KCL, Ray Theories and Application to Sonic Boom

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Abstract. We first review a general formulation of ray theory and write down the conservation forms of the equations of a weakly nonlinear ray theory (WNLRT) and a shock ray theory (SRT) for a weak shock in a polytropic gas. Then we present a formulation of the problem of sonic boom by a maneuvering aerofoil as a one parameter family of Cauchy problems. The system of equations in conservation form is hyperbolic for a range of values of the parameter and has elliptic nature else where, showing that unlike the leading shock, the trailing shock is always smooth.

1. Introduction

A wavefront (or a shock front) Ω_t in m-dimensional space $(x_1, x_2, \cdots, x_m)$ can be recognized as a propagating surface across which rapid (or abrupt) changes in the state of the medium takes place. Accordingly, high frequency approximation (which is automatically satisfied when Ω_t is shock front) for the solution of a system of equations leads to an eikonal equation governing the evolution of Ω_t and a transport equation (or an infinite system of equations for a shock front) for the amplitude w along a ray (or a shock ray for a shock front), [7].[1] For a nonlinear wavefront (or a shock front) the eikonal equation contains the amplitude w and hence the eikonal equation is coupled to the transport equation. This makes the problem of determination of the successive positions of Ω_t difficult. We first examine whether any fields of vectors $\chi(\boldsymbol{x}, t, \boldsymbol{n})$ in $\mathbb{R}^m$ where $\boldsymbol{n}$ is the unit normal to Ω_t, can be treated as a ray velocity.

Let $\Omega_t : \phi(\boldsymbol{x}, t) = 0$ be a moving surface in $\mathbb{R}^m$ where a field of vectors $\boldsymbol{\chi}(\boldsymbol{x}, t, \boldsymbol{n})$ is prescribed with $\boldsymbol{n} = \nabla\phi/|\nabla\phi|$. Then the evolution of Ω_t can be traced with the

[1]We shall frequently refer to various sections of the book, [7], by PP followed by a hypen and the section number. Almost all results in this section are available in, [9] and [5]. The material in the section 2 are taken from the paper, [2].

help of $\boldsymbol{\chi}$ as a ray velocity if, using summation convention,

$$n_\beta n_\gamma \left(n_\beta \frac{\partial}{\partial n_\alpha} - n_\alpha \frac{\partial}{\partial n_\beta} \right) \chi_\gamma = 0,\ \alpha = 1, 2, ..., m \tag{1.1}$$

[8]. When (1.1) is satisfied, $\phi(\boldsymbol{x}, t)$ satisfying the eikonal equation

$$Q := \phi_t + \langle \boldsymbol{\chi}, \nabla \rangle \phi = 0 \tag{1.2}$$

is consistent with the ray equations

$$\frac{dx_\alpha}{dt} = \chi_\alpha,\ \frac{dn_\alpha}{dt} = -n_\beta n_\gamma \left(n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta} \right) \chi_\gamma \tag{1.3}$$

The consistency condition (1.1) is satisfied for a curved wavefront or a shock front in a medium governed by a hyperbolic system of conservation laws, [8].

In terms of the normal velocity $C = -\phi_t/|\nabla\phi| = \langle \boldsymbol{n}, \boldsymbol{\chi} \rangle$, the equation (1.2) becomes

$$\phi_t + C|\nabla\phi| = 0 \tag{1.4}$$

which is used in the level set method (LSM). Since $\boldsymbol{\chi}$ can not be obtained from C, the equation (1.2) is more general than (1.4).

We restrict now to a 2-dimensional curve Ω_t in (x, y)-plane. Let tangential component of $\boldsymbol{\chi}$ be T ie., $T = -n_2\chi_1 + n_1\chi_2$. We define a ray coordinate system (ξ, t) such that ξ = constant is a ray and t = constant is Ω_t. Let $gd\xi$ represent an element of length along Ω_t. Then with $(n_1, n_2) = (\cos\theta, \sin\theta)$, we get a pair of *kinematical conservation laws (KCL)*, [6]; PP-3.3.2)

$$(g \sin\theta)_t + (C\cos\theta - T\sin\theta)_\xi = 0 \tag{1.5}$$

$$(g \cos\theta)_t - (C\sin\theta + T\cos\theta)_\xi = 0 \tag{1.6}$$

It can be shown that: *given χ as a known C^1 function of $\mathbf{x}, t$ and $\mathbf{n}$, the ray equations (1.3) for a moving front Ω_t in two space dimensions are equivalent to the KCL (1.5) and (1.6)*, [8]. KCL being in conservation form, it also admits solutions with shock type of discontinuities in (ξ, t)-plane. These discontinuities, when mapped onto (x, y)-plane with the help of $x_t = \chi_1, y_t = \chi_2$, they give rise to kinks across which the directions of the tangent to a ray and that to the front Ω_t change discontinuosly, (PP-3.3.3). The KCL are physically realistic, they represent conservation of distance in (x, y)-plane. The concept of kink was first introduced by Whitham in 1957, [10], intuitively as he did not have KCL. He called it *Shock-Shock.*

KCL, being only two equations in four quantities g, θ, C and T, is an under determined system. This is expected as KCL is a purely mathematical result and the dynamics of a particular moving curve has not been taken into account in their derivation. We describe here two sets of closure equations. Both of these belong to the case of an isotropic wave propagation, where $T = 0$ i.e., the rays are normal to the front. When a small amplitude curved wave front (across which the physical variables are continuous) or a shock front propagates into a medium at rest and in equilibrium with density $\bar{\rho} = \bar{\rho}_0$, fluid velocity $\bar{\boldsymbol{q}} = \mathbf{0}$ and gas pressure $\bar{p} = \bar{p}_0$, the perturbation on the wavefront or behind the shock front is given by

$$\bar{\rho} = \bar{\rho}_0 + \bar{\rho}_0 w, \bar{\boldsymbol{q}} = a_0(n_1 w, n_2 w), \bar{p} = \bar{p}_0 + \bar{\rho}_0 a_0^2 w \tag{1.7}$$

where w is the non-dimensional amplitude of the perturbation and a_0 is the dimensional sound velocity in the ambient medium (PP-6.1, note that w here is w/a_0 of PP-6.1). The Mach number m of a weakly nonlinear wavefront and M of a shock front are given by

$$m = 1 + \frac{\gamma+1}{2} w, M = 1 + \frac{\gamma+1}{4} w|_s \tag{1.8}$$

where $w|_s$ the value of w on a suitable side (behind a shock for a shock propagating in to the constant state $(\rho_0, \boldsymbol{q} = \mathbf{0}, p_0)$). The nondimensional value of C in (1.4) is m or M as the case may be.

The evolution equations of Ω_t when it is a weakly nonlinear wavefront, are

$$(g \sin\theta)_t + (m \cos\theta)_\xi = 0, (g \cos\theta)_t - (m \sin\theta)_\xi = 0 \tag{1.9}$$

$$\{g(m-1)^2 e^2(m-1)\}_t = 0 \tag{1.10}$$

These are the equations of weakly nonlinear ray theory (WNLRT). The mapping from (ξ, t)-plane to (x, y)-plane is given by the first part of (1.3) i.e., $x_t = m\cos\theta, y_t = m\sin\theta$.

When we choose Ω_t to be a shock front, the closure equations for KCL form an infinite system of equations (PP-Chapters 7 and 9). This infinite system is exact unlike the equation (1.10) derived under the high frequency approximation. However, not only the derivation of the infinite system is too complex and it is too difficult to solve numerically, its solution is non-unique for many interesting problems . By taking the shock to be weak and by truncating the system at a suitable stage, we can construct an approximate shock ray theory, which forms an efficient system of equations for calculation of successive positions of a curved shock in two space dimensions (PP-10.2). Denoting the unit normal to the shock front Ω_t to be $\boldsymbol{N} = (\cos\Theta, \sin\Theta)$, a system of conservation form of the equations for a weak shock Ω_t are two KCL and two additional closure equations, [1]:

$$(G \sin\Theta)_t + (M \cos\Theta)_\xi = 0, (G \cos\Theta)_t - (M \sin\Theta)_\xi = 0 \tag{1.11}$$

$$(G(M-1)^2 e^{2(M-1)})_t + 2M(M-1)^2 e^{2(M-1)} GV = 0 \tag{1.12}$$

$$(GV^2 e^{2(M-1)})_t + GV^3(M+1)e^{2(M-1)} = 0 \tag{1.13}$$

where G is the metric associated with the variable ξ and

$$V = \frac{\gamma+1}{4}\{\langle \boldsymbol{N}, \nabla\rangle w\}|_s \tag{1.14}$$

where the normal derivative $\langle \boldsymbol{N}, \nabla\rangle w$ is first obtained in the region behind the shock if the shock is moving into the undisturbed region and in the region ahead of the shock if it is moving into the disturbed region and then the limit is taken as we approach the shock. The mapping from (ξ, t)-plane can be obtained by integrating the first part of the shock ray equations $x_t = M\cos\Theta, y_t = M\sin\Theta$. (1.11)-(1.13) form the equations of our SRT, which is ideally suited in dealing with many practical problems involving propagation of a curved shock since (i) it has been shown that it gives results which agree well with known exact solutions and experimental results, [4], (ii) it gives sharp geometry of the shock and many finer details of geometrical features of the shock (PP-Chapter 10) , (iii) results obtained by it agree well with those obtained by numerical solutions of full Euler's equations, [1], (iv) it takes considerably less computational time (say less than 10%) compared to the Euler's numerical solution and (v) for a problem like sonic boom, it is difficult to get information in a long narrow region away from the aircraft by Euler's numerical solution, SRT and WNLRT are most suited.

2. Formulation of the problem of sonic boom by a maneuvering aerofoil as a one parameter family of Cauchy problems

Consider a two dimensional unsteady flow produced by a thin maneuvering aerofoil moving with a supersonic velocity along a curved path. We are interested in calculating the sonic boom produced by the aerofoil, the point of observation being far away say at a distance L, from the aerofoil. We use coordinates x, y and time t nondimensionalized by L and the sound velocity a_0 in the ambient medium. In a local rectangular coordinate system (x', y') with origin O' at the nose of the aerofoil and $O'x'$ axis tangential to the path of the nose, which moves along a curve $(X_0(t), Y_0(t))$, let the upper and lower surfaces of the aerofoil be given by

$$(x' = \zeta, y' = b_u(\zeta)) \text{ and } (x' = \zeta, y' = b_l(\zeta)), -d < \zeta < 0 \tag{2.1}$$

respectively. Here d is the nondimensional camber length. We assume that $b'_u(-d) > 0$, $b'_u(0) < 0$, $b'_l(-d) < 0$ and $b'_l(0) > 0$, so that the nose and the tail of the aerofoil are not blunt. We further assume that

$$d = \frac{\bar{d}}{L} = O(\epsilon), O\left\{\frac{max_{-d<\zeta<0} b_u(\zeta)}{d}\right\} = O\left\{\frac{max_{-d<\zeta<0}(-b_l(\zeta))}{d}\right\} = O(\epsilon)$$

where ϵ is a small positive number. Then the amplitude w of the perturbation in the sonic boom also satisfies $w = O(\epsilon)$.

In Fig. 1, we show the geometry of the aerofoil and the sonic boom produced by it at a time t. The sonic boom produced either by the upper or lower surface consists of a leading shock LS: $\Omega_t^{(0)}$ and a trailing shock TS: $\Omega_t^{(-d)}$ and since high frequency approximation is satisfied by the flow between the two shocks, a one parameter family of nonlinear wavefronts $\Omega_t^{(\zeta)}(-d < \zeta < 0, \zeta \neq G)$ originating from the points P_ζ on the aerofoil in between the two shocks. The nonlinear wavefronts produced from the points on the front portion of the aerofoil start interacting with the LS $\Omega_t^{(0)}$ and those from the points near the trailing edge do so with the TS $\Omega_t^{(-d)}$, and after the interaction they keep on disappearing continuously from the flow. These two sets, one interacting with LS and another interacting with TS are

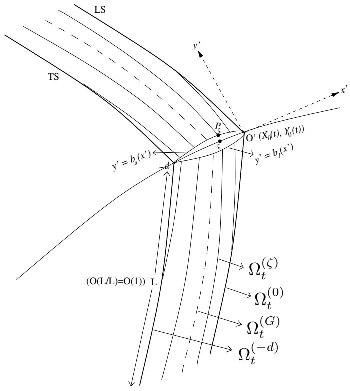

Figure 1: *Sonic boom produced by the upper and lower surfaces: $y' = b_u(x')$ and $y' = b_l(x')$ respectively. The boom produced by either surface consists of a one parameter family of nonlinear wavefronts*

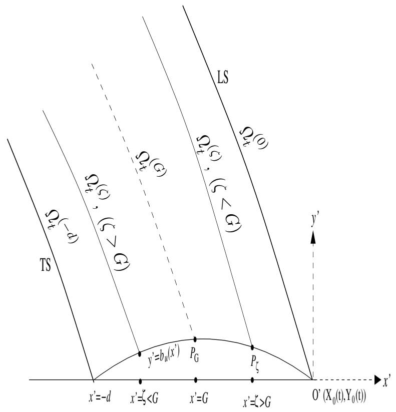

Figure 2: *It An enlarged version of the upper part of the Figure 1 near the aerofoil.*

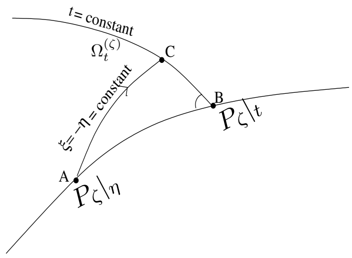

Figure 3: *A formulation of the ray coordinate system (ξ, t) for $\Omega_t(\zeta)$. AB represents the path of a fixed point P_ζ on the aerofoil. A is the position of P_ζ at time η and B that at time t.*

separated by a linear wavefront $\Omega_t^{(G)}$, which originates from a point P_a where the function $b_u(\zeta)$ ($b_l(\zeta)$) are maximum (minimum). Fig. 2 shows an enlarged version of the upper part of the Fig. 1 near the aerofoil. This is simply an enlarged version of Fig. 1 as high frequency approximation is not valid near the aerofoil.

Let us introduce a ray coordinate system (ξ, t) for $\Omega_t^{(\zeta)}$. The front $\Omega_t^{(\zeta)}$ at a given time t can be obtained as the locus of the tip of the rays (at time t) in (x, y)-plane starting from all positions $P_\zeta|_\eta$ of P_ζ at times $\eta < t$ as shown in Fig. 3. Therefore, a value of η identifies a ray and we choose

$$\xi = -\eta, \eta \leq t \tag{2.2}$$

for $\Omega_t^{(\zeta)}$ from the upper surface (for lower surface we need to choose $\zeta = \eta, \eta \leq t$). When $\xi = -\eta = t$, the points A, B and C in the Fig.3 coincide. Hence the base point P_ζ of $\Omega_t^{(\zeta)}$, which lies on the upper surface of the aerofoil, corresponds to a point, which lies on the line $\xi + t = 0$ in the (ξ, t)-plane.

The nonlinear wavefront $\Omega_t^{(\zeta)}, (-d < \zeta < 1, \zeta \neq G)$ satisfies the system (1.9)-(1.10). The Cauchy data on $\xi + t = 0$ to solve this system, can be determined from the inviscid flow condition on the surface of the aerofoil. Retaining only the leading order terms, this is [2].

$$m(\xi, -\xi) = m_0(\xi) := 1 - \frac{(\gamma+1)(\dot{X}_0^2 + \dot{Y}_0^2) b_u'(\zeta)}{2(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}} \tag{2.3}$$

$$g(\xi, -\xi) = g_0(\xi) := (\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}} \tag{2.4}$$

$$\theta(\xi, -\xi) = \theta_0(\xi) := \frac{\pi}{2} + \psi - \sin^{-1}\{1/(\dot{X}_0^2 + \dot{Y}_0^2)^{\frac{1}{2}}\} \tag{2.5}$$

where $\psi = \tan^{-1}\left\{\dot{Y}_0/\dot{X}_0\right\}$. Since $b_u'(\zeta) < 0$ and $b_u'(\zeta) > 0$ for $G < \zeta < 1$ and $-d < \zeta < G$ respectively, $m_0 > 1$ on P_ζ for $G < \zeta < 1$ and $m_0 < 1$ on P_ζ for $-d < \zeta < G$. This can be used to argue that

$$m > 1 \text{ on } \Omega^{(\zeta)}, G < \zeta < 0 \text{ and } m < 1 \text{ on } \Omega^{(\zeta)}, -d < \zeta < G \tag{2.6}$$

Since the eigenvalues of the system (1.9)-(1.10) are

$$\lambda_1 = \sqrt{(m-1)/(2g^2)}, \lambda_2 = 0, \lambda_3 = \sqrt{(m-1)/(2g^2)} \tag{2.7}$$

we get a Cauchy problem for a hyperbolic system for $\Omega_t^{(\zeta)}, G < \zeta < 0$ and an elliptic system for $\Omega_t^{(\zeta)}, -d < \zeta < G$ (we call it elliptic even though $\lambda_2 = 0$ is real).

The derivation of the Cauchy data on $\xi + t = 0$ for the system (1.11)-(1.13) governing the evolution of the shock fronts $\Omega_t^{(-d)}$ and $\Omega_t^{(0)}$ is far more complex. We quote from, [2], the leading order terms in this Cauchy data

$$M(\xi, -\xi) = M_0(\xi) := 1 - \frac{(\gamma+1)(\dot{X}_0^2 + \dot{Y}_0^2) b_u'(\xi)}{4(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}}} \tag{2.8}$$

$$G(\xi, -\xi) = G_0(\xi) := (\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{\frac{1}{2}} \tag{2.9}$$

$$\Theta(\xi, -\xi) = \Theta_0(\xi) := \frac{\pi}{2} + \psi - \sin^{-1}\{1/(\dot{X}_0^2 + \dot{Y}_0^2)^{\frac{1}{2}}\} \tag{2.10}$$

$$V(\xi, -\xi) = V_0(\xi) := \frac{\gamma+1}{4}\{\Omega_{P_{(-d)}} w_0(\xi) - \mathcal{F}(-d, t)\} \tag{2.11}$$

where

$$\begin{aligned}
\Omega_{P_{(-d)}} &= \frac{(\dot{X}_0 \ddot{X}_0 + \dot{Y}_0 \ddot{Y}_0)}{2g(\dot{X}_0^2 + \dot{Y}_0^2)(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{1/2}} + \frac{\dot{X}_0 \ddot{Y}_0 - \dot{Y}_0 \ddot{X}_0}{g \dot{X}_0^2} \\
\mathcal{F}(\zeta, t) &= \frac{(\dot{X}_0^2 + \dot{Y}_0^2) b_u''(\zeta)}{(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{1/2}}\{\mathcal{X}_0(t)\} - \frac{(\dot{X}_0^2 + \dot{Y}_0^2 - 2)(\dot{X}_0 \ddot{X}_0 + \dot{Y} \ddot{Y}_0)}{(\dot{X}_0^2 + \dot{Y}_0^2 - 1)^{3/2}} b_u'(\zeta) \\
\mathcal{X}_0 &= X_0 \cos\psi + Y_0 \sin\psi
\end{aligned}$$

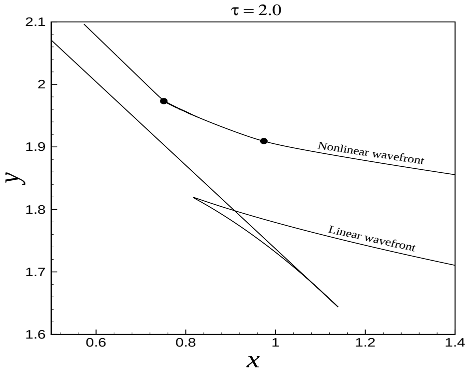

Figure 4: *Sonic boom wavefront at $t = 2$ from the leading edge of an accelerating aerofoil moving in a straight path. Kinks on the nonlinear wavefront are shown by dots. The initial Mach number is 1.8 and the acceleration is 10 in the line interval (1, 1/2).*

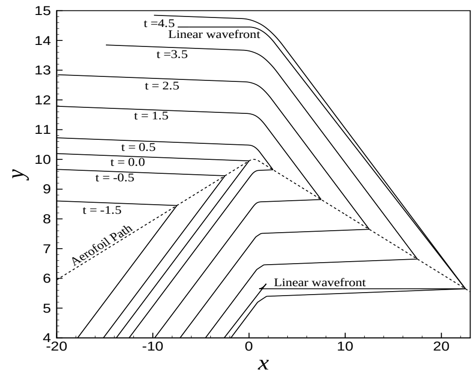

Figure 5: *The nonlinear wavefront from the leading edge of an aerofoil moving with a constant Mach number 5 along a path concave downwards with $b'_u(0) = -0.01$.*

Again $M > 1$ on $\Omega_t^{(0)}$ and $M < 1$ on $\Omega_t^{(-d)}$ so that the two eigenvalues $\Lambda_1 = \sqrt{(M-1)/2G^2}, \Lambda_2 = -\sqrt{(M-1)/2G^2}$ of (1.11)-(1.13) are real for $\Omega_t^{(0)}$ and purely imaginary for $\Omega_t^{(-d)}$. The other two eigenvalues are $\Lambda_{11} = 0, \Lambda_{12} = 0$. Thus, for the LS we get a Cauchy problem for a system which is hyperbolic and for the TS we get it for a system which has elliptic nature.

We have numerically solved the system (1.9)-(1.10) with Cauchy data (2.3)-(2.5) for $\zeta = 0$. This nonlinear wavefront from the leading edge is immediately annihilated by the shock $\Omega_t^{(0)}$. We have solved the system (1.11)-(1.13) with data (2.8)-(2.11) and we find that the geometric shape of nonlinear wavefront is not only topologically same as that of $\Omega_t^{(0)}$ but is very close to it. Hence the nonlinear wavefront from the leading edge gives valuable information about $\Omega_t^{(0)}$. We present some results in Fig 4 and Fig 5.

We note that for an accelerating aerofoil along a straight line, the linear wavefront from the nose develops fold in the caustic region but the nonlinear wavefront does not fold and has a pair of kinks. For a supersonic aerofoil moving on a highly curved path (curved downwards), the nonlinear wavefront from the upper surface is smooth but that from the lower surface has a pair of kinks. The most interesting result seen from our new formulation of the sonic boom problem is the elliptic nature of the equations governing $\Omega_t^{(-d)}$. This implies that whatever may be the flight path and acceleration of the aerofoil, the trailing shock $\Omega_t^{(-d)}$ must be smooth. All these features, which we obtain from our theory are seen in the Euler's numerical solution of Inoue, Sakai and Nishida (1997). Just two of their results have been reproduced in Fig. 6 with their permission.

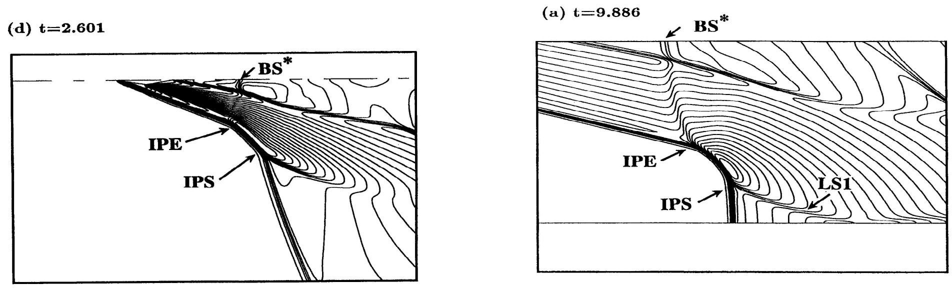

Figure 6: *Numerical solutions of Euler equations for the flow over a diamond-shaped projectile moving from right to left with Mach number sarting from 1.2 to 4. The Figures are reproduced from Inoue, Sakai and Nishida (1997) with the permission of the authors.*

The aerofoil in Fig 6 is moving from right to left and accelerates from a Mach number 1.2 to 4.0. The figure depicts the boom produced by the lower surface of a diamond shaped aerofoil. We note that the leading shock (LS) from the front edge tends to focus and develops two kinks (marked by IPE and IPS). Though a property like smoothness of the trailing shock is hard to see in and predict from a numerical result, we note that Fig 6 shows the trailing shock to be smooth.

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References

[1] S. Baskar and Phoolan Prasad, *Propagation of curved shock fronts using shock ray theory and comparison with other theories*, *J. Fluid Mech.*, **523**, (2005), 171-198.
[2] S. Baskar and Phoolan Prasad, *Formulation of the sonic boom problem by a manuevering aerofoil as a one parameter family of Cauchy problems*, Preprint / 2005, Department of Mathemtatics, IISc, Bangalore.
[3] O. Inoue, T. Sakai and M. Nishida, *Focusing shock wave generated by an accelerating projectile*, Fluid Dynamics Research, **21**, (1997), 403 - 416.
[4] N. K. R. Kevlahan, *The propagation of weak shocks in non-uniform flow*, *J. Fluid Mech.* **327**, (1996), 167-197.
[5] A. Monica and Phoolan Prasad, *Propagation of a curved weak shock*, J. Fluid Mech., **434**, (2001), pp. 119 - 151.
[6] Phoolan Prasad, *Formation and propagation of singularities on a nonlinear wavefront and shockfront.* J.Indian Institute of Science, Bangalore (Special volume of Fluid Mechanics), 75, (1995), 517-535.
[7] Phoolan Prasad, *Nonlinear hyperbolic waves in multi-dimensions*, Chapman and Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics - **121**, (2001).
[8] Phoolan Prasad, *Equivalence of ray equations and kinematical conservation laws*, Preprint / 2005, Department of mathematics, IISc, Bangalore.
[9] Phoolan Prasad and K. Sangeeta, *Numerical simulation of converging nonlinear wavefronts*, J. Fluid Mech., **385**, (1999), 1 - 20.
[10] G. Whitham, *Linear and nonlinear waves*, John Wiley & Sons, New York, (1974).

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