MATHEMATICS OLYMPIAD 2023
Mathematics Association IIT Bombay

Code

A

Maximum Marks: 60
Time: 2 Hours

Please read the following instructions carefully before attempting the question paper.

- Do not open this booklet until instructed by the invigilators.

- Questions must be answered on **C**andidate **R**esponse **S**heet (**CRS**) by darkening the appropriate bubbles (marked a, b, c, d or $0 - 9$) using blue/black ball pen.

- Write your registration number, your name, and the Question Paper code (**A**, **B**, **C** or **D**) in the space provided on the CRS.

- The question paper consists of two sections: **Section 1** and **Section 2**.

- For Section 1:
  - There are **15 Objective Questions** each of which has **exactly one correct answer**.
  - Each correct answer will be awarded **+2** marks and each incorrect answer will be awarded **−1** mark.
  - Darken **exactly one bubble** for each question using black/blue ball pen.

- For Section 2:
  - There are **15 Numerical Answer Type Questions** of which the answer is **an integer between 000 and 999 (both inclusive)**.
  - Each correct answer will be awarded **+2** marks and there will be no negative marking.
  - Darken **three bubbles for each digit** for each question using black/blue ball pen, e.g., 9 should be bubbled as 009, 10 should be bubbled as 010, 831 should be bubbled as 831.

- In case of ties, the following order will be followed to rank the candidates:
  1. Higher number of correct asterisk(∗) questions.
  2. Lower number of wrong answers in Section 1.
  3. Higher score in Section 2.

- This question paper contains **10** printed pages. Please check all pages and report if there is any discrepancy.

- You won't be allowed to leave the examination hall during the exam.

Name: _____

Roll Number: _____

_____
Invigilator's signature

Notation

- $\mathbb{N}$ = the set of all natural numbers = $\{1, 2, 3, \ldots\}$.

- $\mathbb{Z}$ = the set of all integers = $\{0, -1, 1, -2, 2, \ldots\}$.

- $\mathbb{R}$ = the set of all real numbers.

- $x \in A$ means "x is an element of the set A" or "x belongs to the set A".

- For a complex number $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, $\mathrm{Re}(z) = a$ = real part of z and $\mathrm{Im}(z) = b$ = imaginary part of z.

- For $a, b \in \mathbb{R}$,

  (i) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
  (ii) $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$.
  (iii) $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.
  (iv) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

- The base$-b$ representation of a number n is the unique representation $a_k a_{k-1} \cdots a_1 a_0$ where each digit a_i is in between 0 and $b - 1$ (both inclusive), and

$$n = a_0 b^0 + a_1 b^1 + \cdots + a_{k-1} b^{k-1} + a_k b^k.$$

  In particular, the base-2 (binary) representation of a number uses only the digits 0 and 1. For instance, the binary representation of the decimal number 29 is 11101 because

$$29 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4.$$

Section 1

1. Assume that the quadratic equations

$$ax^2 + bx + c = 0$$

and

$$dx^2 + ex + f = 0$$

have exactly one solution α in common, then $\alpha = $ _____.

(a) $\dfrac{af - cd}{bd - ae}$

(b) $\dfrac{bf - ce}{bd - ae}$

(c) $\dfrac{af - cd}{bf - ce}$

(d) None of the above

2. Let $r > 0$ be a fixed real number. Let $a, b \in \mathbb{R}$ be such that $b < 0$ and the roots of the polynomial $p(x) = x^3 + ax + b$ lie on the unique circle with center z_0 and radius r in the complex plane. Then the set of all possible values of $\text{Re}(z_0)$ is _____.

(a) $(-r, r)$ (b) $(-r/3, r/3)$ (c) $(-r, r/3)$ (d) $(-r/3, r)$

3. Let $a_0, a_1, \ldots, a_{13}$ be such that

$$1 - x + x^2 - x^3 + \cdots + x^{12} - x^{13} = a_0 + a_1 \cdot y + \cdots + a_{13} \cdot y^{13}$$

where $y = x + 1$. Then $a_2 = $ _____.

(a) -1001 (b) -364 (c) 364 (d) 1001

4. Consider a circle with center O. Let PT be the tangent to the circle at T and R be the point diametrically opposite to T. See Figure 1.
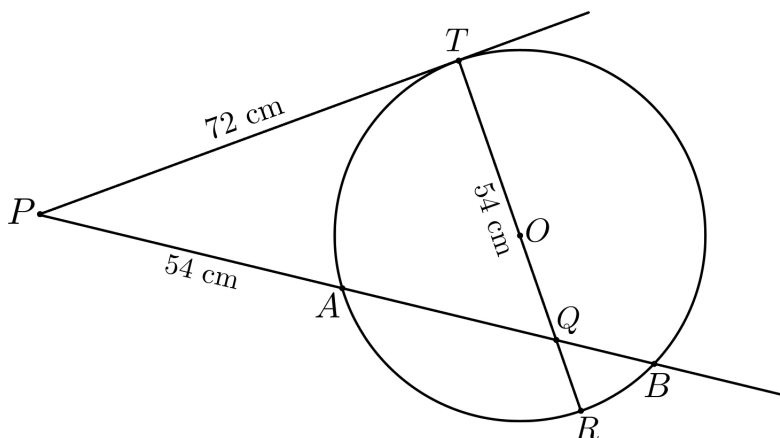

Figure 1

If $PT = 72$ cm, $PA = 54$ cm and $TQ = 54$ cm, then the radius of the circle is _____ cm.

(a) 27   (b) 28   (c) 29   (d) 30

5. On one flip of a coin if you get heads you win 1 Rupee, else you lose 3 Rupees. After exactly 4^{2023} flips, which of the following is possible?

    (a) You win 2023 Rupees.

    (b) You win 2024 Rupees.

    (c) You lose 2022 Rupees.

    (d) You lose $4^{2023} + 2$ Rupees.

6. Consider four concentric circles p, q, r, s with center O, as shown in Figure 2. Assume that tangents to the circles r, q, p form chords of length 5 cm, 7 cm, 5 cm on the circles s, s, r respectively. Then the length of the chord of circle q tangent to circle p is _____.
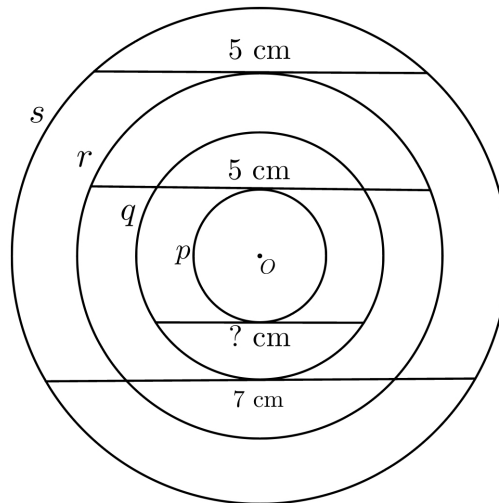
Figure 2

    (a) 1 cm

    (b) 2 cm

    (c) 3 cm

    (d) $2\sqrt{5} - \sqrt{7}$ cm

7. Consider the following statements.

    I. For each integer $n > 1$ there exist $x, y, z \in \mathbb{Z}$ such that $\dfrac{4}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

    II. There is an odd integer $n > 1$ for which there do not exist $x, y, z \in \mathbb{Z}$ such that $\dfrac{4}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

    III. There are infinitely many integers $n > 1$ for which there exist $x, y, z \in \mathbb{N}$ such that $\dfrac{4}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

    Then

    (a) Only statement I is correct

    (b) Only statement II is correct

    (c) Only statements I and III are correct

    (d) Only statements II and III are correct

8. In the Cartesian plane, what is the length of the shortest path from $(0,0)$ to $(8,6)$ that does not go inside the circle $(x-4)^2 + (y-3)^2 = \dfrac{75}{4}$?

   (a) $\dfrac{5}{2} + 2\sqrt{3}\pi$ (b) $5 + 2\sqrt{3}\pi$ (c) $\dfrac{5}{2} + \dfrac{5\pi}{\sqrt{3}}$ (d) $5 + \dfrac{5\pi}{\sqrt{3}}$

9. Consider the following statements.

   I. Any continuous function $f : (0,1) \to \mathbb{R}$ can be extended to a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in (0,1)$.

   II. Any continuous function $f : [0,1] \to \mathbb{R}$ can be extended to a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in [0,1]$.

   III. For any continuous function $f : [0,1] \to [0,1]$, there exists an $x \in [0,1]$ such that $f(x) = x$.

   IV. For any continuous function $f : (0,1) \to (0,1)$, there exists an $x \in (0,1)$ such that $f(x) = x$.

   Which of the following is true?

   (a) Only statements II and III are correct

   (b) Only statements I and IV are correct

   (c) Only statements I, II and III are correct

   (d) All four statements are correct

10. Let C_1 and C_2 be circles whose centers are 12 cm apart, and whose radii are 2 cm and 4 cm respectively. Let x cm^2 be the area of the region formed by all points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY. Then the largest integer which is less than or equal to x is _____.

    (a) 25 (b) 26 (c) 27 (d) 28

11. Let $f(x) = x^4 + 21x^3 + 56x^2 + 84x + 7$. Consider the following statements.

    S : $f(x)$ can be expressed as a product of two polynomials $p(x)$ and $q(x)$ with integer coefficients such that deg $(p(x)) = 2$ and deg $(q(x)) = 2$.

    T : $f(x)$ can be expressed as a product of two polynomials $p(x)$ and $q(x)$ with integer coefficients such that deg $(p(x)) = 1$ and deg $(q(x)) = 3$.

    (a) Both S and T are true (b) S is true, but T is false

    (c) T is true, but S is false (d) Both S and T are false

12. A polynomial P with real coefficients is called *good* if it satisfies

    $$P(x^2 + 1) = P(x)^2 + 1 \quad \text{for all } x \in \mathbb{R}.$$

    For which of the following numbers n does there exist a good polynomial with $P(1) = n$?

    (a) 677 (b) 578 (c) 626 (d) 512

13. An octagon is inscribed inside a circle with center O. All the angles except $\angle A$ and $\angle B$ are given in Figure 3. Then $\angle B - \angle A$ is
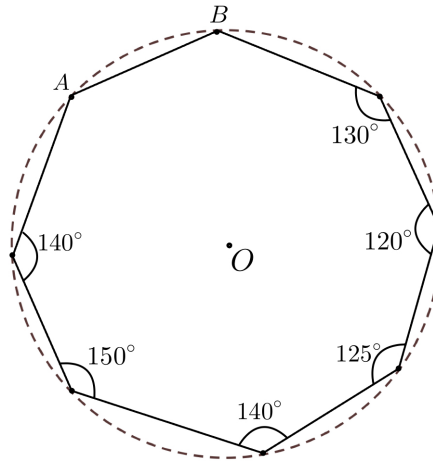

Figure 3

(a) $0°$ (b) $5°$ (c) $10°$ (d) $15°$

14. Mahavir plays a game on the number line. He starts at zero. At each second, he either stays where he is, or jumps some distance left or right. The distance he can jump at second s is 3^s units (he does not have to jump every second, but when he does, he jumps exactly 3^s units left or right). His time starts at $s = 0$. His goal is to reach the point 2023. He has devised a path to do the same in the minimum amount of time. How many times will he jump left on this path?

(a) 1   (b) 2   (c) 3   (d) 4

15. The value of $\displaystyle\lim_{n\to\infty} \frac{2 + 22 + \ldots + \overbrace{22\ldots2}^{(n \text{ times})}}{10^n}$ is _____.

(a) $\dfrac{2}{81}$ (b) $\dfrac{20}{81}$

(c) $\dfrac{2}{9}$ (d) None of the above

Section 2

1. Three people, Adityendra, Deep, Omkar, can finish a job in x, y, z days respectively, where x, y, z are positive integers. It is known that even if they work together, they can't finish the job in one day or less. Suppose the minimum number of days required for them to complete the job together is $\dfrac{p}{q}$, where $\gcd(p, q) = 1$. Then find the value of $p + q$.

2. Let r be the number of real solutions and l be the largest real solution of the equation $|x^2 - 9| + |x - 4| + |x + 16| - 27 = 0$. Then find $l + r$.

3. Let $a_1, a_2, \ldots, a_{100} \in [0, 2)$ be real numbers such that

$$\sum_{i=1}^{100} (a_i + a_i^3) = 100.$$

Find the minimum possible value of

$$\sum_{i=1}^{100} \frac{1}{2 - a_i}.$$

4. The product of two of the four roots of the following equation

$$x^4 - 6x^3 + kx^2 + 30x - 63 = 0$$

is -21. Find k^2.

5. Assume that $x, y \in \mathbb{R}$ satisfy the following system of equations given by

$$2x - 3y = \frac{x^2 + y^2}{5},$$
$$3x + 2y = \frac{8}{3}\sqrt{x^2 + y^2 + 4}.$$

If $x^2 + y^2 \leq 21$, determine $x + y$.

6. What is the maximum number of regions in which three dimensional space can be divided by four planes?

7. Consider a regular tetrahedron such that the length of the altitude from a vertex to the opposite face is 240 cm. We construct the in-sphere, that is the sphere tangent to all four faces of the tetrahedron internally. Then we construct a sphere tangent to the in-sphere and three faces of the tetrahedron internally, see Figure 4. Find the radius of this smaller sphere.
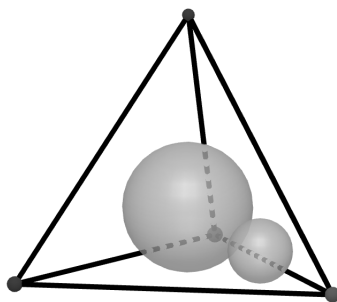

Figure 4

8. Let m_1 be the largest 4 digit number such that $2157m_1 + 8834n_1 = 1$ for some integer n_1, and let m_2 be the largest 4 digit number such that $2157n_2 + 8834m_2 = 1$ for some integer n_2. Then find the last three digits of $m_2 - m_1$.

9. Find the sum of the numbers in the 12$^{\text{th}}$ row in Figure 5.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 3 & & & \\
 & & 4 & & 5 & & 6 & & \\
 & 7 & & 8 & & 9 & & 10 & \\
\end{array}
$$

$-\quad-\quad-\quad-\quad-\quad-\quad-\quad-\quad-$
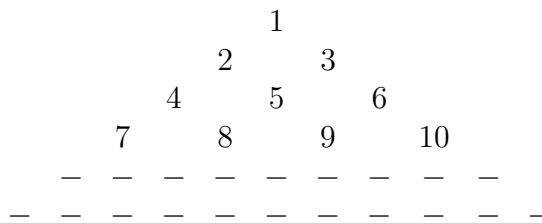
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Figure 5

10. When some standard six-sided dice are rolled, the probability of obtaining a sum of 2023 is greater than zero and is same as the probability of obtaining a sum of S. What is the smallest possible value of S?

11. Recall that the chess knight moves in an 'L' shape, that is, it can move two squares in any direction vertically followed by one square horizontally, or two squares in any direction horizontally followed by one square vertically. The chess knights are placed as in Figure 6 and we want to swap the positions of the black and white knights using only legal moves. Any knight of any colour can be moved in a turn, and two knights cannot occupy the same square. Knights cannot move out of the grid. What is the minimum number of moves needed?

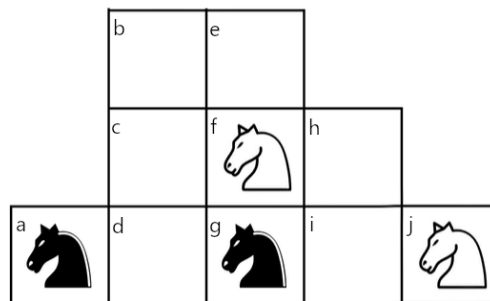Hint: Think about the connections between squares that knights can move between.

Figure 6

12. We can write

$$
\sum_{k=1}^{2023} \frac{1}{k} = \frac{a}{b}
$$

where a, b are coprime positive integers. Let m be the number of trailing zeroes of b when it is written in binary, and let n be the number of trailing zeroes of b when it is written in decimal. Find the value of $m + n$.

(Trailing zeroes of a number are the zeroes that appear at the end when it is written out, e.g., 10100 has 2 trailing zeroes and 230500060 has one trailing zero.)

13.[*] It is given that 3^{2023} is a 966 digit number with 1 being the leftmost digit in its decimal representation. Find the number of positive integers $n \leq 2023$ such that 9 is the leftmost digit in the decimal representation of 3^n.

14.* Madhur starts with a list of 24 numbers with the first entry being 100, last entry being 200 and all other entries being zero. Then he performs a sequence of moves with each move consisting of picking two distinct integers i, j such that $1 \leq i, j \leq 24$ and adding i to the jth entry and subtracting j from the ith entry. The absolute value of a list of numbers is defined as the square root of sum of squares of numbers in the list. Find the minimum possible absolute value of the list that Madhur can reach.

15.* Let $ABCD$ be a cyclic quadrilateral which is also a trapezoid such that AB and CD are parallel. Let AE be the angle bisector of $\angle CAD$ intersecting CD at E and let M be the midpoint of DC. Suppose that $D - E - M - C$ is the position of points on the line segment DC as shown in Figure 7 and $AB > 2EM$. If $CD = 2\sqrt{3}$ cm, $BM = \sqrt{5 + 2\sqrt{3}}$ cm, $EM = 2 - \sqrt{3}$ cm and r is the radius of the circle, then $r^2 + (AB)^2 + (CD)^2 = $ _____ cm². $\left(\text{You may use } \tan(15°) = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}. \right)$
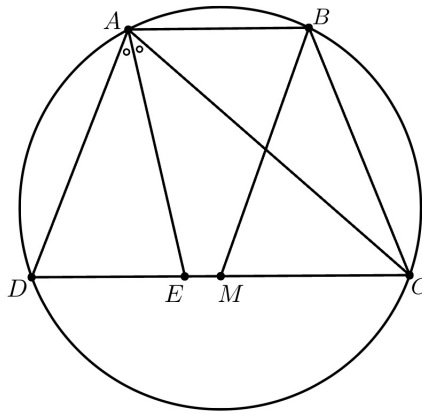

Figure 7

Space for rough work