^2 dx = 1 \tag{10}$$

we call it an orthonormal family.

Example 3 We have seen that the family $\{\frac{e^{\imath nx}}{\sqrt{2\pi}}\}$ is an orthonormal family on $[-\pi, \pi]$. Similarly,

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{2\pi}}, \frac{\sin 2x}{\sqrt{2\pi}}, \cdots\}$$

is also an orthonormal family on $[-\pi, \pi]$.

Definition 5 Given an integrable function f on $[a, b]$ we define

$$c_j(f) := \int_a^b f(t)\overline{\phi_j(x)}dx \tag{11}$$

to be the Fourier coefficient of f with respect to the family $\{\phi_j\}$. Moreover the formal sum $\sum_j c_j(f)\phi_j(x)$ is then called the Fourier series of f with respect to $\{\phi_j\}$. And we express this by

$$f(x) \sim \sum_j c_j(f)\phi_j(x).$$

For any two integrable functions, f, g on $[a, b]$, let us write

$$\langle f, g\rangle = \int_a^b f\bar{g}dx.$$

Also let us write

$$\|f\| := ||f||_2 := \sqrt{\langle f, f\rangle}.$$

Theorem 9 **Pythagorus theorem:** *If* $\langle f, g\rangle = 0$ *then*

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

Proof: Direct.

Theorem 10 **Least Square Approximation** *Let* f *be an integrable function on* $[a, b]$. *Let* $\{\phi_n\}$ *be an orthonormal system and*

$$s_n(x) := \sum_{m=1}^{n} c_m\phi_m(x)$$

be the n^{th} partial sum of the Fourier series of f. Then for all

$$t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$$

we have

$$\int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx \qquad (12)$$

with equality holding iff $\gamma_m = c_m$, for all $1 \leq m \leq n$.

Proof: Check that $f - s_n$ is orthogonal to $s_n - t_n$ and use the above theorem to conclude that

$$\|f - t_n\|^2 = \|f - s_n\|^2 + \|s_n - t_n\|^2.$$

This proves (10). As for the last part, repeated application of Pythagoras yields

$$\|s_n - t_n\|^2 = \sum_{m=1}^{n} |c_m - \gamma_m|^2$$

from which the conclusion follows.

Theorem 11 Bessel's Inequality: *For any integrable function f on $[a, b]$ if $f \sim \sum_m c_m \phi_m$ then*

$$\sum_n |c_n|^2 \leq \|f\|^2$$

Proof: Putting $t_m = 0$ in the proof of the above theorem, we first obtain that $f - s_n$ is orthogonal to s_n. (Or do this directly afresh). Again by Pythagorus theorem, we get

$$\|f\|^2 = \|f - s_n\|^2 + \|s_n\|^2.$$

The conclusion follows. ♠

In particular, we have the so called

Theorem 12 Lebesgue-Riemann theorem: *For any integrable function f on $[-\pi, \pi]$ the sequence of Fourier coefficients converges to* $0:$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos kt\,dt = 0; \quad \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin kt\,dt = 0. \tag{13}$$

Proof: Bessel's inequality implies that $\lim_{n\to\pm\infty} c_n = 0$ and we also have $c_n = \bar{c_n}$. The above two quantities are nothing but $\frac{c_n+\bar{c_n}}{2}$ and $\frac{\bar{c_n}-c_n}{2}$, the real and imaginary parts of c_n. ♠

Lecture 30

Theorem 13 Parseval's Theorem: *Let f, g be integrable functions with period 2π. Put*

$$f(x) \sim \sum_{-\infty}^{\infty} c_m e^{\imath mx}; \ g(x) \sim \sum_{-\infty}^{\infty} \gamma_m e^{\imath mx}.$$

Then
(i) $\displaystyle\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-s_N(f;x)|^2dx = 0.$
(ii) $\displaystyle\frac{1}{2\pi}\int_a^b f(x)\overline{g(x)}dx = \sum_{-\infty}^{\infty} c_m\gamma_m.$
(iii) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{-\infty}^{\infty}|c_m|^2.$

Proof: We shall denote by $\|h\|_2 = \left(\frac{1}{2\pi}\int_a^b |h(x)|^2dx\right)^{1/2}$. Since f is integrable and $f(-\pi) = f(\pi)$, from a previous exercise 2.2, given $\epsilon > 0$, we have a continuous 2π-periodic function h such that

$$\|f - h\|_2 < \epsilon.$$

By the theorem 8 above, there is a trigonometric polynomial

$$P = \sum_{-N}^{N}\gamma_m e^{\imath mx}$$

of degree N, say, such that $|P(x) - h(x)| < \epsilon$ for all $x \in [-\pi, \pi]$ and hence $\|P - h\|_2 < \epsilon$.

Let us use a slightly modified notation: for any $g \in \mathcal{R}(\alpha)[-\pi, \pi]$,

$$s_n(g) := \sum_{-n}^{n} c_k(g) e^{\imath kx}$$

By Least Square Approximation, it follows that

$$\|h - s_n(h)\|_2 \leq \|h - P\|_2 < \epsilon, \text{ for } n \geq N.$$

Also Bessel's inequality, we have,

$$\|s_n(h) - s_n(f)\|_2 = \|s_n(h - f)\|_2 \leq \|h - f\|_2 < \epsilon.$$

Finally by Triangle inequality, we have

$$\|f - s_n(f)\|_2 \leq \|f - h\|_2 + \|h - s_n(h)\|_2 + \|s_n(h) - s_n(f)\|_2 < 3\epsilon$$

for all $n \geq N$. This proves (i).

To prove (ii), we first observe that at finite sum level, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f) \bar{g} dx = \frac{1}{2\pi} \sum_{-N}^{N} \int_{-\pi}^{\pi} c_n e^{\imath nx} \overline{g(x)} dx = \sum_{-N}^{N} c_n \bar{\gamma}_n.$$

Therefore, using Schwarz's inequality, we get

$$\left| \int f\bar{g} - \int s_N \bar{g} \right| \leq \int |f - s_n| \, |g| \leq \left(\int |f - s_N|^2 \right)^{1/2} \left(\int |g|^2 \right)^{1/2}.$$

Letting $N \to \infty$ we get (ii).

(iii) follows from (ii) by putting $g = f$. ♠

Convergence problem for Trigonometric Series.

We shall now on deal with only trigonometric series and consider functions f with period 2π which are Riemann integrable over $[-\pi, \pi]$.

Consider the trigonometric polynomial with all its coeffients equal to 1. (By analogy, this plays the role of the polynomial which is the n^{th} partial sum of the geometric series for $(1-x)^{-1}$.) The trigonometric polynomial

$$D_N(x) = \sum_{-N}^{N} e^{\imath nx}$$

is called the Dirichlet's kernel. Multiplying it by $e^{\imath x} - 1$ we get

$$(e^{\imath x} - 1)D_N(x) = e^{\imath(N+1)x} - e^{-\imath Nx}.$$

Multiplying further by $e^{-\imath x/2}$ we get

$$2\imath \sin(x/2) D_N(x) = 2\imath \sin(N + 1/2)x.$$

Therefore

$$D_N(x) = \frac{\sin(N+1/2)x}{\sin x/2}. \tag{14}$$

Another interesting property of Dirichlet's kernel is that

$$\int_{-\pi}^{\pi} D_n(t)dt = 2\pi. \tag{15}$$

Given any $f \in \mathcal{R}(\alpha)[-\pi,\pi]$ we can rewrite $s_N(f)$ in terms of Dirichlet's kernel:

$$\begin{aligned}
s_N(f)(x) &= \sum_{-N}^{N} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(t)e^{-\imath nt}dt\right) e^{\imath nx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{-N}^{N} e^{\imath n(x-t)}\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt \\
&= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)D_n(s)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)D_n(s)ds
\end{aligned}$$

the last equality being the result of periodicity of the integrand.

We shall now prove a local convergence theorem:

Theorem 14 *Suppose for some x, there exist $\delta > 0, M < \infty$ such that*

$$|f(x+t) - f(x)| \leq M|t|, \quad t \in (-\delta, \delta). \tag{16}$$

Then

$$\lim_{N\to\infty} s_N(f,x) = f(x).$$

Proof: Put

$$g(t) = \begin{cases} \frac{f(x-t)-f(x)}{\sin(t/2)} & 0 < |t| < \pi \\ 0, & t = 0. \end{cases}$$

We first note that $g \in \mathcal{R}(\alpha) \in [-\pi, \pi]$. [Let us prove that g satisfies R-condition in $[0, \pi]$ the proof for the interval $[-\pi, 0]$ being the same. Given $\epsilon > 0$ we can choose $\delta_1 > 0$ such that $|t/\sin(t/2)| < 2$. Now choose $\delta_2 = \min\{\delta, \delta_1, \epsilon/8M\}$. Now observe that in $[\delta_2, \pi]$, g is integrable and hence we can find a partition $P := \{\delta_2 = a_1 < a_2 < \cdots a_n = \pi\}$ in which g satisfies Riemann's condition for $\epsilon/2$. It then follows that for the partition $Q := \{0 < \delta_2 = a_1 < \cdots < a_n\}$, g satisfies Riemann's condition in the interval $[0, \pi]$ for ϵ.]

Using (15) we get

$$\begin{aligned} & s_N(f;x) - f(x) \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]D_n(t)\, dt \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) sin(t/2) D_n(t)\, dt \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(N+1/2)t\, dt \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)[\sin(t/2)\cos N(t) + \cos(t/2)\sin N(t)]dt \\ = & \alpha_N + \beta_N \end{aligned}$$

where α_N and β_N are respectively real part of the N^{th} Fourier coefficient of $g(t)\sin(t/2)$ and the imaginary part of the N^{th} Fourier coefficient of $g(t)\cos(t/2)$. Because of (16) both these functions are Riemann integrable functions in the closed interval. Therefore, by Lebesgue Riemann (12), it follows that $\alpha_N \to 0, \beta_N \to 0$ as $N \to \infty$. ♠

Remark 8 It follows that if $f \in \mathcal{C}^2$ then it satisfies (16) and hence the Fourier series is convergent. However, by carrying out integration by parts twice and using Weierstrass's majorant criterion, one can directly prove that the Fourier series is uniformly convergent to a function g. But then term-by-term integration is valid and hence it follows that the function g is equal to f.

Lemma 4 Let $g \in \mathcal{R}(\alpha)[0, \pi]$. Then

$$\lim_{N\to\infty} \int_0^\pi g(s) \sin[(N + 1/2)s]ds = 0. \tag{17}$$

Proof: Extend g to all over $[-\pi, \pi]$ by defining $g(t) = 0$ for $t \in [\pi, 0)$. Then $g \in \mathcal{R}(\alpha)[-\pi, \pi]$ and we have

$$\int_0^\pi g(s) \sin[(N + 1/2)s]ds = \int_{-\pi}^\pi g(s) \sin[(N + 1/2)s]ds.$$

Use the fact

$$\sin[(N + 1/2)s] = \sin Ns \cos(s/2) + \cos Ns \sin s/2$$

and appeal to the theorem 12. ♠

Theorem 15 *Let $f \in \mathcal{R}(\alpha)[-\pi, \pi]$ and let $x \in [-\pi, \pi]$. Assume that $f(x^\pm), f'(x^\pm)$ exist. Then the Fourier series for f at x will converge to $[f(x^+) + f(x^-)]/2$.*

Proof: The hypothesis $f'(x^+), f(x^-)$ exist implies that f satisfies the following Lipschitz condtions:

$$|f(x + t) - f(x^+)| \le Mt, \quad \text{for } 0 \le t \le \delta$$

and

$$|f(x-t) - f(x^-)| \leq Mt, \quad \text{for } 0 \leq t \leq \delta$$

for some $M, \delta > 0$.

Now we use the property $D_N(x) = D_n(-x)$ to see that

$$s_N(f) = \frac{1}{2\pi}\int_0^\pi [f(x+s) + f(x-s)]D_N(s)ds.$$

Therefore

$$\begin{aligned} & s_N(f,x) - \frac{f(x^+) + f(x^-)}{2} \\ = & \frac{1}{2\pi}\int_0^\pi [f(x+s) + f(x-s) - f(x^+) - f(x^-)]D_N(s)ds \\ \leq & \frac{1}{2\pi}\int_0^\pi (f(x+s) - f(x^+))D_N(s)ds + \frac{1}{2\pi}\int_0^\pi (f(x-s) - f(x^-))D_N(s)ds \\ = & \frac{1}{2\pi}\int_0^\pi g_+(s)\sin[(N+1/2)s]ds + \frac{1}{2\pi}\int_0^\pi g_-(s)\sin[(N+1/2)s]ds \end{aligned}$$

where $g_\pm$ are defined in a similar way as in the proof of the above theorem:

$$g_\pm(s) = \begin{cases} \frac{f(x\pm s)-f(x^\pm)}{\sin(s/2)}, & 0 < s \leq \pi; \\ 0, & s = 0. \end{cases}$$

Exactly as in the above theorem, it follows that $g_\pm \in \mathcal{R}(\alpha)[0, \pm\pi]$. By the lemma above each of the terms on the rhs converge to 0 and we are through. ♠

(C,1) Summability of Fourier series

Given $f \in \mathcal{R}(\alpha)[-\pi, \pi]$, let us discuss the $(C,1)-$summability of the series

$$\sum c_n(f;x)e^{-\imath nx}.$$

We consider the sequence

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_n(f;x)$$

and ask the question under what conditions

$$\lim_{n\to\infty} \sigma_n(x) = f(x)?.$$

Thus it is natural to consider the sequence of sums,

$$K_n(x) = \frac{1}{n}\sum_{0}^{n-1} D_k(x).$$

These functions are called Fejer kernels. We have

$$K_n(x) = \frac{1}{n\sin(x/2)}\sum_{k=0}^{n-1}\sin(k+1/2)x = \frac{\sin^2 nx/2}{2n\sin^2(x/2)}.$$

Also observe that from (15), it follows that

$$\int_{-\pi}^{\pi} K_n(x)dx = 2\pi. \tag{18}$$

Theorem 16 *Let $f \in \mathcal{R}(\alpha)[-\pi,\pi]$ and $x \in (-\pi,\pi)$ be such that f is continuous at x. Then the Fourier series of $f(x)$ is $(C,1)-$convergent to $f(x)$ at x.*

Proof: We have to show that $\sigma_n(x) \to f(x)$. As before, this is the same as showing

$$\lim_{n\to\infty}\int_0^{\pi}[f(x+t)+f(x-t)-2f(x)]K_n(t)dt = 0.$$

By continuity of f at x we can find $0 < \delta < |\pi - x|$ such that for $t \le \delta$ we have

$$|f(x+t)+f(x-t)-2f(x)| < \epsilon/2.$$

On the other hand, for $t \ge \delta$ we have

$$K_n(t) = \frac{\sin^2(nt/2)}{2n\sin^2(t/2)} \le \frac{1}{2n\sin^2(\delta/2)}$$

and hence for sufficiently large n we can make

$$\left|\int_{\delta}^{\pi}[f(x+t)+f(x-t)-2f(x)]K_n(t)\right| \le \frac{2\pi M}{n\sin^2(\delta/2)} < \epsilon/2.$$

The theorem follows. ♠

Remark 9 If x is one of the end points $\pm\pi$ then the continuity of f at x should be interpreted to mean that $f(-\pi) = f(\pi)$ and the extended function defined by $f(x+2\pi) = f(x)$ all over $\mathbb{R}$, should be continuous at $x = \pi$. With this meaning the above arguments go through in this case also. Further, if f is continuous on the whole of $[-\pi, \pi]$ (and $f(-\pi) = f(\pi)$ then the choise of δ in the above proof can be made independent of x and so is the choice of n. This yields:

Theorem 17 *Let f be a periodic continuous function. Then the Fourier series of f is uniformly $(C, 1)$-convergent to f all over $\mathbb{R}$.*

Exercise 3

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a non constant function such that

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

(i) If f is continuous at $x = 0$ show that it is continuous on $\mathbb{R}$.
(ii) Determine all such continuous f.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a non constant function such that

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

(i) If f is continuous at $x = 0$ show that it is continuous on $\mathbb{R}$.
(ii) Determine all such continuous f.

3. Apply Parseval's theorem to the function $f(x) = x,\ 0 \le x < 2\pi$ and obtain the value of $\sum_0^\infty \frac{1}{n^2}$.

4. Prove that on $[-\pi, \pi]$

$$(\pi - |x|)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Evaluate $\sum_0^\infty \frac{1}{n^2}$; $\quad \sum_0^\infty \frac{1}{n^4}$.

5. **Integration by Parts:** Let α be an increasing function on $[a, b]$. Suppose $f(x) = F'(x)$ on $[a, b]$. Then

$$\int_a^b \alpha(x) f(x) dx = F(b)\alpha(b) - F(a)\alpha(a) - \int_a^b F d\alpha.$$