MA 108 - Ordinary Differential Equations

Santanu Dey

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
dey@math.iitb.ac.in

April 7, 2014

Outline of the lecture

- Laplace Transforms
- Examples
- Properties

Examples

Let's now see how to solve IVP's using Laplace transforms.
Solve

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0.$$

Apply Laplace transform through out to obtain :

$$L(y'') - L(y') - 2L(y) = 0.$$

Assume that there is a solution y such that
(i) y'' is piecewise continuous on all $[0, \alpha]$
(ii) y, y' are of exponential order.
Thus,

$$(s^2 L(y) - sy(0) - y'(0)) - (sL(y) - y(0)) - 2L(y) = 0.$$

Thus,

$$(s^2 - s - 2)L(y) - s + 1 = 0.$$

So,

$$L(y)(s) = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

The rhs is

$$\frac{1}{3}L(e^{2t}) + \frac{2}{3}L(e^{-t}).$$

Thus,

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Remark: If you had done this via characteristic polynomial, first you would have got

$$y(t) = c_1 e^{2t} + c_2 e^{-t},$$

and you would evaluate c_1, c_2 from the constraints.

Example

Solve $y'' - 2y' + 5y = 8\sin t - 4\cos t,\ y(0) = 1,\ y'(0) = 3.$
Taking Laplace transforms,

$$L(y'') - 2L(y') + 5L(y) = \frac{8}{s^2+1} - \frac{4s}{s^2+1}$$

$$(s^2L(y) - sy(0) - y'(0)) - 2(sL(y) - y(0)) + 5L(y) = \frac{8-4s}{s^2+1}.$$

Using the initial conditions,

$$L(y)(s^2-2s+5)-s-3+2 = \frac{8-4s}{s^2+1} \Longrightarrow L(y)(s^2-2s+5) = \frac{4(2-s)}{s^2+1}+s+1$$

This yields

$$L(y) = \frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}.$$

That is, $y = L^{-1}\left(\frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}\right).$

Example contd..

$$\frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C(s-1)+D}{(s-1)^2+2^2}$$

That is,

$$(As+B)(s^2-2s+5)+(C(s-1)+D)(s^2+1) = s^3+s^2-3s+9.$$

This yields, $A=0,\ B=2,\ C=1,\ D=0.$
Hence,

$$y = L^{-1}\left(\frac{2}{s^2+1}\right) + L^{-1}\left(\frac{s-1}{(s-1)^2+2^2}\right) = 2\sin t + e^t \cos 2t.$$

Property 5: Integration

Let f be piecewise continuous and suppose there exist K, $t_0 > 0$ and $\alpha \in \mathbb{R}$ such that

$$|f(t)| \leq Ke^{\alpha t},$$

for $t \geq t_0 > 0$. Also, let $L(f) = F(s)$. Then,

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{F(s)}{s}, \qquad \text{for } s > \alpha.$$

Proof: We need to show that

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}L(f),$$

for $s > \alpha$, where $|f(t)| \leq Ke^{\alpha t}$ for $t \geq t_0 > 0$. We may assume that $\alpha > 0$. Set

$$g(t) = \int_0^t f(\tau) d\tau.$$

Then,

$$g'(t) = f(t)$$

except at the points of discontinuities of $f(t)$.

Hence $g'(t)$ is piecewise continuous. Also $g(t)$ is of exponential order. Hence,

$$L(f) = L(g') = sL(g) - g(0) = sL(g),$$

for $s > \alpha$. Thus,

$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}L(f).$$

Example

If $L(f) = \dfrac{1}{s(s^2+\omega^2)}$, find $f(t)$.

Solution:

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f).$$

We know,

$$L\left(\frac{\sin\omega\tau}{\omega}\right) = \frac{1}{s^2+\omega^2}.$$

$$\begin{aligned} f(t) &= L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = L^{-1}\left(\frac{1}{s}\,\frac{1}{s^2+\omega^2}\right) \\ &= \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau = -\frac{\cos\omega\tau}{\omega^2}\bigg|_0^t = \frac{1}{\omega^2}(1-\cos\omega t). \end{aligned}$$

$$L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) = L^{-1}\left(\frac{1}{s}\,\frac{1}{s(s^2+\omega^2)}\right)$$

$$= \int_0^t \frac{1-\cos\omega\tau}{\omega^2}\,d\tau = \frac{1}{\omega^2}\left(\tau - \frac{\sin\omega\tau}{\omega}\right)_0^t = \frac{1}{\omega^2}\left(t - \frac{\sin\omega t}{\omega}\right).$$

Property 6 : Differentiation of Laplace transforms

Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of exponential order. Then,

$$F'(s) = -L(tf(t)).$$

Also,

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Then,

$$\begin{aligned} \frac{dF(s)}{ds} &= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t) dt \\ &= \int_0^\infty -te^{-st} f(t) dt \\ &= -L(tf(t)). \end{aligned}$$

Exercise : Prove for nth derivative.

Examples

We know that

$$L(\cos \beta t) = \frac{s}{s^2 + \beta^2}, \; L(\sin \beta t) = \frac{\beta}{s^2 + \beta^2}.$$

Therefore, using

$$F'(s) = -L(tf(t)),$$

$$L(t \cos \beta t) = -\frac{d}{ds}\left(\frac{s}{s^2 + \beta^2}\right) = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2},$$

and

$$L(t \sin \beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2 + \beta^2}\right) = \frac{2s\beta}{(s^2 + \beta^2)^2}.$$

Inverse Laplace Transforms - Examples

$$L(t\sin\beta t) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}.$$

Thus,

$$L^{-1}\left(\frac{s}{(s^2+\beta^2)^2}\right) = \frac{t\sin\beta t}{2\beta}.$$

Thus, from Property 5 $\left(L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},\ \text{for } s > \alpha\right)$

$$L^{-1}\left(\frac{1}{(s^2+\beta^2)^2}\right) = \int_0^t \frac{\tau\sin\beta\tau}{2\beta}d\tau,$$

and from Property 4 $\left(L(f') = sL(f) - f(0)\right)$,

$$\frac{d}{dt}\left(\frac{t\sin\beta t}{2\beta}\right) = L^{-1}\left(\frac{s^2}{(s^2+\beta^2)^2}\right).$$