MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Cauchy-Euler equations
- Non-homogeneous equations

Cauchy-Euler Equations

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The roots are

$$m_1, m_2 = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

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$$v(x) = \int \frac{e^{-\int \frac{a}{x}dx}}{x^{1-a}}dx = \int \frac{dx}{x} = \ln x$$

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Thus the general solution is given by

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Non-homogeneous Second Order Linear ODE's

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$$y'' + p(x)y' + q(x)y = r(x)$$

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Can we relate the solutions of the above two DE's?

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Summary: In order to find the general solution of a non-homogeneous DE, we need to

- get one particular solution of the non-homogeneous DE
- get the general solution of the corresponding homogeneous DE.