

FACTORIZATIONS OF CONTRACTIONS

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ABSTRACT. The celebrated Sz.-Nagy and Foias theorem asserts that every pure contraction is unitarily equivalent to an operator of the form $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ where \mathcal{Q} is a M_z^* -invariant subspace of a \mathcal{D} -valued Hardy space $H_{\mathcal{D}}^2(\mathbb{D})$, for some Hilbert space \mathcal{D} .

On the other hand, the celebrated theorem of Berger, Coburn and Lebow on pairs of commuting isometries can be formulated as follows: a pure isometry V on a Hilbert space \mathcal{H} is a product of two commuting isometries V_1 and V_2 in $\mathcal{B}(\mathcal{H})$ if and only if there exists a Hilbert space \mathcal{E} , a unitary U in $\mathcal{B}(\mathcal{E})$ and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that (V, V_1, V_2) and $(M_z, M_{\Phi}, M_{\Psi})$ on $H_{\mathcal{E}}^2(\mathbb{D})$ are unitarily equivalent, where

$$\Phi(z) = (P + zP^{\perp})U^* \quad \text{and} \quad \Psi(z) = U(P^{\perp} + zP) \quad (z \in \mathbb{D}).$$

In this context, it is natural to ask whether similar factorization result hold true for pure contractions. The purpose of this paper is to answer this question. More particularly, let T be a pure contraction on a Hilbert space \mathcal{H} and let $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T for some canonical $\mathcal{Q} \subseteq H_{\mathcal{D}}^2(\mathbb{D})$. Then $T = T_1T_2$, for some commuting contractions T_1 and T_2 on \mathcal{H} , if and only if there exists $\mathcal{B}(\mathcal{D})$ -valued polynomials φ and ψ of degree ≤ 1 such that \mathcal{Q} is a joint $(M_{\varphi}^*, M_{\psi}^*)$ -invariant subspace,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}} \quad \text{and} \quad (T_1, T_2) \cong (P_{\mathcal{Q}}M_{\varphi}|_{\mathcal{Q}}, P_{\mathcal{Q}}M_{\psi}|_{\mathcal{Q}}).$$

Moreover, there exists a Hilbert space \mathcal{E} and an isometry $V \in \mathcal{B}(\mathcal{D}, \mathcal{E})$ such that

$$\varphi(z) = V^*\Phi(z)V \quad \text{and} \quad \psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}),$$

where the pair (Φ, Ψ) , as defined above, is the Berger, Coburn and Lebow representation of a pure pair of commuting isometries on $H_{\mathcal{E}}^2(\mathbb{D})$.

1. INTRODUCTION

Let \mathcal{H} be a Hilbert space and V be an isometry on \mathcal{H} . It is a classical result, due to von Neumann and Wold (cf. [12]), that V is unitarily equivalent to $M_z \oplus U$ where M_z is the shift operator on \mathcal{E} -valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D})$, for some Hilbert space \mathcal{E} , and U is a unitary operator on \mathcal{H}_u , where

$$\mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H}.$$

We say that V is *pure* if $\mathcal{H}_u = \{0\}$, or, equivalently, if $V^{*m} \rightarrow 0$ in the strong operator topology (that is, $\|V^{*m}h\| \rightarrow 0$ as $m \rightarrow \infty$ for all $h \in \mathcal{H}$). Pure isometry, that is, shift operator on vector-valued Hardy space plays an important role in the study of general operators that stems from the following result (see [12, 13]):

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Theorem 1.1. (*Sz.-Nagy and Foias*) *Let T be a pure contraction on a Hilbert space \mathcal{H} . Then T and $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ are unitarily equivalent, where \mathcal{Q} is a closed M_z^* -invariant subspace of a vector-valued Hardy space $H_{\mathcal{D}}^2(\mathbb{D})$.*

Here the \mathcal{D} -valued Hardy space over \mathbb{D} , denoted by $H_{\mathcal{D}}^2(\mathbb{D})$, is defined by

$$H_{\mathcal{D}}^2(\mathbb{D}) := \left\{ f = \sum_{k \in \mathbb{N}} \eta_k z^k \in \mathcal{O}(\mathbb{D}, \mathcal{D}) : \eta_j \in \mathcal{D}, j \in \mathbb{N}, \|f\|^2 := \sum_{k \in \mathbb{N}} \|\eta_k\|^2 < \infty \right\}.$$

Recall that a contraction T on a Hilbert space \mathcal{H} is *pure* (cf. [16]) if $T^{*m} \rightarrow 0$ as $m \rightarrow \infty$ in the strong operator topology. Also note that, in the above theorem, one can represent the coefficient Hilbert space \mathcal{D} as $\overline{\text{ran}}(I - TT^*)$ (see [12]).

In contrast with the von-Neumann and Wold decomposition theorem for isometries, the structure of commuting n -tuples of isometries, $n \geq 2$, is much more complicated and very little, in general, is known (see [2, 3, 4, 5, 6, 10, 11, 17, 18, 14]). However, for pure pairs of commuting isometries, the problem is more tractable.

A pair of commuting isometries (V_1, V_2) on a Hilbert space \mathcal{H} is said to be *pure* if V_1V_2 is a pure isometry, that is,

$$\bigcap_{m=0}^{\infty} V_1^m V_2^m \mathcal{H} = \{0\}.$$

With this as motivation, a pair of commuting contractions (T_1, T_2) is said to be *pure* if T_1T_2 is a pure contraction.

The concept of pure pair of commuting isometries introduced by Berger, Coburn and Lebow [5] is an important development in the study of representation and Fredholm theory for C^* -algebras generated by commuting isometries. They showed that a pair of commuting isometries (V_1, V_2) on a Hilbert space \mathcal{H} is pure if and only if there exists a Hilbert space \mathcal{E} , a unitary U in $\mathcal{B}(\mathcal{E})$ and an orthogonal projection in $\mathcal{B}(\mathcal{E})$ such that (V_1, V_2) on \mathcal{H} and (M_{Φ}, M_{Ψ}) on $H_{\mathcal{E}}^2(\mathbb{D})$ are jointly unitarily equivalent, where

$$(1.1) \quad \Phi(z) = (P + zP^{\perp})U^* \quad \text{and} \quad \Psi(z) = U(P^{\perp} + zP) \quad (z \in \mathbb{D}).$$

Moreover

$$M_{\Phi}M_{\Psi} = M_{\Psi}M_{\Phi} = M_z,$$

and V_1V_2 on \mathcal{H} and M_z on $H_{\mathcal{E}}^2(\mathbb{D})$ are unitarily equivalent (see also [4, 9]). More precisely, if $\Pi : \mathcal{H} \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$ denote the unitary map, implemented by the Wold and von Neumann decomposition of the pure isometry V_1V_2 with $\mathcal{E} = \text{ran}(I - V_1V_2V_1^*V_2^*)$ (cf. [16]), then

$$\Pi V_1 = M_{\Phi}\Pi, \quad \text{and} \quad \Pi V_2 = M_{\Psi}\Pi.$$

In what follows, for a triple (\mathcal{E}, U, P) as above we let $\Phi, \Psi \in H_{\mathcal{B}(\mathcal{E})}^{\infty}(\mathbb{D})$ denote the isometric multipliers as defined in (1.1). We refer to such a pure pair of isometries (M_{Φ}, M_{Ψ}) simply as the triple (\mathcal{E}, U, P) .

Our work is motivated by the following equivalent interpretations of the Berger, Coburn and Lebow's characterizations of pure pairs of commuting isometries:

(I) Let (V_1, V_2) be a pure pair of commuting isometries and let M_z on $H_{\mathcal{E}}^2(\mathbb{D})$ be the von Neumann and Wold decomposition representation of V_1V_2 . Then there exists a unitary U and

an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that the representations of V_1 and V_2 in $\mathcal{B}(H_{\mathcal{E}}^2(\mathbb{D}))$ are given by M_{Φ} and M_{Ψ} , respectively.

(II) Let (X, Y) be a pair of commuting isometries in $\mathcal{B}(H_{\mathcal{E}}^2(\mathbb{D}))$. Moreover, let X and Y are Toeplitz operators [12] with analytic symbols from $H_{\mathcal{B}(\mathcal{E})}^{\infty}(\mathbb{D})$. Then $M_z = XY$ if and only if there exists a unitary U and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that $(X, Y) = (M_{\Phi}, M_{\Psi})$.

In this paper we shall obtain similar results for pure pairs of commuting contractions acting on Hilbert spaces. More specifically, summarizing Theorems 3.1, 3.2 and 4.1 and Corollary 4.2, we have the following: Let T be a pure contraction and (T_1, T_2) be a pair of commuting contractions on a Hilbert space \mathcal{H} . Let \mathcal{Q} be the Sz.-Nagy and Foias representation of T , that is, \mathcal{Q} is a M_z^* -invariant subspace of a vector-valued Hardy space $H_{\mathcal{D}}^2(\mathbb{D})$ and T and $P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ are unitarily equivalent (see Theorem 1.1 and Section 2) where $\mathcal{D} = \overline{\text{ran}}(I - TT^*)$. Then the following are equivalent:

(i) $T = T_1T_2$.

(ii) There exists a triple (\mathcal{E}, U, P) and a joint $(M_z^*, M_{\Phi}^*, M_{\Psi}^*)$ -invariant subspace $\tilde{\mathcal{Q}}$ of $H_{\mathcal{E}}^2(\mathbb{D})$ such that

$$T_1 \cong P_{\tilde{\mathcal{Q}}}M_{\Phi}|_{\tilde{\mathcal{Q}}}, T_2 \cong P_{\tilde{\mathcal{Q}}}M_{\Psi}|_{\tilde{\mathcal{Q}}}, T \cong P_{\tilde{\mathcal{Q}}}M_z|_{\tilde{\mathcal{Q}}} \text{ and } M_{\Phi}M_{\Psi} = M_{\Psi}M_{\Phi} = M_z.$$

In other words, (T_1, T_2, T) on \mathcal{H} dilates to $(M_{\Phi}, M_{\Psi}, M_z)$ on $H_{\mathcal{E}}^2(\mathbb{D})$.

(iii) There exists $\mathcal{B}(\mathcal{D})$ -valued polynomials φ and ψ of degree ≤ 1 such that \mathcal{Q} is a joint $(M_{\varphi}^*, M_{\psi}^*)$ -invariant subspace,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}},$$

and

$$(T_1, T_2) \cong (P_{\mathcal{Q}}M_{\varphi}|_{\mathcal{Q}}, P_{\mathcal{Q}}M_{\psi}|_{\mathcal{Q}}).$$

In particular, if $T = T_1T_2$ then the Sz.-Nagy and Foias representations of T_1 and T_2 on \mathcal{Q} are given by $P_{\mathcal{Q}}M_{\varphi}|_{\mathcal{Q}}$ and $P_{\mathcal{Q}}M_{\psi}|_{\mathcal{Q}}$, respectively. Moreover, it turns out that the pair (M_{φ}, M_{ψ}) is closely related to pure pair of isometries in the following sense: there exists a triple (\mathcal{E}, U, P) and an isometry $V \in \mathcal{B}(\mathcal{D}, \mathcal{E})$ such that

$$\varphi(z) = V^*\Phi(z)V \text{ and } \psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}).$$

The plan of the paper is the following. Section 2 contain some preliminaries and a key dilation result. In Section 3, we prove that a pure pair of commuting contractions always dilates to a pure pair of commuting isometries. Our construction is more explicit for pairs of contractions with finite dimensional defect spaces. In Section 4, by using a factorization result of dilation maps, we obtain explicit representations of factors of a pure contraction in its corresponding Sz.-Nagy and Foias space.

2. PRELIMINARIES

In this section, we set notation and definitions and discuss some preliminaries. Also we prove a basic dilation result in Theorem 2.1. This result will play a fundamental role throughout the remainder of the paper.

Let T be a contraction on a Hilbert space \mathcal{H} (that is, $\|Tf\| \leq \|f\|$ for all $f \in \mathcal{H}$ or, equivalently, if $I_{\mathcal{H}} - TT^* \geq 0$) and let \mathcal{E} be a Hilbert space. Then M_z on $H_{\mathcal{E}}^2(\mathbb{D})$ is called an isometric dilation of T (cf. [16]) if there exists an isometry $\Gamma : \mathcal{H} \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$ such that

$$\Gamma T^* = M_z^* \Gamma.$$

Similarly, a pair of commuting operators (U_1, U_2) on a Hilbert space \mathcal{K} is said to be a dilation of a commuting pair of operators (T_1, T_2) on a Hilbert space \mathcal{H} if there exists an isometry $\Gamma : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$\Gamma T_j^* = U_j^* \Gamma \quad (j = 1, 2).$$

Note that, in this case, $\mathcal{Q} := \text{ran} \Gamma$ is a joint (U_1^*, U_2^*) -invariant subspace of \mathcal{K} and

$$T_j \cong P_{\mathcal{Q}} U_j|_{\mathcal{Q}} \quad (j = 1, 2).$$

Now let T be a contraction on a Hilbert space \mathcal{H} . Set

$$\mathcal{D}_T = \overline{\text{ran}}(I_{\mathcal{H}} - TT^*), \quad D_T = (I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}.$$

If in addition, T is pure then M_z on $H_{\mathcal{D}_T}^2(\mathbb{D})$, induced by the isometry $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_T}^2(\mathbb{D})$, is an isometric dilation of T (cf. [16]), where

$$(2.1) \quad (\Pi h)(z) = D_T(I_{\mathcal{H}} - zT^*)^{-1}h \quad (z \in \mathbb{D}, h \in \mathcal{H}).$$

In particular, this yields a proof of Theorem 1.1 that every pure contraction is unitarily equivalent to the compression of M_z to a M_z^* -invariant closed subspace of a vector-valued Hardy space.

It is also important to note that the above dilation is minimal, that is,

$$(2.2) \quad H_{\mathcal{D}_T}^2(\mathbb{D}) = \overline{\text{span}}\{z^m \Pi f : m \in \mathbb{N}, f \in \mathcal{H}\},$$

and hence unique in an appropriate sense (see [12] and also the Factorization theorem, Theorem 4.1, in [15]).

Our considerations will also rely on the techniques of transfer functions (cf. [7]). Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces, and

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

be a unitary operator. Then the $\mathcal{B}(\mathcal{H}_1)$ -valued analytic function τ_U on \mathbb{D} defined by

$$\tau_U(z) := A + zB(I - zD)^{-1}C \quad (z \in \mathbb{D}),$$

is called the *transfer function* of U . Using $U^*U = I$, a standard and well known computation yields (cf. [7])

$$(2.3) \quad I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2)C^*(I - \bar{z}D^*)^{-1}(I - zD)^{-1}C \quad (z \in \mathbb{D}).$$

However, in this paper, we will mostly deal with transfer functions corresponding to unitary

matrices of the form $U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. In this case, it follows (see (2.3)) from the identity

$$I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2)C^*C \quad (z \in \mathbb{D})$$

that τ_U is a $\mathcal{B}(\mathcal{H}_1)$ -valued inner function [12].

Now let (T_1, T_2) be a pair of commuting contractions. Since

$$(I_{\mathcal{H}} - T_1 T_1^*) + T_1(I_{\mathcal{H}} - T_2 T_2^*) T_1^* = T_2(I_{\mathcal{H}} - T_1 T_1^*) T_2^* + (I_{\mathcal{H}} - T_2 T_2^*),$$

it follows that

$$\|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2 = \|D_{T_1} T_2^* h\|^2 + \|D_{T_2} h\|^2 \quad (h \in \mathcal{H}).$$

Thus

$$U : \{D_{T_1} h \oplus D_{T_2} T_1^* h : h \in \mathcal{H}\} \rightarrow \{D_{T_1} T_2^* h \oplus D_{T_2} h : h \in \mathcal{H}\}$$

defined by

$$(2.4) \quad U(D_{T_1} h, D_{T_2} T_1^* h) = (D_{T_1} T_2^* h, D_{T_2} h) \quad (h \in \mathcal{H}),$$

is an isometry. This operator will play a very important role in the sequel.

We now formulate the main theorem of this section, a result which will play a very important part in our considerations later on. Here the proof is similar in spirit to the main dilation result of [7].

Let \mathcal{H} and \mathcal{E} be Hilbert spaces and let (S, T) be a pair of commuting contractions on \mathcal{H} . Let T be pure and $V \in \mathcal{B}(\mathcal{D}_T, \mathcal{E})$ be an isometry. Then the isometric dilation of T , $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_T}^2(\mathbb{D})$ as defined in (2.1), allows us to define an isometry $\Pi_V \in \mathcal{B}(\mathcal{H}, H_{\mathcal{E}}^2(\mathbb{D}))$ by setting

$$\Pi_V := (I_{H^2(\mathbb{D})} \otimes V)\Pi.$$

It is easy to check that

$$\Pi_V T^* = (M_z^* \otimes I_{\mathcal{E}})\Pi_V,$$

and hence we conclude that M_z on $H_{\mathcal{E}}^2(\mathbb{D})$ is an isometric dilation of T . In particular, $\mathcal{Q} = \Pi_V \mathcal{H}$ is a M_z^* -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D})$ and $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$.

Theorem 2.1. *With notations as above, let*

$$U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_S \rightarrow \mathcal{E} \oplus \mathcal{D}_S,$$

be a unitary and

$$U(VD_T h, D_S T^* h) = (VD_T S^* h, D_S h) \quad (h \in \mathcal{H}).$$

We denote by $\Phi(z) = A^* + zC^*B^*$ the transfer function of U^* . Then Φ is a $\mathcal{B}(\mathcal{E})$ -valued inner function and

$$\Pi_V S^* = M_{\Phi}^* \Pi_V.$$

In particular, $\mathcal{Q} = \Pi_V \mathcal{H}$ is a joint (M_z^*, M_{Φ}^*) -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D})$ and

$$T^* \cong M_z^*|_{\mathcal{Q}} \quad \text{and} \quad S^* \cong M_{\Phi}^*|_{\mathcal{Q}}.$$

Proof. We only need to prove that $\Pi_V S^* = M_{\Phi}^* \Pi_V$. Now for each $h \in \mathcal{H}$ we have the equality

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} VD_T h \\ D_S T^* h \end{bmatrix} = \begin{bmatrix} VD_T S^* h \\ D_S h \end{bmatrix},$$

that is,

$$VD_T S^* h = AVD_T h + BD_S T^* h, \quad \text{and} \quad D_S h = CVD_T h.$$

This implies

$$VD_T S^* = AVD_T + BCVD_T T^*.$$

Now if $n \geq 1$, $h \in \mathcal{H}$ and $\eta \in \mathcal{E}$, then

$$\begin{aligned} \langle M_{\mathbb{F}}^* \Pi_V h, z^n \eta \rangle &= \langle (I \otimes V) D_T (I - zT^*)^{-1} h, (A^* + zC^* B^*)(z^n \eta) \rangle \\ &= \langle VD_T T^{*n} h, A^* \eta \rangle + \langle VD_T T^{*(n+1)} h, C^* B^* \eta \rangle \\ &= \langle (AVD_T + BCVD_T T^*)(T^{*n} h), \eta \rangle \\ &= \langle VD_T S^*(T^{*n} h), \eta \rangle. \end{aligned}$$

On the other hand, since

$$\langle \Pi_V S^* h, z^n \eta \rangle = \langle VD_T (I - zT^*)^{-1} S^* h, z^n \eta \rangle = \langle (VD_T S^*)(T^{*n} h), \eta \rangle,$$

we get $\Pi_V S^* = M_{\mathbb{F}}^* \Pi_V$. This completes the proof. \square

3. DILATING TO PURE ISOMETRIES

In this section we prove that a pure pair of commuting contractions dilates to a pure pair of commuting isometries. We describe the construction of dilations more explicitly in the case of finite dimensional defect spaces.

Theorem 3.1. *Let (T_1, T_2) be a pure pair of commuting contractions on \mathcal{H} and $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Then (T_1, T_2) dilates to a pure pair of commuting isometries.*

Proof. Set $\mathcal{E} := \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$ and $T := T_1 T_2$. Let $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_T}^2(\mathbb{D})$ be the isometric dilation of T as defined in (2.1). Consequently,

$$(3.1) \quad \Pi_V := (I_{H^2(\mathbb{D})} \otimes V) \Pi : \mathcal{H} \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$$

is an isometric dilation of T , and hence $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ where $\mathcal{Q} = \Pi_V \mathcal{H}$ is a M_z^* -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D})$ (see the proof of Theorem 2.1). Now observe that the equality

$$I - TT^* = I - T_1 T_2 T_1^* T_2^* = (I - T_1 T_1^*) + T_1 (I - T_2 T_2^*) T_1^*,$$

implies that the operator $V \in \mathcal{B}(\mathcal{D}_T, \mathcal{E})$ defined by

$$V(D_T h) = (D_{T_1} h, D_{T_2} T_1^* h) \quad (h \in \mathcal{H}),$$

is an isometry. Let $\iota_j : \mathcal{D}_{T_j} \rightarrow \mathcal{E}$, $j = 1, 2$, be the inclusion maps, defined by

$$\iota_1(h_1) = (h_1, 0) \quad \text{and} \quad \iota_2(h_2) = (0, h_2) \quad (h_1 \in \mathcal{D}_{T_1}, h_2 \in \mathcal{D}_{T_2}).$$

Then $P := \iota_2 \iota_2^* \in \mathcal{B}(\mathcal{E})$ is the orthogonal projection onto \mathcal{D}_{T_2} , that is,

$$P(h_1, h_2) = (0, h_2) \quad ((h_1, h_2) \in \mathcal{E}).$$

Thus, $\iota_1 \iota_1^* = P^\perp$ is the orthogonal projection onto \mathcal{D}_{T_1} , and so

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_{T_1} \rightarrow \mathcal{E} \oplus \mathcal{D}_{T_1}$$

is a unitary. Now since $\dim \mathcal{E} < \infty$, it follows that the isometry U , as defined in (2.4), extends to a unitary, denoted again by U , on \mathcal{E} . In particular, there exists a unitary operator U on \mathcal{E} such that

$$U(D_{T_1}T_2^*h, D_{T_2}h) = (D_{T_1}h, D_{T_2}T_1^*h) \quad (h \in \mathcal{H}).$$

Then

$$U_1 = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} = \begin{bmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{bmatrix},$$

is a unitary operator in $\mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_1})$. Moreover, for all $h \in \mathcal{H}$, we have

$$\begin{aligned} U_1(V(D_T h), D_{T_1}T^*h) &= U_1(D_{T_1}h, D_{T_2}T_1^*h, D_{T_1}T_1^*T_2^*h) \\ &= (U(D_{T_1}T_1^*T_2^*h, D_{T_2}T_1^*h), D_{T_1}h) \\ &= (D_{T_1}T_1^*h, D_{T_2}T_1^*h, D_{T_1}h) \\ &= (V(D_T T_1^*h), D_{T_1}h). \end{aligned}$$

Consequently, by Theorem 2.1 we have

$$\Pi_V T_1^* = M_\Phi^* \Pi_V,$$

where

$$\Phi(z) = PU^* + z\iota_1\iota_1^*U^* = (P + zP^\perp)U^* \quad (z \in \mathbb{D}),$$

is the transfer function of the unitary operator U_1^* . Similarly, if we define a unitary $U_2 \in \mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_2})$ by

$$U_2 = \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^\perp U^* & \iota_2 \\ \iota_2^* U^* & 0 \end{bmatrix},$$

then

$$U_2(V(D_T h), D_{T_2}T^*h) = (V(D_T T_2^*h), D_{T_2}h) \quad (h \in \mathcal{H}),$$

and hence by Theorem 2.1, we have

$$\Pi_V T_2^* = M_\Psi^* \Pi_V,$$

where

$$\Psi(z) = UP^\perp + zU\iota_2\iota_2^* = U(P^\perp + zP),$$

is the transfer function for the unitary operator U_2^* . This completes the proof that the pair of commuting pure isometries (M_Φ, M_Ψ) on $H_\mathcal{E}^2(\mathbb{D})$ corresponding to the triple (\mathcal{E}, U, P) dilates (T_1, T_2) . \square

We will now go on to give a proof of the general result. The proof is essentially the same as the previous theorem except the constructions of unitary operators and inclusion maps.

Theorem 3.2. *Let (T_1, T_2) be a pure pair of commuting contractions on \mathcal{H} . Then (T_1, T_2) dilates to a pure pair of commuting isometries.*

Proof. Let $\dim \mathcal{D}_{T_1} = \infty$, or $\dim \mathcal{D}_{T_2} = \infty$ and \mathcal{D} be an infinite dimensional Hilbert space. Set $\mathcal{E} := (\mathcal{D} \oplus \mathcal{D}_{T_1}) \oplus \mathcal{D}_{T_2}$. We now define inclusion maps $\iota_1 : \mathcal{D} \oplus \mathcal{D}_{T_1} \rightarrow \mathcal{E}$ and $\iota_2 : \mathcal{D}_{T_2} \rightarrow \mathcal{E}$ by

$$\iota_1(h, h_1) = (h, h_1, 0) \quad \text{and} \quad \iota_2 h_2 = (0, 0, h_2), \quad (h \in \mathcal{D}, h_1 \in \mathcal{D}_{T_1}, h_2 \in \mathcal{D}_{T_2})$$

respectively, and an isometric embedding $V \in \mathcal{B}(\mathcal{D}_T, \mathcal{E})$ by

$$VD_T h = (0, D_{T_1} h, D_{T_2} T_1^* h) \quad (h \in \mathcal{H}).$$

We also define the orthogonal projection P by $P = \iota_2 \iota_2^*$. Therefore

$$P(h_1, h_2, h_3) = (0, 0, h_3) \quad ((h_1, h_2, h_3) \in \mathcal{E}).$$

Finally, since

$$U_{\mathcal{D}}(0_{\mathcal{D}}, D_{T_1} h, D_{T_2} T_1^* h) = (0_{\mathcal{D}}, D_{T_1} T_2^* h, D_{T_2} h) \quad (h \in \mathcal{H}),$$

defines an isometry from $\{0_{\mathcal{D}}\} \oplus \{D_{T_1} h \oplus D_{T_2} T_1^* h : h \in \mathcal{H}\}$ to $\{0_{\mathcal{D}}\} \oplus \{D_{T_1} T_2^* h \oplus D_{T_2} h : h \in \mathcal{H}\}$, we can therefore extend $U_{\mathcal{D}}$ to a unitary, denoted again by $U_{\mathcal{D}}$, acting on \mathcal{E} . With these notations we define unitary operators

$$U_1 = \begin{bmatrix} U_{\mathcal{D}} P & U_{\mathcal{D}} \iota_1 \\ \iota_1^* & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{T_1})) \quad \text{and} \quad U_2 = \begin{bmatrix} P^{\perp} U_{\mathcal{D}}^* & \iota_2 \\ \iota_2^* U_{\mathcal{D}}^* & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_2}).$$

The rest of the proof proceeds in the same way as in Theorem 3.1. This completes the proof. \square

The main inconvenience of our approach seems to be the nonuniqueness of the triple (\mathcal{E}, U, P) . This issue is closely related to the nonuniqueness of Ando dilation [1] and solutions of commutant lifting theorem [8].

4. FACTORIZATIONS

Let (T_1, T_2) be a pair of commuting contractions on \mathcal{H} and $T = T_1 T_2$ be a pure contraction. Then by Theorem 1.1 we can realize T as $P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ where $\mathcal{Q} = \text{ran} \Pi = \Pi \mathcal{H}$ is the Sz.-Nagy and Foias model space and $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_T}^2(\mathbb{D})$ is the minimal isometric dilation of T (see (2.2)).

In this section we will show that T_1 and T_2 can be realized as compressions of two $\mathcal{B}(\mathcal{D}_T)$ -valued polynomials of degree ≤ 1 in the Sz.-Nagy and Foias model space \mathcal{Q} of the pure contraction T .

Our method involves a ‘‘pull-back’’ technique (see [15]) of the pure pair of isometric dilation, as obtained in Theorems 3.1 and 3.2, to the Sz.-Nagy and Foias minimal isometric dilation.

Let $\Pi_V : \mathcal{H} \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$ be the isometric dilation as in Theorems 3.1 and 3.2, that is,

$$\Pi_V T_1^* = M_{\Phi}^* \Pi_V \quad \text{and} \quad \Pi_V T_2^* = M_{\Psi}^* \Pi_V.$$

Now by canonical factorization of dilations (cf. Theorem 4.1 [15]), there exists a unique isometry $\tilde{V} \in \mathcal{B}(\mathcal{D}_T, \mathcal{E})$ such that the following diagram commutes

$$\begin{array}{ccc}
 & & H_{\mathcal{D}_T}^2(\mathbb{D}) \\
 & \nearrow \Pi & \downarrow I \otimes \tilde{V} \\
 \mathcal{H} & \xrightarrow{\quad} & H_{\mathcal{E}}^2(\mathbb{D}) \\
 & \Pi_V &
 \end{array}$$

that is

$$\Pi_V = (I \otimes \tilde{V})\Pi.$$

By uniqueness of \tilde{V} and the definition of Π_V (see (3.1)), we have that

$$V = \tilde{V}.$$

Thus

$$\Pi T_1^* = (I \otimes V^*)M_{\Phi}^*(I \otimes V)\Pi = M_{\varphi}^*\Pi,$$

where

$$\varphi(z) = V^*\Phi(z)V \quad (z \in \mathbb{D}).$$

Similarly, we derive

$$\Pi T_2^* = (I \otimes V^*)M_{\Psi}^*(I \otimes V)\Pi = M_{\psi}^*\Pi,$$

where

$$\psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}).$$

In particular, $\text{ran } \Pi$ is a joint $(M_{\varphi}^*, M_{\psi}^*)$ -invariant subspace and by construction of Π it follows that

$$\Pi T^* = M_z^*\Pi,$$

and $\text{ran } \Pi$ is also a M_z^* -invariant subspace of $H_{\mathcal{D}_T}^2(\mathbb{D})$. We have thus proved the following theorem.

Theorem 4.1. *Let T be a pure contraction on a Hilbert space \mathcal{H} and let $T \cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T as in Theorem 1.1. If $T = T_1T_2$, for some commuting pair of contractions (T_1, T_2) on \mathcal{H} , then there exists $\mathcal{B}(\mathcal{D}_T)$ -valued polynomials φ and ψ of degree ≤ 1 such that \mathcal{Q} is a joint $(M_{\varphi}^*, M_{\psi}^*)$ -invariant subspace and*

$$(T_1, T_2) \cong (P_{\mathcal{Q}}M_{\varphi}|_{\mathcal{Q}}, P_{\mathcal{Q}}M_{\psi}|_{\mathcal{Q}}).$$

In particular,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}}.$$

It is important to note that $P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}}$, even though, in general

$$\varphi\psi \neq \psi\varphi.$$

A reformulation of Theorem 4.1 is the following:

Corollary 4.2. *Let T be a pure contraction on a Hilbert space \mathcal{H} and let $T \cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T as in Theorem 1.1. Then $T = T_1T_2$, for some commuting pair of contractions (T_1, T_2) on \mathcal{H} , if and only if there exists $\mathcal{B}(\mathcal{D}_T)$ -valued polynomials φ and ψ of degree ≤ 1 such that \mathcal{Q} is a joint $(M_{\varphi}^*, M_{\psi}^*)$ -invariant subspace,*

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\varphi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\varphi}|_{\mathcal{Q}},$$

and

$$(T_1, T_2) \cong (P_{\mathcal{Q}}M_{\varphi}|_{\mathcal{Q}}, P_{\mathcal{Q}}M_{\psi}|_{\mathcal{Q}}).$$

Moreover, there exists a triple (\mathcal{E}, U, P) and an isometry $V \in \mathcal{B}(\mathcal{D}_T, \mathcal{E})$ such that

$$\varphi(z) = V^*\Phi(z)V \text{ and } \psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}).$$

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REFERENCES

- [1] T. Ando, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged), 24 (1963), 88-90.
- [2] H. Bercovici, R.G. Douglas and C. Foias, *Canonical models for bi-isometries*, A panorama of modern operator theory and related topics, 177-205, Oper. Theory Adv. Appl., 218, Birkhauser/Springer Basel AG, Basel, 2012.
- [3] H. Bercovici, R.G. Douglas and C. Foias, *Bi-isometries and commutant lifting*, Characteristic functions, scattering functions and transfer functions, 51-76, Oper. Theory Adv. Appl., 197, Birkhauser Verlag, Basel, 2010.
- [4] H. Bercovici, R.G. Douglas and C. Foias, *On the classification of multi-isometries*, Acta Sci. Math. (Szeged) 72 (2006), 639-661.
- [5] C.A. Berger, L.A. Coburn and A. Lebow, *Representation and index theory for C^* -algebras generated by commuting isometries*, J. Funct. Anal. 27 (1978), no. 1, 51-99.
- [6] K. Bickel and G. Knese, *Canonical Agler decompositions and transfer function realizations*, Trans. Amer. Math. Soc. 368 (2016), 6293-6324.
- [7] B.K. Das and J. Sarkar, *Ando dilations, von Neumann inequality, and distinguished varieties*, arXiv:1510.04655.
- [8] C. Foias and A. E. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhauser Verlag, Basel, 1990.
- [9] D. Gaspar and P. Gaspar, *Wold decompositions and the unitary model for biisometries*, Integral Equations Operator Theory, 49 (2004), 419-433.
- [10] K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. 53 (2004), 205-222.
- [11] W. He, Y. Qin and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J., 64 (2015), 1-19.
- [12] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space. North-Holland*, Amsterdam-London, 1970.
- [13] B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. (Szeged), 15 (1953), 87-92.
- [14] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.
- [15] J. Sarkar, *Operator theory on symmetrized bidisc*, Indiana Univ. Math. J. 64 (2015), 847-873.

- [16] J. Sarkar, *An Introduction to Hilbert module approach to multivariable operator theory*, Operator Theory, (2015) 969–1033 , Springer.
- [17] R. Yang, *The core operator and congruent submodules*, J. Funct. Anal. 228 (2005), 469-489.
- [18] R. Yang, *Hilbert-Schmidt submodules and issues of unitary equivalence*, J. Operator Theory 53 (2005), 169-184.

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