MA 108 - Ordinary Differential Equations

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Outline of the lecture

1. Basic Concepts
2. Separable DE

Differential equations

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where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Check list : If the dependent variable is y, derivatives occur upto first degree only, no products of y and/or its derivatives are there.

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Linear ODE of first order.

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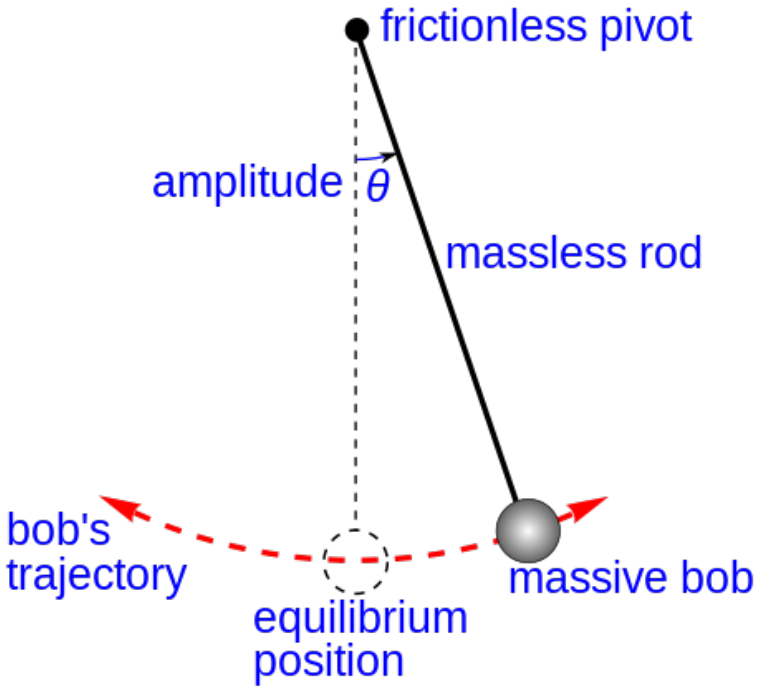

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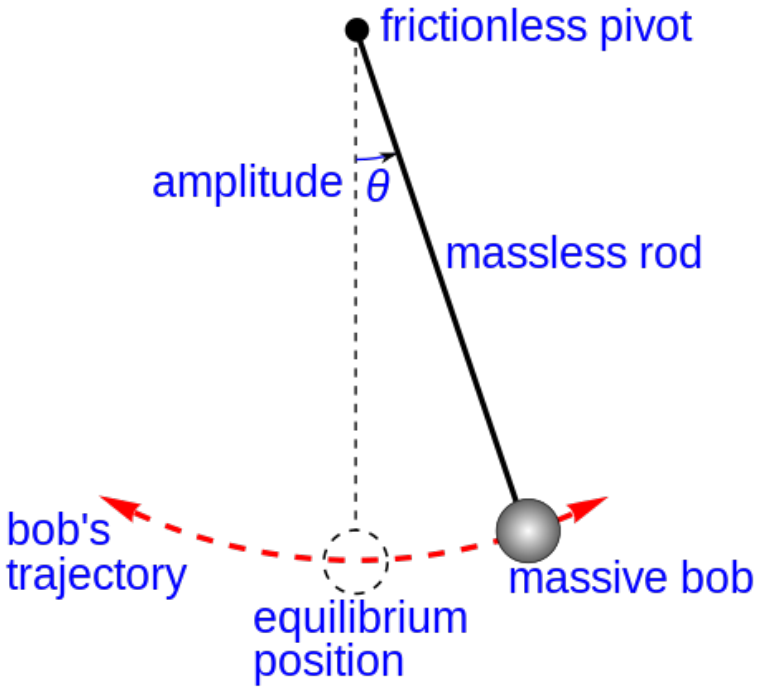

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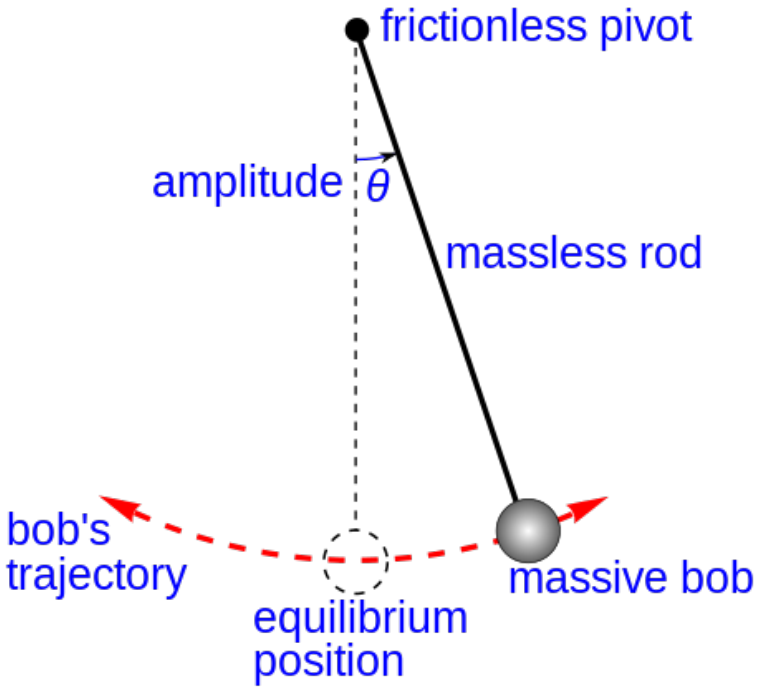

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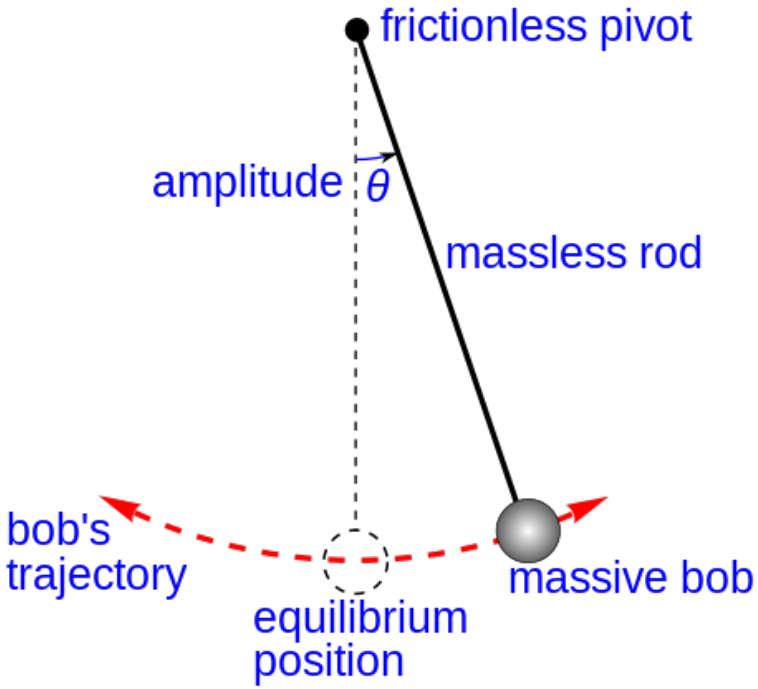

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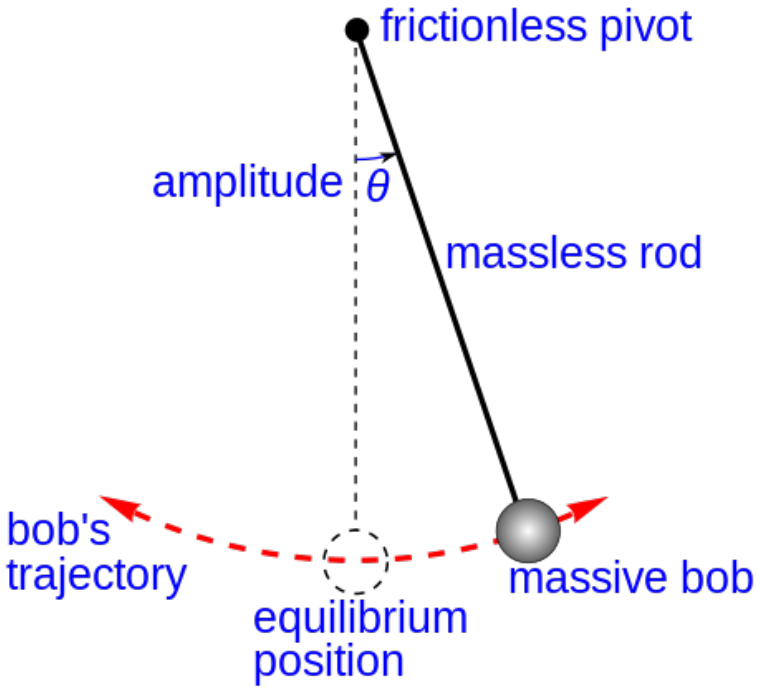

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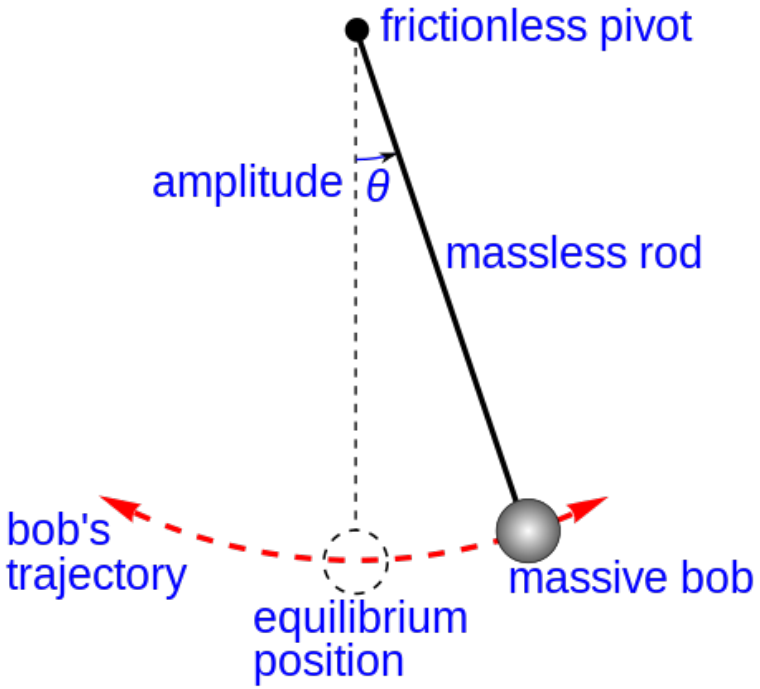

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$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \text{ or } y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

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$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \text{ or } y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

Definition

A explicit solution of the ODE $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ on the interval $\alpha < x < \beta$ is a function $\phi(x)$ such that $\phi', \phi'', \cdots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f(x, \phi, \phi', \cdots, \phi^{(n-1)}),$$

for every x in $\alpha < x < \beta$.

Implicit solution & Formal solution

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A relation $g(x, y) = 0$ is called an implicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ if this relation defines at least one function $\phi(x)$ on an interval $\alpha < x < \beta$, such that, this function is an explicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ in this interval.

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Examples :

1. $x^2 + y^2 - 25 = 0$ is an implicit solution of $x + yy' = 0$ in $-5 < x < 5$, because it defines two functions

$$\phi_1(x) = \sqrt{25 - x^2},\ \phi_2(x) = -\sqrt{25 - x^2}$$

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2. Consider $x^2 + y^2 + 25 = 0 \Longrightarrow x + yy' = 0 \Longrightarrow y' = -\frac{x}{y}$. We say $x^2 + y^2 + 25 = 0$ formally satisfies $x + yy' = 0$. But it is NOT an implicit solution of DE as this relation doesn't yield ϕ which is an explicit solution of the DE on any real interval I.

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Definition

Initial value problem (IVP) : A DE along with an initial condition is an IVP.

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is a particular solution to the above ODE with the given initial condition.

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We first start with a few methods for finding out the solution of first order ODEs, discuss the geometric meaning of solutions and then proceed to study existence-uniqueness results.

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Note: This method many times gives us an implicit solution and not necessarily an explicit one!