

DISTINGUISHED VARIETIES AND THE NEVANLINNA–PICK PROBLEM ON THE SYMMETRIZED BIDISK

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ABSTRACT. We study the uniqueness of the solutions of a solvable Pick interpolation problem in the symmetrized bidisk

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\}.$$

The uniqueness set is the largest set in \mathbb{G} where all the solutions to a solvable Pick problem coincide. There is a canonical construction of an algebraic variety, which coincides with the uniqueness set in \mathbb{G} . The algebraic variety is called the uniqueness variety. A solvable Pick problem is called extremal, if it has no solutions of supremum norm (over \mathbb{G}) less than one. We show that if an N -point extremal Pick problem is such that none of the $(N - 1)$ -point subproblems is extremal, then the uniqueness variety contains a distinguished variety that contains all the initial nodes. Here, a distinguished variety is an algebraic variety that intersects the domain \mathbb{G} and exits through its distinguished boundary. The proof of the first main result requires a thorough understanding of distinguished varieties. Indeed, this article is as much a study of the uniqueness varieties as it is about distinguished varieties. We obtain a complete algebraic and geometric characterizations of distinguished varieties.

1. INTRODUCTION

Given a data set $\mathfrak{D} = \{(\lambda_j, w_j) : 1 \leq j \leq N\}$, where $\lambda_1, \lambda_2, \dots, \lambda_N$ are distinct points of the open unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and w_1, w_2, \dots, w_N are some points in \mathbb{D} , the original Pick problem asks *if there exists an analytic function $f : \mathbb{D} \rightarrow \mathbb{D}$ that interpolates the data, i.e.,*

$$f(\lambda_j) = w_j \quad \text{for } j = 1, 2, \dots, N. \tag{1.1}$$

Pick showed in [44] that such a function (referred to as an *interpolant*) exists if and only if the *Pick matrix* $\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - w_i \overline{w_j}} \right]$ is positive semi-definite. Pick also showed that f is unique if and only if the matrix (4.1) has rank less than N , which is further equivalent to the existence of a Blaschke function of degree less than N interpolating the data. Later in [40, 41], Nevanlinna gave a characterization of all the interpolants. There have been numerous versions of this classical interpolation problem beginning with Abrahamse's work [1] for multiply connected domains. In the paper [46], Sarason introduced a new paradigm for solving the interpolation problem (1.1) that paved the way for a several variable generalization of the problem. Indeed, Sarason's approach was adapted in the work [2] by Agler for solving the problem (1.1) in the setting of polydisk - see

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also the work of Ball and Trent [16] for a simpler proof of Agler's solution and for a parametrization of all interpolating functions. Recently, this program is carried out first in [10] and then in [22] in the setting of the *symmetrized bidisk*

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\}. \quad (1.2)$$

Theorem 4.4 below describes a necessary and sufficient condition for solvability of a given Pick problem $\mathfrak{D} = \{(\lambda_j, w_j) : 1 \leq j \leq N\}$ in \mathbb{G} , i.e., when the *initial nodes* λ_j come from \mathbb{G} and, as in the classical case, the *target nodes* w_j belong to \mathbb{D} . Also see [22, Theorem 4.2] and [10, Theorem 5.1] for two other variations of necessary and sufficient conditions.

This paper assumes that a given Pick problem in \mathbb{G} is solvable and is concerned with the uniqueness of its solutions. We shall have use of the following terminologies.

Definition 1.1. For a solvable data in \mathbb{G} , the *uniqueness set* is the largest set in \mathbb{G} on which all the solutions agree. An interpolation problem is said to be *extremal*, if it is solvable but there is no interpolant of supremum norm over \mathbb{G} less than one.

If a Pick problem is not extremal, then there cannot be a unique solution. In fact, *the uniqueness set for a non-extremal problem is just the set of the initial nodes* - see Observation 4.2 for a brief discussion. Therefore in order to investigate the uniqueness of the solutions of a Pick problem, one must start with an extremal problem. However, unlike the classical situation, an extremal Pick problem in \mathbb{G} may or may not have a unique solution - see Examples 4.9 and 4.10 below. For the particular case of 2-point problems, Theorem 5.4 of [36] completely describes when the solution is unique.

With the lack of uniqueness in general, it is therefore reasonable to study the uniqueness set. When the solution to a given Pick problem is unique, then obviously the uniqueness set is whole \mathbb{G} ; when the solution is not unique, the size of the uniqueness set, however, falls by at least a dimension. More precisely, *for a solvable data in \mathbb{G} , the uniqueness set coincides with an algebraic variety (i.e., the common zero set of a collection of polynomials) in \mathbb{G}* . Moreover, for a solvable data, there is a canonical way to construct the algebraic variety - see Observation 4.1 for a brief discussion. The unique algebraic variety will be referred to as the *uniqueness variety*. The driving force producing the uniqueness variety is the following result.

Theorem 1.2. *There is always a rational solution to a solvable Pick problem in \mathbb{G} . Moreover, the rational solution can be obtained so that it is unimodular a.e. (with respect to the Lebesgue measure) on the symmetrized torus, i.e.,*

$$b\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{T}\}. \quad (1.3)$$

The statement remains true even when the target data w_j are square matrices; in this case the word 'unimodular' is replaced by 'unitary matrices' - see [7] for a similar result for \mathbb{D}^2 . The proof of Theorem 1.2 makes crucial use of the realization theory for bounded analytic functions on \mathbb{G} . The realization theory for \mathbb{G} is developed in the recent papers [10] and [22] via different approaches; we borrow the technique from [10] to prove Theorem 1.2. We shall detail on these topics and prove Theorem 1.2 in Section 3.

A certain type of algebraic varieties - starting with the work of Rudin [45] - has been studied by various mathematicians around the globe from the operator theoretic point

of view [5, 6, 20, 17, 24, 27, 42], the function theoretic point of view [24, 30, 32, 47] and the geometric point of view [34, 48].

Definition 1.3. Given a bounded domain Ω in \mathbb{C}^2 , a *distinguished variety with respect to Ω* is the zero set of a two-variable polynomial ξ such that

$$Z(\xi) \cap \Omega \neq \emptyset \quad \text{and} \quad Z(\xi) \cap \partial\Omega = Z(\xi) \cap b\Omega, \quad (1.4)$$

where $\partial\Omega$ is the topological boundary of Ω and $b\Omega$ denotes the distinguished boundary of Ω (i.e., the Šilov boundary with respect to the uniform algebra $\mathcal{A}(\Omega)$ of functions holomorphic in Ω and continuous on $\bar{\Omega}$ - see [11, Chapter 9]).

This terminology is due to Agler and McCarthy [5], where they were inexorably led to distinguished varieties with respect to \mathbb{D}^2 while studying bivariate matrices; see also the work [25]. The distinguished boundary of \mathbb{G} is the symmetrized torus (1.3) - see [8, Theorem 2.4].

The first main result of this paper is the following.

Theorem 1.4. *Let $\mathfrak{D} = \{(\lambda_j, w_j) : 1 \leq j \leq N\}$ be an extremal Pick problem in \mathbb{G} such that none of the $(N - 1)$ -point subproblems is extremal. Then the uniqueness variety contains a distinguished variety with respect to \mathbb{G} that contains the initial nodes.*

The proof of the above theorem requires a thorough understanding of the geometry of the distinguished varieties with respect to \mathbb{G} . Indeed, this paper is as much a study of the uniqueness varieties corresponding to solvable Pick problems as it is about the distinguished varieties. Since symmetrized bidisk is the only domain with respect to which the distinguished varieties will be studied in this paper, we omit the mention of the domain from now on when referring to a distinguished variety.

A *numerical contraction* F is a linear operator acting on a Hilbert space \mathcal{H} whose *numerical radius* is not greater than one, i.e.,

$$\nu(F) := \sup\{|\langle Fh, h \rangle| : \|h\| = 1\} \leq 1. \quad (1.5)$$

In a remarkable extension of the seminal work [5], Pal and Shalit showed in [43] that *if a $d \times d$ matrix F is a strict numerical contraction (i.e., $\nu(F) < 1$), then*

$$\mathcal{W}_F := \{(s, p) \in \mathbb{C}^2 : \det(F^* + pF - sI) = 0\} \quad (1.6)$$

is a distinguished variety. What about the $\nu(F) = 1$ case? Simple examples such as $F = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ show that \mathcal{W}_F may or may not be a distinguished variety when $\nu(F) = 1$. The mystery intensifies as one learns the converse direction: *every distinguished variety is of the form (1.6) for some matrix F with $\nu(F) \leq 1$ (not strict).* See Theorem 3.5 of [43] for this characterization of distinguished varieties. This, rather fortuitous connection between numerical contractions and distinguished varieties stems from an earlier work [21], where the authors introduced a fundamental tool whose contribution in the theory of symmetrized bidisk has been extraordinary. In an effort to better understand this connection, the recent work [20] showed that in the above characterization of distinguished varieties, the numerical contraction F can be chosen (in both the directions) from a considerably smaller class of numerical contractions, viz.,

$$\{PU + U^*P^\perp : P \text{ an orthogonal projection, } U \text{ a unitary acting on } \mathbb{C}^d, d \geq 1\}.$$

See [20, Lemmas 3.10 and 6.2] to see that the above is a proper subclass of numerical contractions and [20, Theorem 6.1] for the refinement of the Pal–Shalit result. The mystery however remained: examples of projections P and unitaries U can be found to show that $\mathcal{W}_{PU+U^*P^\perp}$ may or may not be distinguished when $\nu(PU + U^*P^\perp) = 1$, see Examples 6.4 and 6.5 in [20].

This paper resolves the issue. If an operator has no reducing subspace where it is unitary, then it is called *completely non-unitary (c.n.u.)*. In this paper, the notation $M_d(\mathbb{C})$ is used to denote the algebra of $d \times d$ complex matrices.

Theorem 1.5. *Let F in $M_d(\mathbb{C})$ be a numerical contraction (not necessarily strict). Then \mathcal{W}_F as in (1.6) is a distinguished variety if and only if F is completely non-unitary. Moreover, each irreducible component of \mathcal{W}_F intersects \mathbb{G} , when F is a c.n.u. numerical contraction. Conversely, every distinguished variety with each irreducible component intersecting \mathbb{G} is of the form (1.6) for a c.n.u. numerical contraction F .*

This result is contained in Theorem 2.5 - the second main result of this paper - that exhibit some other geometric equivalent conditions for \mathcal{W}_F to be a distinguished variety. This result in turn gives a new proof of Proposition 4.1 of [34], where Knese produces an elegant geometric characterization of distinguished varieties with respect to \mathbb{D}^2 .

We then characterize the pairs (P, U) of orthogonal projections P and unitary U acting on finite dimensional Hilbert spaces so that $PU + U^*P^\perp$ is completely non-unitary. This in turn gives a (P, U) -version of Theorem 1.5 - see Theorem 2.17.

This work is inspired by the seminal work [5] of Agler and McCarthy, where they proved a \mathbb{D}^2 -version of Theorem 1.4. An alert reader with the wisdom that a distinguished variety with respect to \mathbb{G} is just the symmetrization of a distinguished variety with respect to \mathbb{D}^2 and vice versa (see Lemma 3.1 in [43]), may wonder if there is a quick way to establish Theorem 1.4 from its \mathbb{D}^2 -analogue. It appears to be inaccessible. While an extremal Pick problem in \mathbb{D}^2 gives an extremal problem in \mathbb{G} via the symmetrization map $\pi : (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2)$, the converse is not true: a data in \mathbb{G} may be extremal but it may not give an extremal problem when pulled back to \mathbb{D}^2 via π . Consider for example the data $\{((0, 0), 0), ((0, 1/2), 1/2)\}$ in \mathbb{G} . We shall show in Example 4.11 that this problem is extremal. The points $(1/\sqrt{2}i, -1/\sqrt{2}i)$ and $(-1/\sqrt{2}i, 1/\sqrt{2}i)$ are the only two pre-images of $(0, 1/2)$ under π . But none of the problems $\{((0, 0), 0), ((-\frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}}i), \frac{1}{2})\}$ and $\{((0, 0), 0), ((\frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}i), \frac{1}{2})\}$ in \mathbb{D}^2 is extremal because the functions $(z_1, z_2) \mapsto \pm \frac{1}{\sqrt{2}}i z_1$ are solutions of norm less than one.

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2. DISTINGUISHED VARIETIES AND NUMERICAL CONTRACTIONS

The purpose of this section is to explore the connection between numerical contractions and distinguished varieties. We begin with the following well-known result about the numerical range of a matrix F , i.e., the set

$$W(F) := \{\langle Fh, h \rangle : \|h\| = 1\}.$$

Theorem 2.1 (See Theorem 5.1-9 in [29]). *If an eigenvalue β of a $d \times d$ matrix F belongs to $\partial W(F)$, the topological boundary of $W(F)$, then F is unitarily equivalent to $\begin{bmatrix} \beta I_{\mathbb{C}^r} & 0 \\ 0 & F' \end{bmatrix}$, $r \leq d$ such that β is not an eigenvalue of F' .*

This result has an important consequence: Consider the spectrum $\sigma(F)$ of a numerical contraction F . If it contains a unimodular member β_1 , then it must belong to the boundary of $W(F)$ and therefore by Theorem 2.1, F must be unitarily equivalent to a matrix of the form $\begin{bmatrix} \beta_1 I_{\mathbb{C}^{r_1}} & 0 \\ 0 & F_1 \end{bmatrix}$. Now consider $\sigma(F_1)$ and repeat the process until we arrive at a matrix F' whose spectrum does not contain any unimodular member. Note that F' would then obviously be a c.n.u. numerical contraction. Consequently, we have the following result, which is perhaps well-known.

Proposition 2.2. *Every numerical contraction acting on a finite dimensional Hilbert space decomposes into a direct sum of a unitary and a c.n.u. numerical contraction.*

For a square matrix F and a complex number p , we are interested in the linear pencil $F^* + pF$. We begin with a couple of elementary facts about these linear pencils each of which makes crucial contribution in the proof of the main result of this section.

Lemma 2.3. *Let F be a $d \times d$ matrix and $p \in \mathbb{C}$ be such that $|p| \neq 1$.*

- (1) *Then for $0 \leq r \leq d$, $F = \begin{bmatrix} \alpha I_{\mathbb{C}^r} & 0 \\ 0 & A \end{bmatrix}$ if and only if $F^* + pF = \begin{bmatrix} \mu I_{\mathbb{C}^r} & 0 \\ 0 & B \end{bmatrix}$; and*
- (2) *If F is a numerical contraction, then*

$$W(F^* + pF) \cap \{\beta + \bar{\beta}p : |\beta| = 1\} \subseteq \partial W(F^* + pF).$$

Proof of (1). The ‘only if’ part is obvious. For the other part, let $u, v \in \mathbb{C}^d$ be such that $\langle (F^* + pF)u, v \rangle = 0 = \langle (F^* + pF)v, u \rangle$. This, together with the fact that p is not unimodular, readily implies that both $\langle Fu, v \rangle$ and $\langle Fv, u \rangle$ must be zero. Thus by choosing u, v from the members of the standard orthonormal basis vectors of \mathbb{C}^d , we see that (1) follows.

Proof of (2). It is enough to show that any neighbourhood of $\beta + \bar{\beta}p$ contains a point that is not in $W(F^* + pF)$. For every $\epsilon > 0$, there is a small enough $\delta > 0$ such that the point $(1 + \delta/2)(\beta + \bar{\beta}p)$ is in the ϵ -ball around $\beta + \bar{\beta}p$, but the point does not belong to $W(F^* + pF)$ because if it does, then there would exist a point $\alpha \in \mathbb{D}$ such that $(1 + \delta/2)(\beta + \bar{\beta}p) = \alpha + \bar{\alpha}p$, which violates the assumption that p is not unimodular. This completes the proof. \square

The closure of the symmetrized bidisk \mathbb{G} is denoted by Γ , i.e.,

$$\Gamma := \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\}.$$

Among many characterizations of \mathbb{G} , the following are of particular importance.

Theorem 2.4 (See Theorem 2.1 in [8]). *The following are equivalent for s, p in \mathbb{C} :*

- (i) $(s, p) \in \mathbb{G}$;
- (ii) $|s - \bar{s}p| < 1 - |p|^2$;
- (iii) $|p| < 1$ and there exists a unique $\beta \in \mathbb{D}$ such that $s = \beta + \bar{\beta}p$. Moreover, β is given by $\beta = \frac{s - \bar{s}p}{1 - |p|^2}$;
- (iv) $|s| < 2$ and for all $\alpha \in \mathbb{D}$, $|\frac{2\alpha p - s}{2 - \alpha s}| < 1$.

The above theorem, especially part (iii), will be used throughout the paper. The following is the main result of this section.

Theorem 2.5. *For a numerical contraction F in $M_d(\mathbb{C})$, the set*

$$\mathcal{W}_F = \{(s, p) \in \mathbb{C}^2 : \det(F^* + pF - sI) = 0\} = \{(s, p) \in \mathbb{C}^2 : (s, p) \in \sigma(F^* + pF, pI)\}$$

is an algebraic variety that always intersects Γ . Moreover, the following are equivalent:

- (1) \mathcal{W}_F is a distinguished variety;
- (2) F is completely non-unitary, or equivalently, $\sigma(F) \cap \mathbb{T} = \emptyset$;
- (3) $\mathcal{W}_F \subset \pi(\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2)$, where $\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$; and
- (4) Every irreducible component of \mathcal{W}_F intersects \mathbb{G} .

Conversely, if \mathcal{W} is any distinguished variety such that each of its irreducible components meets \mathbb{G} , then there exists a c.n.u. numerical contraction F such that $\mathcal{W} = \mathcal{W}_F$.

We make a couple of remarks before we prove this theorem.

Remark 2.6. Let F in $M_d(\mathbb{C})$ be any numerical contraction. Then by Proposition 2.2, $F = F_0 \oplus F_u$, where F_0 is a completely non-unitary matrix and F_u is a unitary matrix. It is easy to note that $\mathcal{W}_F = \mathcal{W}_{F_0} \cup \mathcal{W}_{F_u}$ and that for some $0 \leq r \leq d$,

$$\mathcal{W}_{F_u} = \bigcup_{j=1}^r \{(\beta_j + \overline{\beta_j}p, p) : \beta_j \in \mathbb{T}, p \in \mathbb{C}\}.$$

Therefore in view of part (iii) of Theorem 2.4, we have $\mathcal{W}_F \cap \mathbb{G} = \mathcal{W}_{F_0} \cap \mathbb{G}$.

Remark 2.7. One of the important consequences of Theorem 2.5 is that if \mathcal{W} is a distinguished variety with an irreducible component not intersecting \mathbb{G} , then there is no numerical contraction F such that $\mathcal{W} = \mathcal{W}_F$. A concrete example demonstrating this phenomenon is the variety

$$\mathcal{W} = \{(s, p) \in \mathbb{C}^2 : (s^2 - 4p)(p - 1) = 0\}.$$

This is a distinguished variety. The irreducible component $\{(s, p) \in \mathbb{C}^2 : p - 1 = 0\}$ does not intersect \mathbb{G} . There is, in fact, no matrix F such that $\mathcal{W} = \mathcal{W}_F$. Because if there was an F , then in view of the joint spectrum representation of \mathcal{W}_F :

$$\mathcal{W}_F = \{(s, p) \in \mathbb{C}^2 : (s, p) \in \sigma(F^* + pF, pI)\},$$

$\sigma(F^* + F)$ would have to contain the set $\{z_1 + z_2 : z_1 z_2 = 1\}$. This is absurd because the spectrum of a matrix is a finite set.

Proof of Theorem 2.5. That \mathcal{W}_F is an algebraic variety follows from the fact that the determinant is always a polynomial; for this, F does not even have to be a numerical contraction. To see that \mathcal{W}_F intersects Γ , note that \mathcal{W}_F contains the points $(\sigma(F^*), 0)$, which is contained in Γ because $\sigma(F^*) \subset \overline{\mathbb{D}}$.

For the moreover part, we prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1). The equivalence stated in item (2) is a consequence of Theorem 2.1. Indeed, if a c.n.u. numerical contraction F has a unimodular eigenvalue, then Theorem 2.1 shows that F must decompose into a matrix of the form $\beta I_{\mathbb{C}^r} \oplus F'$ for some $1 \leq r$. This is not possible because F is a c.n.u. numerical contraction. The other direction is straightforward.

Proof of (1) \Rightarrow (2): This is easy to see. Indeed, if there was a unitary U such that F is unitarily equivalent to $U \oplus F'$, then \mathcal{W}_F would contain points of the form $(\beta + \bar{\beta}p, p)$ such that $(\beta, p) \in \mathbb{T} \times \mathbb{D}$, which would violate the distinguished property of \mathcal{W}_F .

Proof of (2) \Rightarrow (3): This is the non-trivial part of the proof and has some important implications that we shall record as we progress with the proof.

To establish the containment in (3), it is equivalent to show that \mathcal{W}_F does not intersect

$$R_1 := \pi(\mathbb{D} \times \mathbb{T}) \cup \pi(\mathbb{T} \times \mathbb{D}) \cup \pi(\mathbb{E} \times \mathbb{T}) \cup \pi(\mathbb{T} \times \mathbb{E})$$

and

$$R_2 := \pi(\mathbb{D} \times \mathbb{E}) \cup \pi(\mathbb{E} \times \mathbb{D}).$$

Let us suppose on the contrary that $(s, p) = \pi(z_1, z_2) \in \mathcal{W}_F \cap \pi(\mathbb{D} \times \mathbb{E})$. This means that there exists a norm-unit vector v such that with $\beta = \langle F^*v, v \rangle$,

$$z_1 + z_2 = \beta + \bar{\beta}z_1z_2,$$

or equivalently,

$$z_2 = \frac{\beta - z_1}{1 - \bar{\beta}z_1}.$$

Since F is a numerical contraction, $\beta \in \overline{\mathbb{D}}$, and since $z_1 \in \mathbb{D}$, the above representation of z_2 implies that it must belong to $\overline{\mathbb{D}}$ contradicting the assumption that $z_2 \in \mathbb{E}$. Similar argument works for the case $\pi(\mathbb{E} \times \mathbb{D})$. Thus \mathcal{W}_F does not intersect R_2 . Note that we have not used the completely non-unitary property of F to rule out R_2 ; the analysis works for any numerical contractions.

To show that \mathcal{W}_F does not intersect R_1 , we begin with the following characterization of R_1 .

Lemma 2.8. *A pair (s, p) is in R_1 if and only if p is not in \mathbb{T} and $\frac{s - \bar{s}p}{1 - |p|^2}$ is in \mathbb{T} .*

Proof. If $(s, p) = (\zeta + z, \zeta z) \in R_1$ such that $\zeta \in \mathbb{T}$ and $|z| \neq 1$, then obviously $|p| \neq 1$. Straightforward computation yields $s - \bar{s}p = \zeta(1 - |p|^2)$, proving the forward assertion. The converse follows immediately from the equality

$$s = \frac{s - \bar{s}p}{1 - |p|^2} + \frac{\bar{s} - \bar{s}\bar{p}}{1 - |p|^2}p.$$

□

Now let us suppose on the contrary that $(s, p) \in \mathcal{W}_F \cap R_1$. Then there exists a norm-unit vector v such that $(F^* + pF)v = sv$. Thus s is an eigenvalue of $F^* + pF$, and with $\beta = \langle F^*v, v \rangle$ we have $s = \beta + \bar{\beta}p$. Since $|p| \neq 1$, we must have $\beta = \frac{s - \bar{s}p}{1 - |p|^2}$. By Lemma 2.8, $\beta \in \mathbb{T}$. Thus by part (2) of Lemma 2.3, $s = \beta + \bar{\beta}p$ must belong to $\partial W(F^* + pF)$. Since s is an eigenvalue of $F^* + pF$, by Theorem 2.1, there exists a positive integer $r \leq d$, and a unitary τ such that

$$\tau(F^* + pF)\tau = \begin{bmatrix} sI_{C^r} & 0 \\ 0 & G \end{bmatrix}$$

for some matrix G whose spectrum does not contain s . It turns out that we do not need the spectrum assertion on G . Now apply part (1) of Lemma 2.3 for $\tau^*F\tau$ to

obtain

$$\tau^* F \tau = \begin{bmatrix} \bar{\beta} I_{\mathbb{C}^r} & 0 \\ 0 & F' \end{bmatrix} \quad (2.1)$$

for some matrix F' . The scalar in the (11)-entry of $\tau^* F \tau$ has to be $\bar{\beta}$ because $s = \beta + \bar{\beta}p$. But the fact that $\beta \in \mathbb{T}$ contradicts the hypothesis that F is completely non-unitary. This completes the proof of (2) \Rightarrow (3).

Proof of (3) \Rightarrow (4): This follows from the fact that no algebraic variety can be entirely contained in $\mathbb{T}^2 \cup \mathbb{E}^2$. Indeed, if $Z(\xi) \subset \mathbb{T}^2 \cup \mathbb{E}^2$ for a polynomial ξ in $C[z_1, z_2]$, then ξ cannot be a polynomial of only one of the variables. Now choose $z_1 \in \mathbb{D}$ and consider the polynomial $\xi_{z_1}(z_2) = \xi(z_1, z_2)$. Since \mathbb{C} is an algebraically closed field, ξ_{z_1} must vanish at some point. This violates the containment $Z(\xi) \subset \mathbb{T}^2 \cup \mathbb{E}^2$. Now the proof follows from the understanding that a variety in $(z_1 + z_2, z_1 z_2)$ -coordinate is the symmetrization of a variety in (z_1, z_2) -coordinate - see Lemma 3.1 in [43].

Proof of (4) \Rightarrow (1): If every irreducible component of \mathcal{W}_F meets \mathbb{G} , then clearly F must be completely non-unitary. We have already proved that in such case, \mathcal{W}_F is contained in $\pi(\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2)$. This, together with the fact that no algebraic variety is entirely contained in $\mathbb{T}^2 \cup \mathbb{E}^2$ ensure the non-emptiness of $\mathcal{W}_F \cap \mathbb{G}$. The rest follows from the definition of a distinguished variety.

For the converse part, it is enough to assume that \mathcal{W} is an irreducible variety because if it is not, then one considers the direct sum of the numerical contractions obtained for each irreducible component.

So given an irreducible distinguished variety \mathcal{W} , we have to find a c.n.u. numerical contraction F such that $\mathcal{W} = \mathcal{W}_F$. The idea goes back to the work [5] of Agler and McCarthy, which was adapted in the setting of the symmetrized bidisk to obtain an F by Pal and Shalit [43, Theorem 3.5]. We sketch the proof for completeness. The first step is to construct a Hardy-type reproducing kernel Hilbert space $H(k)$ on $\mathcal{W} \cap \mathbb{G}$ that originates from a regular Borel measure on the boundary $\mathcal{W} \cap b\mathbb{G}$. The existence of the measure is guaranteed by a result from the theory of Riemann surfaces. Consider the pair (M_s, M_p) of multiplication by coordinate functions on $H(k)$. It turns out (see Lemma 3.4 in [43]) that the closed symmetrized bidisk Γ is a spectral set for (M_s, M_p) (the so called Γ -contraction, a terminology due to Agler and Young [9]), i.e.,

$$\|f(M_s, M_p)\| \leq \sup\{|f(s, p)| : (s, p) \in \Gamma\}$$

for every polynomial f in two variables. Note that this makes M_p a contraction. Invoke Theorem 4.2 in [21] that says that there is a unique numerical contraction $F : \text{Ran}(I - M_p M_p^*) \rightarrow \text{Ran}(I - M_p M_p^*)$ such that

$$M_s^* - M_s M_p^* = (I - M_p M_p^*)^{\frac{1}{2}} F (I - M_p M_p^*)^{\frac{1}{2}}.$$

Pal and Shalit showed that $\text{Ran}(I - M_p M_p^*)$ is finite dimensional (thus F is a matrix) and that $\mathcal{W} \cap \mathbb{G} = \mathcal{W}_F \cap \mathbb{G}$.

While there is no reason to believe that $\nu(F) < 1$, we note that F can be replaced by its c.n.u. part harmlessly. This is because the unitary part of F has no contribution in the intersection $\mathcal{W}_F \cap \mathbb{G}$. Now, we know that if F is c.n.u., then every irreducible component of \mathcal{W}_F meets \mathbb{G} . Since $\mathcal{W} \cap \mathbb{G} = \mathcal{W}_F \cap \mathbb{G}$, and \mathcal{W} is irreducible, \mathcal{W}_F must be

irreducible. So we have arrived at a situation where two irreducible varieties coincide in \mathbb{G} . This forces the two varieties to be the same. This completes the proof. \square

During the course of the proof of (2) \Rightarrow (3), there emerged an interesting fact that we record below as a proposition. Note that to arrive at the conclusion (2.1), all we used is that F is a numerical contraction and $(s, p) \in \mathcal{W}_F$ such that $|p| \neq 1$ and $\frac{s-\bar{s}p}{1-|p|^2} \in \mathbb{T}$. Thus we have already proved (1) \Rightarrow (2) of the following result; the implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are obvious.

Proposition 2.9. *Let $F \in M_d(\mathbb{C})$ be a numerical contraction and $\beta \in \mathbb{T}$. Then the following are equivalent:*

- (1) \mathcal{W}_F contains a point of the form $(\beta + \bar{\beta}p, p)$ such that $|p| \neq 1$;
- (2) F is unitarily equivalent to $\begin{bmatrix} \beta I_{\mathbb{C}^r} & 0 \\ 0 & F' \end{bmatrix}$ for some matrix F' ; and
- (3) $\mathcal{W}_\beta \subseteq \mathcal{W}_F$.

Remark 2.10. We believe that items (2) and (3) of Proposition 2.9 are equivalent even in the case when $|\beta| \neq 1$. This is easily seen to be true when F is normal for example. Indeed, the containment $\mathcal{W}_\beta \subset \mathcal{W}_F$ implies that

$$\det[(F^* - \beta I) + p(F - \bar{\beta}I)] = 0 \quad \text{for all } p \in \mathbb{C}. \quad (2.2)$$

If F is normal, then so is $F^* - \beta I$ and hence is diagonalizable, which readily implies that $F^* - \beta I$ must have 0 as an eigenvalue. This establishes item (2). Note that in this case, we used the information (2.2) just for $p = 0$.

Remark 2.11. If an algebraic variety \mathcal{V} is contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$, then as explained in the proof of (3) \Rightarrow (4) of Theorem 2.5, every irreducible component of \mathcal{V} meets \mathbb{D}^2 . In Proposition 4.1 of [34], Knese elegantly showed that the converse is also true: if \mathcal{V} is a distinguished variety with respect to \mathbb{D}^2 each of whose irreducible components meets \mathbb{D}^2 , then $\mathcal{V} \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. The proof involves tools of complex analysis such as the residue theorem and the Schwarz reflection principle. Theorem 2.5 gives a new proof of this result involving linear algebra. Indeed, let \mathcal{V} be a distinguished variety with respect to \mathbb{D}^2 such that each irreducible component intersects \mathbb{D}^2 . Then $\mathcal{W} = \pi(\mathcal{V})$ is a distinguished variety with each irreducible component intersecting \mathbb{G} . Apply the converse of Theorem 2.5 to conclude that $\mathcal{W} = \mathcal{W}_F$ for some c.n.u. numerical contraction F . Then item (3) of Theorem 2.5 implies that $\pi(\mathcal{V}) \subset \pi(\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2)$. Now pull back this containment to obtain Knese's characterization.

Definition 2.12. A set $X \subset \mathbb{C}^2$ is said to have the *distinguished property*, if its closure satisfies the following boundary constraint:

$$\overline{X} \cap \partial\Gamma = \overline{X} \cap b\mathbb{G}.$$

Therefore by definition, a distinguished variety has the distinguished property. Such sets have appeared before in [24] while studying the totally Abelian analytic Toeplitz operators.

Given a numerical contraction F , an interesting consequence of Theorem 2.5 is that the intersection $\mathcal{W}_F \cap \mathbb{G}$ always enjoys the distinguished property, while the variety \mathcal{W}_F need not have the distinguished property (precisely when F is not c.n.u.).

Corollary 2.13. *Given a numerical contraction F , the set $\mathcal{W}_F \cap \mathbb{G}$ always satisfies the distinguished property.*

Proof. Note that if F' is the c.n.u. part of F , then $\mathcal{W}_F \cap \mathbb{G} = \mathcal{W}_{F'} \cap \mathbb{G}$ because the component corresponding to the unitary part never intersects the open symmetrized bidisk. Since $\mathcal{W}_{F'}$ is a distinguished variety and each of its irreducible components meets \mathbb{G} , we have

$$\overline{\mathcal{W}_{F'} \cap \mathbb{G}} \cap \partial\Gamma = \mathcal{W}_{F'} \cap \partial\Gamma = \mathcal{W}_{F'} \cap b\mathbb{G} = \overline{\mathcal{W}_{F'} \cap \mathbb{G}} \cap b\mathbb{G}.$$

□

The royal variety $\mathcal{R} = \{(s, p) \in \mathbb{C}^2 : s^2 - 4p = 0\}$ is a special distinguished variety. It has proved to be of great importance in understanding the complex geometry of \mathbb{G} , see for example [3, Theorem 1.4] and [18, Theorem 2.6]. The mystery surrounding the $\nu(F) = 1$ case led to the belief among the experts of the field that in case when $\nu(F) = 1$ and \mathcal{W}_F is a distinguished variety, then \mathcal{W}_F must contain the royal variety. The following example shows that this need not be true.

Example 2.14. Let $F = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$. Then a simple calculation reveals that $\nu(F) = 1$.

Also note that

$$\mathcal{W}_F = \{(s, p) \in \mathbb{C}^2 : ((1+p) - 2s)^2 - 4p = 0\}.$$

By part (2) of Theorem 2.5, \mathcal{W}_F is a distinguished variety. But \mathcal{W}_F does not contain the royal variety \mathcal{R} because for example $(0, 0)$ is in \mathcal{R} but not in \mathcal{W}_F .

Consider the family of $d \times d$ matrices of the form

$$\mathcal{N} = \{PU + U^*P^\perp : P \text{ is projection, } U \text{ a unitary on } \mathbb{C}^d\}.$$

This is a family of numerical contraction by Lemma 3.10 of [20]. Moreover, \mathcal{N} is a significantly smaller family of numerical contractions. Indeed, a 2×2 matrix with eigenvalues of distinct absolute values does not belong to \mathcal{N} ; see [20, Lemma 6.2] for more details. The following theorem was proved in [20].

Theorem 2.15. *Let P be an orthogonal projection and U be a unitary acting on a finite dimensional Hilbert space such that $\nu(PU + U^*P^\perp) < 1$. Then $\mathcal{W}_{PU+U^*P^\perp}$ is a distinguished variety. Conversely, if \mathcal{W} is any distinguished variety, then there is an orthogonal projection P and a unitary U acting on a finite dimensional Hilbert space such that $\mathcal{W} \cap \mathbb{G} = \mathcal{W}_{PU+U^*P^\perp} \cap \mathbb{G}$.*

This result makes it natural to ask for a (P, U) -version of Theorem 2.5. This requires the knowledge of exactly when a numerical contraction of the form $PU + U^*P^\perp$ is completely non-unitary. The following lemma provides it.

Lemma 2.16. *For an orthogonal projection P and a unitary U acting on \mathbb{C}^d , $PU + U^*P^\perp$ is c.n.u. if and only if there is no non-zero space $\mathcal{H} \subseteq \mathbb{C}^d$ such that U decomposes as the direct sum*

$$U = \begin{bmatrix} W_1 & & \\ & W_2 & \\ & & W \end{bmatrix} : \begin{bmatrix} P\mathcal{H} \\ P^\perp\mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} P\mathcal{H} \\ P^\perp\mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix}, \quad (2.3)$$

for some unitaries W_1, W_2 and W acting on the respective spaces.

Proof. Suppose there is a non-zero Hilbert space \mathcal{H} such that U decomposes as stated. Then with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H} \oplus \mathcal{H}^\perp$, we have

$$P = \begin{bmatrix} I & & \\ & 0 & \\ & & Q \end{bmatrix} \quad \text{and} \quad P^\perp = \begin{bmatrix} 0 & & \\ & I & \\ & & Q^\perp \end{bmatrix},$$

wherer Q is the projection $P|_{\mathcal{H}^\perp}$. Now a simple matrix computation yields

$$PU + U^*P^\perp = \begin{bmatrix} W_1 & & \\ & W_2^* & \\ & & QW + W^*Q^\perp \end{bmatrix}.$$

Note that since \mathcal{H} is non-zero, at least one of the spaces $P\mathcal{H}$ and $P^\perp\mathcal{H}$ must be non-zero. Therefore $PU + U^*P^\perp$ must not be completely non-unitary. Note that this direction works for infinite dimensional Hilbert spaces also. We shall use the finite dimensionality in the other direction.

Suppose that there is a non-zero space $\mathcal{H} \subseteq \mathbb{C}^d$ that decomposes $PU + U^*P^\perp$ and $PU + U^*P^\perp|_{\mathcal{H}}$ is unitary. Suppose

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \text{Ran } P \\ \text{Ran } P^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran } P \\ \text{Ran } P^\perp \end{bmatrix}.$$

This gives

$$PU + U^*P^\perp|_{\mathcal{H}} = \begin{bmatrix} A & B + C^* \\ 0 & D^* \end{bmatrix} : \begin{bmatrix} P\mathcal{H} \\ P^\perp\mathcal{H} \end{bmatrix} \rightarrow \begin{bmatrix} P\mathcal{H} \\ P^\perp\mathcal{H} \end{bmatrix}$$

a unitary, which (by a simple matrix computation and finite dimensionality) implies that $W_1 := A|_{P\mathcal{H}} : P\mathcal{H} \rightarrow P\mathcal{H}$ and $W_2 := D|_{P^\perp\mathcal{H}} : P^\perp\mathcal{H} \rightarrow P^\perp\mathcal{H}$ are unitaries. This justifies the decomposition (2.3) of U . \square

The following result is now an easy consequence of Theorem 2.5, Theorem 2.15 and the fact that $PU + U^*P^\perp$ is a numerical contraction.

Theorem 2.17. *For an orthogonal projection P and a unitary U acting on \mathbb{C}^d , the set $\mathcal{W}_{PU+U^*P^\perp}$ is an algebraic variety that always intersects Γ . Moreover, the following are equivalent:*

- (1) $\mathcal{W}_{PU+U^*P^\perp}$ is a distinguished variety;
- (2) $PU + U^*P^\perp$ is completely non-unitary, or equivalently, $\sigma(PU + U^*P^\perp) \cap \mathbb{T} = \emptyset$;
- (3) U does not decompose as in (2.3) for any non-zero subspace $\mathcal{H} \subseteq \mathbb{C}^d$;
- (4) $\mathcal{W}_{PU+U^*P^\perp} \subset \pi(\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2)$; and
- (5) Every irreducible component of $\mathcal{W}_{PU+U^*P^\perp}$ intersects \mathbb{G} .

*Conversely, if W is any distinguished variety such that each of its irreducible components meets \mathbb{G} , then there exist a projection P and a unitary U such that $PU + U^*P^\perp$ is c.n.u. and $\mathcal{W} = \mathcal{W}_{PU+U^*P^\perp}$.*

3. RATIONAL INNER FUNCTIONS ON THE SYMMETRIZED BIDISK

If Ψ on \mathbb{D}^2 is an operator-valued bounded analytic function, then it is not difficult to see that the radial limit

$$\Psi(\zeta, \eta) := \lim_{r \rightarrow 1^-} \Psi(r\zeta, r\eta)$$

exists almost everywhere in \mathbb{T}^2 . Since every function on \mathbb{G} can be viewed as a symmetric function on \mathbb{D}^2 , it follows that every bounded analytic function on \mathbb{G} has a radial limit almost everywhere in $b\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{T}\}$. An operator-valued bounded analytic function Ψ on \mathbb{G} is said to be *inner*, if the radial limits are isometric operators a.e. in $b\mathbb{G}$.

The objective of this section is to prove Theorem 1.2, which will be used in the following section. The main tool is an elegant representation formula for bounded analytic functions. The classical version of this formula states that for two Hilbert spaces \mathcal{E} and \mathcal{F} , a function $\Psi : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{F})$ is contractive analytic if and only if there is an auxiliary Hilbert space \mathcal{H} and a unitary operator $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \mathcal{H} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F} \\ \mathcal{H} \end{bmatrix}$ such that

$$\Psi(z) = A + zB(I - zD)^{-1}C.$$

This is called the realization formula for Ψ . This result has been extended to polydisk [2] and various other generalities [12, 13, 14, 28]. The realization formula for bounded analytic functions on \mathbb{G} was found in [10] and [22] by different methods. In both the approaches, the rational functions

$$\varphi(\alpha, s, p) = \frac{2\alpha p - s}{2 - \alpha s} \quad \text{for } \alpha \in \mathbb{D} \quad \text{and} \quad (s, p) \in \mathbb{G}$$

have played a significant role. For a fixed $(s, p) \in \mathbb{G}$, the function $\varphi(\cdot, s, p)$ is analytic in \mathbb{D} and continuous on $\overline{\mathbb{D}}$. Thus for a contractive operator τ , the operator

$$\varphi(\tau, s, p) := (2\tau p - s)(2 - \tau s)^{-1}$$

is well-defined - see for example [39, Sec. 2, Ch. III]. Moreover, the supremum of $\varphi(\cdot, s, p)$ over \mathbb{D} is not greater than one by item (iv) of Theorem 2.4, therefore $\varphi(\tau, s, p)$ is a contraction. The realization formula in the setting of \mathbb{G} is the following.

Theorem 3.1. *A function $\Psi : \mathbb{G} \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{F})$ is contractive analytic if and only if there exist a Hilbert space \mathcal{H} and unitary operators*

$$\tau : \mathcal{H} \rightarrow \mathcal{H} \quad \text{and} \quad U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \mathcal{H} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F} \\ \mathcal{H} \end{bmatrix} \quad (3.1)$$

such that

$$f(s, p) = A + B\varphi(\tau, s, p)(I - D\varphi(\tau, s, p))^{-1}C. \quad (3.2)$$

We show that a rational inner function on the symmetrized bidisk has a *finite dimensional realization formula*, i.e., when the auxiliary Hilbert space \mathcal{H} in Theorem 3.1 is of finite dimension. Such a result is known for the bidisk, which we state below. See the papers [15, 23, 26, 33, 37] for various proofs and generalizations.

Theorem 3.2. *Let Ψ be a $d \times d$ matrix-valued analytic function on \mathbb{D}^2 . Then the following are equivalent:*

- (RI) Ψ is a rational inner function;
- (AD) There exist analytic functions $F_1, F_2 : \mathbb{D}^2 \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_j})$ such that

$$I - \Psi(w)^*\Psi(z) = (1 - \bar{w}_1 z_1)F_1(w)^*F_1(z) + (1 - \bar{w}_2 z_2)F_2(w)^*F_2(z);$$

(RF) *There exist positive integers d_1, d_2 , and a unitary*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix}$$

such that

$$\Psi(z) = A + B\mathcal{E}_z(I - D\mathcal{E}_z)^{-1}C$$

where $\mathcal{E}_z = z_1P_1 + z_2P_2$ and P_1, P_2 are orthogonal projections of $\mathbb{C}^{d_1+d_2}$ onto \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively.

Moreover, there is a minimal choice of the integers d_1, d_2 in item (RF) above, viz., if $\det \Psi = \tilde{g}/g$, then $(d_1, d_2) = \deg \tilde{g}$.

A subset \mathbb{F} of \mathbb{D}^2 is said to be *symmetric*, if $(z_2, z_1) \in \mathbb{F}$, whenever $(z_1, z_2) \in \mathbb{F}$. Given a symmetric subset \mathbb{F} of \mathbb{D}^2 , a function g on $\mathbb{F} \times \mathbb{F}$ is said to be *doubly symmetric* if it is symmetric in both the coordinates, i.e., denoting $z^\sigma := (z_2, z_1)$ for $z = (z_1, z_2)$,

$$g(z, w) = g(z^\sigma, w) = g(z, w^\sigma) = g(z^\sigma, w^\sigma)$$

for all $z, w \in \mathbb{F}$.

The following lemma was proved for the scalar-valued functions in [10, Lemma 2.3], and it was observed that the proof works for the operator-valued case as well. We have observed that the proof also works for functions acting on an arbitrary symmetric subset \mathbb{F} of \mathbb{D}^2 instead of the whole domain. The technique, however, is exactly the same as the one for the scalar case. Therefore we omit the proof.

Lemma 3.3. *Let \mathbb{F} be a symmetric subset of \mathbb{D}^2 and $G : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{C}^{d \times d}$ be a doubly symmetric function such that there exist positive integers d_1, d_2 and functions $F_j : \mathbb{F} \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_j})$ such that*

$$G(z, w) = (1 - \overline{w_1}z_1)F_1(w)^*F_1(z) + (1 - \overline{w_2}z_2)F_2(w)^*F_2(z) \quad (3.3)$$

for all $z, w \in \mathbb{F}$. Then there exist a unitary operator τ on $\mathbb{C}^{d_1+d_2}$ and a function $F : \pi(\mathbb{F}) \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_1+d_2})$ such that for all $z, w \in \mathbb{F}$,

$$G(z, w) = F(t, q)^*(I - \varphi(\tau, t, q)^*\varphi(\tau, s, p))F(s, p), \quad (3.4)$$

where $(s, p) = \pi(z)$ and $(t, q) = \pi(w)$.

The following theorem shows that the rational inner functions on \mathbb{G} are exactly the ones that have a finite dimensional realization formula.

Theorem 3.4. *Let Ψ be a $d \times d$ matrix-valued analytic function on \mathbb{G} . Then the following are equivalent:*

- (RI) Ψ is rational and inner function;
- (AD) There exist positive integers d_1, d_2 , a unitary operator

$$\tau : \begin{bmatrix} \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix}$$

and an analytic function $F : \mathbb{G} \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_1+d_2})$ such that

$$I - \Psi(t, q)^*\Psi(s, p) = F(t, q)^*(I - \varphi(\tau, t, q)^*\varphi(\tau, s, p))F(s, p); \quad (3.5)$$

(RF) *There exist positive integers d_1, d_2 and unitary operators*

$$\tau : \begin{bmatrix} \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1} \\ \mathbb{C}^{d_2} \end{bmatrix}$$

such that

$$\Psi(s, p) = A + B\varphi(\tau, s, p)(I - D\varphi(\tau, s, p))^{-1}C. \quad (3.6)$$

Moreover, writing $\det \Psi \circ \pi = \tilde{g}/g$, the integers in items (AD) and (RF) can be chosen to be $(d_1, d_2) = \deg \tilde{g}$; and this is the minimal possible integers for such a representation.

Proof. The strategy is to follow the path (RI) \Rightarrow (AD) \Rightarrow (RF) \Rightarrow (RI).

For (RI) \Rightarrow (AD), the idea is to apply Theorem 3.2 to the function $\tilde{\Psi} : \mathbb{D}^2 \rightarrow \mathbb{C}^{d \times d}$ defined as

$$\tilde{\Psi}(z_1, z_2) = \Psi(z_1 + z_2, z_1 z_2) \text{ for all } (z_1, z_2) \in \mathbb{D}^2.$$

Clearly, $\tilde{\Psi}$ is rational and inner. Therefore by (RI) \Rightarrow (AD) of Theorem 3.2, there exist positive integers d_1, d_2 and analytic functions $F_j : \mathbb{D}^2 \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_j})$, $j = 1, 2$, such that

$$I - \tilde{\Psi}(w)^* \tilde{\Psi}(z) = (1 - \overline{w_1} z_1) F_1(w)^* F_1(z) + (1 - \overline{w_2} z_2) F_2(w)^* F_2(z)$$

for all $z, w \in \mathbb{D}^2$. Since the function $(z, w) \mapsto I - \tilde{\Psi}(w)^* \tilde{\Psi}(z)$ is doubly symmetric on $\mathbb{D}^2 \times \mathbb{D}^2$, by Lemma 3.3 there exist a unitary operator τ on $\mathbb{C}^{d_1+d_2}$ and an analytic map $F : \mathbb{G} \rightarrow \mathcal{B}(\mathbb{C}^{d_1+d_2}, \mathbb{C}^d)$ such that for all $z, w \in \mathbb{D}^2$,

$$I - \tilde{\Psi}(w)^* \tilde{\Psi}(z) = F(t, q)^*(I - \varphi(\tau, t, q)^* \varphi(\tau, s, p)) F(s, p) = I - \Psi(t, q)^* \Psi(s, p),$$

where $(s, p) = \pi(z)$ and $(t, q) = \pi(w)$. This establishes (AD).

(AD) \Rightarrow (RF) Rearrange equation 3.5 to obtain

$$I + F(t, q)^* \varphi(\tau, t, q)^* \varphi(\tau, s, p) F(s, p) = \Psi(t, q)^* \Psi(s, p) + F(t, q)^* F(s, p),$$

which, by a standard lurking isometry argument implies that there exists a unitary matrix,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix}$$

such that for all $(s, p) \in \mathbb{G}$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ \varphi(\tau, s, p) F(s, p) \end{bmatrix} = \begin{bmatrix} \Psi(s, p) \\ F(s, p) \end{bmatrix},$$

or equivalently,

$$\begin{aligned} A + B\varphi(\tau, s, p)F(s, p) &= \Psi(s, p) \\ C + D\varphi(\tau, s, p)F(s, p) &= F(s, p). \end{aligned}$$

From these two equations, one easily eliminates $F(s, p)$ to obtain

$$\Psi(s, p) = A + B\varphi(\tau, s, p)(I - D\varphi(\tau, s, p))^{-1}C.$$

(RF) \Rightarrow (RI) The rationality of Ψ is easily read off from the expression (3.6), while the assertion of innerness follows from a straightforward computation that leads to the identity

$$I - \Psi(s, p)^* \Psi(s, p) = C^* (I - \varphi(\tau, s, p)^* D^*)^{-1} (I - \varphi(\tau, s, p)^* \varphi(\tau, s, p)) (I - D \varphi(\tau, s, p))^{-1} C$$

for every $(s, p) \in \mathbb{G}$. Now a straightforward computation shows that $\varphi(\tau, s, p)$ is unitary whenever (s, p) is in $b\mathbb{G}$ and τ is a unitary. Thus in view of the above equation, the assertion in (RI) follows. The ‘moreover’ part follows from that of Theorem 3.2. \square

Finally, the takeaway of this section is the following result.

Theorem 3.5. *A solvable matrix Nevanlinna–Pick problem on the symmetrized bidisk has a rational inner solution.*

Proof. Consider an N -point solvable matrix Nevanlinna–Pick problem with initial data $(s_1, p_1), (s_2, p_2), \dots, (s_N, p_N)$ in \mathbb{G} and final data M_1, M_2, \dots, M_N in the closed operator-norm unit ball of $d \times d$ complex matrices. For each $j = 1, 2, \dots, N$, let $z_j \in \mathbb{D}^2$ be such that $\pi(z_j) = (s_j, p_j)$. Consider the following (at most $2N$ -point) matrix Nevanlinna–Pick interpolation problem in \mathbb{D}^2 :

$$z_j \mapsto M_j \quad \text{and} \quad z_j^\sigma \mapsto M_j \quad \text{for each } j = 1, 2, \dots, N.$$

By hypothesis, this interpolation problem is solvable and thus by [7, Corollary 2.13], there exists a rational inner function $\Psi : \mathbb{D}^2 \rightarrow \mathbb{C}^{d \times d}$ such that

$$\Psi(z_j) = M_j = \Psi(z_j^\sigma) \quad \text{for each } j = 1, 2, \dots, N.$$

Apply Theorem 3.2 to the rational inner function Ψ to get positive integers d_1, d_2 and functions $F_j : \mathbb{D}^2 \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_j})$ for $j = 1, 2$ such that

$$I - \Psi(w)^* \Psi(z) = (1 - \bar{w}_1 z_1) F_1(w)^* F_1(z) + (1 - \bar{w}_2 z_2) F_2(w)^* F_2(z).$$

Consider the finite subset $\mathbb{F} = \{z_j, z_j^\sigma : j = 1, 2, \dots, N\}$ of \mathbb{D}^2 . Define $G : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{C}^{d \times d}$ by

$$G(z, w) = I - \Psi(w)^* \Psi(z).$$

Since G is doubly symmetric, invoke Lemma 3.3 to get a unitary τ on $\mathbb{C}^{d_1+d_2}$ and a function $F : \pi(\mathbb{F}) \rightarrow \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d_1+d_2})$ such that

$$I - M_i^* M_j = F(s_i, p_i)^* (I - \varphi(\tau, s_i, p_i)^* \varphi(\tau, s_j, p_j)) F(s_j, p_j)$$

for each $j = 1, 2, \dots, N$. Rearrange the above equation to obtain

$$I + F(s_i, p_i)^* \varphi(\tau, s_i, p_i)^* \varphi(\tau, s_j, p_j) F(s_j, p_j) = M_i^* M_j + F(s_i, p_i)^* F(s_j, p_j). \quad (3.7)$$

This readily implies that the operator defined as

$$\begin{bmatrix} I_{\mathbb{C}^d} \\ \varphi(\tau, s_j, p_j) F(s_j, p_j) \end{bmatrix} \xi \mapsto \begin{bmatrix} M_j \\ F(s_j, p_j) \end{bmatrix} \xi \quad (3.8)$$

from

$$\text{span} \left\{ \begin{bmatrix} I \\ \varphi(\tau, s_j, p_j) F(s_j, p_j) \end{bmatrix} \xi : \xi \in \mathbb{C}^d, j = 1, 2, \dots, N \right\} \subset \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix}$$

onto

$$\text{span} \left\{ \begin{bmatrix} M_j \\ F(s_j, p_j) \end{bmatrix} \xi : \xi \in \mathbb{C}^d, j = 1, 2, \dots, N \right\} \subset \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix}$$

is unitary. By extending this unitary to $\begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix}$ and denoting by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^d \\ \mathbb{C}^{d_1+d_2} \end{bmatrix}$$

we define $\Phi : \mathbb{G} \rightarrow \mathbb{C}^{d \times d}$ by

$$\Phi(s, p) = A + B\varphi(\tau, s, p)(I - D\varphi(\tau, s, p))^{-1}C.$$

By (RF) \Rightarrow (RI) of Theorem 3.4, Φ so defined is rational and inner. Moreover, Φ interpolates the data because of (3.8). \square

4. EXTREMAL PROBLEMS AND THE UNIQUENESS VARIETY

The objective of this section is to prove Theorem 1.4. We begin with a couple of elementary observations.

Observation 4.1. *The uniqueness set corresponding to a solvable data in \mathbb{G} coincides with an algebraic variety.*

Proof. Let $\mathfrak{D} = \{((s_j, p_j), w_j) : 1 \leq j \leq N\}$ be a solvable data in \mathbb{G} and \mathcal{S} be the uniqueness set. Let (s, p) be any point in $\mathbb{G} \setminus \mathcal{S}$. Then by definition of the uniqueness set, there exists a pair of interpolants f and f' so that $w = f(s, p) \neq f'(s, p) = w'$. Since the data $\mathfrak{D} \cup \{((s, p), w)\}$ and $\mathfrak{D} \cup \{((s, p), w')\}$ are solvable, by Theorem 3.5, there exist rational solutions to both of these problems. Consequently, \mathcal{S} is contained in the common zero set of the differences of these rational interpolants, which is an algebraic variety; denote this algebraic variety by $\mathcal{W}_{(s,p)}$. Repeat the analysis for all $(s, p) \in \mathbb{G} \setminus \mathcal{S}$ and denote the intersection of $\mathcal{W}_{(s,p)}$ by \mathcal{W} . Obviously, \mathcal{W} is an algebraic variety that coincides with \mathcal{S} in \mathbb{G} , i.e., $\mathcal{S} = \mathbb{G} \cap \mathcal{W}$. \square

Observation 4.2. *The uniqueness set for a non-extremal data $\mathfrak{D} = \{((s_j, p_j), w_j) : 1 \leq j \leq N\}$ in \mathbb{G} is just the set $\{(s_j, p_j) : 1 \leq j \leq N\}$.*

Proof. Let f be a solution of supremum norm less than 1 and $\eta \neq 0$ be any complex number. Choose $\epsilon > 0$ small enough so that for every $0 < \delta \leq \epsilon$ the function

$$h_\delta(s, p) = f(s, p) + \delta \prod_{r=1}^N [(s - s_r) + \eta(p - p_r)],$$

which obviously interpolates, has supremum norm no greater than one. Now for (s, p) to be in the uniqueness set we must have $\prod_{r=1}^N [(s - s_r) + \eta(p - p_r)] = 0$, which implies that there is an $r = 1, 2, \dots, N$, so that $(s - s_r) + \eta(p - p_r) = 0$. Note that if $p = p_r$, then obviously $s = s_r$. If $p \neq p_r$, then we must have $\eta = -(s - s_r)/(p - p_r)$. So we choose η other than these points. Hence the observation is made. \square

Let us recall that a data $\{(\lambda_j, w_j) : 1 \leq j \leq N\}$ in \mathbb{D} is solvable if and only if the Pick matrix

$$\left[\frac{1 - w_i \bar{w}_j}{1 - \lambda_i \bar{\lambda}_j} \right]_{i,j=1}^N$$

is positive semi-definite. One way of interpreting this result is that the solvability of a Pick problem in \mathbb{D} is equivalent to checking the positivity of $1 - w_i \bar{w}_j$ against the Szegő

kernel \mathbb{S} of \mathbb{D} , i.e., $\mathbb{S}(\lambda_i, \lambda_j) := (1 - \lambda_i \bar{\lambda}_j)^{-1}$. This point of view makes it possible to approach the Pick problem in other cases such as the multi-connected domains [1], the polydisk [2, 4], the distinguished varieties [30] and the symmetrized bidisk. However, one must consider a family of kernels instead of just one.

Definition 4.3. A *kernel* on $\mathcal{G} \subseteq \mathbb{G}$ is a function $k : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$ such that for every choice of finitely many points $\{\lambda_j\}_{j=1}^N$ in \mathcal{G} , the matrix $[k(\lambda_i, \lambda_j)]_{i,j=1}^N$ is positive semi-definite and the diagonal entries are non-zero.

An *admissible kernel* k on $\mathcal{G} \subseteq \mathbb{G}$ is a kernel such that the pair (M_s, M_p) of multiplication by coordinate functions on $H(k)$ has Γ (the closure of \mathbb{G}) as a spectral set, i.e.,

$$\|f(M_s, M_p)\| \leq \sup\{|f(s, p)| : (s, p) \in \Gamma\}$$

for every polynomial f in two variables.

In the definition above, $H(k)$ is the reproducing kernel Hilbert space corresponding to the kernel k - see [4, Chapter 2]. An example of admissible kernel on \mathbb{G} is given by

$$((s, p), (t, q)) \mapsto \frac{1}{(1 - p\bar{q})^2 - (s - \bar{t}p)(\bar{t} - s\bar{q})}.$$

This kernel is known to have all the properties to be referred to as the Szegő kernel for \mathbb{G} - see the papers [38, 22, 19]. One of the criteria for the solvability of a Pick problem in \mathbb{G} is the following.

Theorem 4.4 (See [22]). *A Pick data $\{(\lambda_j, w_j) : 1 \leq j \leq N\}$ in \mathbb{G} is solvable if and only if for every admissible kernel k on \mathbb{G} , the matrix*

$$[(1 - w_i \bar{w}_j)k(\lambda_i, \lambda_j)]_{i,j=1}^N \quad (4.1)$$

is positive semi-definite.

The lemma below does the heavy lifting in the proof of Theorem 1.4. The proof of part (iii) is due to John E. McCarthy from a private communication.

Lemma 4.5. *Let $\Lambda = \{(s_j, p_j) : j = 1, 2, \dots, N\}$ be a subset of \mathbb{G} and k be an admissible kernel on Λ . Then there is a distinguished variety \mathcal{W} containing Λ and an admissible kernel K on $\mathcal{W} \cap \mathbb{G}$ that extends k . Moreover, the distinguished variety and the extension can be obtained with the following bonuses:*

- (i) *there is a completely non-unitary numerical contraction F such that $\mathcal{W} = \mathcal{W}_F$;*
- (ii) *the extension can be made so that the extended kernel K is given by*

$$K((s, p), (t, q)) = \frac{1}{1 - p\bar{q}} \langle u(t, q), u(s, p) \rangle = \langle \mathbb{S}_q \otimes u(t, q), \mathbb{S}_p \otimes u(s, p) \rangle,$$

where \mathbb{S} is the Szegő kernel for \mathbb{D} and $u(s, p) \in \text{Ker}(F + \bar{p}F^ - \bar{s}I)$; and*

- (iii) *the kernel vectors $u(s, p)$ can be chosen so that for each (s_j, p_j) , there exist q_j functions $\alpha_1, \alpha_2, \dots, \alpha_{q_j}$ acting on a neighbourhood V_j of \bar{p}_j so that for each $z \in V_j$ and $l \in \{1, 2, \dots, q_j\}$, $\alpha_l(z)$ belongs to $\sigma(F + \bar{p}_j F^*)$,*

$$\alpha_l(z) \rightarrow \bar{s}_j \quad \text{and} \quad \sum_{l=1}^{q_j} u(\overline{\alpha_l(z)}, \bar{z}) \rightarrow u(s_j, p_j) \quad \text{as} \quad z \rightarrow \bar{p}_j.$$

Proof. Let k be an admissible kernel on the set Λ , i.e., the pair (M_s, M_p) of multiplication by coordinate functions on $H(k) = \text{span}\{k_{(s_j, p_j)} : j = 1, 2, \dots, N\}$ has Γ as a spectral set (i.e., a Γ -contraction). Let F' on $\mathcal{F}' = \text{Ran}(I - M_p M_p^*)^{1/2}$ be the numerical contraction given by [21, Theorem 4.2] applied to (M_s^*, M_p^*) , i.e., F' is the unique numerical contraction on \mathcal{F}' (the so called fundamental operator) such that

$$M_s^* - M_s M_p^* = D_{M_p^*} F' D_{M_p^*}, \quad (4.2)$$

where $D_{M_p^*} = (I - M_p M_p^*)^{1/2}$. Since $(s_j, p_j) \in \mathbb{G}$, we have $|p_j| < 1$ for each j and therefore $M_p^{*n} \rightarrow 0$ strongly as $n \rightarrow \infty$. Using this convergence, one can compute that the operator $\Pi : H(k) \rightarrow H^2(\mathcal{F}')$ given by

$$\Pi h = \sum_{n \geq 0} z^n D_{M_p^*} M_p^{*n} h \quad \text{for all } h \in H(k) \quad (4.3)$$

is an isometry. The following intertwining relations was established in the proof of [21, Theorem 4.6]:

$$\Pi(M_s^*, M_p^*) = (M_{F'^* + zF'}^*, M_z^*)\Pi. \quad (4.4)$$

Using the first intertwining relation, we compute

$$\begin{aligned} \overline{s_j} \Pi k_{(s_j, p_j)} &= \Pi M_s^* k_{(s_j, p_j)} = M_{F'^* + zF'}^* \Pi k_{(s_j, p_j)} = M_{F'^* + zF'}^* \sum_{n \geq 0} z^n \overline{p_j}^n D_{M_p^*} k_{(s_j, p_j)} \\ &= \sum_{n \geq 0} z^n \overline{p_j}^n F' D_{M_p^*} k_{(s_j, p_j)} + \sum_{n \geq 0} z^n \overline{p_j}^{n+1} F'^* D_{M_p^*} k_{(s_j, p_j)} \\ &= \sum_{n \geq 0} z^n \overline{p_j}^n (F' + \overline{p_j} F'^*) D_{M_p^*} k_{(s_j, p_j)}. \end{aligned}$$

Equating the coefficients we thus obtain

$$(F' + \overline{p_j} F'^* - \overline{s_j} I) D_{M_p^*} k_{(s_j, p_j)} = 0 \quad \text{for each } j = 1, 2, \dots, N.$$

Let $F' = \begin{bmatrix} F & 0 \\ 0 & F_u \end{bmatrix} : \begin{bmatrix} \mathcal{F} \\ \mathcal{F}_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F} \\ \mathcal{F}_u \end{bmatrix}$ be the decomposition of F' into the c.n.u. part F and the unitary part F_u . We pause here to note that none of the vectors $D_{M_p^*} k_{(s_j, p_j)}$ has a non-zero component in \mathcal{F}_u . Indeed, if there was a non-zero vector v such that $(F_u + \overline{p_j} F_u^* - \overline{s_j} I)v = 0$ for some j , then we would have $s_j = \beta + \overline{\beta} p_j$ for some $\beta \in \mathbb{T}$. This contradicts part (iii) of Theorem 2.4 because (s_j, p_j) comes from \mathbb{G} . Therefore we must have

$$(F + \overline{p_j} F^* - \overline{s_j} I) D_{M_p^*} k_{(s_j, p_j)} = 0 \quad \text{for each } j = 1, 2, \dots, N, \quad (4.5)$$

which readily implies that the points (s_j, p_j) belong to

$$\mathcal{W}_F = \{(s, p) \in \mathbb{C}^2 : \det(F^* + pF - sI) = 0\}, \quad (4.6)$$

which is a distinguished variety by Theorem 2.5.

To produce an admissible extension of k to $\mathcal{W}_F \cap \mathbb{G}$, we pick an (s, p) in $(\mathcal{W}_F \cap \mathbb{G}) \setminus \Lambda$ and choose $u(s, p)$ to be *any* vector in $\text{Ker}(F + \overline{p} F^* - \overline{s} I)$ and set $u(s_j, p_j) = D_{M_p^*} k_{(s_j, p_j)}$ for each $j = 1, 2, \dots, N$. We shall prove that the function $K : (\mathcal{W}_F \cap \mathbb{G}) \times (\mathcal{W}_F \cap \mathbb{G}) \rightarrow \mathbb{C}$ given by

$$K((s, p), (t, q)) = \langle \mathbb{S}_q \otimes u(t, q), \mathbb{S}_p \otimes u(s, p) \rangle_{H^2 \otimes \mathcal{F}}$$

is an admissible kernel that extends k . That it extends k follows from the following easy computation:

$$\begin{aligned} K((s_i, p_i), (s_j, p_j)) &= \frac{1}{(1 - p_i \bar{p}_j)} \langle D_{M_p^*} k_{(s_j, p_j)}, D_{M_p^*} k_{(s_i, p_i)} \rangle \\ &= \frac{1}{(1 - p_i \bar{p}_j)} \langle (I - M_p M_p^*) k_{(s_j, p_j)}, k_{(s_i, p_i)} \rangle = k((s_i, p_i), (s_j, p_j)). \end{aligned}$$

To prove the admissibility of K , we show that the pair of multiplication by coordinate functions (M_s, M_p) on $H(K) = \overline{\text{span}}\{K_{(s,p)} : (s,p) \in \mathcal{W}_F \cap \mathbb{G}\}$ is a Γ -contraction. To that end, we first observe that $H(K)$ is unitarily equivalent to

$$\mathcal{R} = \overline{\text{span}}\{\mathbb{S}_p \otimes u(s, p) : (s, p) \in \mathcal{W}_F \cap \mathbb{G}\} \subseteq H^2 \otimes \mathcal{F}$$

via the unitary given by

$$\tau : \sum_{l=1}^d c_l K_{(s_l, p_l)} \mapsto \sum_{l=1}^d c_l \mathbb{S}_{p_l} \otimes u(s_l, p_l) \quad \text{where } (s_l, p_l) \in \mathcal{W}_F \cap \mathbb{G}.$$

We show that (M_s^*, M_p^*) on $H(K)$ is unitarily equivalent to $(M_{F^*+zF}^*, M_z^*)|_{\mathcal{R}}$ via the unitary τ . This is easily achieved by doing the following computation on the kernel functions:

$$\begin{aligned} M_{F^*+zF}^* \tau K_{(s,p)} &= M_{F^*+zF}^* (\mathbb{S}_p \otimes u(s, p)) = \mathbb{S}_p \otimes F u(s, p) + \bar{p} \mathbb{S}_p \otimes F^* u(s, p) \\ &= \mathbb{S}_p \otimes (F + \bar{p} F^*) u(s, p) \\ &= \bar{s} \mathbb{S}_p \otimes u(s, p) = \tau M_s^* K_{(s,p)}. \end{aligned}$$

Note that pairs of the form $(M_{F^*+zF}^*, M_z^*)$ are known to be Γ -contractions whenever F is a numerical contraction; see [9, Theorem 2.4]. Since restrictions and the adjoint of a Γ -contraction are again Γ -contractions, admissibility of K is established. This completes the proof of items (i) and (ii).

Proof of item (iii) requires some ideas in perturbation theory for the eigenvalue problem: a study of how the eigenvalues and the eigenvectors of a matrix-valued analytic function change subject to a small perturbation. The results that we shall use can be found in the books [35, 31]. Although we shall apply the results to a very simple matrix-valued function, viz., $z \mapsto F + zF^*$, we briefly discuss the general theory below.

Let Ψ be an $N \times N$ matrix-valued polynomial (the theory extends to the analytic case as well but we restrict ourselves to just polynomials). It follows from the discussion in [35, Sec. 13, Ch. 5] that the function

$$z \mapsto \#\sigma(\Psi(z))$$

is constant, say $r \leq N$, except at only a finite number of points - the so called *exceptional points*, which may of course be empty. The eigenvalues of $\Psi(z)$ - the so called *algebraic functions* - are the roots of the algebraic equation

$$\det(\Psi(z) - wI) = (-1)^N w^N + g_1(z) w^{N-1} + \cdots + g_{n-1}(z) w + g_N(z) = 0.$$

The algebraic functions are (branches of) analytic functions with the exceptional points as the possible branch points. To demonstrate, consider the two examples

$$\begin{bmatrix} 1 & z \\ z & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix}.$$

For the first example, the eigenvalues are $\pm(1+z^2)^{1/2}$, with the exceptional points $\pm i$ as the branch points; for the second example, the eigenvalues are the entire functions $\pm z$, and 0 is the only exceptional point.

For a non-exceptional point z , let $\alpha_1(z), \alpha_2(z), \dots, \alpha_r(z)$ be the distinct eigenvalues of $\Psi(z)$. Let $P_j(z)$ be the projection (i.e., idempotent but not necessarily self-adjoint) onto the generalized eigenspace corresponding to the eigenvalue $\alpha_j(z)$. The projections $P_j(z)$ are called the *eigenprojections*. It is known (see [31, Sec. I-3,4 and Sec. II-4]) that

$$P_i(z)P_j(z) = P_j(z)P_i(z) = \delta_{ij}P_i(z) \quad \text{and} \quad \sum_{j=1}^r P_j(z) = I_N.$$

Therefore, with the *eigennilpotent* defined as

$$D_j(z) = (\Psi(z) - \alpha_j(z)I)P_j(z)$$

it follows that

$$\Psi(z) = \sum_{j=1}^r \alpha_j(z)P_j(z) + \sum_{j=1}^r D_j(z).$$

The above decomposition is referred to as the canonical decomposition of Ψ ; see the discussion in [31, Sec. I-4 and II-4].

Let z_0 be any complex number (possibly an exceptional point) and w_0 be an eigenvalue of $\Psi(z_0)$. Let l be the algebraic multiplicity of w_0 . Let $\epsilon > 0$ be small enough so that $\mathbb{D}(w_0, \epsilon)$, the disk of radius ϵ around w_0 , does not contain any other eigenvalue of $\Psi(z_0)$. Then by the ‘‘Theorem on the continuity of the roots’’ in [35, Page 122], it is possible to find a $\delta = \delta(\epsilon) > 0$ so that for every z in $\mathbb{D}(z_0, \delta)$, the disk of radius δ around z_0 , there are *exactly* l eigenvalues of $\Psi(z)$, say $\alpha_1(z), \alpha_2(z), \dots, \alpha_l(z)$ lying inside $\mathbb{D}(w_0, \epsilon)$. In other words, every l -fold eigenvalue of $\Psi(z_0)$ branches off into exactly l simple roots $\alpha_1(z), \alpha_2(z), \dots, \alpha_l(z)$ such that for each j , $\alpha_j(z) \rightarrow w_0$ as $z \rightarrow z_0$ (compare it with the examples above). Define the operator

$$P(z) = \frac{1}{2\pi i} \int_{\mathbb{D}(w_0, \epsilon)} (\zeta - \Psi(z))^{-1} d\zeta. \quad (4.7)$$

Then the crux of the matter is that $z \mapsto P(z)$ is projection-valued and holomorphic near z_0 . Furthermore, it is equal to the sum of the eigenprojections corresponding to all the eigenvalues that are inside $\mathbb{D}(w_0, \epsilon)$, i.e.,

$$P(z) = \sum_{j=1}^l P_j(z). \quad (4.8)$$

It should, however, be noted that while P is holomorphic near z_0 , the eigenprojections P_j may not even be defined at the exceptional points; see for example [31, Example II-1.12]. For the results stated above, see [31, Prob. I-5.9 and Sec. II-4].

Let us now consider the example $\Psi(z) = F + zF^*$ for some matrix F . It can be checked that $\Psi(z)$ is normal matrix for each z on the unit circle. Therefore by Theorem II-1.10 in Kato [31], $D_j(z) = 0$ for every z except possibly the exceptional ones. In other words, $\Psi(z)$ is diagonalizable at every z except possibly the exceptional points, i.e., for each j , $\text{Ran } P_j(z)$ equals to the eigenspace corresponding to the j -th eigenvalue.

Let us recall that the vectors $u(s, p)$ in item (ii) were just any vectors in the kernel of $F + \bar{p}F^* - \bar{s}I$. To prove item (iii), we have to make certain choices of such vectors. To that end, let us pick the point (s_1, p_1) ; the analysis for the other nodes remains the same. Note that \bar{s}_1 is an eigenvalue of $\Psi(\bar{p}_1)$ with $u(s_1, p_1) = D_{M_p^*}k_{(s_1, p_1)}$ as an eigenvector. We apply the above analysis for the choice $(z_0, w_0) = (\bar{p}_1, \bar{s}_1)$ to obtain, for a sufficiently small $\epsilon > 0$, a $\delta > 0$ such that for every z in $\mathbb{D}(\bar{p}_1, \delta) \cap \mathbb{D}$, there exist points $\alpha_1(z), \alpha_2(z), \dots, \alpha_l(z)$ in $\mathbb{D}(\bar{s}_1, \epsilon) \cap \sigma(\Psi(z))$, i.e., for each $j = 1, 2, \dots, l$,

$$\alpha_j(z) \rightarrow \bar{s}_1 \quad \text{as } z \rightarrow \bar{p}_1. \quad (4.9)$$

With the projection P as defined in (4.7), we then have $P(\bar{p}_1)u(s_1, p_1) = u(s_1, p_1)$. For each z in $\mathbb{D}(\bar{p}_1, \delta) \cap \mathbb{D}$, set

$$v(z) = P(z)u(s_1, p_1) = \sum_{j=1}^l P_j(z)u(s_1, p_1) = \sum_{j=1}^l v_j(z),$$

where $v_j(z) := P_j(z)u(s_1, p_1)$. Now if necessary we can choose smaller δ so that $\mathbb{D}(\bar{p}_1, \delta) \cap \mathbb{D}$ does not contain any exceptional point of Ψ (except possibly \bar{p}_1 itself). Therefore we can assume that $\Psi(z)$ is diagonalizable for each $z \in \mathbb{D}(\bar{p}_1, \delta) \cap \mathbb{D}$. Therefore the vectors $v_j(z)$ are just the eigenvectors corresponding to $\alpha_j(z)$, i.e.,

$$(\Psi(z) - \alpha_j(z)I)v_j(z) = (F + zF^* - \alpha_j(z)I)v_j(z) = 0 \quad (4.10)$$

for every $j = 1, 2, \dots, l$ and each $z \in \mathbb{D}(\bar{p}_1, \delta) \cap \mathbb{D}$. This together with (4.9) imply that the points $(\overline{\alpha_j(z)}, \bar{z})$ are in $\mathcal{W}_F \cap \mathbb{G}$ and converge to (s_1, p_1) as $z \rightarrow \bar{p}_1$. The proof of item (iii) now follows if we set $u(\overline{\alpha_j(z)}, \bar{z}) = v_j(z)$. \square

Remark 4.6. The substance of Lemma 4.5 above is not that there is a distinguished variety containing the finite set Λ ; it is that there is a distinguished variety \mathcal{W} to which a given admissible kernel on Λ can be admissibly extended to $\mathcal{W} \cap \mathbb{G}$. Indeed, given a finite subset $\{(s_j, p_j) : 1 \leq j \leq N\}$ of \mathbb{G} , there is a quick way to get a distinguished variety containing these points. For example, consider the diagonal matrix $F = \text{diag}(\beta_j)$. By part (iii) of Theorem 2.4, this is a strict contraction and hence a strict numerical contraction. One can show using the relation between (s_j, p_j) and β_j given in part (iii) of Theorem 2.4 that the distinguished variety \mathcal{W}_F contains all the points (s_j, p_j) .

Given a solvable data $\{(\lambda_j, w_j) : 1 \leq j \leq N\}$ in \mathbb{G} , an admissible kernel k is called *active*, if the (positive semi-definite) matrix $[(1 - w_i \bar{w}_j)k(\lambda_i, \lambda_j)]$ has a non-zero vector in its null space. There is an easy way (if one is lucky) to tell if a given solvable data is extremal.

Lemma 4.7. *A solvable Pick problem on the symmetrized bidisk is extremal if and only if it has an active kernel.*

The proof of this lemma goes along the same lines as that of Lemma 4.4 in [5]; thus we omit it. In the following, we shall use the following convention of notation

$$k((s_i, p_i), (s_j, p_j)) =: k_{ij}.$$

An N -point extremal Pick problem in \mathbb{G} is said to be *minimal*, if none of the $(N - 1)$ -point subproblems is extremal.

Theorem 4.8. *Given a minimal extremal Pick problem $\{(s_j, p_j) \rightarrow w_j\}_{j=1}^N$ on the symmetrized bidisk with \mathcal{U} as its uniqueness variety, there is a distinguished variety \mathcal{W} such that*

$$\{(s_1, p_1), (s_2, p_2), \dots, (s_N, p_N)\} \subset \mathcal{W} \subseteq \mathcal{U}. \quad (4.11)$$

Proof. Since the Pick problem is extremal, by Lemma 4.7, there is an active kernel k , i.e., k is an admissible kernel such that the corresponding Pick matrix has a non-trivial kernel. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N)^T$ be a non-zero vector such that

$$[(1 - w_i \bar{w}_j) k_{ij}] \gamma = 0. \quad (4.12)$$

Note that no component of γ can be zero, for otherwise, an $(N-1)$ -point sub-problem would have an active kernel, and by Lemma 4.7, this $(N-1)$ -point Pick problem would be extremal, which contradicts the minimality of the problem.

Invoke Lemma 4.5 to obtain a distinguished variety \mathcal{W} containing each of (s_j, p_j) and an admissible kernel K on $\mathcal{W} \cap \mathbb{G}$ extending k with additional properties (i), (ii) and (iii). We shall show that possibly a subvariety of this distinguished variety satisfies the containments in (4.11). We shall establish the second containment only in the domain \mathbb{G} . Note that this suffices because every irreducible component of \mathcal{W} intersects \mathbb{G} . The strategy is to show that if (s_{N+1}, p_{N+1}) is any point in $\mathcal{W} \cap \mathbb{G}$ other than the nodes (s_j, p_j) and w_{N+1} is complex number assumed by some interpolant at (s_{N+1}, p_{N+1}) , then w_{N+1} must not depend on the interpolant.

This will take a fair bit of analysis. We start with the observation that since the Pick problem $\{(s_j, p_j) \rightarrow w_j\}_{j=1}^{N+1}$ is solvable (by the choice of the point w_{N+1}) and K is admissible, by Theorem 4.4, we have

$$[(1 - w_i \bar{w}_j) K_{ij}]_{i,j=1}^{N+1} \succeq 0,$$

which in particular implies that

$$\langle [(1 - w_i \bar{w}_j) K_{ij}] \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \rangle \geq 0, \quad (4.13)$$

where γ is as in (4.12) and δ is any complex number. Unfolding the LHS of (4.13) and making use of (4.12) we get

$$2 \operatorname{Re} \left[\bar{\delta} \sum_{j=1}^N (1 - w_{N+1} \bar{w}_j) K_{N+1,j} \gamma_j \right] + |\delta|^2 (1 - |w_{N+1}|^2) K_{N+1,N+1} \geq 0. \quad (4.14)$$

Since the above inequality holds for all $\delta \in \mathbb{C}$, it follows from a routine analysis that

$$\sum_{j=1}^N (1 - w_{N+1} \bar{w}_j) K_{N+1,j} \gamma_j = 0,$$

which, after a rearrangement of terms, leads to

$$w_{N+1} \left(\sum_{j=1}^N \bar{w}_j K_{N+1,j} \gamma_j \right) = \sum_{j=1}^N K_{N+1,j} \gamma_j. \quad (4.15)$$

This clearly yields an expression of w_{N+1} depending only on the interpolation data and the point (s_{N+1}, p_{N+1}) , viz.,

$$w_{N+1} = \frac{\sum_{j=1}^N K_{N+1,j} \gamma_j}{\sum_{j=1}^N \bar{w}_j K_{N+1,j} \gamma_j}, \quad (4.16)$$

provided that the denominator does not vanish identically.

In the remainder of the proof, we make sure that at each node (s_j, p_j) , there is at least one sheet of \mathcal{W} passing through (s_j, p_j) that contains a sequence converging to (s_j, p_j) at each term of which the denominator of (4.16) is non-zero. Then every interpolant agrees along the sequence and thus by the Identity Theorem, every interpolant coincides throughout the sheet. We shall then take the union of those irreducible components of \mathcal{W} that contains these sheets and discard the other components of \mathcal{W} , if there is any. Note that the resulting variety will possibly be a sub-variety of \mathcal{W} which would nevertheless satisfy (4.11).

We consider the node (s_1, p_1) ; the analysis for the other nodes remains the same. Item (iii) of Lemma 4.5 will come handy now. By part (iii) of Lemma 4.5, there are q_1 sequences $\{(s_{n,i}, p_n)\}_{n=1}^\infty$, $i = 1, 2, \dots, q_1$ in $\mathcal{W} \cap \mathbb{G}$, each of which converges to (s_1, p_1) and

$$\sum_{i=1}^{q_1} u(s_{n,i}, p_n) \rightarrow u(s_1, p_1) \quad \text{as } n \rightarrow \infty. \quad (4.17)$$

Suppose on the contrary that there is no sheet passing through (s_1, p_1) which contains a sequence converging to (s_1, p_1) and the denominator of (4.16) vanishes at each of the points in the sequence. Then there is a neighbourhood of (s_1, p_1) in \mathbb{G} such that the denominator vanishes identically in the intersection of the neighbourhood with every sheet passing through (s_1, p_1) . This means that the denominator would vanish for the choice $(s_{N+1}, p_{N+1}) = (s_{n,i}, p_n)$ for every n and i , i.e.,

$$0 = \sum_{j=1}^N \bar{w}_j K((s_j, p_j), (s_{n,i}, p_n)) \gamma_j = \left\langle \sum_{j=1}^N \bar{w}_j \gamma_j \langle \mathbb{S}_{\bar{p}_j}, \mathbb{S}_{\bar{p}_n} \rangle u(s_j, p_j), u(s_{n,i}, p_n) \right\rangle.$$

Since this is true for every $i = 1, 2, \dots, q_1$ and $n \in \mathbb{N}$, in view of the convergence (4.17), we therefore have

$$\sum_{j=1}^N \bar{w}_j K_{1,j} \gamma_j = 0. \quad (4.18)$$

The equality above jeopardizes the minimality of the extremal problem $\{\lambda_j \rightarrow w_j\}_{j=1}^N$. Indeed, if ϵ is any non-zero number, then the interpolation problem $\{\lambda_j \rightarrow w'_j\}_{j=1}^N$ where $w'_1 = w_1 + \epsilon$ and $w'_j = w_j$ for $j \geq 2$ is not solvable. This is easily seen because,

with the admissible kernel K and the non-zero vector γ as in (4.12), we have

$$\begin{aligned} \langle [((1 - w'_i \overline{w'_j}) K_{ij})] \gamma, \gamma \rangle &= \sum_{i,j=1}^N (1 - w'_i \overline{w'_j}) K_{ij} \gamma_j \overline{\gamma_i} \\ &= \sum_{i,j=1}^N (1 - w_i \overline{w_j}) K_{ij} \gamma_j \overline{\gamma_i} - 2 \operatorname{Re} \left[\epsilon \overline{\gamma_1} \sum_{j=1}^N \overline{w_j} K_{1,j} \gamma_j \right] - |\epsilon|^2 K_{11} |\gamma_1|^2 \\ &= -|\epsilon|^2 K_{11} |\gamma_1|^2 < 0. \quad [\text{by 4.12 and 4.18}] \end{aligned}$$

Here we are using the fact that K does not vanish at the diagonal entries and that none of the entries of γ is zero. From the fact that the problem $\{\lambda_j \rightarrow w'_j\}_{j=1}^N$ is not solvable for any $\epsilon \neq 0$, we conclude that the value w_1 at the node λ_1 is uniquely determined by the other values at $\lambda_2, \lambda_3, \dots, \lambda_N$. This makes the $(N-1)$ -point problem $\{\lambda_j \rightarrow w_j\}_{j=2}^N$ extremal, contradicting the minimality of the problem.

Therefore, equation (4.16) gives a unique representation formula for a function interpolating the data along some sheets. This was to be proved. \square

Below we consider several examples of Pick problems and compute their uniqueness varieties explicitly. First we consider an example of a 2-point problem with a unique solution and hence is extremal.

Example 4.9. Consider the data $\{((-i/\sqrt{2}, 0), 0), ((1/\sqrt{2}, 0), 1/\sqrt{2})\}$ in \mathbb{G} . Define the holomorphic map $f : \mathbb{D} \rightarrow \mathbb{G}$ by setting $\lambda \rightarrow (\lambda + a(\lambda), \lambda \cdot a(\lambda))$, where a is the automorphism of \mathbb{D} given by $a(\lambda) = i(\sqrt{2}\lambda - 1)(\sqrt{2} - \lambda)^{-1}$. It is easy to see that $f(0) = (-i/\sqrt{2}, 0)$ and $f(1/\sqrt{2}) = (1/\sqrt{2}, 0)$. Therefore by Schwarz lemma any $F : \mathbb{G} \rightarrow \mathbb{D}$ interpolating the data is a left inverse of f , and vice versa. On the other hand, since $a(\lambda) = \lambda$ has a double root on \mathbb{T} viz., $h = (1-i)/\sqrt{2}$, by Theorem 5.4 of [36], f has a unique left inverse. Moreover, the left inverse is given in Lemma 5.7 of [36], which in this case simplifies to the formula

$$F(s, p) = \frac{\sqrt{2}(i + \sqrt{2}s - (1 + 2i)p)}{(3i + 1) + \sqrt{2}(1 - i)s - (1 - i)p}.$$

Since f has a unique left inverse, the solution is unique and hence the uniqueness set is whole \mathbb{G} .

Now we consider two examples of 2-point extremal problems such that the solution is not unique. The non-uniqueness of the solutions can also be explained by Theorem 5.4 of [36].

Example 4.10. Consider the data $\{((0, 0), 0), ((1, 1/4), 1/2)\}$ in \mathbb{G} . While both the functions $\frac{s}{2}$ and $-\frac{2p-s}{2-s}$ solve the problem, it is extremal because if $f : \mathbb{G} \rightarrow \mathbb{D}$ is any function solving the problem, then the holomorphic function $\tilde{f} : \mathbb{D} \rightarrow \mathbb{D}$ defined as

$$\tilde{f}(z) = f(2z, z^2) \tag{4.19}$$

interpolates the data $\{(0, 0), (1/2, 1/2)\}$ in \mathbb{D} . Thus, by Schwarz lemma we have

$$\tilde{f}(z) = f(2z, z^2) = z \text{ for all } z \in \mathbb{D}. \tag{4.20}$$

Therefore f has the supremum norm over \mathbb{G} one.

Since the Pick problem does not have a unique solution, it is worthwhile to compute its uniqueness variety. We show that the uniqueness variety corresponding to this Pick problem is the royal variety

$$\mathcal{R} = \{(2z, z^2) : z \in \mathbb{C}\}. \quad (4.21)$$

Indeed, the analysis above shows that any two interpolants coincide on the set $\{(2z, z^2) : z \in \mathbb{D}\}$. This shows that the set is contained in the uniqueness set. Conversely, if (s, p) is in the uniqueness set, then, in particular, the two interpolants $\frac{s}{2}$ and $-\frac{2p-s}{2-s}$ would agree at (s, p) , which holds if and only if $s^2 = 4p$. Thus the uniqueness set must be contained in

$$\{(s, p) \in \mathbb{G} : s^2 = 4p\} = \{(2z, z^2) : z \in \mathbb{D}\}.$$

Therefore the uniqueness variety must be \mathcal{R} .

Example 4.11. The data $((0, 0), 0), ((0, 1/2), 1/2)$ is extremal and its uniqueness variety is given by $\{(s, p) : s = 0\}$. If f is any solution, then $\tilde{f} : \mathbb{D} \rightarrow \mathbb{D}$ given as $\tilde{f}(z) = f(0, z)$ solves the Pick data $\{(0, 0), (1/2, 1/2)\}$ in \mathbb{D} . Thus by the Schwarz lemma, we have

$$\tilde{f}(z) = f(0, z) = z \text{ for all } z \in \mathbb{D}. \quad (4.22)$$

This shows that the problem is extremal. It is evident from (4.22) that any solution of the given data has to agree on the set $\{(0, z) : z \in \mathbb{D}\} = \{(s, p) \in \mathbb{G} : s = 0\}$. For the other containment, note that both the functions p and $(2p - s)/(2 - s)$ solve the data. They agree at $(s, p) \in \mathbb{G}$ if and only if $s = 0$. This establishes the claim.

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